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"Dynamical" interactions and gauge invariance

R. Saar,[1] S. Groote,[1,2] I. Ots,[1,*] and H. Liivat[1]

[1]*Loodus-ja Tehnoloogiateaduskond, Füüsika Instituut, Tartu Ülikool, Riia 142, 51014 Tartu, Estonia*
[2]*Institut für Physik der Johannes-Gutenberg-Universität, Staudinger Weg 7, 55099 Mainz, Germany*
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In order to avoid known, long-standing problems with higher-spin interactions, the electromagnetic field is introduced "dynamically" by using a nonsingular, Lorentz-type transformation, acting as adjoint representation on the Poincaré algebra of the free theory. In doing so, Lorentz transformation and local gauge transformation are placed on the same foundations, leading to the phase transition as a consequence of the gauge transformation. The procedure is exemplified in the case of plane waves for the Dirac-type equation and the Rarita-Schwinger equation.

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I. INTRODUCTION

Understanding the higher-spin interactions[1] is a long-standing problem. However, in spite of its 70 year history, the main goal—the construction of a consistent higher-spin theory, even for the electromagnetic interaction which ought to be the simplest case—has not been achieved yet.

The theory of higher-spin interactions has never belonged to the "mainstream" theories. The field has been cultivated by groups of enthusiasts. On the other hand, the theory of higher-spin interactions is needed for solving many mainstream problems. It is related to the standard model (SM) in several ways. By introducing the massive spin-one gauge bosons into the theory one also introduces higher-spin problems into the standard model. Difficulties appear for instance in scattering processes with the charged gauge bosons $W^{\pm}$ in the initial or final state, or in constructing three-vertex gauge boson self-interactions. A consistent higher-spin interaction theory is also needed in chromodynamics. Quantum chromodynamics (QCD) does not yet allow one to describe low-energy hadronic processes in terms of underlying quark-gluon dynamics. Because of this one has to use a more phenomenological approach in terms of hadronic fields. However, one of the basic problems here is the treatment of hadrons with higher spins [2].

Also for theories beyond the SM one needs a better understanding of ordinary higher-spin field theory. String theory for instance is free of many higher-spin problems and due to this it is believed that it can consistently describe quantum gravity. A reason behind this consistent behavior is that string theories contain an infinite tower of all spin states. But at the same time serious troubles exist in the physical interpretation of the string theories. The existence of a consistent higher-spin interaction theory would help in better understanding the physics behind the string theory. It is believed that if a breakthrough in understanding the basic problems of the ordinary higher-spin field theory happens, it might become a fashionable topic [3].

The investigations of higher-spin fields started in the 1930s of the last century with papers by Dirac [4], Wigner [5], Fierz and Pauli [6], and were followed by the works of Rarita and Schwinger [7], Bargmann and Wigner [8], and others [9–15]. The difficulties in higher-spin physics revealed themselves when one tried to couple higher-spin fields to an electromagnetic field. In the 1960s concrete defects of the higher-spin interaction theory were found. Johnson and Sudarshan [16] and Schwinger [17] demonstrated that in the case of minimal electromagnetic coupling some of the anticommutation relations become indefinite. It appeared that the defects were also present on the classical level. Velo and Zwanziger [18] and Shamaly and Capri [19] showed that in an external electromagnetic field there appeared acausal (superluminal) modes of propagation.[2] Afterward other defects—bad high-energy behavior of the amplitudes, various algebraic problems etc.—were found. Since the 1960s of the last century much work was done to solve the problems, but no result which one can call a breakthrough was obtained in the framework of ordinary field theory. In the case of higher-spin electromagnetic interactions investigations of the last two decades have moved in two directions. One part of the community develops the theory on the ground of

[*]Deceased.

[1]In this paper states with spin one and higher are considered as higher-spin states. This concept is not universally accepted. For part of the investigators "higher spin" means $s \geq 3/2$. The specialists in supergravity updated the convention of the higher spin to be even $s \geq 5/2$ [1]. Nevertheless, at least in the standard model the troubles start already from the value $s = 1$. Therefore it seems that the convention $s \geq 1$ as the higher-spin region is more justified than the other ones.

[2]Earnestly, as shown by Cox [20] the constraint analysis leading to these acausal pathologies is incomplete. On the contrary, in the complete constraint analysis a new tier of constraints occurs for the critical external field values, reducing the pathology to the field-induced change of the degrees of freedom. Because of these field-dependent constraints the analysis of acausal models is very complicated.

the minimal electromagnetic coupling, and the other part searches for a consistent theory by using nonminimal couplings.

A. Problems of higher-spin interactions

Difficulties in higher-spin physics are generic to all field theoretical descriptions of relativistic higher-spin particles. They are related to the fact that the covariant higher-spin field has more components than necessary to describe the spin degrees of freedom of the physical particle. To get rid of redundant degrees of freedom one must set up constraints between the field components. If the interactions are introduced consistently with the free-field theory, the number of independent field components remains unchanged. Otherwise the free theory constraints may be violated and unphysical degrees of freedom become involved. The possibility to construct consistent higher-spin theories with gauge invariant couplings was first pointed out by Weinberg and Witten [21]. However, the realization of this scenario is beset with difficulties. Even though certain progress in understanding of a higher-spin interaction theory has been made [3,22], up to now no general prescription for the construction of a consistent higher-spin field theory for any spin has been found.

In order to put constraints on the field components of the free theory it is reasonable to use symmetry principles. The interacting theory then has to obey similar symmetry requirements as the corresponding free theory or, even better, preserves the symmetries of the free theory. Space-time properties of a system under consideration are due to symmetries under the Poincaré group. In fact, the very definition and characterization of distinct species of elementary particles are provided by the set of inequivalent irreducible projective unitary representations of the space-time symmetry group $\mathcal{P}_{1,3}$, the Poincaré group. According to conventional understanding of a particle, its physical states of definite mass and spin, labeled by the moment p^μ and the helicity λ, arise from the irreducible representation of this symmetry group. The irreducible unitary representations of the Poincaré group are characterized by the eigenvalues of the two Casimir operators P^2 and W^2 of the Lie algebra $p_{1,3}$,

$$P^2|m, s\rangle = m^2|m, s\rangle, \qquad W^2|m, s\rangle = -m^2 s(s+1)|m, s\rangle. \tag{1}$$

The independent components of the Pauli-Lubanski pseudovector W^μ with $[P_\mu, W_\nu] = 0$ form the Lie algebra of the little group of fixed momentum p^μ. For every irreducible unitary representation of the little group one can derive a corresponding irreducible induced representation of the Poincaré group labeled by (m, s), i.e. by the eigenvalues of the Casimir operators in Eq. (1). Notice that the procedure of deriving induced representations [23,24] corresponds very well to the physical idea of first determining the internal degrees of freedom (the helicity) of the system and then all its possible states of motion. In this sense the natural identification of elementary particle systems is the direct geometric transition from space-time to the system under consideration.

The identification of elementary particle systems and irreducible representations of the Poincaré group finds its physical limitations in the description of interacting systems and internal quantum numbers of composite systems. Since gauge symmetry is a fundamental concept in quantum electrodynamics, all physical quantities and dynamical equations of particles have to be gauge invariant. However, if gauge invariance is realized by minimal coupling, Poincaré invariance is violated at least for the theory of higher-spin fields ($s \geq 1$) in its realization as first-order equations. The deficits occur both on the classical level (acausality and algebraic inconsistency) as well as on the quantum level (indefiniteness of antimutation relations). A lot of work has been done to solve these problems, and a consistent model for the spin-$3/2$ field in its realization as second order equation (projector formalism) is proposed by Napsuciale *et al.* in Ref. [25].

