

Invariants of collective neutrino oscillationsY. Pehlivan,^{1,2,*} A. B. Balantekin,^{3,†} Toshitaka Kajino,^{2,4,‡} and Takashi Yoshida^{4,§}¹*Mimar Sinan Fine Arts University, Istanbul 34349, Turkey*²*National Astronomical Observatory of Japan 2-21-1 Osawa, Mitaka, Tokyo, 181-8588, Japan*³*Department of Physics, University of Wisconsin—Madison, Madison, Wisconsin 53706, USA*⁴*Department of Astronomy, University of Tokyo, Tokyo 113-0033, Japan*

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We consider the flavor evolution of a dense neutrino gas by taking into account both vacuum oscillations and self-interactions of neutrinos. We examine the system from a many-body perspective as well as from the point of view of an effective one-body description formulated in terms of the neutrino polarization vectors. We show that, in the single angle approximation, both the many-body picture and the effective one-particle picture possess several constants of motion. We write down these constants of motion explicitly in terms of the neutrino isospin operators for the many-body case and in terms of the polarization vectors for the effective one-body case. The existence of these constants of motion is a direct consequence of the fact that the collective neutrino oscillation Hamiltonian belongs to the class of Gaudin Hamiltonians. This class of Hamiltonians also includes the (reduced) BCS pairing Hamiltonian describing superconductivity. We point out the similarity between the collective neutrino oscillation Hamiltonian and the BCS pairing Hamiltonian. The constants of motion manifest the exact solvability of the system. Borrowing the well established techniques of calculating the exact BCS spectrum, we present exact eigenstates and eigenvalues of both the many-body and the effective one-particle Hamiltonians describing the collective neutrino oscillations. For the effective one-body case, we show that spectral splits of neutrinos can be understood in terms of the adiabatic evolution of some quasiparticle degrees of freedom from a high-density region where they coincide with flavor eigenstates to the vacuum where they coincide with mass eigenstates. We write down the most general consistency equations which should be satisfied by the effective one-body eigenstates and show that they reduce to the spectral split consistency equations for the appropriate initial conditions.

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I. INTRODUCTION

Collective oscillations of neutrinos is the process by which neutrino-neutrino scattering contributes to the flavor evolution of a sufficiently dense neutrino gas in a way which is somewhat similar to the well-known matter-enhanced neutrino oscillations in the Sun. Such high neutrino densities are believed to be achieved in the core-collapse supernovae [1–5], in the Early Universe [6–9], and possibly in other astrophysical sites. The contribution of the neutrino-neutrino scattering to the flavor evolution differs from that of ordinary matter scattering due to the fact that in the former the scattered and the scattering particles are of the same kind, giving rise to exchange type forward scattering terms [10,11]. These terms couple the flavor evolutions of neutrinos with different energies and turn the study of the system into a nonlinear many-body problem. A *mean-field* type effective one-particle approximation to this problem was proposed in Refs. [10,11] and has been widely adopted in the subsequent studies. However, it became evident that analytical solutions of the resulting nonlinear evolution

equations were needed in order to explore the full range of physical scenarios. Recent analytical studies of various special cases, such as the limit in which neutrino-neutrino scattering potential becomes dominant, revealed several situations in which neutrinos with different energies oscillate collectively [5,12–20].

An algebraic approach to the problem was worked out in Ref. [21] from a many-body point of view. In this approach the flavor evolution of the many-body system is formulated as an SU(2) or SU(3) coherent state path integral for two or three flavors, respectively. The evolution operator for the entire system is calculated using both the saddle point and the operator product linearization approximations and these two methods were shown to yield the same answer. Such an approach is useful in providing a framework to look for exact solutions or systematic approximations.

There is an increasingly growing literature studying the collective neutrino oscillations (see, e.g., Refs. [22–47]). There are, however, several excellent recent reviews that may serve as a starting point in exploring this literature [22,24,25].

The algebraic approach proposed in Ref. [21] is helpful in exploring the hidden symmetries of the system. Hamiltonian describing collective neutrino oscillations possesses an $SU(N)_f$ rotation symmetry in the neutrino flavor

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space [21,26,27]. Various collective modes, including spectral swappings or splittings arise from this symmetry even in the inhomogeneous or anisotropic environments [26]. One expects that such a complex nonlinear system may exhibit further symmetries. Indeed, several authors noted the presence of various conserved quantities in collective neutrino oscillations [20,28]. More recently, it was shown that collective oscillations that maintain coherence can be classified by a number of linearly-independent functions [23], implying that scalar products of a unique linear combination of the original polarization vectors are conserved. The goal of this paper is to further explore symmetries and conserved quantities associated with the collective neutrino oscillation Hamiltonian.

Our study of the symmetries of the neutrino Hamiltonian is based on the observation that the neutrino-neutrino forward scattering Hamiltonian has the form of a spin-exchange interaction as was pointed out by many authors earlier. Here, the spin does not refer to the intrinsic spin of the neutrino but to the so called *neutrino isospin* which is defined by introducing a multiplet of neutrino states. Interactions of this type are also encountered in many-body systems with pair coupling where the role of the spin is played by the so called *quasispin*. Examples include the residual pairing interaction between nucleons in the nuclear shell model and the pairing of valence electrons in the BCS theory of superconductivity [48]. The fact that the pairing Hamiltonian is exactly solvable, which hints at the existence of symmetries and associated constants of motion, was pointed out as early as 1963 by Richardson [49]. These constants of motion were later identified by Gaudin [50,51] and others [52] (for a review, see Refs. [53,54]).

Here, we consider both the exact many-body Hamiltonian and the effective one-body Hamiltonian describing collective oscillations of neutrinos. We formulate the effective one-body picture in terms of the random phase approximation (RPA) method whereby the intrinsic consistency requirements are manifested as equations of motion of the neutrino polarization vectors. Our analysis includes both the vacuum oscillations and the self-interactions of neutrinos. We show that, under the single angle approximation, both the exact many-body and the effective one-particle pictures possess many constants of motion which were not so far carefully studied. We express these constants of motion in terms of the neutrino isospin operators in the case of the exact many-body Hamiltonian and in terms of the neutrino polarization vectors in the case of the RPA evolution. In both cases, the constants of motion depend on the parameter which couples the flavor evolution of neutrinos with different energies manifesting the existence of associated dynamical symmetries.

We give analytical expressions for the exact eigenstates and energy eigenvalues of both the many-body Hamiltonian and the effective one-particle Hamiltonian. To achieve this goal, we use the method of Bethe ansatz [55]. For the RPA

Hamiltonian we use the method of Bogoliubov transformations to bring the system into a diagonal form in terms of noninteracting *quasiparticle* states. We show that the quasiparticle picture is useful in offering a formal and intuitive description of the spectral splits which were reported to occur in various numerical simulations when the neutrinos adiabatically evolve from a region of high neutrino density to the vacuum. Our explanation is complementary to the one offered in Ref. [20] in terms of the neutrino polarization vectors. A preliminary account of our results was given in [29].

The organization of this paper is as follows: In Sec. II, we briefly review the concept of neutrino isospin and write down the many-body Hamiltonian describing the collective neutrino oscillations in terms of the isospin operators. We also present the many-body constants of motion in terms of the isospin operators in this section. In Sec. III, we present the exact many-body eigenstates and eigenvalues which are found in an analytical way using the method of Bethe ansatz. In Sec. IV, we apply the method of RPA to bring the many-body neutrino Hamiltonian into its effective one-particle form and we briefly review how the RPA consistency requirements yield the time evolution equations of the neutrino polarization vectors. We write down the constants of motion of the RPA evolution in terms of the polarization vectors. In this section, we also consider the eigenstates and eigenvalues of the RPA Hamiltonian and write down the RPA consistency equations for these eigenstates. At the end of this section we give a brief interpretation of the spectral splits in terms of the adiabatic evolution of quasiparticles. The first four sections of this paper deal only with neutrinos in order to keep the formulas simple and emphasize the underlying physics. We include the antineutrinos in Sec. V and describe how the results of the earlier sections, including the invariants, are generalized in this case. We conclude the paper in Sec. VI by elaborating the connection between our invariants and other invariants described in the literature.

II. THE ISOSPIN FORMULATION OF THE PROBLEM

A. Mass and flavor isospin operators

In this paper, we consider the mixing between two neutrino flavors. Without loss of generality, we can take one of these flavors to be ν_e and the other to be an orthogonal flavor state that we denote by ν_x . In other words, ν_x can be either ν_μ or ν_τ or a normalized combination of them. We denote the fermion operator for an α flavor neutrino with momentum \mathbf{p} by $a_\alpha(\mathbf{p})$ where $\alpha = e, x$.¹ The global rotation

¹In general, additional quantum numbers are needed in order to distinguish neutrinos with the same momentum. But to keep our formulas simple we do not explicitly include additional quantum numbers in our notation. Instead, one can view \mathbf{p} as a multiple index like $(\mathbf{p}, s_1, s_2, \dots)$.

$$\begin{aligned} a_e(\mathbf{p}) &= \cos\theta a_1(\mathbf{p}) + \sin\theta a_2(\mathbf{p}) \\ a_x(\mathbf{p}) &= -\sin\theta a_1(\mathbf{p}) + \cos\theta a_2(\mathbf{p}) \end{aligned} \quad (2.1)$$

relates them to the fermion operators $a_i(\mathbf{p})$ for the corresponding mass eigenstates ν_i for $i = 1, 2$.

Following the earlier literature, we introduce the *flavor isospin operators*

$$\begin{aligned} J_{\mathbf{p}}^+ &= a_e^\dagger(\mathbf{p})a_x(\mathbf{p}), & J_{\mathbf{p}}^- &= a_x^\dagger(\mathbf{p})a_e(\mathbf{p}), \\ J_{\mathbf{p}}^z &= \frac{1}{2}(a_e^\dagger(\mathbf{p})a_e(\mathbf{p}) - a_x^\dagger(\mathbf{p})a_x(\mathbf{p})), \end{aligned} \quad (2.2)$$

which obey the usual $SU(2)$ commutation relations

$$[J_{\mathbf{p}}^+, J_{\mathbf{q}}^-] = 2\delta_{\mathbf{p}\mathbf{q}}J_{\mathbf{p}}^z, \quad [J_{\mathbf{p}}^z, J_{\mathbf{q}}^\pm] = \pm\delta_{\mathbf{p}\mathbf{q}}J_{\mathbf{p}}^\pm, \quad (2.3)$$

such that we have as many orthogonal $SU(2)$ *flavor isospin algebras* as the number of neutrinos. It follows from the definitions given in Eq. (2.2) that each flavor isospin algebra is realized in the spin-1/2 representation and that the electron neutrino is taken to be isospin up. We note, however, that sometimes the opposite convention for the isospin doublet is used (see, for example, Ref. [21]).

We introduce the following summation convention for all quantities labeled by neutrino momentum \mathbf{p} :

$$A_\omega \equiv \sum_{|\mathbf{p}|=p} A_{\mathbf{p}} \quad \text{and} \quad A \equiv \sum_{\omega} A_\omega. \quad (2.4)$$

Here ω is the vacuum oscillation frequency for a neutrino with energy p . It is given by

$$\omega = \frac{\delta m^2}{2p} \quad (2.5)$$

with $\delta m^2 = m_2^2 - m_1^2$ where m_i is the mass of the state ν_i . In the case of flavor isospin operators, for example, \vec{J}_ω represents the total flavor isospin operator of all neutrinos with the same vacuum oscillation frequency ω whereas \vec{J} represents the total flavor isospin operator of all neutrinos. Since the operators \vec{J}_ω and \vec{J} are sums of individual $SU(2)$ operators, their components also obey the $SU(2)$ commutation relations. The corresponding total flavor isospin quantum numbers can take several values. Note that in Eq. (2.4) and in all subsequent equations sums are to be taken over all occupied neutrino states. Also note that we use boldface letters to indicate vectors in momentum space (e.g. \mathbf{p}) and arrows to indicate vectors in flavor space (e.g. \vec{J}). Throughout this paper, we use both the Cartesian basis (x, y, z) and the cylindrical basis $(+, -, z)$ for vectors in flavor space. These two bases are related by

$$J_\pm = J_x \pm iJ_y. \quad (2.6)$$

One can similarly introduce the *mass isospin operators*

$$\begin{aligned} \mathcal{J}_{\mathbf{p}}^+ &= a_1^\dagger(\mathbf{p})a_2(\mathbf{p}), & \mathcal{J}_{\mathbf{p}}^- &= a_2^\dagger(\mathbf{p})a_1(\mathbf{p}), \\ \mathcal{J}_{\mathbf{p}}^z &= \frac{1}{2}(a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) - a_2^\dagger(\mathbf{p})a_2(\mathbf{p})), \end{aligned} \quad (2.7)$$

which also obey the $SU(2)$ commutation relations given in Eq. (2.3). As can be seen from the definitions given in Eq. (2.7), we take the mass eigenstate ν_1 to be isospin up. The total mass isospin operators are defined as in Eq. (2.4).

