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Classical noncommutative electrodynamics with external source

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In a $U(1)_\star$-noncommutative gauge field theory we extend the Seiberg-Witten map to include the (gauge-invariance-violating) external current and formulate—to the first order in the noncommutative parameter–gauge-covariant classical field equations. We find solutions to these equations in the vacuum and in an external magnetic field, when the 4-current is a static electric charge of a finite size a, restricted from below by the elementary length. We impose extra boundary conditions, which we use to rule out all singularities, $1/r$ included, from the solutions. The static charge proves to be a magnetic dipole, with its magnetic moment being inversely proportional to its size a. The external magnetic field modifies the long-range Coulomb field and some electromagnetic form factors. We also analyze the ambiguity in the Seiberg-Witten map and show that at least to the order studied here it is equivalent to the ambiguity of adding a homogeneous solution to the current-conservation equation.

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I. INTRODUCTION

Noncommutative (NC) field theories, based on a profound revision of the most fundamental properties of the space-time achieved by introducing an elementary length, play a challenging role in modern theoretical physics. These theories do not need a lengthy introduction; we refer the interested reader to the review papers [1,2]. The present paper is devoted to a construction of an NC extension of electrodynamics in its most classical sense. It is notable that the resulting electrodynamics is, already at the classical level, a nonlinear theory, rich in properties. For instance, it possesses the birefringence and photon splitting in external fields [3]. But, in contrast to other nonlinear theories, e.g., the classical Yang-Mills theory, or the Born-Infeld electrodynamics, or else quantum electrodynamics (QED) after radiative corrections in it are taken into account, the classical NC electrodynamics is, besides, intrinsically anisotropic. We shall demonstrate below that the classical NC electrodynamics reproduces also other interesting features, known [4,5] in QED. Moreover, we establish that electrostatic charge with its density homogeneously distributed over a finite-size sphere carries a magnetic moment depending on its size. Hence, an idea of the NC magnetic moment of the proton appears. In an external magnetic field we find a modification of the Coulomb law at large distances from the charge of a sort completely alien to QED. This effect may be referred to as a macroscopic manifestation of the elementary length at large distances. A study of these and the like classical phenomena is inevitable, as long as possible observable consequences of noncommutativity are to be looked for.

In the present paper we do not quantize the electromagnetic field. The charge carriers are represented through the currents rather than through elementary fields. This latter task, namely, the introduction of currents, will appear to be rather nontrivial. It is very well known that there are severe restrictions on the gauge groups and their representations so that the gauge transformations might form a closed algebra on the NC plane [6]. To overcome this difficulty one either uses the Seiberg-Witten (SW) map [7], or makes the gauge transformations twisted [8,9]. None of these are strictly speaking necessary in NC electrodynamics, since the $U(1)$ gauge group can be nicely deformed to a $U(1)_\star$ group. Therefore, many papers define noncommutative electrodynamics as a $U(1)_\star$ gauge theory,[1] see, e.g., [11–14]. Nevertheless, various aspects of the SW map were developed for the NC $U(1)$ theories [15,16]. Since the electromagnetic potential after the SW map has the standard gauge transformation properties, this map facilitates the analysis of phenomenological predictions of NC theories [17–20]. Also, the SW map has interesting

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[1]There is another, less frequently used terminology, see [10], according to which this deformation is called the Moyal modification.

effects on renormalizability of NC field theories even in the $U(1)$ case [16,21–23].

In this work we study, to the lowest order of noncommutativity, a NC Maxwell theory in the presence of sources. Clearly, if the phenomenological analysis involves a comparison of solutions in a commutative theory to noncommutative corrections, it is essential that both commutative and noncommutative fields have the same transformation properties under the gauge group. For example, the electric and magnetic fields must be gauge-invariant. In other words, for such applications one has to introduce commutative fields into a noncommutative field theory.[2] This is precisely what the SW map does.