In summarizing, one can conclude that if the problem is investigated by group theoretical methods of space-time symmetries of interacting systems, symmetries of interacting systems lead to the general covariance group in case of a charged particle moving in an external electromagnetic field. As a consequence, the group theoretical definition of an elementary particle can be extended to the case where an external field is present. Even though the Poincaré group is not a subgroup of the general covariance group [26,27], this point of view is of help to solve the problem.

In this paper we use a higher-spin electromagnetic interaction theory developed by us earlier, based on the "dynamical" representation of the Poincaré algebra as a dynamical principle which leads to a nonminimal coupling. The representations are constructed from the generators of the free Poincaré algebra and the external field in such a way that the new, field-dependent generators obey the commutation relations of free Poincaré algebra. Introducing the interactions in this way preserves the Poincaré symmetry of the free theory and, hopefully, also the number of degrees of freedom of the free theory. The dynamical theory has achieved success in constructing causal spin-$3/2$ equations [28] and for justifying the value of gyromagnetic ratio $g = 2$ for any spin [29].

The paper is organized as follows. In Sec. II we set up our conventions related to the Poincaré group $\mathcal{P}_{1,3}$. In Sec. III we explain how the electromagnetic field A can be introduced by using a nonsingular transformation $\mathcal{V}(A)$, specified by Lorentz-type gauge invariance. Realizations are shown for the particular example of plane waves, leading to local phase transformation via $e^{iq\lambda}$. In Sec. IV we treat the Rarita-Schwinger equation. Section V contains our conclusions and an outlook on future work.

II. THE POINCARÉ GROUP

Relativistic field theories are based on the invariance under the Poincaré group $\mathcal{P}_{1,3}$ (known also as the inhomogeneous Lorentz group $I\mathcal{L}$ [5,23,30–37]). This group is obtained by combining Lorentz transformations Λ and space-time translations a_T,

$$(a, \Lambda) \equiv a_T\Lambda: \mathbb{E}_{1,3} \ni x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{2}$$

The group's composition law $(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2)$ generates the semidirect structure of $\mathcal{P}_{1,3}$,

$$\mathcal{P}_{1,3} = \mathcal{T}_{1,3} \odot \mathcal{L},$$

where $\mathcal{T}_{1,3}$ is the Abelian group of space-time translations (i.e. the additive group $\mathbb{R}^4$) and $\mathcal{L} = \{\Lambda: \det\Lambda = +1, \Lambda^0{}_0 \geq 1\}$ is the proper orthochronous Lorentz group[3] acting on the Minkowski space $\mathbb{E}_{1,3}$ with metric

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1).$$

The condition of the metric to be invariant under Lorentz transformations Λ takes the form

$$\Lambda^\mu{}_\rho \eta_{\mu\sigma} \Lambda^\sigma{}_\nu = \eta_{\rho\nu}. \tag{3}$$

In order to set up the conventions used in this paper, in the following we deal with the properties of representations of the Lorentz group in more detail.

A. Transformation of covariant functions

Under the Lorentz transformation $\Lambda \in \mathcal{L}$ the covariant functions ψ transform according to a representation $\tau(\Lambda)$ of the Lorentz group [5,8,12–15,23,30–33] where the diagram

$$\begin{array}{lccc} \psi: & x \in \mathbb{E}_{1,3} & \longrightarrow & \psi(x) \\ \tau(\Lambda)\downarrow & \downarrow\Lambda & & \downarrow T(\Lambda) \\ \tau(\Lambda)\psi: & \Lambda x & \longrightarrow & T(\Lambda)\psi(x) \end{array}$$

is commutative, i.e.

$$\psi^\Lambda(\Lambda x) \equiv (\tau(\Lambda)\psi)(\Lambda x) = T(\Lambda)\psi(x). \tag{4}$$

The map $T: \Lambda \to T(\Lambda)$ is a finite-dimensional representation of $\mathcal{L}$. If we parametrize the element $\Lambda \in \mathcal{L}$ by $\Lambda(\omega) = \exp(-\frac{1}{2}\omega_{\mu\nu}e^{\mu\nu})$ where the Lorentz generators are given by

$$(e_{\mu\nu})^\rho{}_\sigma = -\eta_\mu{}^\rho \eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_\nu{}^\rho$$

and $\omega^{\mu\nu} = -\omega^{\nu\mu}$ are six independent parameters, the parametrization of T reads

$$T(\Lambda(\omega)) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}s^{\mu\nu}\right).$$

The Lorentz group $\mathcal{L}$ is noncompact. As a consequence, all unitary representations are infinite dimensional. In order to avoid this, we introduce the concept of H unitarity (see e.g. Ref. [23] and references therein). A finite representation T is called H unitary if there exists a nonsingular Hermitian matrix $H = H^\dagger$ so that

$$T^\dagger(\Lambda)H = HT^{-1}(\Lambda) \Leftrightarrow s^\dagger_{\mu\nu}H = Hs_{\mu\nu}. \tag{5}$$

Notice that a H-unitary metric is always indefinite, so that the inner product $\langle , \rangle$ generated by H is sesquilinear sharing the Hermiticity condition $\langle\psi, \varphi\rangle = \langle\varphi, \psi\rangle^*$. The most famous case of H unitarity is given in the Dirac theory of spin-1/2 particles where $H = \gamma^0$.

B. Transformation of operators

The transformation $\tau(\Lambda)$ in Eq. (4) is a covariant transformation for the operator $\mathcal{O}$ [26,38] acting on the ψ space of covariant functions[4] if the diagramm

$$\begin{array}{lccc} \mathcal{O}\psi: & x & \longrightarrow & (\mathcal{O}\psi)(x) \\ \tau(\Lambda)\downarrow & \downarrow\Lambda & & \downarrow T(\Lambda) \\ \tau(\Lambda)(\mathcal{O}\psi): & \Lambda x & \longrightarrow & T(\Lambda)(\mathcal{O}\psi)(x) \end{array}$$

is commutative, i.e.

$$(\tau(\Lambda)\mathcal{O}\tau^{-1}(\Lambda))(\Lambda x)(\tau(\Lambda)\psi)(\Lambda x) = T(\Lambda)\mathcal{O}(x)\psi(x). \tag{6}$$

Using Eq. (4) we obtain

$$(\tau(\Lambda)\mathcal{O}\tau^{-1}(\Lambda))(\Lambda x)T(\Lambda)\psi(x) = T(\Lambda)\mathcal{O}(x)\psi(x).$$

Notice that the covariance of the transformation embodies only the property of equivalence of reference systems. The covariant operator $\mathcal{O}$ is invariant under the transformation (4) if in addition $\mathcal{O}\tau(\Lambda) = \tau(\Lambda)\mathcal{O}$. As a consequence we obtain the commutative diagram

$$\begin{array}{lccc} \mathcal{O}\psi: & x & \longrightarrow & (\mathcal{O}\psi)(x) \\ \tau(\Lambda)\downarrow & \downarrow\Lambda & & \downarrow T(\Lambda) \\ \mathcal{O}(\tau(\Lambda)\psi): & \Lambda x & \longrightarrow & T(\Lambda)(\mathcal{O}\psi)(x) \end{array}$$

or

$$\mathcal{O}(\Lambda x)T(\Lambda)\psi(x) = T(\Lambda)\mathcal{O}(x)\psi(x) \tag{7}$$

which means

$$\mathcal{O}(\Lambda x)T(\Lambda) = T(\Lambda)\mathcal{O}(x) \tag{8}$$

[3] In order to simplify the notation, in the following we refer to the proper orthochronous Lorentz group and proper orthochronous Lorentz transformation as Lorentz group and Lorentz transformation, respectively.