Eqs. (2.1) imply that the particle operators in the mass and flavor bases are related by a unitary transformation. The most general unitary transformation operator in flavor space can be written as

$$U_{\mathbb{P}} = e^{\sum_{\mathbf{p}} z_{\mathbf{p}} \mathcal{J}_{\mathbf{p}}^+} e^{\sum_{\mathbf{p}} \ln(1+|z_{\mathbf{p}}|^2) \mathcal{J}_{\mathbf{p}}^z} e^{-\sum_{\mathbf{p}} z_{\mathbf{p}} \mathcal{J}_{\mathbf{p}}^-} \quad (2.8)$$

where $z_{\mathbf{p}} = e^{i\delta_{\mathbf{p}}} \tan\theta_{\mathbf{p}}$ is a complex parameter. The effect of such a general transformation on particle operators is

$$\begin{aligned} U_{\mathbb{P}}^\dagger a_1(\mathbf{p}) U_{\mathbb{P}} &= \cos\theta_{\mathbf{p}} a_1(\mathbf{p}) - e^{i\delta_{\mathbf{p}}} \sin\theta_{\mathbf{p}} a_2(\mathbf{p}) \\ U_{\mathbb{P}}^\dagger a_2(\mathbf{p}) U_{\mathbb{P}} &= e^{-i\delta_{\mathbf{p}}} \sin\theta_{\mathbf{p}} a_1(\mathbf{p}) + \cos\theta_{\mathbf{p}} a_2(\mathbf{p}), \end{aligned} \quad (2.9)$$

i.e., each momentum mode is rotated by a different angle and acquires a different (relative) phase. The transformation from mass to flavor basis given in Eqs. (2.1) is a special case of Eqs. (2.9) for which the rotation angle is equal to the vacuum mixing angle and the phase is zero for all momentum modes. In other words, we can write

$$a_e(\mathbf{p}) = U^\dagger a_1(\mathbf{p}) U \quad \text{and} \quad a_x(\mathbf{p}) = U^\dagger a_2(\mathbf{p}) U, \quad (2.10)$$

where

$$U = e^{z\mathcal{J}^+} e^{\ln(1+|z|^2)\mathcal{J}^z} e^{-z\mathcal{J}^-} \quad (2.11)$$

with $z_{\mathbf{p}} = z = \tan\theta$ for all \mathbf{p} . Note that we use the subscript \mathbb{P} in denoting the operator $U_{\mathbb{P}}$ defined in Eq. (2.8) in order to emphasize that it imposes a different rotation on each momentum mode. Although the operator U is only a special case of $U_{\mathbb{P}}$, it carries no such index because it induces the same transformation on all momentum modes. Also note that the rotation parameter $z_{\mathbf{p}}$ is not subject to the summation rule introduced in Eq. (2.4). Instead, its index simply indicates its dependence on momentum.

The operator $U_{\mathbb{P}}$ given in Eq. (2.8) also induces a transformation on the isospin operators as follows:

$$\begin{aligned} U_{\mathbb{P}}^\dagger \mathcal{J}_{\mathbf{p}}^z U_{\mathbb{P}} &= \cos 2\theta_{\mathbf{p}} \mathcal{J}_{\mathbf{p}}^z + \frac{1}{2} e^{i\delta_{\mathbf{p}}} \sin 2\theta_{\mathbf{p}} \mathcal{J}_{\mathbf{p}}^+ \\ &\quad + \frac{1}{2} e^{-i\delta_{\mathbf{p}}} \sin 2\theta_{\mathbf{p}} \mathcal{J}_{\mathbf{p}}^-, \\ U_{\mathbb{P}}^\dagger \mathcal{J}_{\mathbf{p}}^+ U_{\mathbb{P}} &= \cos^2 \theta_{\mathbf{p}} \mathcal{J}_{\mathbf{p}}^+ - e^{-i\delta_{\mathbf{p}}} \sin 2\theta_{\mathbf{p}} \mathcal{J}_{\mathbf{p}}^z \\ &\quad - e^{-2i\delta_{\mathbf{p}}} \sin^2 \theta_{\mathbf{p}} \mathcal{J}_{\mathbf{p}}^-, \\ U_{\mathbb{P}}^\dagger \mathcal{J}_{\mathbf{p}}^- U_{\mathbb{P}} &= \cos^2 \theta_{\mathbf{p}} \mathcal{J}_{\mathbf{p}}^- - e^{i\delta_{\mathbf{p}}} \sin 2\theta_{\mathbf{p}} \mathcal{J}_{\mathbf{p}}^z - e^{2i\delta_{\mathbf{p}}} \sin^2 \theta_{\mathbf{p}} \mathcal{J}_{\mathbf{p}}^+. \end{aligned} \quad (2.12)$$

In particular, mass and flavor isospin operators are related by

$$\vec{J}_{\mathbf{p}} = U^\dagger \vec{\mathcal{J}}_{\mathbf{p}} U \quad (2.13)$$

which implies

$$\begin{aligned}
J_{\mathbf{p}}^z &= \cos 2\theta \mathcal{J}_{\mathbf{p}}^z + \sin 2\theta \frac{\mathcal{J}_{\mathbf{p}}^+ + \mathcal{J}_{\mathbf{p}}^-}{2}, \\
J_{\mathbf{p}}^+ &= \cos^2 \theta \mathcal{J}_{\mathbf{p}}^+ - \sin 2\theta \mathcal{J}_{\mathbf{p}}^z - \sin^2 \theta \mathcal{J}_{\mathbf{p}}^-, \\
J_{\mathbf{p}}^- &= \cos^2 \theta \mathcal{J}_{\mathbf{p}}^- - \sin 2\theta \mathcal{J}_{\mathbf{p}}^z - \sin^2 \theta \mathcal{J}_{\mathbf{p}}^+.
\end{aligned} \quad (2.14)$$

The inverse transformation from flavor to mass basis can be found by substituting $-\theta$ in place of θ in Eq. (2.14). Note that Eqs. (2.11) and (2.13) follow from the fact that the total isospin operator \vec{J} is the generator of global rotations in flavor space.

B. The Hamiltonian and the quantum invariants

The many-body Hamiltonian describing the vacuum oscillations of a group of neutrinos is given by

$$H_\nu = \sum_{\mathbf{p}} \left(\frac{m_1^2}{2p} a_1^\dagger(\mathbf{p}) a_1(\mathbf{p}) + \frac{m_2^2}{2p} a_2^\dagger(\mathbf{p}) a_2(\mathbf{p}) \right). \quad (2.15)$$

One can write this Hamiltonian in terms of the neutrino isospin operators defined in Sec. II A. First note that since the neutrinos only exchange their momenta with the forward scattering, the total number of neutrinos in each momentum mode is constant. As a result, the term

$$\sum_{\mathbf{p}} \frac{m_1^2 + m_2^2}{4p} (a_1^\dagger(\mathbf{p}) a_1(\mathbf{p}) + a_2^\dagger(\mathbf{p}) a_2(\mathbf{p})) \quad (2.16)$$

is proportional to identity for a given number of particles and can be subtracted from the Hamiltonian without any consequences. Subtracting this term from Eq. (2.15) and using the definitions given in Eqs. (2.5) and (2.7) together with the transformation given in Eqs. (2.14) one finds

$$H_\nu = \sum_{\omega} \frac{\delta m^2}{2p} \vec{B} \cdot \vec{J}_{\omega}. \quad (2.17)$$

Here \vec{B} is the unit vector which points in the *mass direction*. Its components are given by

$$\vec{B} = (0, 0, -1)_{\text{mass}} = (\sin 2\theta, 0, -\cos 2\theta)_{\text{flavor}} \quad (2.18)$$

in mass and flavor bases, respectively.

In writing the Hamiltonian which describes the self-refraction of neutrinos, one should take into account the fact that there are two kinds of neutrino-neutrino scattering diagrams which add up coherently during the neutrino propagation. One of them is the forward scattering diagram in which there is no momentum transfer and both neutrinos remain in their original states after the scattering. These diagrams give rise to a diagonal refraction potential in the flavor basis, similar to the Mikheyev-Smirnov-Wolfenstein (MSW) potential. The other one is the exchange diagram in which the neutrinos exchange their states after the scattering. These diagrams give rise to a nondiagonal refraction potential in the flavor basis which is the source of the nonlinearity of the neutrino self-refraction problem [10,11,30]. Taking into account the contribution of both

kinds of diagrams mentioned above, the effective many-body Hamiltonian describing the self-refraction of a dense neutrino gas can be written as [56]

$$\begin{aligned}
H_{\nu\nu} &= \frac{G_F}{\sqrt{2}V} \sum_{\mathbf{p}} \sum_{\mathbf{q}} (1 - \cos \vartheta_{\mathbf{p}\mathbf{q}}) [a_e^\dagger(\mathbf{p}) a_e(\mathbf{p}) a_e^\dagger(\mathbf{q}) a_e(\mathbf{q}) \\
&\quad + a_x^\dagger(\mathbf{p}) a_x(\mathbf{p}) a_x^\dagger(\mathbf{q}) a_x(\mathbf{q}) + a_x^\dagger(\mathbf{p}) a_e(\mathbf{p}) a_e^\dagger(\mathbf{q}) a_x(\mathbf{q}) \\
&\quad + a_e^\dagger(\mathbf{p}) a_x(\mathbf{p}) a_x^\dagger(\mathbf{q}) a_e(\mathbf{q})].
\end{aligned} \quad (2.19)$$

Here V is the quantization volume and $\vartheta_{\mathbf{p}\mathbf{q}}$ is the angle between the momentum modes \mathbf{p} and \mathbf{q} . The factor $1 - \cos \vartheta_{\mathbf{p}\mathbf{q}}$ guarantees that those neutrinos traveling in the same direction do not undergo scattering.

One can write the neutrino self-refraction Hamiltonian in terms of the neutrino isospin operators defined in Sec. II A as follows:

$$H_{\nu\nu} = \frac{\sqrt{2}G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos \vartheta_{\mathbf{p}\mathbf{q}}) \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}. \quad (2.20)$$

Here, Eq. (2.20) can be directly obtained from Eq. (2.19) by using the definitions given in Eq. (2.2) and discarding those terms proportional to identity. Equation (2.20) tells us that in the language of flavor isospins the neutrino-neutrino interaction takes the form of a spin-exchange interaction. This is an expected result because the effective Hamiltonian given in Eq. (2.19) consists only of those neutrino-neutrino interactions in which neutrinos either keep or exchange their momenta.

The flavor evolution of a dense neutrino gas (in the absence of any other background) is described by the sum of the vacuum oscillation term and self-interaction term

$$H = \sum_{\omega} \omega \vec{B} \cdot \vec{J}_{\omega} + \mu \sum_{\mathbf{p},\mathbf{q}} (1 - \cos \vartheta_{\mathbf{p}\mathbf{q}}) \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}. \quad (2.21)$$

Here we defined

$$\mu = \frac{\sqrt{2}G_F}{V}. \quad (2.22)$$

In this study, we adopt what is commonly referred to as the ‘‘single angle approximation,’’ i.e., we will assume that the term involving $\cos \vartheta_{\mathbf{p}\mathbf{q}}$ in the Hamiltonian averages to zero so that the neutrinos traveling in different directions which are otherwise identical undergo the same flavor evolution. With this simplification, the total Hamiltonian becomes

$$H = -\sum_{\omega} \omega \mathcal{J}_{\omega}^z + \mu \vec{J} \cdot \vec{J} \quad (2.23a)$$

$$= \sum_{\omega} \omega \vec{B} \cdot \vec{J}_{\omega} + \mu \vec{J} \cdot \vec{J}. \quad (2.23b)$$

Note that the self-interaction term has the same form in both mass and flavor bases because these two bases are related by a *global* rotation as described in Eq. (2.13) which leaves *all* scalar products invariant.

It has been pointed out by many authors that the Hamiltonian given in Eq. (2.23) is analogous to the Hamiltonian of an *interacting spin system*, i.e., a group of spins interacting with a position-dependent *external magnetic field* and with each other via spin-exchange interaction. In the mass basis the external magnetic field points in the $-z$ direction whereas in flavor basis it points in the direction of the unit vector $\vec{B} = (\sin 2\theta, 0, -\cos 2\theta)$. This analogy makes it easier to see that the length of each isospin is conserved:

$$L_\omega = \vec{J}_\omega \cdot \vec{J}_\omega \quad [H, L_\omega] = 0. \quad (2.24)$$

Similarly, the total isospin component in the direction of the external magnetic field is also conserved:

$$C_0 = \vec{B} \cdot \vec{J} \quad [H, C_0] = 0. \quad (2.25)$$

It is worth noting that the Hamiltonian in Eq. (2.23) also appears in connection with the pairing problem. For example, in the BCS theory of superconductivity [48] the (reduced) pairing interaction between the valence electrons is described by the Hamiltonian²

$$H_{\text{BCS}} = \sum_k 2\epsilon_k t_k^z - GT^+ T^-. \quad (2.26)$$

Here G is a constant which represents the strength of the pairing interaction. Within the context of the BCS model, it is assumed that $G > 0$ leading to an attractive pairing and making the formation of Cooper pairs energetically favorable. ϵ_k are the degenerate single-particle energy levels which can be occupied by pairs of spin-up and spin-down electrons (i.e., the Cooper pairs). The operators

$$t_k^+ = c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger, \quad t_k^- = c_{k\downarrow} c_{k\uparrow} \quad \text{and} \quad t_k^z = \frac{1}{2}(c_{k\uparrow}^\dagger c_{k\uparrow} + c_{k\downarrow}^\dagger c_{k\downarrow} - 1) \quad (2.27)$$

are called *quasispin operators* and they also obey the same $SU(2)$ commutation relations as given in Eq. (2.3). In the quasispin scheme, a single particle state k has quasi-spin-up if it is occupied by a pair and quasi-spin-down if it is not. The operator

$$\vec{T} = \sum_k \vec{t}_k \quad (2.28)$$

represents the total quasispin. As can be easily verified, the third component of the total quasispin is a constant of motion, i.e.,

²The most general (i.e., nonreduced) pairing Hamiltonian has the form

$$H_{\text{BCS}} = \sum_k 2\epsilon_k t_k^z - G \sum_{k,k'} c_{kk'} t_k^+ t_{k'}^-,$$

where the dimensionless coefficients $c_{kk'}$ lead to a state-dependent pairing strength similar to the coefficient $1 - \cos \vartheta_{\text{pq}}$ in the neutrino Hamiltonian.

$$[H_{\text{BCS}}, T^z] = 0, \quad (2.29)$$

which is analogous to the conservation of C_0 mentioned in Eq. (2.25). From Eq. (2.27) we see that $2T^z$ is equal to the total number of electron pairs in the system.