[4] We have to impose the action on covariant functions because in the case of higher spins the relations between operators we obtain are valid only as weak conditions.

on the ψ space. The invariance is a symmetry of the physical system and implies the conservation of currents. In particular, the symmetry transformations leave the equations of motion form invariant.

C. The Lie algebra

While the Lorentz transformation $T(\Lambda)$ changes the wave function ψ itself as well as the argument of this function [cf. Eq. (4)], the proper Lorentz transformation $\tau(\Lambda)$ causes a change of the wave function only. On the ground of infinitesimal transformations, this change is performed by a substantial variation. Starting from an arbitrary infinitesimal coordinate transformation $\Lambda(\delta\omega)$: $x^\mu \to x^\mu + \delta\omega^{\mu\nu}x_\nu$, the substantial variation is given by Ref. [12]

$$\delta_0\psi(x) \equiv \psi^\Lambda(x) - \psi(x) = -\frac{i}{2}\delta\omega^{\rho\sigma}M_{\rho\sigma}\psi(x),$$

where $M_{\rho\sigma} = \ell_{\rho\sigma} + s_{\rho\sigma}$, and $\ell_{\rho\sigma} = i(x_\rho\partial_\sigma - x_\sigma\partial_\rho)$. The corresponding finite proper Lorentz transformation can be written as

$$\tau(\Lambda(\omega)) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right),$$

and the multiplicative structure of the group generates the adjoint action

$$\mathrm{Ad}_{\tau(\Lambda)}\colon M_{\mu\nu} \to \tau^{-1}(\Lambda)M_{\mu\nu}\tau(\Lambda) = \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma M_{\rho\sigma}. \quad (9)$$

Because of Eq. (5) the generators $s_{\rho\sigma}$ fulfill $s^\dagger_{\rho\sigma}H = Hs_{\rho\sigma}$. They depend on the spin of the field but not on the coordinates x_μ. Therefore, we have $[\ell_{\mu\nu}, s_{\rho\sigma}] = 0$. If a generic element of the translation group is written as

$$\exp(+ia_\mu P^\mu),$$

the commutator relations of the Lie algebra are given by

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}),\\
[M_{\mu\nu}, P_\rho] &= i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu), \qquad [P_\mu, P_\nu] = 0.
\end{aligned} \quad (10)$$

The Casimir operators of the algebra are $P^2 = P_\mu P^\mu$ and $W^2 = W_\mu W^\mu$, where

$$W^\mu = +\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}M_{\nu\rho}P_\sigma$$

is the Pauli-Lubanski pseudovector, $[P_\mu, W_\nu] = 0$. In coordinate representation we have $P_\mu = i\partial_\mu$, and the finite Poincaré transformation has the form

$$\tau(a,\Lambda)\colon \psi(x) \to (\tau(a,\Lambda)\psi)(x) = T(\Lambda)\psi(\Lambda^{-1}(x-a)). \quad (11)$$

This relation constitutes the Lorentz-Poincaré connection [35]. While the representation T generally generates a reducible representation of $\mathcal{P}_{1,3}$, the spectra of the Casimir operators P^2 and W^2 determine the mass and spin content of the system.

D. Linear wave equation

As a rule the Lorentz-Poincaré connection is realized by the relativistic wave equations. If the relativistic wave equation transforms as a finite-dimensional representation of the Lorentz group by Eq. (4), it contains spins exceeding the desired physical spins. In order that the solutions of the field equation correspond to a particle with a definite spin, the equation must act like a projection operator to pick out the desired spin components, i.e. to select the corresponding irreducible representation of the Poincaré group.

The wave equation we consider has the form

$$\mathcal{D}(\partial)\psi(x) \equiv (i\gamma^\mu\partial_\mu - \rho)\psi(x) = 0, \quad (12)$$

where ψ is an N-component function, γ^μ ($\mu = 0, 1, 2, 3$), and ρ are $N \times N$ matrices independent of x. Following Bhabha's conception [9], it is "... logical to assume that the fundamental equations of the elementary particles must be first-order equations of the form (12) and that all properties of the particles must be derivable from these without the use of any further subsidiary conditions."[5] Therefore, different from Napsuciale's approach [25] we start from first order differential equations.

The principle of relativity states that a change of the reference frame cannot have implications for the motion of the system. This means that Eq. (12) is invariant under Lorentz transformations. Equivalently, the Lorentz symmetry of the system means the covariance and form invariance of Eq. (12) under the transformation in Eq. (4), i.e. the transformed wave equation is equivalent to the old one. Therefore, we require that every solution $\psi^\Lambda(\Lambda x)$ of the transformed equation $\mathcal{D}^\Lambda(\Lambda\partial)\psi^\Lambda(\Lambda x) = 0$ can be obtained as Lorentz transformation of the solution $\psi(x)$ of Eq. (12) in the original system and that the solutions in the original and transformed systems are in one-to-one correspondence. The explicit form of the covariance follows from Eq. (6),

$$\begin{aligned}
\mathcal{D}^\Lambda(\Lambda\partial)\psi^\Lambda(\Lambda x) &= (\tau(\Lambda)\mathcal{D}\tau^{-1}(\Lambda))(\Lambda\partial)(\tau(\Lambda)\psi)(\Lambda x)\\
&= T(\Lambda)\mathcal{D}(\partial)\psi(x) = 0,
\end{aligned} \quad (13)$$

leading to the explicit Lorentz transformations

[5] In order to avoid confusion, we have to emphazise that in citing Bhabha [9] we do not imply Eq. (12) to be the Bhabha equation. Instead, the equation is a generalization of the Dirac equation, obeying the Lorentz conditions explained in the following.

$$\gamma'^{\mu} = \Lambda^{\mu}{}_{\rho} T(\Lambda)\gamma^{\rho} T^{-1}(\Lambda), \qquad \rho' = T(\Lambda)\rho T^{-1}(\Lambda).$$

The Lorentz invariance is given by the substitution

$$\mathcal{D}(\partial)\psi(x) = 0 \xrightarrow{\text{Eq. (4)}} \mathcal{D}(\partial)\psi^{\Lambda}(x) = 0$$

or

$$T^{-1}(\Lambda)\gamma^{\mu} T(\Lambda) = \Lambda^{\mu}{}_{\rho}\gamma^{\rho}, \qquad T^{-1}(\Lambda)\rho T(\Lambda) = \rho.$$

The difference of the original and transformed wave equation is given by the wave equation where the wave function ψ is replaced by the substantial variation $\delta_0\psi$, $\mathcal{D}(\partial)\delta_0\psi(x) = 0$. As a consequence we obtain $[\mathcal{D}, M^{\rho\sigma}] = 0$ or

$$[\gamma^{\mu}, s^{\rho\sigma}] = i(\eta^{\mu\rho}\gamma^{\sigma} - \eta^{\mu\sigma}\gamma^{\rho}), \qquad [\rho, s^{\rho\sigma}] = 0. \tag{14}$$

An excellent discussion of such matrices γ can be found in Refs. [9,10,12,39–42]. The Hermiticity of the representation T in Eq. (5) implies the Hermiticity of Eq. (12). Including a still unspecified Hermitian matrix H the Hermiticity condition reads $\mathcal{D}(\partial)^{\dagger} H \overset{!}{=} (\mathcal{D}(\partial)H)^{\dagger} = -H\mathcal{D}(-\partial)$ or

$$\gamma^{\mu\dagger} H = H\gamma^{\mu}, \qquad \rho H = H\rho. \tag{15}$$

Writing $\bar{\psi} = \psi^{\dagger} H$, one obains the adjoint equation

$$\bar{\psi}\mathcal{D}(-\overleftarrow{\partial}) = \bar{\psi}(-i\gamma^{\mu}\overleftarrow{\partial}_{\mu} - \rho) = -(H\mathcal{D}(\partial)\psi)^{\dagger} = 0. \tag{16}$$

III. INTRODUCTION OF THE EXTERNAL FIELD

Because of the arguments given earlier, it may be reasonable to introduce the external field directly into the Poincaré algebra. To do so one has to transform the generators of the Poincaré group to be dependent on the external field in such a way that the new, field-dependent generators obey the commutation relations (10). As proposed by Chakrabarti [43] and Beers and Nickle [44], the simplest way to build such a field-dependent algebra is to introduce the external field by a nonsingular transformation

$$\begin{aligned}\mathrm{Ad}_{\mathcal{V}(A)}\colon p_{1,3} \rightarrow p^{d}_{1,3}(A) &= \mathcal{V}(A) p_{1,3} \mathcal{V}^{-1}(A)\\ &= p_{1,3} + [\mathcal{V}(A), p_{1,3}]\mathcal{V}^{-1}(A).\end{aligned} \tag{17}$$

More explicitly, the transformed operators

$$\begin{aligned}\Pi_{\mu}(A) &= P^{\mu} + [\mathcal{V}(A), P_{\mu}]\mathcal{V}^{-1}(A),\\ \xi_{\mu}(A) &= x^{\mu} + [\mathcal{V}(A), x_{\mu}]\mathcal{V}^{-1}(A),\\ \sigma_{\mu\nu}(A) &= s_{\mu\nu} + [\mathcal{V}(A), s_{\mu\nu}]\mathcal{V}^{-1}(A),\\ \mu_{\mu\nu}(A) &= \xi_{\mu}(A)\Pi_{\nu}(A) - \xi_{\nu}(A)\Pi_{\mu}(A) + \sigma_{\mu\nu}(A)\end{aligned} \tag{18}$$

must satisfy the commutation relations of the Poincaré algebra. In the case of a particular external electromagnetic field A, the external field can be introduced by using an evolution operator $\mathcal{V}(A)$, called the dynamical representation [28,29]. By analogy with the free particle case one can realize this representation on the solution space of relativistically invariant equations. Expressing the operators explicitly in terms of free-field operators, one obtains the dynamical interaction. Applying for instance the operator $\mathcal{V}(A)$ to Eq. (12) one obtains

$$\mathcal{V}(A)\colon \mathcal{D}(\partial)\psi(x) = 0 \rightarrow \mathcal{D}^{d}(\partial, A)\Psi(x, A) = 0, \tag{19}$$

where $\mathcal{D}^{d}(\partial, A) = \mathcal{V}(A)\mathcal{D}(\partial)\mathcal{V}^{-1}(A)$ and

$$\Psi(x, A) = \mathcal{V}(A)\psi(x) \tag{20}$$

(here and in the following we skip the argument x for Ψ and the argument ∂ for $\mathcal{D}^{d}$). Having introduced the external gauge field A, we introduce gauge covariance on the same foundation as Lorentz covariance in Eq. (4), i.e. by claiming that the diagram

$$\begin{array}{cccc}\Psi\colon & A & \longrightarrow & \Psi(A)\\ g(\lambda)\downarrow & \downarrow\lambda & & \downarrow G(\lambda)\\ \Psi^{\lambda}\colon & A^{\lambda} = A + \partial\lambda & \longrightarrow & G(\lambda)\Psi(A)\end{array}$$

is commutative,

$$\Psi^{\lambda}(A + \partial\lambda) = G(\lambda)\Psi(A). \tag{21}$$

According to Eq. (7), the dynamical interaction $\mathcal{D}^{d}$ is gauge invariant under the gauge transformation $A \rightarrow A^{\lambda} \equiv A + \partial\lambda$ if the diagram

$$\begin{array}{cccc}\mathcal{D}^{d}\Psi\colon & A & \longrightarrow & \mathcal{D}^{d}(A)\Psi(A)\\ \downarrow & \downarrow\lambda & & \downarrow G(\lambda)\\ \mathcal{D}^{d}\Psi^{\lambda}\colon & A + \partial\lambda & \longrightarrow & G(\lambda)\mathcal{D}^{d}(A)\Psi(A)\end{array}$$

is commutative, i.e.

$$\mathcal{D}^{d}(A + \partial\lambda)\Psi^{\lambda}(A + \partial\lambda) = G(\lambda)\mathcal{D}^{d}(A)\Psi(A). \tag{22}$$

Together with Eq. (21) we obtain $\mathcal{D}^{d}(A + \partial\lambda)G(\lambda)\Psi(A) = G(\lambda)\mathcal{D}^{d}(A)\Psi(A)$ or

$$\mathcal{D}^{d}(A + \partial\lambda)G(\lambda) = G(\lambda)\mathcal{D}^{d}(A) \tag{23}$$

on the ψ space. Note that up to now we have not specified the explicit shape of the finite-dimensional representation $G\colon \lambda \rightarrow G(\lambda)$ of the gauge group.

A. Specifying $\mathcal{V}(A)$ by gauge invariance

At this point we specify $\mathcal{V}(A)$ by two claims [45]. Because of gauge symmetry as a fundamental principle the dynamical transformation $\mathcal{V}(A)$ has to be compatible with the gauge transformation. Therefore, we first claim the gauge invariance in Eq. (23) not only for the operator $\mathcal{D}^{d}$ but for the whole dynamical Poincaré algebra $p^{d}_{1,3}(A)$,

$$p^{d}_{1,3}(A + \partial\lambda)G(\lambda) = G(\lambda)p^{d}_{1,3}(A). \tag{24}$$

By using Eq. (17) and multiplying by $G(\lambda)^{-1}$ from the right we obtain

$$\mathcal{V}(A+\partial\lambda)p_{1,3}\mathcal{V}^{-1}(A+\partial\lambda) = G(\lambda)\mathcal{V}(A)p_{1,3}(G(\lambda)\mathcal{V}(A))^{-1}. \quad (25)$$

This means that the first claim is fulfilled if $\mathcal{V}(A+\partial\lambda) = G(\lambda)\mathcal{V}(A)$. On the other hand, with Eqs. (20) and (21) we obtain

$$\mathcal{V}^{\lambda}(A+\partial\lambda)\psi(x) = G(\lambda)\mathcal{V}(A)\psi(x) \quad (26)$$

and, therefore, $\mathcal{V}^{\lambda} = \mathcal{V}$ on the ψ space. To summarize, by the first claim the gauge symmetry determines the gauge properties of $\mathcal{V}(A)$ and, therefore, the gauge properties of the interacting field $\Psi(A)$,

$$\Psi(A) \to \Psi(A+\partial\lambda) = G(\lambda)\Psi(A). \quad (27)$$

The second claim is that the dynamical transformation operator $\mathcal{V}(A)$ should be of a Lorentz type, i.e. for the generators $s_{\mu\nu}$ of the Poincaré algebra $p_{1,3}$ one has

$$\mathcal{V}(A)s^{\mu\nu}\mathcal{V}^{-1}(A) = V^{\mu}{}_{\rho}(A)V^{\nu}{}_{\sigma}(A)s^{\rho\sigma} \quad (28)$$

which is a local extension of Eq. (9). $V_{\mu\nu}(A) = V_{\mu\nu}(x, A)$ is the local Lorentz transformation generated by the external field A and obeying

$$V_{\mu\rho}(A)V^{\mu}{}_{\sigma}(A) = V_{\rho\mu}(A)V_{\sigma}{}^{\mu}(A) = \eta_{\rho\sigma}. \quad (29)$$

If such a local Lorentz transformation exists, the problem is solved.