The Hamiltonians given in Eqs. (2.23a) and (2.26) and are very similar. In fact, if we make the substitutions $\omega \rightarrow 2\epsilon_k$ and $\vec{J}_p \rightarrow \vec{t}_k$ and keep in mind that the ‘‘missing’’ term $GT^z(T^z - 1)$ in Eq. (2.26) is proportional to identity for a given number of pairs (i.e., has no effect on the evolution), then we see that the two models are mathematically identical up to an overall minus sign (remember that μ is by definition a positive quantity). The correspondence between these two models can be summarized as follows: Single particle states with energy ϵ_k in the BCS model correspond to the neutrino states with oscillation frequency ω . Both sets of states are multiply degenerate. A state ϵ_k which is occupied (respectively, unoccupied) by a pair in the BCS model corresponds to a neutrino state with oscillation frequency ω occupied by a neutrino in ν_1 (respectively, ν_2) mass eigenstate in the neutrino model.

It was first shown by Richardson in 1963 that the BCS Hamiltonian given in Eq. (2.26) can be diagonalized analytically with the method of algebraic Bethe ansatz [49,55]. Later work by Gaudin [50,51] and others [52–54] revealed that the integrability of the Hamiltonian in Eq. (2.26) derives from the existence of a set of *constants of motion* or *quantum invariants*. These results can be easily carried over from BCS model to the neutrino model. For example, it can be shown that the neutrino Hamiltonian given in Eq. (2.23) has the following constants of motion:

$$h_\omega = \vec{B} \cdot \vec{J}_\omega + 2\mu \sum_{\omega' (\neq \omega)} \frac{\vec{J}_\omega \cdot \vec{J}_{\omega'}}{\omega - \omega'}. \quad (2.30)$$

The operators given in Eq. (2.30) are known as the *Gaudin magnet Hamiltonians*. It is straightforward to show that these operators commute with one another and with the Hamiltonian given in Eq. (2.23), i.e.,

$$[h_\omega, h_{\omega'}] = 0 \quad \text{and} \quad [H, h_\omega] = 0 \quad (2.31)$$

are satisfied for every ω and ω' . Note that the invariants given in Eq. (2.30) are independent from one another and from the invariants mentioned in Eq. (2.24). However, the invariant C_0 mentioned in Eq. (2.25) is the sum of the Gaudin magnet Hamiltonians

$$C_0 = \sum_\omega h_\omega \quad (2.32)$$

and thus is not an independent invariant. The Hamiltonian itself is also a linear combination of the invariants given in Eqs. (2.24) and (2.30):

$$H = \sum_\omega \omega h_\omega + \sum_\omega L_\omega. \quad (2.33)$$

This tells us that if the neutrinos occupy Ω different energy modes, then the system has 2Ω independent constants of

motion given in Eqs. (2.24) and (2.30). These invariants can be expressed in several alternative forms which may be useful in different applications. In Appendix A, we present two alternative ways of writing down the invariants.

From the definitions given in Eq. (2.7), one can see that the constant C_0 can be written as

$$C_0 = -\mathcal{J}^z = \frac{\hat{N}_2 - \hat{N}_1}{2} \quad (2.34)$$

where

$$\hat{N}_1 = \sum_{\mathbf{p}} a_1^\dagger(\mathbf{p})a_1(\mathbf{p}) \quad \text{and} \quad \hat{N}_2 = \sum_{\mathbf{p}} a_2^\dagger(\mathbf{p})a_2(\mathbf{p}) \quad (2.35)$$

denote the total particle number operators for the mass eigenstates ν_1 and ν_2 , respectively. Note that in this paper we denote the particle *number operators* with hats (e.g., \hat{N}_i) and the corresponding particle *numbers* without hats (e.g., N_i). Together with the fact that the total number of neutrinos $N = N_1 + N_2$ is constant, the conservation of C_0 mentioned in Eq. (2.25) ensures that the Hamiltonian preserves the number of neutrinos in each mass eigenstate, i.e., we have

$$[H, \hat{N}_i] = 0, \quad (2.36)$$

for $i = 1, 2$.

Although C_0 , which is a combination of the invariants h_ω , can be simply expressed in terms of particle number operators as in Eq. (2.34), this is an exceptional situation. Apart from this particular case, the invariants h_ω or their combinations cannot be written in terms of the particle number operators. This is evident from the fact that the terms $\vec{J}_\omega \cdot \vec{J}_{\omega'}$ in Eq. (2.30) are diagonal in neither the mass basis nor the flavor basis and that these terms disappear only in the particular combination given in Eq. (2.32).

III. DIAGONALIZATION OF THE HAMILTONIAN

The existence of constants of motion is a manifestation of exact solvability of the pairing Hamiltonian given in Eq. (2.23). As mentioned earlier in previous sections, the pioneering work in this direction was that of Richardson who showed that the application of the Bethe ansatz method to the pairing Hamiltonian yields its exact eigenstates and eigenvalues in an analytical way. This exact solvability of the pairing Hamiltonian has been studied and exploited extensively thereafter both in the context of the nuclear shell model and in the context of BCS theory. In this section, we will review this procedure for a self-interacting neutrino gas.

The Bethe ansatz method, if applicable to a problem, usually reveals a general and simple *functional form* for the many-body eigenstates. In the case of the self-interacting neutrino gas, the many-body eigenstates turn out to resemble those of an harmonic oscillator in that they can be obtained by repeated application of a step operator on a lowest weight state. The crucial difference is that in the neutrino case the step operator is parametrized by a

complex number. This parameter takes on different values at each step and should be determined by solving the equations of Bethe ansatz as described in more detail below. The number of equations that one is required to solve in order to find the *full spectrum exactly* is of the order of the number of particles. For a dense neutrino gas, this is clearly unfeasible. However, if an approximate method of determining the Bethe ansatz variables can be found, then the simple step operator form of the eigenstates allows one to write down an (approximate) evolution operator for the system. Alternatively, one can consider a relatively small number of neutrinos occupying a very small volume so as to yield a large density. Mimicking the conditions in a dense environment in this way allows one to explore many-body physics with the exact eigenstates. One can then increase the number of neutrinos and the volume that they occupy keeping the density constant. The resulting limit of the Bethe ansatz equations have been studied extensively in the context of the BCS model (see, for example, Refs. [53,57–61] and the references therein) and in connection with matrix models that appear in 2D gravity [62]. This limit will be reviewed in Sec. III B.

We would like to note that a study of the general functional form of the eigenstates as revealed by the Bethe ansatz method can itself give insight into the collective behavior of the system. For example, it was shown in Ref. [63] that the expectation value of the step operator in the effective one-particle approximation to the pairing problem is the generating function for the canonical variables of the system and that the exact analytical solutions of the corresponding RPA equations of motion can be obtained in terms of these variables.

A. The Bethe ansatz method

The eigenstates of the neutrino Hamiltonian given in Eq. (2.23) can be easily found in the two opposite limits of the parameter μ defined in Eq. (2.22).

- (i) As μ approaches to zero, neutrinos occupy a larger and larger volume and neutrino-neutrino scattering becomes negligible. In this limit, the Hamiltonian consists only of vacuum oscillations given in Eq. (2.17). The eigenstates in this limit are simply the tensor products of mass eigenstates for individual neutrinos. Speaking in terms of the interacting spin system analogy, in this limit the spins interact only with the external magnetic field which is in the $-z$ direction in mass basis. The eigenstates are those in which each spin is either aligned or antialigned with the external magnetic field, i.e.,

$$\begin{aligned} & |\nu_1 \nu_1 \nu_1 \dots\rangle, & |\nu_2 \nu_1 \nu_1 \dots\rangle, & |\nu_1 \nu_2 \nu_1 \dots\rangle, \\ & |\nu_1 \nu_1 \nu_2 \dots\rangle, & \dots & \end{aligned} \quad (3.1)$$

- (ii) As μ approaches to infinity, neutrinos are crowded into a smaller and smaller volume and eventually

the neutrino-neutrino scattering term becomes dominant. In this limit, one can ignore the vacuum oscillations and write the Hamiltonian as

$$H_\infty \equiv \lim_{\mu \rightarrow \infty} H = \mu \vec{J} \cdot \vec{J}. \quad (3.2)$$

The *total isospin quantum number* j is a scalar quantity so that we can use it without any references to mass or flavor bases. Note that j can take several values starting from 0 or $1/2$ (depending on whether we have an even or odd number of neutrinos) up to $j_{\max} = N/2$ where N is the total number of neutrinos. In the limit where $\mu \rightarrow \infty$, both the total mass isospin states $|j, m\rangle_m$ and the total flavor isospin states $|j, m\rangle_f$ are eigenstates of the Hamiltonian with the same energy, i.e.,

$$\begin{aligned} H_\infty |j, m\rangle_m &= \mu j(j+1) |j, m\rangle_m \quad \text{and} \\ H_\infty |j, m\rangle_f &= \mu j(j+1) |j, m\rangle_f. \end{aligned} \quad (3.3)$$

It is clear from Eq. (2.13) that these two sets of states are related by

$$|j, m\rangle_f = U^\dagger |j, m\rangle_m. \quad (3.4)$$

Since the operator U involves only the total isospin operators [see Eq. (2.11)] it cannot change the representation in which a state lives, i.e., it preserves the value of j in accordance with the above arguments.

To illustrate the use of Eq. (3.4), let us consider a state in which all neutrinos are ν_e . Since this corresponds to having all flavor isospins up, this state is the highest weight state of the total flavor isospin algebra, i.e.,

$$|\nu_e \nu_e \nu_e \dots\rangle = |j_{\max}, j_{\max}\rangle_f. \quad (3.5)$$

This state can be converted to mass basis using Eqs. (2.11) and (3.4). The result is

$$\begin{aligned} |j_{\max}, j_{\max}\rangle_f &= U^\dagger |j_{\max}, j_{\max}\rangle_m \\ &= \sum_{m=-j_{\max}}^{j_{\max}} (\cos\theta)^{j_{\max}-m} (\sin\theta)^{j_{\max}+m} \\ &\quad \times \sqrt{\frac{(2j_{\max})!}{(j_{\max}-m)!(j_{\max}+m)!}} |j_{\max}, m\rangle_m. \end{aligned} \quad (3.6)$$

We see that the state $|j_{\max}, j_{\max}\rangle_f$ is a linear combination of the states $|j_{\max}, m\rangle_m$, which all live in the $j = j_{\max}$ representation and have the same energy in the limit where $\mu \rightarrow \infty$.

We next present the eigenstates and eigenvalues of the collective neutrino oscillation Hamiltonian given in Eq. (2.23) away from those two limits, i.e., for $0 < \mu < \infty$. We begin by noting that Eq. (2.36) implies that all

eigenstates of the collective oscillation Hamiltonian are also eigenstates of the number operators \hat{N}_1 and \hat{N}_2 , i.e., each many-body eigenstate has a definite number of neutrinos in ν_1 and ν_2 states.

In general, the total isospin states $|j, m\rangle_m$ and $|j, m\rangle_f$ mentioned in Eq. (3.3) are no longer eigenstates of the Hamiltonian for a finite value of μ but the highest and lowest weight states $|j, \pm j\rangle_m$ of the total mass isospin continue to be eigenstates. Let us consider, for example, the states in which all neutrinos occupy the same mass eigenstate, i.e., they are all ν_1 or all ν_2 . Speaking in the language of isospin, this is equivalent to having all mass isospins up or all mass isospins down, respectively. These situations, respectively, correspond to the total isospin states $|j_{\max}, j_{\max}\rangle_m$ and $|j_{\max}, -j_{\max}\rangle_m$:

$$\begin{aligned} |j_{\max}, j_{\max}\rangle_m &= \prod_{\mathbf{p}} a_1^\dagger(\mathbf{p}) |0\rangle \quad \text{and} \\ |j_{\max}, -j_{\max}\rangle_m &= \prod_{\mathbf{p}} a_2^\dagger(\mathbf{p}) |0\rangle. \end{aligned} \quad (3.7)$$

One can easily show that these states are eigenstates of the Hamiltonian given in Eq. (2.23) with the respective energies

$$\begin{aligned} E_{(+j_{\max})} &= -\sum_{\omega} \frac{\omega N_{\omega}}{2} + \mu j_{\max} (j_{\max} + 1) \quad \text{and} \\ E_{(-j_{\max})} &= \sum_{\omega} \frac{\omega N_{\omega}}{2} + \mu j_{\max} (j_{\max} + 1). \end{aligned} \quad (3.8)$$

Here we denote the total number of neutrinos in an individual energy mode by N_{ω} . The other highest and lowest weight states $|j, \pm j\rangle_m$ of the total mass isospin are also eigenstates of the Hamiltonian. To see this, let us first consider the total isospin quantum number j_{ω} of an energy mode which can take several values from 0 or $1/2$ (depending on whether we have an even or odd number of neutrinos in this mode) up to $N_{\omega}/2$, possibly with many multiplicities. For each energy mode, we can write

$$\mathcal{J}_{\omega}^z |j_{\omega}, \pm j_{\omega}\rangle_m = \pm j_{\omega} |j_{\omega}, \pm j_{\omega}\rangle_m. \quad (3.9)$$

Therefore the states

$$\begin{aligned} |j, j\rangle_m &\equiv |j_{\omega_1}, j_{\omega_1}\rangle_m \otimes |j_{\omega_2}, j_{\omega_2}\rangle_m \otimes \dots \otimes |j_{\omega_{\Omega}}, j_{\omega_{\Omega}}\rangle_m \\ |j, -j\rangle_m &\equiv |j_{\omega_1}, -j_{\omega_1}\rangle_m \otimes |j_{\omega_2}, -j_{\omega_2}\rangle_m \otimes \dots \otimes |j_{\omega_{\Omega}}, -j_{\omega_{\Omega}}\rangle_m \end{aligned} \quad (3.10)$$

are simultaneous eigenstates of all \mathcal{J}_{ω}^z [see Eq. (3.9)] as well as the quadratic operator $\vec{\mathcal{J}} \cdot \vec{\mathcal{J}}$ [see Eq. (3.3)]. As a result, the states given in Eqs. (3.10) are eigenstates of the Hamiltonian (2.23) for all values of μ with the respective energies

$$\begin{aligned}
E_{(+j)} &= -\sum_{\omega} \omega j_{\omega} + \mu j(j+1) \quad \text{and} \\
E_{(-j)} &= \sum_{\omega} \omega j_{\omega} + \mu j(j+1).
\end{aligned}
\tag{3.11}$$

In Eqs. (3.10), we used Ω to denote the number of different energy modes. In this equation the total angular momentum $j = j_{p_1} + j_{p_2} + \dots + j_{p_{\Omega}}$ may come with many multiplicities because in general more than one combination of $j_{p_1}, j_{p_2}, \dots, j_{p_{\Omega}}$ may correspond to the same sum j . Such multiplicities are inherent in the addition of angular momenta and one usually introduces additional quantum numbers to distinguish the resulting degenerate total angular momentum states. However, in this paper we avoid introducing such quantum numbers to keep the formulas readable. Since the energies given in Eq. (3.11) depend only on j_{ω} and j , this omission does not lead to confusion in what follows. Finally note that Eqs. (3.7) and (3.8) correspond the *unique* special case of Eqs. (3.10) and (3.11) in which we have $j_{\omega} = N_{\omega}/2$ for all energy modes and thus $j = j_{\max}$.