B. Solution for plane waves

It is not easy to construct the Lorentz transformation $V_{\mu\nu}(A)$ in general. In a sequel of this paper we deal with this problem in detail. In order to show that such a solution exists, in the following we give an example for an explicit realization of the local Lorentz transformation $V_{\mu\nu}(A)$. As first shown by Taub [46], in the case of a plane-wave field we obtain

$$V_{\mu\nu}(A) = \eta_{\mu\nu} - \frac{q}{k_P}(k_\mu A_\nu - k_\nu A_\mu) - \frac{q^2}{2k_P^2}A^2 k_\mu k_\nu, \quad (30)$$

where q is the electric charge of the particle.[6] The plane wave is characterized by its lightlike propagation vector k_μ, $k^2 = 0$, and its polarization vector a^μ such that

$$a^2 = -1, \qquad ka = 0. \quad (31)$$

The operator $k_P \equiv k_\mu P^\mu$ commutes with any other operator and has a special role in the theory. For particles with nonzero mass one has $k_\mu P^\mu \neq 0$. Therefore, for the plane wave the operator $1/k_P$ is well defined for the plane-wave solution ψ_P of the Klein-Gordon equation. In all other cases, $1/k_P$ is assumed to exist (for a further discussion see the Appendix).

We write $A_\mu(\xi) = a_\mu f(\xi)$, where the variable $\xi = k_\mu x^\mu$ can be used in place of the proper time. From Eq. (31) one obtains the conditions

$$\partial_\mu A^\mu = k_\mu \frac{dA^\mu(\xi)}{d\xi} = k_\mu A^{\mu\prime}(\xi) = 0, \qquad k_\mu A^\mu = 0, \quad (32)$$

where we used $A'_\mu(\xi) = dA_\mu(\xi)/d\xi$, while the field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = k_\mu A'_\nu(\xi) - k_\nu A'_\mu(\xi) = F_{\mu\nu}(\xi)$$

satisfies

$$\partial_\mu F^{\mu\nu} = k_\mu F^{\mu\nu}(\xi) = 0, \qquad F_{\mu\nu}F^{\nu}{}_{\rho} = -k_\mu k_\rho (A'(\xi))^2. \quad (33)$$

It turns out that Eq. (30) can be written as

$$V_{\mu\nu}(A) = \exp\left(-\frac{q}{k_P}G\right)_{\mu\nu}, \quad (34)$$

where $G_{\mu\nu} = k_\mu A_\nu - k_\nu A_\mu$. Note that the exponential series truncates after the second order term. In addition one obtains

$$V_{\mu\nu}(A) = V_{\nu\mu}(A) + \frac{2q}{k_P}G_{\nu\mu}, \quad V_{\mu\nu}(A)k^\nu = V_{\nu\mu}(A)k^\nu = k_\mu, \quad [P_\mu, V_{\rho\sigma}(A)] = -i\frac{q}{k_P}k_\mu F_{\rho\sigma} - i\frac{q^2}{k_P^2}(AA')k_\mu k_\rho k_\sigma. \quad (35)$$

From the second equation in (35) one concludes that $V_{\mu\nu}(A)$ is an element of the (local) little group $\mathcal{L}g(\xi)$ of the propagation vector k_μ. It is easy and interesting to see that $V_{\mu\nu}(A)$ generates a gauge transformation on A_μ,

$$V_{\mu\nu}(A)A^\nu = A_\mu + \partial_\mu \lambda_V(\xi), \quad \lambda_V(\xi) = -\frac{q}{k_P}\int_{\xi_0}^{\xi} d\xi' A^2(\xi'), \quad (36)$$

and that the field $F_{\mu\nu}$ is invariant under this gauge transformation,

$$V^{\mu}{}_{\rho}(A)V^{\nu}{}_{\sigma}(A)F^{\rho\sigma} = F^{\mu\nu}. \quad (37)$$

Therefore, the local Lorentz transformation $V_{\mu\nu}(A)$ is a symmetry. Notice that the local Lorentz transformation (30) has been rederived many times [50–52] and widely exploited often in the context of its physical implications. In particular, at the classical level the solutions of the Lorentz form equation can be expressed in terms of these local transformations (30). Therefore, in the plane-wave case $V_{\mu\nu}(A)$ plays the role of an evolution operator.

[6] Note that the plane-wave solution of the Dirac equation was found more than 70 years ago by Volkov [47] and extended later on to a field of two beams of electromagnetic radiation [48,49]. However, these approaches did not make use of the nonsingular transformation $\mathcal{V}(A)$.

The realization of $\mathcal{V}(A)$ can be achieved by the nonsingular transformation $\mathcal{V}(A) = \mathcal{V}_0(A)\mathcal{V}_s(A)$, where

$$\mathcal{V}_0(A) = \exp\left\{-i\int \frac{d\xi}{2k_P}(2q(AP) - q^2A^2)\right\},$$
$$\mathcal{V}_s(A) = \exp\left\{-\frac{iq}{2k_P}G_{\mu\nu}s^{\mu\nu}\right\}. \tag{38}$$

It has to be mentioned that the evolution operator $\mathcal{V}(A)$ may be chosen to be H unitary according to the representation T in Eq. (5), i.e.

$$\mathcal{V}^{\dagger}(A)H = H\mathcal{V}^{-1}(A).$$

Collecting the results obtained, the generators of the interacting Poincaré algebra $p_{1,3}$ have the form

$$\Pi_\mu(A) = P_\mu + k_\mu\frac{q}{2k_P}(qA^2 - 2AP - \not{F}),$$
$$\sigma_{\mu\nu}(A) = s_{\mu\nu} - \frac{q}{k_P}\Big(\frac{q}{2k_P}A^2(\eta_{\mu\rho}k_\nu - \eta_{\nu\rho}k_\mu)k_\sigma + \eta_{\mu\rho}(k_\nu A_\sigma - k_\sigma A_\nu) - \eta_{\nu\rho}(k_\mu A_\sigma - k_\sigma A_\mu) - \frac{q}{k_P}(k_\mu A_\nu - k_\nu A_\mu)k_\rho A_\sigma\Big)s^{\rho\sigma}, \tag{39}$$

$$\xi_\mu(A) = x_\mu - \frac{q}{2k_P}\left[x_\mu, \int d\xi(qA^2 - 2AP) - \not{G}\right],$$

where $\not{F} \equiv F_{\mu\nu}s^{\mu\nu}$ and $\not{G} \equiv G_{\mu\nu}s^{\mu\nu}$. The transformed first Casimir operator $\Pi^2(A)$ reads

$$\Pi^2(A) = D^2(A) - q\not{F}, \tag{40}$$

where $D_\mu(A) = P_\mu - qA_\mu$. The explicit form of the transformed Pauli-Lubanski vector $\Omega_\mu(A)$ is

$$\Omega_\mu(A) = W_\mu - \frac{q}{2k_P}\epsilon_{\mu\nu\rho\sigma}\Big\{\eta^{\nu\alpha}\Big(\frac{q}{2k_P}A^2k^\rho k^\beta + G^{\rho\beta}\Big) - \frac{q}{2k_P}G^{\nu\rho}G^{\alpha\beta}\Big\}s_{\alpha\beta}P^\sigma + \frac{q}{4k_P}\epsilon_{\mu\nu\rho\sigma}k^\sigma\eta^{\nu\alpha}\Big(\eta^{\rho\beta} - \frac{2q}{k_P}G^{\rho\beta}\Big) \times s_{\alpha\beta}(qA^2 - 2AP - \not{F}) \tag{41}$$

which yields the transformed second Casimir operator

$$\Omega^2(A) = -\frac{1}{2}s^2D^2 + s^{\sigma\alpha}s_{\sigma\beta}D_\alpha D^\beta + \frac{1}{2}qs^2F - \frac{q}{2k_P}\{(k_\alpha s^{\alpha\sigma}\not{F})s_{\sigma\beta} + s_{\sigma\beta}(k_\alpha s^{\alpha\sigma}\not{F})\}D^\beta + \frac{q^2}{4k_P}(k_\alpha s^{\alpha\sigma}\not{F})(k^\beta s_{\beta\sigma}\not{F}) - \frac{iq}{2k_P}(k^\alpha \not{F} s_{\alpha\beta})D^\beta - \frac{iq}{2k_P}(k_\alpha s^{\alpha\sigma})(k^\beta s_{\beta\sigma}\not{F}'). \tag{42}$$

C. A nonminimal coupling

Considering the nonsingular transformation of Dirac-type wave equation

$$\mathcal{V}(A): (\gamma^\mu P_\mu - m)\psi = 0 \rightarrow (\Gamma^\mu(A)\Pi_\mu(A) - m)\Psi(A) = 0, \tag{43}$$

with the help of Eq. (38) the dynamical counterparts to the operator $P_\mu = i\partial_\mu$ can be calculated to be $\Pi_\mu(A) = \mathcal{V}(A)P_\mu\mathcal{V}^{-1}(A)$,

$$P_\mu \rightarrow \Pi_\mu(A) = P_\mu + k_\mu\frac{q}{2k_P}(qA^2 - 2AP - \not{F}), \tag{44}$$

$$P^2 \rightarrow \Pi^2(A) = (P - qA)^2 - q\not{F} \tag{45}$$

($\not{F} \equiv s^{\mu\nu}F_{\mu\nu}$) while the dynamical counterpart to γ^μ is given by $\Gamma^\mu(A) = \mathcal{V}(A)\gamma^\mu\mathcal{V}^{-1}(A)$,

$$\Gamma^\mu(A) = V^\mu{}_\nu(A)\gamma^\nu = \gamma^\mu - \frac{q}{k_P}\Big(\frac{q}{2k_P}A^2k^\mu k^\nu + G^{\mu\nu}\Big)\gamma^\nu. \tag{46}$$

In terms of $\Pi_\mu(A)$ and $\Gamma^\mu(A)$ we have

$$\mathcal{D}^d(A)\Psi(A) = (\Gamma^\mu(A)\Pi_\mu(A) - m)\Psi(A) = 0. \tag{47}$$

However, expressed in terms of $D_\mu = P_\mu - qA_\mu$ and γ^μ, we obtain

$$\mathcal{D}^d(A)\Psi(A) \equiv \Big(\gamma^\mu D_\mu - \frac{q}{2k_P}\not{k}\not{F} - m\Big)\Psi(A) = 0, \tag{48}$$

where $\not{k} \equiv \gamma^\mu k_\mu$. This interaction is nonminimal. However, as we have shown before, it is determined completely by the claim of gauge invariance.

Note that due to the antimutation of the γ matrices, in the spin-1/2 case the dynamical interaction in Eq. (48) reduces to the minimal coupling. However, in order to obtain the correct values of the gyromagnetic factor, in some cases the (phenomenological) Pauli term $\gamma_\mu\gamma_\nu F^{\mu\nu}$ has to be added by hand to the minimal coupling of the Dirac equation (see also Ref. [53], p. 109). In the case of plane waves the exact solution of this (supplemented) Dirac equation as given by Chakrabarti [43] obeys the same gauge invariance condition $\Psi(A + \partial\lambda) = G(\lambda)\Psi(A)$. This property is found also in the book by Fried [54].

Moreover, in the path integral representation of the (nonrelativistic) Schrödinger quantum mechanics the Feynman propagator for any external electromagnetic field (as an operator $\mathcal{O}$ on the wave function Ψ) is gauge invariant, i.e. the diagram

$$\begin{array}{llcr} \mathcal{O}\Psi: & A & \rightarrow & \mathcal{O}(A)\Psi(A) \\ \downarrow & \downarrow\lambda & & \downarrow G(\lambda) \\ \mathcal{O}\Psi^\lambda: & A + \partial\lambda & \rightarrow & G(\lambda)\mathcal{O}(A)\Psi(A) \end{array}$$

is commutative, $\mathcal{O}(A+\partial\lambda)G(\lambda)=G(\lambda)\mathcal{O}(A)$ due to Eq. (21) on the ψ space, and the wave function transforms as (27).

D. Local phase transformation

For the physical quantities k_μ and $F_{\mu\nu}$ in the model introduced before in Eqs. (35) and (37) the external (unquantized) field is acting on the particle without reaction of the particle on the field. The identification of the elementary particle system with the Poincaré group invariants in Eqs. (40) and (42) leads to the equations

$$(P^2-m^2)\psi=0\rightarrow(\Pi^2(A)-m^2)\Psi = (D^2-qF_{\mu\nu}s^{\mu\nu}-m^2)\Psi=0, \tag{49}$$

$$(W^2+m^2s(s+1))\psi=0\rightarrow(\Omega^2(A)+m^2s(s+1))\Psi=0. \tag{50}$$

These two equations must be satisfied by any field in the presence of the plane-wave field. As a consequence of Eq. (49) the gyromagnetic factor is $g=2$ and the Bargmann-Michel-Telegdi equation for the four-polarization vector of the particle takes its simplest form in the proper time frame of the particle [29].

Finally, as a consequence of the explicit form (38), the associated transformation of the evolution operator $\mathcal{V}(A)$ under the local gauge transformation for the plane-wave field,

$$A_\mu(\xi)\rightarrow A_\mu(\xi)+\partial_\mu\lambda(\xi) \tag{51}$$

becomes

$$\mathcal{V}(A)\rightarrow\mathcal{V}(A+\partial\lambda)=e^{-iq\lambda}\mathcal{V}(A). \tag{52}$$

We conclude that the phase transformation is a *consequence* of the gauge transformation. This should hold not only for the particular case of plane waves as analyzed explicitly in this section but also for a general solution $\mathcal{V}(A)$.

IV. THE RARITA-SCHWINGER EQUATION IN THE FRAMEWORK OF A DYNAMICAL INTERACTION

The spin-3/2 field may be described entirely in terms of the vector-bispinor Ψ_μ corresponding to the representation of the proper Lorentz group

$$(\tfrac{1}{2},\tfrac{1}{2})\otimes((\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2}))=(1,\tfrac{1}{2})\oplus(\tfrac{1}{2},1)\oplus(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2}). \tag{53}$$

The transformation rule according to Eq. (4) is

$$(\tau(\lambda)\psi)_\mu(p)=\Lambda_{\mu\nu}T_D(\Lambda)\psi^\nu(\Lambda^{-1}p), \tag{54}$$

where $T_D(\Lambda)$ is the Dirac representation of the Lorentz group. The generators of the representation are

$$s_{\mu\nu}=-ie_{\mu\nu}\otimes\mathbb{1}_D+\mathbb{1}_P\otimes s_{D\mu\nu} = i(-\tfrac{1}{2}\eta_{\mu\nu}+E_{\mu\nu}\otimes\mathbb{1}_D-E_{\nu\mu}\otimes\mathbb{1}_D+\tfrac{1}{2}\mathbb{1}_P\otimes\gamma_\mu\gamma_\nu), \tag{55}$$

where the indices P and D stand for the Proca and Dirac parts of the direct product in Eq. (53). Here the 16 matrices $E_{\mu\nu}$ generate the Weyl's basis of the set of 4×4 matrices,

$$(E_{\mu\nu})_{\rho\sigma}=\eta_{\mu\rho}\eta_{\nu\sigma},\qquad E_{\mu\nu}E_{\rho\sigma}=\eta_{\nu\rho}E_{\mu\sigma},$$

and $e_{\mu\nu}=-E_{\mu\nu}+E_{\nu\mu}$ for the Lorentz generators of the vector representation. The SO_3 decomposition of the representation (53) is

$$2D^{(3/2)}\oplus4D^{(1/2)}. \tag{56}$$

Therefore, the representation of the Poincaré group contains spins 3/2 and 1/2. The Pauli-Lubanski vector reads

$$W_\mu=i\epsilon_{\mu\rho\sigma\nu}(E^{\rho\sigma}\otimes\mathbb{1}_D+\tfrac{1}{4}\mathbb{1}_P\otimes\gamma^\rho\gamma^\sigma)P^\nu \tag{57}$$

and its square

$$W^2=-\frac{15}{4}P^2+P^2(E^{\mu\nu}\otimes\gamma_\mu\gamma_\nu) + P_\mu P^\nu(E^{\mu\rho}\otimes\gamma_\rho\gamma_\nu+E^{\rho\mu}\otimes\gamma_\nu\gamma_\rho). \tag{58}$$