The eigenstates of the Hamiltonian other than the highest and lowest weight states $|j, \pm j\rangle_m$ can be found with the Bethe ansatz technique. This method is based on writing down a trial state depending on a set of unknown parameters which are called *Bethe ansatz variables*. The values of these parameters are determined subject to the requirement that the state be an eigenstate of the Hamiltonian. For the particular problem at hand, the trial state is formed with the help of the *Gaudin operators*

$$\mathcal{Q}^{\pm}(\xi) = \sum_{\omega} \frac{\mathcal{J}_{\omega}^{\pm}}{\omega - \xi}.
\tag{3.12}$$

Here ξ is a complex parameter which will later play the role of a Bethe ansatz variable. In order to demonstrate the method of Bethe ansatz, let us consider the state $|j, -j\rangle_m$ defined in Eq. (3.10). This state contains $N_1 = N/2 - j$ neutrinos in mass eigenstate ν_1 and $N_2 = N/2 + j$ neutrinos in mass eigenstate ν_2 . The operator $\mathcal{Q}^+(\xi)$ turns one ν_2 into ν_1 such that the state

$$|\xi\rangle \equiv \mathcal{Q}^+(\xi)|j, -j\rangle_m
\tag{3.13}$$

contains one more ν_1 and one less ν_2 . It is, in fact, a linear superposition of many such states depending on the unknown parameter ξ . In order to find for which value(s) of ξ the state in Eq. (3.13) is an eigenstate, we act on it with the Hamiltonian given in Eq. (2.23). The result is

$$\begin{aligned}
H\mathcal{Q}^+(\xi)|j, -j\rangle_m &= (E_{(-j)} - \xi - 2\mu j)\mathcal{Q}^+(\xi)|j, -j\rangle_m \\
&\quad - \left(1 + 2\mu \sum_{\omega} \frac{-j_{\omega}}{\omega - \xi}\right)\mathcal{J}^+|j, -j\rangle_m.
\end{aligned}
\tag{3.14}$$

Here $E_{(-j)}$ is the energy of the state $|j, -j\rangle_m$ given in Eq. (3.11). In order for the state $\mathcal{Q}^+(\xi)|j, -j\rangle_m$ to be an

eigenstate of the Hamiltonian, we should choose ξ in such a way that the second term on the right-hand side of Eq. (3.14) vanishes. This tells us that ξ should obey

$$\sum_{\omega} \frac{-j_{\omega}}{\omega - \xi} = -\frac{1}{2\mu}.
\tag{3.15}$$

Equation (3.15) is called a *Bethe ansatz equation* and in general it has several solutions. It is clear from Eq. (3.14) that every ξ which satisfies Eq. (3.15) gives us an eigenstate in the form of Eq. (3.13) with the energy

$$E(\xi) = E_{(-j)} - \xi - 2\mu j.
\tag{3.16}$$

One can extend this line of thought to find more eigenstates of the Hamiltonian. In the most general case, the Gaudin operator plays the role of a one-parameter step operator, i.e., the eigenstates can be obtained by its repeated application on a lowest weight state but the parameter ξ takes on different values at each step. It can be shown that a state in the form

$$|\xi_1, \xi_2, \dots, \xi_{\kappa}\rangle \equiv \mathcal{Q}^+(\xi_1)\mathcal{Q}^+(\xi_2)\dots\mathcal{Q}^+(\xi_{\kappa})|j, -j\rangle_m
\tag{3.17}$$

is an eigenstate of the Hamiltonian with the energy

$$E(\xi_1, \xi_2, \dots, \xi_{\kappa}) = E_{(-j)} - \sum_{\alpha=1}^{\kappa} \xi_{\alpha} - \kappa\mu(2j - \kappa + 1),
\tag{3.18}$$

if the Bethe ansatz variables $\xi_1, \xi_2, \dots, \xi_{\kappa}$ obey the *Bethe ansatz equations*

$$\sum_{\omega} \frac{-j_{\omega}}{\omega - \xi_{\alpha}} = -\frac{1}{2\mu} + \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{\kappa} \frac{1}{\xi_{\alpha} - \xi_{\beta}}.
\tag{3.19}$$

Eqs. (3.19) form a set of κ complex equations in κ complex variables that have to be simultaneously satisfied for every $\alpha = 1, 2, \dots, \kappa$. As stated earlier, the state $|j, -j\rangle_m$ contains respectively $N_1 = N/2 - j$ and $N_2 = N/2 + j$ neutrinos in mass eigenstates ν_1 and ν_2 . Since each one of the Gaudin operators $\mathcal{Q}^+(\xi_{\alpha})$ transforms one ν_2 into ν_1 , corresponding occupancies for the state in Eq. (3.17) are $N_1 = N/2 - j + \kappa$ and $N_2 = N/2 + j - \kappa$. It should be noted that if we add one more Gaudin operator to the state in Eq. (3.17) to transform one more neutrino from ν_2 to ν_1 , then the Bethe ansatz equations (3.19) are modified and become a *new* set of $\kappa + 1$ equations in $\kappa + 1$ variables. This new set of equations are different from the earlier ones with κ variables and have different solutions which could be denoted as $(\check{\xi}_1, \check{\xi}_2, \dots, \check{\xi}_{\kappa}, \check{\xi}_{\kappa+1})$. Each time one adds one more Gaudin operator to Eq. (3.17) to find new eigenstates with different occupancies, one has to solve a new set of coupled algebraic equations. Note that the example described in Eqs. (3.13), (3.14), (3.15), and (3.16) is a special case of the general scheme described in Eqs. (3.17), (3.18), and (3.19) with $\kappa = 1$. In this particular case, the sum on the right-hand side of Eq. (3.19) vanishes because there is only one Bethe ansatz variable.

It is obvious from the above remarks that it becomes more and more difficult to find eigenstates as we flip more and more neutrinos from ν_2 into ν_1 . However, a symmetry transformation between the eigenstates of the Hamiltonian is helpful in reducing the number of Bethe ansatz equations that has to be solved. In order to present this symmetry, let us first observe that the operator

$$T = e^{-i\pi\mathcal{J}^x} \quad (3.20)$$

transforms ν_1 and ν_2 neutrinos into each other. In particular, it exchanges the states $|j, -j\rangle_m$ and $|j, j\rangle_m$ defined in Eqs. (3.10)

$$T^\dagger |j, -j\rangle_m = |j, j\rangle_m \quad (3.21)$$

and it also transforms the isospin operators as

$$T^\dagger \mathcal{J}_\omega^\pm T = \mathcal{J}_\omega^\mp \quad \text{and} \quad T^\dagger \mathcal{J}_\omega^z T = -\mathcal{J}_\omega^z. \quad (3.22)$$

As a result, the Hamiltonian given in Eq. (2.23a) is transformed as

$$T^\dagger H T = H' = \sum_\omega \omega \mathcal{J}_\omega^z + \vec{\mathcal{J}} \cdot \vec{\mathcal{J}}. \quad (3.23)$$

Elementary quantum mechanics tells us that if a state $|\psi'\rangle$ is an eigenstate of the Hamiltonian $H' = T^\dagger H T$ with a particular energy, then the state $|\psi\rangle = T|\psi'\rangle$ is an eigenstate of the Hamiltonian H with the same energy. Therefore, if we find the eigenstates and eigenvalues of the Hamiltonian H' with the method presented above (by making the change $\omega \rightarrow -\omega$) and then transform the results with the operator T , we arrive at the eigenstates of the Hamiltonian H given in (2.23). In this way, one can show that the states

$$|\xi'_1, \xi'_2, \dots, \xi'_\kappa\rangle \equiv \mathcal{Q}^-(\xi'_1) \mathcal{Q}^-(\xi'_2) \dots \mathcal{Q}^-(\xi'_\kappa) |j, j\rangle_m \quad (3.24)$$

are eigenstates of the Hamiltonian H with the energy

$$E(\xi'_1, \xi'_2, \dots, \xi'_\kappa) = E_{(+j)} + \sum_{\alpha=1}^{\kappa} \xi'_\alpha - \kappa\mu(2j - \kappa + 1), \quad (3.25)$$

if the variables $\xi'_1, \xi'_2, \dots, \xi'_\kappa$ obey the Bethe ansatz equations

$$\sum_\omega \frac{-j_\omega}{\omega - \xi'_\alpha} = \frac{1}{2\mu} + \sum_{\substack{\beta=1 \\ (\beta \neq \alpha)}}^{\kappa} \frac{1}{\xi'_\alpha - \xi'_\beta}. \quad (3.26)$$

These Bethe ansatz equations are similar to the ones given in Eq. (3.19) except that the sign of the constant term is reversed. The term $E_{(+j)}$ which appears in Eq. (3.25) is the energy of the state $|j, j\rangle_m$ given in Eq. (3.11). The state $|j, j\rangle_m$ which is defined in Eq. (3.10) contains respectively $N_1 = N/2 + j$ and $N_2 = N/2 - j$ neutrinos in mass eigenstates ν_1 and ν_2 . Each Gaudin operator $\mathcal{Q}^-(\xi'_\alpha)$ transforms one ν_1 into ν_2 so that the corresponding occupancies of the state in Eq. (3.24) are $N_1 = N/2 + j - \kappa$ and $N_2 = N/2 - j + \kappa$. We see that the occupancies of the mass eigenstates ν_1 and ν_2 are reversed in the state

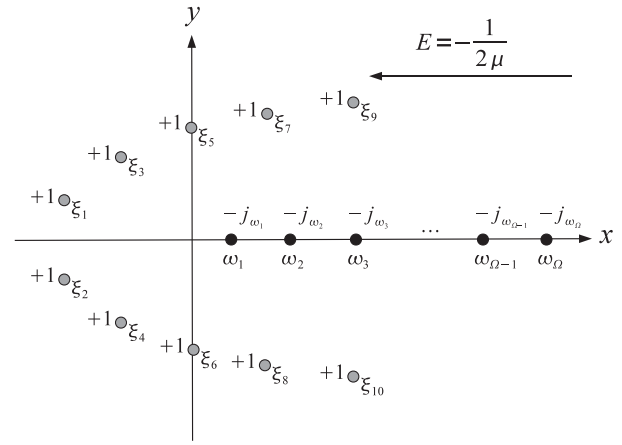


FIG. 1. The Bethe ansatz equations (3.19) can be viewed as the stability conditions for a group of point charges in two dimensions as shown here.

given in Eq. (3.24) as compared to the state given in Eq. (3.17) but both states are reached by solving the same number of equations. Therefore, depending on the occupancies of the mass eigenstates ν_1 and ν_2 , either the method presented in Eqs. (3.17), (3.18), and (3.19) or the method presented in Eqs. (3.24), (3.25), and (3.26) is more economical to use.

B. Electrostatic analogy

The Bethe ansatz equations given in Eqs. (3.19) and (3.26) can be viewed as the stability conditions for a group of point charges in two dimensions³ [59]. Let us consider Eqs. (3.19) for which the electrostatic model is depicted in Fig. 1. There are two kinds of charges in this scheme: the *fixed negative charges* and the *free positive charges*. For each energy mode we have a fixed point charge of magnitude $-j_\omega$ pinned at the point ω on the real axis and for each Bethe ansatz variable we have a free charge of magnitude $+1$ whose equilibrium position yields ξ_α . There is also a constant and uniform electric field in the $-x$ direction as shown in the figure.

In two dimensions, one can use a complex coordinate $z = x + iy$ instead of two real coordinates (x, y) . The electrostatic potential at the point z created by a charge q sitting at the point z_0 is proportional to $q \ln|z - z_0|$. As a result, the total electrostatic energy of the charge configuration shown in Fig. 1 is proportional to

$$V \propto \frac{1}{2\mu} \sum_\alpha \text{Re}(\xi_\alpha) - \frac{1}{2\mu} \sum_\omega j_\omega \text{Re}(\omega) - \frac{1}{2} \sum_{\substack{\alpha, \beta \\ (\alpha \neq \beta)}} \ln|\xi_\alpha - \xi_\beta| - \frac{1}{2} \sum_{\substack{\omega, \omega' \\ \omega \neq \omega'}} j_\omega j_{\omega'} \ln|\omega - \omega'| + \sum_{\alpha, \omega} j_\omega \ln|\xi_\alpha - \omega|. \quad (3.27)$$

³Alternatively, one can imagine infinite lines of charge in three dimensions perpendicular to a plane.

The free charges come to an equilibrium when the electrostatic potential energy reaches a local minimum. The minimum energy condition is obtained by setting

$$\frac{\partial V}{\partial \xi_\alpha} = 0, \quad (3.28)$$

for every $\alpha = 1, 2, \dots, \kappa$. It is a straightforward calculation to show that Eqs. (3.27) and (3.28) lead to the Bethe ansatz equations (3.19) which now tell that the total force on each free charge is zero. A similar electrostatic picture for the Bethe ansatz equations (3.26) can be obtained by reversing the direction of the electric field in Fig. 1. Note that since the solutions of Bethe ansatz equations come in complex conjugate pairs, the organization of the free charges is always symmetric with respect to x axis.

The electrostatic analogy described here provides an intuitive picture and can be particularly useful in finding the solutions of Bethe ansatz equations in the limit of a large number of neutrinos. In this scheme, the total amount of fixed charge on the x axis is equal to $-j$. Although j takes all values between 0 and $N/2$, for most eigenstates its value is very large. The total amount of free charges is given by

$$\kappa = j + (N_1 - N_2)/2. \quad (3.29)$$

Here we assume that j and κ are of same order of magnitude which is true for most eigenstates.

Suppose that we start with a small number of neutrinos in a very small volume so that the neutrino density is in the regime where the neutrino-neutrino interactions are important. In this case we have a small amount of charge on the x axis, a few free charges for a typical eigenstate and a small external electric field in the $-x$ direction. Note that, according to the definition of μ given in Eq. (2.22), the external electric field $-1/2\mu$ is proportional to the volume occupied by the neutrinos. Now, suppose that we increase the number of neutrinos and the volume which they occupy by keeping the density constant. In this case the external electric field grows proportionally while the total fixed charge on the x axis and the number of free charges also increase. For a realistic neutrino spectrum, fixed charges form a continuous charge distribution on the x axis. However, for the sake of this discussion, we assume that they are combined into discrete energy bins such that each j_ω is of the order of j . Numerical solutions of Bethe ansatz equations suggest that in this limit the free charges form (piecewise) continuous distributions. Such solutions have been worked out in the context of the electron pairing in superconductors [53,57–61]. An intuitive way of dealing with this limit was introduced in Ref. [58] and is based on an expansion of the total electrostatic field in powers of $1/\kappa$. In what follows we closely follow this reference.

Let us begin by considering the total electrostatic field of the system in Fig. 1. It is given by

$$F(z) = -\frac{1}{2\mu} + \sum_{\omega} \frac{-j_{\omega}}{z - \omega} + \sum_{\alpha=1}^{\kappa} \frac{1}{z - \xi_{\alpha}}. \quad (3.30)$$

Using the Bethe ansatz equations (3.19), it is straightforward to show that the electrostatic field obeys the following differential equation:

$$\frac{dF}{dz} + F^2 = \frac{1}{2} \sum_{\omega} \frac{j_{\omega}}{(z - \omega)^2} + \left(\sum_{\omega} \frac{-j_{\omega}}{z - \omega} + \frac{1}{2\mu} \right)^2 - \sum_{\omega} \frac{j_{\omega} H(\omega)}{z - \omega}. \quad (3.31)$$

Here $H(\omega)$ is the electric field produced by the free charges at the position of the fixed charge $-j_{\omega}$. It is given by

$$H(\omega) = \sum_{\alpha=1}^{\kappa} \frac{1}{\omega - \xi_{\alpha}}. \quad (3.32)$$

$H(\omega)$ can also be expressed as a contour integral

$$H(\omega) = \frac{1}{2\pi i} \oint_C \frac{F(z)}{\omega - z}. \quad (3.33)$$

Here C is a contour which encloses only those singularities of the field $F(z)$ due to the free charges. The equivalence of Eqs. (3.32) and (3.33) can be shown by direct substitution of the field $F(z)$ given in Eq. (3.30) in Eq. (3.33).

Let us now consider the multipole expansion of the electrostatic field $F(z)$:

$$F(z) = \sum_{m=0}^{\infty} F^{(m)} z^{-m}. \quad (3.34)$$

A direct expansion of Eq. (3.30) gives these multipole moments as

$$F^{(0)}(z) = -\frac{1}{2\mu}, \quad (3.35)$$

$$F^{(1)}(z) = \kappa - j, \quad (3.36)$$

$$F^{(n)}(z) = \sum_{\alpha=1}^{\kappa} \xi_{\alpha}^n - \sum_{\omega} \omega^n j_{\omega} \quad n \geq 2, \quad (3.37)$$

for $|z| > \max(\omega, |\xi_{\alpha}|)$. The zeroth-order moment gives the value of the field at infinity and therefore is equal to the value of the constant field. The first-order moment is equal to the total amount of charge, as expected. By comparing Eqs. (3.37) and (3.18) one can see that the energy of an eigenstate can be found from the second-order moment of the corresponding electrostatic field.

Richardson's method is based on solving the differential equation given in Eq. (3.31) for the leading terms in an expansion of the form

$$F(z) = \sum_{r=0}^{\infty} F_r(z). \quad (3.38)$$

Here the term $F_r(z)$ is assumed to be of the order of κ^{1-r} . In the present case, we will consider only the $r = 0$ term which is of the order of κ . It is reasonable to assume that those terms on the right-hand side of the differential equation which are of the form $\sum_{\omega} j_{\omega}(\dots)$ are of the order of j . Since the volume is increased in proportion with the number of neutrinos, we also treat the external field $-1/2\mu$ as being of the order of j . Finally, Eq. (3.32) tells us that H_{ω} is of the order of κ . Taking these into account and substituting the expansion given in Eq. (3.38) into the differential Eq. (3.31) we find

$$F_0(z)^2 = \left(\sum_{\omega} \frac{j_{\omega}}{(z-\omega)^2} + \frac{1}{2\mu} \right)^2 - \sum_{\omega} \frac{j_{\omega} H_0(\omega)}{z-\omega}, \quad (3.39)$$

where $H_0(\omega)$ is given by

$$H_0(\omega) = \frac{1}{2\pi i} \oint_C \frac{F_0(z)}{\omega - z}. \quad (3.40)$$

Equation (3.39) is an integral equation which includes only the fixed charges as parameters and should be solved to find the leading-order electrostatic field created by the whole configuration including the free charges in their stable configuration. For this reason, it can be thought as a replacement for Bethe ansatz equations. Once a solution of Eq. (3.39) is obtained, one can determine the leading-order contribution to the energy of the corresponding eigenstate from the second-order moment of the field. The locations of the free charges can be found from the singularities (or the branch cuts) of the field other than the fixed charges and these can be used to write down the corresponding eigenstate itself.

However, instead of directly solving Eq. (3.39), a more practical way is to form its solutions on physical grounds. The typical way to proceed is to start with a small number of neutrinos in a small volume and see the way the solutions of Bethe ansatz equations organize themselves as the number of neutrinos are increased while the density is kept at a constant value. As mentioned before, numerical simulations suggest that the free charges coalesce to form (piecewise) continuous charge distributions. One can guess the limiting shape that they will assume possibly in terms of some unknown parameters. These parameters can later be determined either by forming a field $F_0(z)$ which describes such a distribution to the leading order and substituting it in Eq. (3.39) or, in simple situations, directly from self-consistency requirements as described in the following example.

For example, it is known that a solution of Bethe ansatz equations exists in which all free charges organize themselves into a single arc extending from a point a to a point a^* (the parameters to be determined). The situation is similar to the one depicted in Fig. 1 except that the free charges now form a continuous curve. In the case of the BCS model, this solution is known to lead to the BCS ground state. Since the singularities due to the free charges

form a continuum, the field $F_0(z)$ is expected to have a branch cut along the arc such that

$$\frac{1}{2\pi i} \oint_C F_0(z) dz = \kappa \quad (3.41)$$

is satisfied for any closed path C enclosing the free charges. Note that it is reasonable to also demand that the field $F_0(z)$ describes the fixed charges and the external field correctly, i.e.,

$$\lim_{z \rightarrow \omega} (z - \omega) F_0(z) = j_{\omega} \quad \text{and} \quad \lim_{z \rightarrow \infty} F_0(z) = -\frac{1}{2\mu} \quad (3.42)$$

because both j_{ω} and the external field are of the order of j . In this case, the zeroth and the first-order moments given in Eqs. (3.35) and (3.36) will receive no corrections from the higher-order terms in the expansion of electrostatic field given in Eq. (3.38).

In Ref. [58], it is argued that a candidate for the field $F_0(z)$ with these properties is given by

$$F_0(z) = -[(z-a)(z-a^*)]^{1/2} \sum_{\omega} \frac{j_{\omega}}{|\omega-a|(z-\omega)}. \quad (3.43)$$

This field is of the order of j and has a branch cut along an arc which extends from a to a^* . It also has singularities at the points ω and a constant limit as $z \rightarrow \infty$ as required by the above arguments. The first three moments of the field $F_0(z)$ are given by

$$F_0^{(0)} = -\sum_{\omega} \frac{j_{\omega}}{|\omega-a|}, \quad (3.44)$$

$$F_0^{(1)} = \sum_{\omega} \frac{j_{\omega}(\lambda - \omega)}{|\omega-a|}, \quad (3.45)$$

$$F_0^{(2)} = \sum_{\omega} \frac{j_{\omega}(\lambda\omega - \omega^2 - \Delta^2/2)}{|\omega-a|}, \quad (3.46)$$

where $a = \lambda + i\Delta$. One can try to determine the value of the parameter a by demanding that $F_0(z)$ reproduces the zeroth and the first-order moments without any contributions from higher-order fields as mentioned above. Equating the moments in Eqs. (3.44) and (3.45) to those in Eqs. (3.35) and (3.36) one finds the equations

$$\frac{1}{2\mu} = \sum_{\omega} \frac{j_{\omega}}{|\omega-a|} \quad \frac{\lambda}{2\mu} + j - \kappa = \sum_{\omega} \frac{\omega j_{\omega}}{|\omega-a|}, \quad (3.47)$$

whose solution yields the value of $a = \lambda + i\Delta$. It is shown in Ref. [58] that the substitution of $F_0(z)$ given in Eq. (3.43) with the value of a satisfying Eqs. (3.47) shows that it is indeed the correct solution of the differential Eq. (3.39). Note that, in the context of the BCS theory, the parameters Δ and λ are known as the BCS gap and the chemical potential, respectively. Accordingly, Eqs. (3.47) are called the BCS gap and chemical potential equations.

It was already mentioned that the second-order moment of the total electrostatic field is related to the energy of the corresponding eigenstate. Using Eqs. (3.18), (3.37), (3.46), and (3.47), the energy of the eigenstate corresponding to the solution given in Eq. (3.43) can be found as

$$E = E_{(-j)} - \kappa\mu(2j - \kappa + 1) - \sum_{\omega} j_{\omega} \omega \left(1 - \frac{\omega - \lambda}{|\omega - a|}\right) + \frac{\Delta^2}{4\mu} \quad (3.48)$$

to the first order in j . Here $E_{(-j)}$ is given by Eq. (3.11).

Note that the neutrino Hamiltonian given in Eq. (2.23a) has an overall minus sign with respect to the BCS pairing Hamiltonian given in Eq. (2.26). Therefore, although the field $F_0(z)$ given in Eq. (3.43) yields the ground state of the (reduced) BCS pairing Hamiltonian, for the neutrino case, it yields the eigenstate with the highest energy. A study of the general properties of Bethe ansatz states in the continuum limit from the perspective of collective neutrino oscillations is an open problem and will be considered elsewhere.

IV. THE RANDOM PHASE APPROXIMATION (RPA)

A large quantum system such as a self-interacting neutrino gas can be conveniently studied within an effective one-particle approximation whereby it is described in terms of single particles interacting with an average potential created by all other particles in the medium. The requirement that the average potential should evolve in line with the time evolution of individual particles gives rise to a set of consistency equations. In this section, we formulate the effective one-particle description in terms of the RPA method. Our aim is to reach a quantum mechanical effective one-body Hamiltonian which describes the interaction of a single neutrino with its background represented by the classical polarization vectors. In this picture, the well-known time evolution equations of the polarization vectors emerge from the RPA consistency condition. In Sec. IV A, we show that the time evolution equations of the polarization vectors possess several constants of motion which are basically the expectation values of the many-body constants of motion mentioned in Sec. II B. One of the main results of Sec. IV involves the diagonalization of the linearized RPA Hamiltonian which yields a noninteracting (quasiparticle) basis for neutrinos. We show in Sec. IV B that the well-known phenomenon of spectral splits can be viewed as a result of the adiabatic evolution of these quasiparticle degrees of freedom from flavor to mass eigenstates under appropriate conditions.

A. Equations of motion

In the method of RPA, a quadratic operator $\mathcal{O}_1\mathcal{O}_2$ is approximated as

$$\mathcal{O}_1\mathcal{O}_2 \sim \mathcal{O}_1\langle\mathcal{O}_2\rangle + \langle\mathcal{O}_1\rangle\mathcal{O}_2 - \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle, \quad (4.1)$$

where the expectation values should be calculated with respect to a state $|\Psi\rangle$ which satisfies the condition

$$\langle\mathcal{O}_1\mathcal{O}_2\rangle = \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle. \quad (4.2)$$

The state $|\Psi\rangle$ is usually found by solving the resulting RPA consistency equations as illustrated below.