Note that

$$(W^2)^2=-\frac{9}{4}P^2\Big\{-\frac{15}{4}P^2+P^2(E^{\mu\nu}\otimes\gamma_\mu\gamma_\nu) + P_\mu P^\nu(E^{\mu\rho}\otimes\gamma_\rho\gamma_\nu) + P_\mu P^\nu(E^{\rho\mu}\otimes\gamma_\nu\gamma_\rho)+\frac{5}{8}P^2\Big\} = -2s^2P^2\Big(W^2+\frac{s^2-1}{2}P^2\Big)\Big|_{s=3/2} \tag{59}$$

is a pure spin-3/2 object which enables us to construct the Poincaré covariant mass (m) and spin (j) projectors ($j=3/2,\ 1/2$) [25]. The free spin-3/2 particle Rarita-Schwinger equation is given as

$$(P_\nu\gamma^\nu-m)\psi^\mu=0, \tag{60}$$

$$\gamma_\mu\psi^\mu=0. \tag{61}$$

The other constraints

$$(P^2-m^2)\psi^\mu=0, \tag{62}$$

$$P_\mu\psi^\mu=0 \tag{63}$$

turn out to be a consequence of Eqs. (60) and (61). It is interesting to note that the static condition (61) and the dynamic condition (63) together eliminate the spin-1/2 state completely, i.e. the equations

$$(P^2-m^2)\psi^\mu=0,\qquad\gamma_\mu\psi^\mu=P_\mu\psi^\mu=0$$

with ψ_μ transforming according to Eq. (54) gives a theory for spin-3/2 states. Indeed, using the explicit form of W^2 in Eq. (58) it is easy to see that under the constraints (61) and (63) we obtain

$$W^2\psi = -\frac{15}{4}P^2\psi = -s(s+1)P^2\psi|_{s=3/2}. \tag{64}$$

Therefore, Eqs. (60) and (61) describe indeed a single particle of mass m and spin $3/2$.

The dynamical interaction is obtained in the way described in Sec. III. Taking into account the explicit form (55) of the generators $s_{\mu\nu}$, the transformation $\mathcal{V}(A)$ in Eq. (38) becomes

$$\mathcal{V}_{\rm RS}(A) = \exp\Big(-\frac{iq}{k_P}\int(AP - \frac{q}{2}A^2)\Big)(\mathbb{1}_P\otimes\mathbb{1}_D)\Big\{\mathbb{1}_P - \frac{q}{k_P}\Big(G_{\rho\sigma} - \frac{q}{2k_P}(G^2)_{\rho\sigma}\Big)E^{\rho\sigma}\Big\}\otimes\Big\{\mathbb{1}_D + \frac{q}{4k_P}G^{\rho\sigma}\gamma_\rho\gamma_\sigma\Big\}. \tag{65}$$

A straightforward calculation yields

$$P_\mu \to \Pi_\mu(A) = \Big(P_\mu + k_\mu\frac{q}{2k_P}(qA^2 - 2AP)\Big)(\mathbb{1}_P\otimes\mathbb{1}_D) - k_\mu\frac{iq}{k_P}F_{\rho\sigma}(E^{\rho\sigma}\otimes\mathbb{1}_D) - k_\mu\frac{iq}{4k_P}F_{\rho\sigma}(\mathbb{1}_P\otimes\gamma^\rho\gamma^\sigma),$$

$$\begin{aligned} s_{\mu\nu} \to \sigma_{\mu\nu}(A) = {}& -i\Big(\frac{1}{2}\eta_{\mu\nu} + \frac{q}{k_P}G_{\mu\nu}\Big)(\mathbb{1}_P\otimes\mathbb{1}_D) + i\Big\{-\eta_{\mu\rho}\eta_{\nu\sigma} + \frac{q}{k_P}(\eta_{\mu\rho}G_{\nu\sigma} - \eta_{\nu\rho}G_{\mu\sigma}) \\ & - \frac{q^2}{2k_P^2}(\eta_{\mu\rho}(G^2)_{\nu\rho} - \eta_{\nu\rho}(G^2)_{\mu\sigma} + G_{\mu\nu}G_{\rho\sigma})\Big\}\Big(e^{\rho\sigma}\otimes\mathbb{1}_D - \frac{1}{2}\mathbb{1}_P\otimes\gamma^\rho\gamma^\sigma\Big), \end{aligned}$$

$$\begin{aligned} W_\mu \to \Omega_\mu(A) = {}& -\frac{iq}{2k_P}\epsilon_{\mu\nu\rho\sigma}k^\nu A^\rho P^\sigma\mathbb{1}_P\otimes\mathbb{1}_D \\ & - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\Big\{\Big(\eta^\nu{}_\alpha\eta^\rho{}_\beta - \frac{q}{k_P}\eta^\nu{}_\alpha G^\rho{}_\beta + \frac{q^2}{2k_P^2}G^{\nu\rho}G_{\alpha\beta} - \frac{q^2A^2}{2k_P^2}k^\rho k_\beta\eta^\nu{}_\alpha\Big)P^\sigma \\ & + \frac{q}{2k_P}(qA^2 - 2AP)k^\sigma\eta^\nu{}_\alpha\Big(\eta^\rho{}_\beta + \frac{2q}{k_P}A^\rho k_\beta\Big)\Big\}\Big\{e^{\alpha\beta}\otimes\mathbb{1}_D - \frac{1}{2}\mathbb{1}_P\otimes\gamma^\alpha\gamma^\beta\Big\} \\ & + \frac{q}{4k_P}\epsilon_{\mu\nu\rho\sigma}k^\sigma\eta^\nu{}_\alpha\Big(\eta^\rho{}_\beta + \frac{2q}{k_P}A^\rho k_\beta\Big)F_{\lambda\tau}\Big\{e^{\alpha\beta}e^{\lambda\tau}\otimes\mathbb{1}_D - \frac{1}{2}e^{\alpha\beta}\otimes\gamma^\lambda\gamma^\tau \\ & - \frac{1}{2}e^{\lambda\tau}\otimes\gamma^\alpha\gamma^\beta + \frac{1}{4}\mathbb{1}_P\otimes\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\tau\Big\}. \end{aligned} \tag{66}$$