Application of the RPA method to the neutrino Hamiltonian given in Eq. (2.23) yields

$$H \sim H^{\text{RPA}} = \sum_{\omega} \omega \vec{B} \cdot \vec{J}_{\omega} + \mu \vec{P} \cdot \vec{J}. \quad (4.3)$$

In Eq. (4.3), we introduced the *polarization vector* which is defined as

$$\vec{P}_{\mathbf{p}} = 2\langle\vec{J}_{\mathbf{p}}\rangle. \quad (4.4)$$

The total polarization vectors are defined as in Eq. (2.4). In Eq. (4.4), the expectation values are calculated with respect to a state $|\Psi\rangle$ which is assumed to be a tensor product of one-particle states, i.e.,

$$|\Psi\rangle \equiv |\psi(\mathbf{p}_1)\rangle \otimes |\psi(\mathbf{p}_2)\rangle \otimes \dots \otimes |\psi(\mathbf{p}_N)\rangle, \quad (4.5)$$

where

$$|\psi(\mathbf{p})\rangle = \psi_e(\mathbf{p})|v_e\rangle + \psi_x(\mathbf{p})|v_x\rangle. \quad (4.6)$$

The state $|\Psi\rangle$ defined in Eq. (4.5) is an $SU(2)$ coherent state and thus automatically obeys the condition given in Eq. (4.2) [see Ref. [21]]. Substitution of the state $|\Psi\rangle$ in Eq. (4.4) yields the explicit form of the polarization vector as follows⁴:

$$\vec{P}_{\mathbf{p}} = \begin{pmatrix} 2 \operatorname{Re}[\psi_e^*(\mathbf{p})\psi_x(\mathbf{p})] \\ 2 \operatorname{Im}[\psi_e^*(\mathbf{p})\psi_x(\mathbf{p})] \\ |\psi_e(\mathbf{p})|^2 - |\psi_x(\mathbf{p})|^2 \end{pmatrix}. \quad (4.7)$$

The linearized Hamiltonian given in Eq. (4.3) yields the following Heisenberg equations of motion

$$\frac{d}{dt} \vec{J}_{\omega} = -i[\vec{J}_{\omega}, H^{\text{RPA}}] = (\omega \vec{B} + \mu \vec{P}) \times \vec{J}_{\omega}. \quad (4.8)$$

Because of the self-consistency requirement of the RPA method, the polarization vectors should then obey

$$\frac{d}{dt} \vec{P}_{\omega} = (\omega \vec{B} + \mu \vec{P}) \times \vec{P}_{\omega} \quad (4.9)$$

⁴An alternative way to define the neutrino polarization vector is to use the density matrix method. For a single neutrino with momentum \mathbf{p} , the elements of the flavor density matrix are defined as $\rho_{\alpha\beta}(\mathbf{p}) = \langle a_{\beta}^{\dagger}(\mathbf{p})a_{\alpha}(\mathbf{p}) \rangle$. For the state $|\Psi\rangle$ given in Eq. (4.5), this definition amounts to

$$\rho(\mathbf{p}) = \begin{pmatrix} |\psi_e(\mathbf{p})|^2 & \psi_x^*(\mathbf{p})\psi_e(\mathbf{p}) \\ \psi_e^*(\mathbf{p})\psi_x(\mathbf{p}) & |\psi_x(\mathbf{p})|^2 \end{pmatrix}.$$

Then, one can define the polarization vector through the expansion $\rho(\mathbf{p}) = \frac{1}{2}(1 + \vec{\sigma} \cdot \vec{P}_{\mathbf{p}})$ in terms of the Pauli sigma matrices $\vec{\sigma}$. This definition leads to the same result as in Eq. (4.7).

for every ω . Equation (4.9) can be obtained by taking the expectation values of the isospin operators on both sides of Eq. (4.8). Solution of the consistency equations yields the state $|\Psi\rangle$ postulated at Eq. (4.5).

Note that in approximating the exact many-body Hamiltonian with the RPA method, one sacrifices the conservation of the many-body invariants h_ω given in Eq. (2.30) because the RPA Hamiltonian no longer commutes with these operators. However, the expectation values of the operators h_ω with respect to the state $|\Psi\rangle$ continue to be invariant. The expectation values $\langle h_\omega \rangle$ can be easily calculated using Eqs. (4.2) and (4.4). The result is

$$I_\omega = 2\langle h_\omega \rangle = \vec{B} \cdot \vec{P}_\omega + \mu \sum_{\omega' (\neq \omega)} \frac{\vec{P}_\omega \cdot \vec{P}_{\omega'}}{\omega - \omega'}. \quad (4.10)$$

Here a factor of 2 is introduced in the definition of I_ω for convenience. Using the RPA equations of motion (4.9), it is a straightforward calculation to show that

$$\frac{d}{dt} I_\omega = 0. \quad (4.11)$$

The particular combination

$$K_0 \equiv \sum_{\omega} I_\omega = \vec{B} \cdot \vec{P} \quad (4.12)$$

is already pointed out to be a constant of motion by many authors. Investigation of the N -mode coherence in collective neutrino oscillations leads to similar invariants obtained as the scalar product of the polarization vector and its Hilbert transform [23]. The invariants we find *in the RPA formalism* are closely related to the invariants of such N -mode coherence. It should be mentioned out that such invariants were first discussed in the context of BCS dynamics [63].

In the mass basis, we denote the polarization vectors by calligraphic letters as in the case of isospin operators, i.e., $\vec{P}_p = 2\langle \vec{J}_p \rangle$. From Eq. (2.7) we see that the conserved quantity given in Eq. (4.12) is equal to

$$K_0 = -\mathcal{P}^z = \frac{\langle \hat{N}_2 \rangle - \langle \hat{N}_1 \rangle}{2}. \quad (4.13)$$

Together with the conservation of $\langle \hat{N}_1 \rangle + \langle \hat{N}_2 \rangle$, this implies that both $\langle \hat{N}_1 \rangle$ and $\langle \hat{N}_2 \rangle$ are individually conserved under the RPA evolution. Therefore, although the exact many-body Hamiltonian given in Eq. (2.23) strictly conserves the number of particles in each mass eigenstate [see Eq. (2.36)], the RPA evolution conserves only their average values.

B. Stationary solutions and the spectral splits

It has been pointed out in Sec. II B that the BCS pairing Hamiltonian and the collective neutrino oscillation Hamiltonian have the same mathematical structure. This equivalence has been exploited in Sec. III in order to write down the exact eigenstates and eigenvalues of the

many-body neutrino Hamiltonian. In this section, we will consider the eigenstates and eigenvalues of the RPA neutrino Hamiltonian given in Eq. (4.3). These eigenstates correspond to the steady-state solutions of Eqs. (4.9). To this end, we will borrow the method of Bogoliubov transformations from the RPA-BCS model by which the well-known BCS ground state and its excitations are obtained.

Application of the method of RPA to the BCS pairing Hamiltonian given in Eq. (2.26) yields

$$H_{\text{BCS}}^{\text{RPA}} = \sum_k 2\epsilon_k t_k^z - G(\langle T^+ \rangle T^- + T^+ \langle T^- \rangle). \quad (4.14)$$

In order to find the ground state of the RPA-BCS Hamiltonian for a given number of pairs, one introduces a chemical potential through the method of Lagrange variables. In other words, one tries to minimize the Hamiltonian

$$H_{\text{BCS}}^{\text{RPA}} - 2\mu T^z = \sum_k 2(\epsilon_k - \mu) t_k^z - G(\langle T^+ \rangle T^- + T^+ \langle T^- \rangle). \quad (4.15)$$

The excited states are then obtained from the ground state by applying the excitation operators. Although the many-body BCS Hamiltonian H_{BCS} conserves the total pair number $2T^z$ [see Eq. (2.29)], the RPA-BCS Hamiltonian given above conserves only its mean value $2\langle T^z \rangle$. Thus, the BCS ground state and the excited states obtained this way have a constant mean pair number, but they are not eigenstates of the pair number operator, i.e., they have *indefinite* number of pairs.

As pointed out at the end of Sec. IV A, one has a similar situation for neutrino Hamiltonian and we introduce an analogous Lagrange variable which we denote by ω_c :

$$H^{\text{RPA}} + \omega_c \mathcal{J}^z = \sum_{\omega} (\omega_c - \omega) \mathcal{J}_{\omega}^z + \mu \vec{P} \cdot \vec{J}. \quad (4.16)$$

This way one can find the state which minimizes the RPA neutrino Hamiltonian (4.3) for a given value of the constant in Eq. (4.13). Note that, since the Hamiltonian in Eq. (4.16) commutes with the total particle number operator \hat{N} , the total neutrino number N is always well defined. Consequently, the Lagrange multiplier in Eq. (4.16) fixes the values of both $\langle \hat{N}_1 \rangle$ and $\langle \hat{N}_2 \rangle$. In what follows, we will use a method borrowed from Bogoliubov's solution of the RPA-BCS model which yields both the ground and the excited states with these properties [64].

Let us begin by pointing out that the Hamiltonian in Eq. (4.16) can be diagonalized by rotating the flavor space of each energy mode in a different way as described in Sec. II A. Such a rotation can be carried out by the unitary operator given in Eq. (2.8). This time we denote the operator by U_{ω} and its parameter by z_{ω} to emphasize that all neutrinos with a given energy are subject to the same transformation while different energy modes undergo

different transformations. Our aim is to write the Hamiltonian given in Eq. (4.16) in the form

$$H^{\text{RPA}} + \omega_c \mathcal{J}^z = \sum_{\omega} 2\lambda_{\omega} U_{\omega}^{\dagger} \mathcal{J}_{\omega}^z U_{\omega}. \quad (4.17)$$

The first of Eqs. (2.12) tells us that for the equality in Eq. (4.17) to hold, the rotation parameter $z_{\omega} = e^{i\delta_{\omega}} \tan\theta_{\omega}$ should satisfy

$$\begin{aligned} 2\lambda_{\omega} \cos 2\theta_{\omega} &= (\omega_c - \omega) + \mu \mathcal{P}^z, \\ 2\lambda_{\omega} e^{\pm i\delta_{\omega}} \sin 2\theta_{\omega} &= \mu \mathcal{P}^{\pm}. \end{aligned} \quad (4.18)$$

Solving Eqs. (4.18) we find that the phase δ_{ω} is the same for all energy modes and is given by

$$e^{i\delta} = \frac{\mathcal{P}^+}{|\mathcal{P}^+|} \quad (4.19)$$

whereas the rotation angle θ_{ω} depends on the energy and is given by

$$\begin{aligned} \cos\theta_{\omega} &= \sqrt{\frac{1}{2} \left(1 + \frac{\omega_c - \omega + \mu \mathcal{P}^z}{2\lambda_{\omega}} \right)}, \\ \sin\theta_{\omega} &= \sqrt{\frac{1}{2} \left(1 - \frac{\omega_c - \omega + \mu \mathcal{P}^z}{2\lambda_{\omega}} \right)}. \end{aligned} \quad (4.20)$$

The parameter λ_{ω} can also be found from Eqs. (4.18) as

$$\lambda_{\omega} = \frac{1}{2} \sqrt{(\omega_c - \omega + \mu \mathcal{P}^z)^2 + \mu^2 \mathcal{P}^+ \mathcal{P}^-}. \quad (4.21)$$

Equation (4.17) motivates the definition of the Bogoliubov transformed *quasiparticle* operators⁵

$$\begin{aligned} \alpha_1(\mathbf{p}) &= U_{\omega}^{\dagger} a_1(\mathbf{p}) U_{\omega} = \cos\theta_{\omega} a_1(\mathbf{p}) + e^{i\delta} \sin\theta_{\omega} a_2(\mathbf{p}) \\ \alpha_2(\mathbf{p}) &= U_{\omega}^{\dagger} a_2(\mathbf{p}) U_{\omega} = -e^{-i\delta} \sin\theta_{\omega} a_1(\mathbf{p}) + \cos\theta_{\omega} a_2(\mathbf{p}) \end{aligned} \quad (4.22)$$

together with the corresponding isospin

$$\vec{\mathcal{J}}_{\mathbf{p}} = U_{\omega}^{\dagger} \vec{\mathcal{J}}_{\mathbf{p}} U_{\omega}, \quad (4.23)$$

⁵In its broadest sense, a Bogoliubov transformation is any mixture of particle operators which preserves the (anti) commutation relations. However, the term usually suggests the mixing of creation and annihilation operators because in Bogoliubov's approach to BCS theory the electron and hole states are mixed. Although Eq. (4.22) is only a rotation in flavor space (apart from the phases) it can also be viewed as a special Bogoliubov transformation in the broad sense of the term because the rotated particle operators satisfy the same anticommutation relations as the original ones. In that sense, the distinction between the *real* particles and the *quasi*-particles seems somewhat arbitrary because one can see the flavor basis as quasiparticles of the mass basis. However the term quasiparticle is usually reserved for the noninteracting basis in the presence of an interaction between the particles and the background. Here, our usage of the terms *Bogoliubov transformation* and *quasiparticle* is due to the established terminology of the BCS theory.

which we call the α -isospin operators. They are equal to

$$\begin{aligned} \mathcal{J}_{\mathbf{p}}^+ &= \alpha_1^{\dagger}(\mathbf{p}) \alpha_2(\mathbf{p}), & \mathcal{J}_{\mathbf{p}}^- &= \alpha_2^{\dagger}(\mathbf{p}) \alpha_1(\mathbf{p}), \\ \mathcal{J}_{\mathbf{p}}^z &= \frac{1}{2} (\alpha_1^{\dagger}(\mathbf{p}) \alpha_1(\mathbf{p}) - \alpha_2^{\dagger}(\mathbf{p}) \alpha_2(\mathbf{p})). \end{aligned} \quad (4.24)$$

The Hamiltonian given in Eq. (4.17) can now be written as the Hamiltonian of a noninteracting system in terms of the quasiparticles

$$\begin{aligned} H^{\text{RPA}} + \omega_c \mathcal{J}^z &= \sum_{\omega} 2\lambda_{\omega} \mathcal{J}_{\omega}^z \\ &= \sum_{\mathbf{p}} \lambda_{\omega} (\alpha_1^{\dagger}(\mathbf{p}) \alpha_1(\mathbf{p}) - \alpha_2^{\dagger}(\mathbf{p}) \alpha_2(\mathbf{p})). \end{aligned} \quad (4.25)$$

The eigenstates of the Hamiltonian given in Eq. (4.25) are the ones in which all neutrinos occupy α_1 and α_2 states. In particular, the ground state is an “ α_2 condensate.” In the language of α -isospins introduced in Eq. (4.24), all α -isospins point down in the ground state so that it is equal to the lowest weight state

$$|j_{\max}, -j_{\max}\rangle_{\alpha} = \prod_{\mathbf{p}} \alpha_2^{\dagger}(\mathbf{p}) |0\rangle, \quad (4.26)$$

where $j_{\max} = N/2$. The states with higher energies can be obtained by exciting the particles from α_2 to α_1 states which amounts to applying the operator $\mathcal{J}_{\mathbf{p}}^+$ defined in Eq. (4.24). In general, the eigenstates and eigenvalues of the Hamiltonian in Eq. (4.25) can be written as

$$|\psi\rangle = \prod_{\omega} |j_{\omega}, m_{\omega}\rangle_{\alpha} \quad \text{and} \quad E = \sum_{\omega} 2m_{\omega} \lambda_{\omega}. \quad (4.27)$$

In Eqs. (4.26) and (4.27), the subscript α points to the fact that these are the eigenstates of the α -isospin.⁶ The ground state given in Eq. (4.26) is the special case of Eq. (4.27) in which $m_{\omega} = -j_{\omega}$ for all energy modes.

⁶It is evident from Eq. (4.23) that these states are related to the corresponding eigenstates $|j_{\omega}, m_{\omega}\rangle_m$ of the mass isospin by

$$|j_{\omega}, m_{\omega}\rangle_{\alpha} = U_{\omega}^{\dagger} |j_{\omega}, m_{\omega}\rangle_m,$$

which in turn are related to the eigenstates $|j_{\omega}, m_{\omega}\rangle_f$ of the flavor isospin by

$$|j_{\omega}, m_{\omega}\rangle_m = U^{\dagger} |j_{\omega}, m_{\omega}\rangle_f.$$

Therefore, we can write the eigenstates in Eq. (4.27) as

$$|\psi\rangle = \prod_{\omega} |j_{\omega}, m_{\omega}\rangle_{\alpha} = U_{\omega}^{\dagger} \prod_{\omega} |j_{\omega}, m_{\omega}\rangle_m = U_{\omega}^{\dagger} U^{\dagger} \prod_{\omega} |j_{\omega}, m_{\omega}\rangle_f.$$

Note that the transformation described by the operator U_{ω} is not a global rotation. From Eq. (4.20), it is clear that each energy mode ω undergoes a different rotation. As a result, this transformation does not preserve all scalar products. In particular, the total α -isospin quantum number of a state is not always the same as its total mass or flavor isospin quantum number. The state in Eq. (4.26), for example, does not live in the j_{\max} representation of mass or flavor isospin although it lives in the j_{\max} representation of the α -isospin. However, j_{ω} is a well defined quantum number in all three bases because, within each energy mode ω , the transformation induced by U_{ω} is a “global” one.

For consistency, the states in Eq. (4.27) should satisfy

$$2\langle\psi|\mathcal{J}^+|\psi\rangle = \mathcal{P}^+ \quad \text{and} \quad 2\langle\psi|\mathcal{J}^z|\psi\rangle = \mathcal{P}^z. \quad (4.28)$$

It is an easy exercise in algebra to show that

$$2\langle\psi|\mathcal{J}^+|\psi\rangle = 4\sum_{\omega} m_{\omega} \frac{z_{\omega}}{1+|z_{\omega}|^2} = e^{i\delta} \mu |\mathcal{P}^+| \sum_{\omega} \frac{m_{\omega}}{\lambda_{\omega}} \quad (4.29)$$

and

$$2\langle\psi|\mathcal{J}^z|\psi\rangle = 2\sum_{\omega} m_{\omega} \frac{1-|z_{\omega}|^2}{1+|z_{\omega}|^2} = \sum_{\omega} m_{\omega} \frac{\omega - \omega_c + \mu \mathcal{P}^z}{\lambda_{\omega}}. \quad (4.30)$$

In this case, the consistency equations (4.28) become

$$\frac{1}{2\mu} = \sum_{\omega} \frac{m_{\omega}}{2\lambda_{\omega}} \quad \frac{\omega_c}{2\mu} = \sum_{\omega} \frac{m_{\omega}\omega}{2\lambda_{\omega}}. \quad (4.31)$$

The consistency equations given in Eqs. (4.9) differ from those given above in that the former ones are valid for a generic time dependent state whereas the later ones are valid for the eigenstates (i.e., stationary states) of the RPA Hamiltonian. At this point, we would like to note the similarity between Eqs. (3.47) and (4.31). From Eq. (4.27) we see that the eigenstate with the highest energy has $m_{\omega} = j_{\omega}$. For this particular state, the consistency equations (4.31) are the same as those given in Eqs. (3.47) with the substitution $\lambda = \omega_c - \mu \mathcal{P}^z$ and $\Delta = \mu |\mathcal{P}^+|$ together with $\kappa = j + \mathcal{P}^z/2$ [see Eq. (3.29)].

A special but very interesting case of Eqs. (4.31) is obtained in Ref. [20] in connection with the spectral splits of neutrinos which occur if the parameter μ defined in Eq. (2.22) changes in an adiabatic way from ∞ to 0 (also see Ref. [24] for an in-depth review). In this case, the Lagrange multiplier ω_c introduced above plays the role of the split frequency. Here we will show that this phenomenon can also be understood in terms of the adiabatic evolution of quasiparticle states. Namely, we will show that when the neutrino density is high, the quasiparticle states coincide with flavor eigenstates whereas in vacuum they coincide with mass eigenstates. As a result, neutrinos emerging in flavor eigenstates in the high-density region gradually transform into mass eigenstates if the neutrino density decreases in an adiabatic way. The Lagrange multiplier ω_c determines the final energy distribution of mass eigenstates.

In order to see how this phenomenon takes place, let us assume that initially the neutrino density is so high that the $\mu \rightarrow \infty$ limit is practically attained and that all neutrinos initially occupy flavor eigenstates ν_e and ν_x . Note that here it is not assumed that all neutrinos occupy the same flavor state. Neutrinos emanating from the surface of a proto-neutron star after a supernova explosion are believed to meet these initial state assumptions. In $\mu \rightarrow \infty$ limit, Eq. (4.21) gives $2\lambda_{\omega} = \mu |\vec{\mathcal{P}}| = \mu |\vec{P}|$ for all neutrinos. Since we assumed that all neutrinos are in flavor eigenstates, we have $\mathcal{P}^+ = \mathcal{P}^- = 0$, in which case Eqs. (2.14) lead to $\mathcal{P}^z = \cos 2\theta P^z$. Substituting these in Eq. (4.20) we find

$$\cos\theta_{\omega} = \sqrt{\frac{1}{2}\left(1 + \frac{P^z}{|\vec{P}|} \cos 2\theta\right)} \quad \sin\theta_{\omega} = \sqrt{\frac{1}{2}\left(1 - \frac{P^z}{|\vec{P}|} \cos 2\theta\right)}. \quad (4.32)$$

Now, let us assume for a moment that we have more ν_e than ν_x so that $P^z > 0$. In this case, Eq. (4.32) yields $\cos\theta_{\omega} = \cos\theta$, i.e., the angle θ_{ω} is equal to the vacuum mixing angle for all neutrinos. Also note that Eqs. (2.14) lead to $\mathcal{P}^+ = \sin 2\theta P^z$ which tells us that the phase δ defined in Eq. (4.19) is equal to zero for this particular case. These observations lead to the result that $z_{\omega} = z = \tan\theta$ for all neutrinos. In this case, the operator U_{ω} simply becomes equal to the operator U defined in Eq. (2.11). As a result, one arrives at

$$\begin{aligned} \alpha_1(\mathbf{p}) &= U^{\dagger} a_1(\mathbf{p}) U = a_e(\mathbf{p}) \quad \text{and} \\ \alpha_2(\mathbf{p}) &= U^{\dagger} a_2(\mathbf{p}) U = a_x(\mathbf{p}), \end{aligned} \quad (4.33)$$

where we used Eq. (2.10). The result stated in Eq. (4.33) is important because it tells us that in the limit where $\mu \rightarrow \infty$, the quasiparticle operators become equal to the flavor particle operators and thus the state in which all neutrinos occupy flavor states is an eigenstate of the RPA Hamiltonian (4.25) in this limit. Note that, although we assumed that we have initially more ν_e than ν_x , the opposite scenario can be worked out in a similar way and shown to lead to the same conclusion with $\alpha_1(\mathbf{p}) = -a_x(\mathbf{p})$ and $\alpha_2(\mathbf{p}) = a_e(\mathbf{p})$.

Since one of the states $|\psi\rangle$ in Eq. (4.27) represents the initial state of the system, then we have a steady-state case with no time evolution as long as μ stays constant. It is already known that in the limit where $\mu \rightarrow \infty$ the state in which all neutrinos occupy flavor eigenstates is a steady-state solution which is consistent with the above discussion. However, if μ decreases (which happens when the neutrinos emanate from a source), then the quasiparticle operators $\alpha_1(\mathbf{p})$ and $\alpha_2(\mathbf{p})$ change accordingly and the initial state cannot remain as an eigenstate. However, if the change in μ occurs in an adiabatic way, then the state evolves steadily in such a way that it continues to keep its original form in terms of the quasiparticles while the quasiparticle operators $\alpha_1(\mathbf{p})$ and $\alpha_2(\mathbf{p})$ themselves slowly change. In the limit where $\mu \rightarrow 0$, Eq. (4.21) yields

$$2\lambda_{\omega} = |\omega_c - \omega| \quad (4.34)$$

and Eqs. (4.20) lead to

$$\cos\theta_{\omega} = \sqrt{\frac{1}{2}\left(1 + \frac{\omega_c - \omega}{|\omega_c - \omega|}\right)} \quad \sin\theta_{\omega} = \sqrt{\frac{1}{2}\left(1 - \frac{\omega_c - \omega}{|\omega_c - \omega|}\right)}. \quad (4.35)$$

As a result, in $\mu \rightarrow 0$ limit one obtains $\cos\theta_{\omega} = 1$ and $\sin\theta_{\omega} = 0$ for those neutrinos with $\omega < \omega_c$ leading to

$$\alpha_1(\mathbf{p}) = a_1(\mathbf{p}), \quad \alpha_2(\mathbf{p}) = a_2(\mathbf{p}). \quad (4.36)$$

On the other hand, for those neutrinos with $\omega > \omega_c$, the $\mu \rightarrow 0$ limit yields $\cos\theta_{\omega} = 0$ and $\sin\theta_{\omega} = 1$ leading to

$$\alpha_1(\mathbf{p}) = a_2(\mathbf{p}), \quad \alpha_2(\mathbf{p}) = a_1(\mathbf{p}), \quad (4.37)$$

up to some inconsequential phase factors. We see that, a state which starts off with all neutrinos in flavor eigenstates in the $\mu \rightarrow \infty$ limit can adiabatically evolve towards the $\mu \rightarrow 0$ limit in such a way that those neutrinos with $\omega < \omega_c$ and those with $\omega > \omega_c$ end up in the opposite mass eigenstates. For example, if we initially have more ν_e than ν_x , then those electron neutrinos with $\omega < \omega_c$ evolve into the first mass eigenstate while those with $\omega > \omega_c$ evolve into the second mass eigenstate. In this case, the opposite is true for ν_x . Such an evolution leads to a spectral split as discussed in Ref. [20].

In order to find the split frequency ω_c , one should solve the consistency equations given in Eqs. (4.31). Since we assumed that initially all neutrinos are in flavor states, one has $2m_\omega = P_\omega^z = \epsilon_\omega |\vec{P}_\omega|$ where $\epsilon_\omega = \pm 1$ depending on whether P_ω^z initially points up (more ν_e) or down (more ν_x). As a result, the consistency equations given in Eq. (4.31) reduce to

$$1 = \sum_\omega \frac{\epsilon_\omega |\vec{P}_\omega|}{\sqrt{\left(\frac{\omega_c - \omega}{\mu} + P^z\right)^2 + P^+ P^-}} \quad \text{and} \quad (4.38)$$

$$\omega_c = \sum_\omega \frac{\epsilon_\omega \omega |\vec{P}_\omega|}{\sqrt{\left(\frac{\omega_c - \omega}{\mu} + P^z\right)^2 + P^+ P^-}}.$$

These equations were also obtained in Ref. [20] where a method based on the time evolution of polarization vectors is used. The derivation presented here in terms of the evolution of quasiparticles is physically equivalent to the one given in Ref. [20].