The two Casimir invariants of the dynamical Poincaré algebra are

$$P^2 \to \Pi^2(A) = D^2(A) - 2iqF^{\rho\sigma}\{(E_{\rho\sigma}\otimes\mathbb{1}_D) + \tfrac{1}{4}(\mathbb{1}_P\otimes\gamma_\sigma)\}, \tag{67}$$

and

$$\begin{aligned} W^2 \to \Omega^2(A) = {}& \Big(\frac{9}{2} + (e^{\rho\sigma}\otimes\gamma_{\rho\sigma})\Big)D^2 + \frac{1}{2}(-4(E_{\alpha\beta}\otimes\mathbb{1}_D) + e_\alpha{}^\rho\otimes\gamma_{\rho\beta} + e_\beta{}^\rho\otimes\gamma_{\rho\alpha})D^\alpha D^\beta \\ & - \frac{iq}{2k_P}k^\tau F^{\rho\sigma}\Big\{-\frac{3}{2}(h_{\tau\beta}\otimes\gamma_{\rho\sigma}) + (h_{\tau\rho}\otimes\gamma_{\sigma\beta}) - \eta_{\sigma\beta}(h_\rho{}^\alpha\otimes\gamma_{\alpha\tau}) - \frac{i}{2}\epsilon_{\rho\sigma\alpha\beta}(e_\tau{}^\alpha\times\gamma^5)\Big\}D^\beta \\ & + iqF^{\rho\sigma}\Big\{-16(E_{\rho\sigma}\otimes\mathbb{1}_D) - \frac{29}{8}(\mathbb{1}_P\otimes\gamma_{\rho\sigma}) - 6(e^\alpha{}_\sigma\otimes\gamma_{\rho\alpha}) - i\epsilon_{\rho\sigma\alpha\beta}(E^{\alpha\beta}\otimes\gamma_5)\Big\} \\ & - \frac{q}{k_P}k^\alpha k^\beta F'^{\rho\sigma}(E_{\alpha\beta}\otimes\gamma_{\rho\sigma}), \end{aligned} \tag{68}$$

where we used the abbreviations $\gamma_{\mu\nu} \equiv \gamma_\mu\gamma_\nu$ and $h_{\mu\nu} \equiv E_{\mu\nu} + E_{\nu\mu}$. Applying the operator $\mathcal{V}_{\rm RS}(A)$ to the Rarita-Schwinger equations (60) and (61) one obtains

$$\Big\{(D^\mu\gamma_\mu - m)\eta_{\rho\sigma} - \frac{iq}{k_P}(k^\mu\gamma_\mu)F_{\rho\sigma}\Big\}\Psi^\sigma = 0, \tag{69}$$

$$\gamma_\mu\Psi^\mu = 0, \tag{70}$$

where $\Psi(x,A) = \mathcal{V}_{\rm RS}(x,A)\psi(x)$.

Equation (69) is the true equation of motion containing all derivatives $D_\mu\Psi_\sigma$. The static constraint (70) survives the dynamical interaction and eliminates all superfluous spin-1/2 components. As a consequence the other constraints are the Feynman–Gell-Mann equation

$$\{(\not{D}^2 - m^2)\eta_{\mu\rho} - 2iqF_{\mu\rho}\}\Psi^\rho = 0 \tag{71}$$

and the kinematical constraint

$$\left\{D_\mu - \frac{iq}{4k_P}(F^{\rho\sigma}\gamma_\rho\gamma_\sigma)k_\mu\right\}\Psi^\mu = 0. \tag{72}$$

Note that as in the free case the dynamical interaction is algebraically consistent. Moreover, the second order equation (71) describes the causal propagation of waves (assuming the continuity of the first order derivatives of Ψ).

V. CONCLUSIONS AND OUTLOOK

Based on the Lorentz-Poincaré connection we showed that an external electromagnetic field A can be introduced most consistently by using the nonsingular transformation $\mathcal{V}(A)$. Imposing the two claims that the transformation (1) applies not only to the differential operator $\mathcal{D}$ of the equation of motion but to the whole Poincaré algebra, and (2) applied to the generators $s_{\mu\nu}$ of the Poincaré algebra yields a Lorentz-type transformation, the nonsingular transformation $\mathcal{V}(A)$ is uniquely defined. For the case of plane waves we showed this explicitly for the Dirac-type equation and the Rarita-Schwinger equation. The local phase transformation of the covariant functions ψ appears *as a consequence* of the local gauge transformation. This is opposite to the traditional point of view where phase transformation and gauge transformation are imposed simultaneously.

An essential point in our approach is that Lorentz and gauge transformation are placed on the same foundation. Accordingly, the covariant functions in the presence of an external electromagnetic field A have to depend explicitly both on the space-time location x and the field A, $\Psi(x, A)$. The field A, therefore, has to be understood as a coordinate. In a forthcoming publication we will quantize this system. On the other hand, we are inspired by the success of the realization of the nonsingular transformation $\mathcal{V}(A)$ for the plane-wave case. In a sequel of this paper we will generalize this to the more general situation of an arbitrary electromagnetic field A.

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APPENDIX: ALGEBRAIC CONSISTENCY, LOCALITY, AND CAUSALITY

In this Appendix we deal in detail with problems of consistency and causality related to the introduction of an interacting electromagnetic field into higher-spin theories, as first mentioned by Velo and Zwanziger [18].

1. Algebraic inconsistency

The term "algebraic inconsistency" was coined by Velo and Zwanziger [18] and explained explicitly in 1971 [55], where the authors showed that the minimal substitution for the three equations (60)–(62) leads to the unwanted constraint $e\gamma_\mu F^{\mu\nu}\psi_\nu = 0$. As only escape from this "disaster" they proposed the method of Fierz and Pauli [6] where an ansatz for the interaction is used which had to be adjusted to the physical requirements. Following this method via the second order Klein-Gordon equation, they ended up with an additional contribution $O(F)$ to the wave function. The same procedure but based on the first order equation of motion with a more general ansatz is used by Porrati and Rahman [56]. A possible nonminimal action term could be constructed explicitly. The procedure was extended for the application to massive spin-2 bosonic string states by Argyres and Nappi [57], while Porrati *et al.* [58] applied the method to string states with arbitrary high spin and showed that the BRST operator employed by Argyres and Nappi is not necessary. However, all these applications of the Fierz-Pauli method still need a consistency check.

With our method we escape from this necessity because the three equations (60)–(62) are not independent of each other. Instead, the third is a consequence of the first two. In applying the nonsingular transformation $\mathcal{V}_{\rm RS}(A)$ to these two equations we end up with Eqs. (69) and (70). Equations (71) and (72) are a consequence and, therefore, evidently consistent with Eqs. (69) and (70).

2. Locality

Our results should in principal be comparable with the results of Ref. [56]. What makes it difficult to perform this cross-check is the nature of the operator $1/k_P$. As the differential operator $k_P \equiv k_\mu P^\mu$ commutes with all other operators of the representation space of the Poincaré group, so does $1/k_P$. Therefore, according to Schur's lemma both operators are diagonal. As reciprocal of a differential operator, $1/k_P$ need not be local. However, as stressed by Chakrabarti [43], Beers and Nickle [44], and later by Brown and Kowalski [52,59], as applied to eigenstates to the Poincaré group, the operator still turns out to be local and contributes to the local Lorentz transformation $V_{\mu\nu}(A)$. For the near future we hope to overcome these difficulties and perform the comparison with the results given by Porrati and Rahman [56] as well as with Deser *et al.* [60].

3. Causality

Because the Poincaré group takes care of the space-time structure of the result and, therefore, the causality, there is no need to show the causality of the result explicitly. In this context it is worth stressing that the Velo-Zwanziger problem is not the final word. As explained by Cox [20], the constraint analysis of Velo and Zwanziger [18] is not complete because the "true equation of motion" still does not determine the time derivatives. In completing the analysis, instead of acausality Cox finds a loss of degrees of freedom.

Using our method, an explicit check for causality was performed in Ref. [28] by analyzing the characteristic surfaces (see e.g. [61]), as it was employed starting from Velo and Zwanziger (for a detailed explanation see Ref. [62]) up to recent works of Porrati *et al.* [58]. Our result for the normal vector n_μ obeying [28]

$$\Delta(n) = (\tfrac{1}{2})^4 (n^2)^8 \tag{A1}$$

shows that every characteristic surface is a light cone and the propagation, therefore, is causal.

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