V. ANTINEUTRINOS

In this section, we include antineutrinos in the formalism introduced in previous sections. Let us denote the fermion operators for a neutrino with momentum \mathbf{p} in the flavor state $\bar{\nu}_\alpha$ by $\bar{a}_\alpha(\mathbf{p})$ and in the mass state $\bar{\nu}_i$ by $\bar{a}_i(\mathbf{p})$ where $\alpha = e, x$ and $i = 1, 2$. It has been pointed out by many authors that describing antineutrinos with the *rotated* spinor $(-\bar{\nu}_2, \bar{\nu}_1)$ instead of the regular spinor $(\bar{\nu}_1, \bar{\nu}_2)$ is more advantageous (see, for example, Refs. [31,32]), because this leads to a seamless integration of antineutrinos into the formalism making the physics more transparent (see Appendix B). In order to implement this, one can first define the antineutrino isospin operators as in Eqs. (2.2) and (2.7) and then perform a transformation as in Eqs. (2.9) with $\theta_{\mathbf{p}} = \pi/2$ and $\delta_{\mathbf{p}} = 0$ for all momentum modes.⁷ This leads to

⁷Note that for $\theta_{\mathbf{p}} = \pi/2$ the parameter $z_{\mathbf{p}}$ is singular in Eq. (2.8). However, the transformation is still well defined because the operator $U_{\mathbf{p}}$ can be written in several alternative forms. One of them, which is not singular for $\theta_{\mathbf{p}} = \pi/2$, is given by

$$U_{\mathbf{p}} = e^{2\theta_{\mathbf{p}}(\mathcal{J}^x \sin \delta_{\mathbf{p}} + \mathcal{J}^y \cos \delta_{\mathbf{p}})}.$$

$$\bar{a}_1(\mathbf{p}) \equiv -\bar{a}_2(\mathbf{p}), \quad \bar{a}_2(\mathbf{p}) \equiv \bar{a}_1(\mathbf{p}) \quad (5.1)$$

and similarly for the particle operators of flavor states. We denote the corresponding flavor and mass antineutrino isospin operators by $\vec{J}_{\mathbf{p}}$ and $\vec{J}_{\mathbf{p}}$ which are defined as in Eqs. (2.2) and (2.7), respectively.

With the introduction of antineutrinos, we modify the summation convention introduced in Eq. (2.4) as

$$A_\omega \equiv \sum_{|\mathbf{p}|=p} A_{\mathbf{p}}, \quad A_{-|\omega|} \equiv \sum_{|\mathbf{p}|=p} \tilde{A}_{\mathbf{p}} \quad \text{and} \quad A \equiv \sum_\omega A_\omega \quad (5.2)$$

and also introduce

$$\omega = -\frac{\delta m^2}{2p} \quad (5.3)$$

for antineutrinos. Note that the last sum in Eq. (5.2) over ω now includes the negative values for antineutrinos as well as the positive values for neutrinos.

The Hamiltonian describing the vacuum oscillations and self-interactions of neutrinos and antineutrinos in the single angle approximation is given by Eq. (2.23), except that now the new convention introduced in Eq. (5.2) applies [56]. As a result, all of the main results of this paper, including the invariants given in Eqs. (2.30) and (4.10), can be generalized to include antineutrinos by extending the sums over ω to include negative values. We only would like to make a few comments in what follows.

With the inclusion of antineutrinos, the number of invariants is now doubled because one has the invariants h_ω for neutrinos and the invariants $h_{-\omega}$ for antineutrinos. Considering the diagonalization of the Hamiltonian described in Sec. (III A), j now represents the total isospin quantum number of both neutrinos and antineutrinos in accordance with Eq. (5.2). In the case of the electrostatic analogy, one has additional negative charges $-j_{-\omega}$ located on the negative x -axis at positions $-\omega$. In this case, one can still apply the formalism described in Sec. III B by shifting the origin of the coordinate system in Fig. 1. Finally we note that when antineutrinos are treated as described above, their polarization vectors defined as in Eq. (4.4) are given by

$$\vec{P}_{\mathbf{p}} = \begin{pmatrix} -2 \operatorname{Re}[\tilde{\psi}_x^*(\mathbf{p}) \tilde{\psi}_e(\mathbf{p})] \\ -2 \operatorname{Im}[\tilde{\psi}_x^*(\mathbf{p}) \tilde{\psi}_e(\mathbf{p})] \\ |\tilde{\psi}_x(\mathbf{p})|^2 - |\tilde{\psi}_e(\mathbf{p})|^2 \end{pmatrix}. \quad (5.4)$$

This definition involves an overall minus sign in comparison to the one which is frequently used in the literature.

VI. CONCLUSIONS

In this paper, we studied the symmetries and the associated constants of motion of the collective neutrino oscillation Hamiltonian by taking into account both the vacuum oscillations and the self-interactions of neutrinos. We examined the system both from the exact many-body perspective and from the point of view of an effective

one-body description formulated with the application of the RPA method. We showed that, under the single angle approximation, both the many-body and the RPA pictures possess many constants of motion manifesting the existence of associated dynamical symmetries in the system.

The existence of these constants of motion make the system completely integrable in the sense that the exact eigenstates and eigenvalues can be found in an analytical way. We wrote down these eigenstates and eigenvalues for both the exact many-body and the RPA Hamiltonians. In the case of the many-body Hamiltonian, the eigenvalues and eigenstates were found with the application of the Bethe ansatz method and they depend on the solutions of Bethe ansatz equations which are analogous to the equilibrium conditions of an electrostatic system in two dimensions. In the case of the RPA Hamiltonian, the eigenstates and eigenvalues were found by applying a suitable Bogoliubov transformation which brings the Hamiltonian into the form of a noninteracting system in terms of quasiparticles. We wrote down the consistency equations for all RPA eigenstates and showed that these equations reduce to a particular but physically very important case which was studied earlier in Ref. [20] in connection with the spectral splits of neutrinos. We showed that the spectral splits can be understood as the adiabatic evolution of quasiparticle states from a high-density region where they coincide with flavor eigenstates into the vacuum where they coincide with mass eigenstates.

In general, the existence of constants of motion offers practical ways of extracting information even from exceedingly complex systems. Even when the symmetries which guarantee their existence is broken, they usually provide a convenient set of variables which behave in a relatively simple manner depending on how drastic the symmetry breaking factor is. In this paper, we omitted an ordinary matter background and concentrated on a neutrino gas which undergoes vacuum oscillations as well as self-interactions. Although an ordinary background of p , n , e^- and e^+ coexists with neutrinos in most astrophysical sites, it is only the net electron fraction which plays a part in flavor evolution of neutrinos through a diagonal potential in flavor basis which is proportional to $\sqrt{2}G_F|n_{e^-} - n_{e^+}|$. Neglecting this potential can only be justified when the asymmetry of the electron background is small in comparison to the neutrino background. On the other hand, even under the circumstances where the electron background asymmetry cannot be ignored, the constants of motion presented in this paper can still be useful as a set of convenient variables because, since they commute with the *neutrino gas* part of the Hamiltonian, their time variation only come from their commutator with the electron background term.

As was recently illustrated in Ref. [23], existence of such invariants naturally leads to associated N -mode collective neutrino oscillations. These collective oscillations

are closely related to the m -spin solutions presented in Ref. [63] in the context of the BCS model. However, symmetries alone do not guarantee the stability of such collective behavior. Some issues related to the stability of the m -spin solutions of the BCS model have been addressed in Refs. [63,65]. In the case of the collective neutrino phenomena associated with our invariants the question of stability is an open problem and could be illuminated by numerical studies.

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APPENDIX A: ALTERNATIVE WAYS OF EXPRESSING QUANTUM INVARIANTS

In this appendix, we will briefly mention two alternative ways to rewrite the constants of motion mentioned in the text. The first method is based on the following sum of the many-body invariants h_ω and L_ω :

$$C_n = \sum_\omega \omega^n h_\omega + \sum_\omega n \omega^{n-1} L_\omega. \quad (\text{A1})$$

Clearly C_n is a constant of motion for every value of n but here we specifically assume that n is a positive integer. If we define the quantities

$$\vec{Q}_n = \sum_\omega \omega^n \vec{J}_\omega, \quad (\text{A2})$$

then C_n defined in Eq. (A1) can be written as

$$C_n = \vec{B} \cdot \vec{Q}_n + \mu(\vec{Q}_{n-1} \cdot \vec{Q}_0 + \vec{Q}_{n-2} \cdot \vec{Q}_1 + \dots + \vec{Q}_1 \cdot \vec{Q}_{n-2} + \vec{Q}_0 \cdot \vec{Q}_{n-1}). \quad (\text{A3})$$

The invariant C_0 was already mentioned in Eq. (2.32). The Hamiltonian itself is equal to C_1 as can be seen from Eq. (2.33).

A similar method can be used to rewrite the constants of motion of the RPA formalism. One starts from the sum of the RPA invariants I_ω and $|\vec{P}_\omega|^2$ given by

$$K_n = \sum_{\omega} \omega^n I_{\omega} + \frac{1}{2} \sum_{\omega} n \omega^{n-1} |\vec{P}_{\omega}|^2 \quad (\text{A4})$$

which is clearly an invariant for every n . Defining the quantities

$$\vec{P}_n = \sum_{\omega} \omega^n \vec{P}_{\omega}, \quad (\text{A5})$$

it can be shown that K_n is equal to

$$K_n = \vec{B} \cdot \vec{P}_n + \frac{\mu}{2} (\vec{P}_{n-1} \cdot \vec{P}_0 + \vec{P}_{n-2} \cdot \vec{P}_1 + \dots + \vec{P}_1 \cdot \vec{P}_{n-2} + \vec{P}_0 \cdot \vec{P}_{n-1}). \quad (\text{A6})$$

Another alternative way to express the many-body quantum invariants, which is frequently encountered in the literature on Gaudin formalism, is based on the following sum:

$$H(\lambda) = - \sum_{\omega} \frac{h_{\omega}}{\omega - \lambda} + \sum_{\omega} \frac{L_{\omega}}{(\omega - \lambda)^2}. \quad (\text{A7})$$

Here λ is a complex valued parameter and $H(\lambda)$ is clearly a constant of motion for every value of λ . Defining

$$\vec{Q}(\lambda) = \sum_{\omega} \frac{\vec{J}_{\omega}}{\omega - \lambda} \quad (\text{A8})$$

and using Eq. (2.30), one can show that the operator $H(\lambda)$ defined in Eq. (A7) can be written as

$$H(\lambda) = \mathcal{Q}^z(\lambda) + \mu \vec{Q}(\lambda) \cdot \vec{Q}(\lambda). \quad (\text{A9})$$

$H(\lambda)$ is said to form a *one-parameter family of conserved quantities*. It satisfies

$$[H(\lambda), H(\mu)] = 0 \quad \text{and} \quad [H(\lambda), H] = 0 \quad (\text{A10})$$

for every complex value of λ and μ .

In the case of the RPA formalism, one similarly defines the sum of the RPA invariants I_{ω} and $|\vec{P}_{\omega}|^2$ as follows:

$$D(\lambda) = - \sum_{\omega} \frac{I_{\omega}}{\omega - \lambda} + \frac{1}{2} \sum_{\omega} \frac{|\vec{P}_{\omega}|^2}{(\omega - \lambda)^2}. \quad (\text{A11})$$

$D(\lambda)$ is a constant of motion for every complex value of λ . Defining the vectors

$$\vec{\mathcal{P}}(\lambda) = \sum_{\omega} \frac{\vec{P}_{\omega}}{\omega - \lambda}, \quad (\text{A12})$$

one can show that it can be written as

$$D(\lambda) = \mathcal{P}^z(\lambda) + \frac{\mu}{2} \vec{\mathcal{P}}(\lambda) \cdot \vec{\mathcal{P}}(\lambda). \quad (\text{A13})$$

$D(\lambda)$ also forms a one-parameter family of conserved quantities.

APPENDIX B: ANTINEUTRINO HAMILTONIAN

The effective many-body Hamiltonian describing the collective flavor evolution of neutrinos in a dense medium in the presence of neutrino self-interactions was obtained in Ref. [56] by expanding the full neutral current interaction and keeping only the forward scattering terms. The underlying group is $SU(3) \times SU(3)$ representing all three neutrino and antineutrino flavors. Reducing to the $SU(2) \times SU(2)$ subgroup for the two neutrino case yields the following effective many-body Hamiltonian:

$$H = \sum_{\omega > 0} \omega \left(-\cos 2\theta J_{\omega}^0 + \sin 2\theta \frac{J_{\omega}^+ + J_{\omega}^-}{2} - \cos 2\theta \vec{J}_{\omega}^0 \right) + \sin 2\theta \frac{\vec{J}_{\omega}^+ + \vec{J}_{\omega}^-}{2} + \mu \sum_{\mathbf{p}, \mathbf{q}} (1 - \cos \vartheta_{\mathbf{p}\mathbf{q}}) \times (\vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}} + \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}} - 2J_{\mathbf{p}}^0 J_{\mathbf{q}}^0 - J_{\mathbf{p}}^+ J_{\mathbf{q}}^+ - J_{\mathbf{p}}^- J_{\mathbf{q}}^-). \quad (\text{B1})$$

Here \vec{J} are the flavor isospin operators for antineutrinos defined as in Eq. (2.2). They are subject to the same summation convention as \vec{A} given in Eq. (5.2). The terms involving the neutrinos and antineutrinos in the Hamiltonian (B1) are dissimilar. However, if one performs the transformation described at the beginning of Sec. V and defines the antiparticle operators \tilde{a} as in Eq. (5.1), then the corresponding isospin operators are related by

$$(\vec{J}_{\mathbf{p}}^0, \vec{J}_{\mathbf{p}}^+, \vec{J}_{\mathbf{p}}^-) = (-\vec{J}_{\mathbf{p}}^0, -\vec{J}_{\mathbf{p}}^-, -\vec{J}_{\mathbf{p}}^+). \quad (\text{B2})$$

In terms of the transformed antineutrino isospin operators, the Hamiltonian in Eq. (B1) becomes

$$H = \sum_{\omega > 0} \omega \left(-\cos 2\theta J_{\omega}^0 + \sin 2\theta \frac{J_{\omega}^+ + J_{\omega}^-}{2} + \cos 2\theta \vec{J}_{\omega}^0 \right) - \sin 2\theta \frac{\vec{J}_{\omega}^+ + \vec{J}_{\omega}^-}{2} + \mu \sum_{\mathbf{p}, \mathbf{q}} (1 - \cos \vartheta_{\mathbf{p}\mathbf{q}}) (\vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}} + \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}} + 2\vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}). \quad (\text{B3})$$

If we define the oscillation frequencies of antineutrinos with a minus sign as given in Eq. (5.3), then (in the single angle approximation) the total Hamiltonian (B3) takes the form of Eq. (2.23) with the sum over ω extended to include the negative values.

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