In Sec. II, as a preparation to the SW construction, we first study the $U(1)_\star$ gauge theory with currents. We observe, that although the set of equations of motion consisting of the Maxwell equation and the current conservation condition is gauge-covariant, the action is not gauge-invariant. This fact is analogous to a well-known property of non-Abelian ("commutative") Yang-Mills theories and does not mean by itself any internal inconsistency. However, for the SW map it implies that this map has to be performed in equations of motion rather than at the level of the action. We proceed in this way and rederive the SW map for currents [25] in the first order of the noncommutativity parameter. Field equations include potentials along with the field strengths. Their gauge covariance is effectuated via the statement that the gauge-transformed potentials satisfy the same equations within the accuracy adopted in the paper. Moreover, solutions to the equations, from which all potential-containing terms are dropped, satisfy the original equations with the potentials retained.

In Sec. III we consider first-order NC corrections to the field of a static spherically symmetric charge distributed over a sphere of finite size, assuming that the noncommutativity is space-space. It is important that the size of the charge be larger than the elementary length characteristic of the NC theory. In contrast to our previous paper [26], we place the static delocalized charge in a constant and homogeneous external magnetic field. This gives rise to NC corrections to the electrostatic potential, linear with respect to the charge and to the external magnetic field, as well as to appearance of NC magnetic field, produced by the charge, quadratic with respect to its value and independent of the external magnetic field. In Sec. III A we impose boundary conditions onto the field equations that would exclude a singular behavior of their solutions in the origin (where the charge is centered) and find magnetic and electrostatic field, produced by the static charge. In Secs. III B and III Cwe consider other magnetic and electric solutions that are either singular in the origin or do not decrease in the remote region and discuss which solutions should be selected as physical and associated with the charge. It is notable that the physical (regular in the origin) magnetic solution does not have a finite limit if the size of the charge is taken infinitely small. It is not obliged to, indeed, since no size of any physical object may be smaller than the elementary length. In Sec. IV we discuss various peculiarities carried by the solutions found: the magneto-electric effect and, especially, the NC magnetic moment intrinsic to a charged extended particle, which is inversely proportional to its size and, hence, most important [26] for particles that are considered pointlike within the present experimental possibilities, like charged leptons and quarks. Also the corresponding hyperfine splitting caused by this moment if the particle is taken as an atomic nucleus (Sec. IV A) is considered. In Sec. IV B the NC correction to the electrostatic potential is inspected that consists of an anisotropic Coulomb field and of an electric quadrupole contribution. An NC analog of the Zeeman splitting is pointed. In Sec. IV C we propose an extension of the Furry theorem in application to NC electrodynamics that explains, on general basis, the character of the dependence of the magnetic and electrostatic solutions upon powers of the charge, external field and the NC parameter.

It is known, that the SW map is not unique. In Sec. V we show that—to the first order at least—the ambiguity in the SW map for the current derived in Sec. II is precisely the ambiguity of adding a homogeneous solution of the charge-conservation equation.

II. MAXWELL EQUATIONS ON NONCOMMUTATIVE SPACE-TIME

A. $U(1)_\star$ gauge theory

In this paper we work on the Moyal plane, which is (identified with) a space of sufficiently smooth functions on $\mathbb{R}^4$ equipped with the Moyal star product

$$f(x) \star g(x) = f(x) e^{(i/2)\overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} g(x) \tag{1}$$

where the NC parameter $\theta^{\mu\nu}$ is assumed to be constant and small.

We start with the action of a NC $U(1)_\star$ gauge theory

$$\begin{aligned} \check{S} &= \check{S}_A + \check{S}_{jA}, \\ \check{S}_A &= -\frac{1}{16\pi c}\int dx \check{F}_{\mu\nu} \star \check{F}^{\mu\nu}, \\ \check{S}_{jA} &= -\frac{1}{c^2}\int dx \check{j}^\mu \star \check{A}_\mu, \\ \check{F}_{\mu\nu} &= \partial_\mu \check{A}_\nu - \partial_\nu \check{A}_\mu + ig[\check{A}_\mu \overset{\star}{,} \check{A}_\nu], \\ [\check{A}_\mu \overset{\star}{,} \check{A}_\nu] &= \check{A}_\mu \star \check{A}_\nu - \check{A}_\nu \star \check{A}_\mu, \end{aligned} \tag{2}$$

where the coupling constant is $g = e/(\hbar c)$, with e being the elementary charge (for the electron $e = -|e|$). The action $\check{S}_{jA}$ may be obtained from an NC theory with

[2] Stress, that for some applications, like the recent analysis of noncommutative Dirac quantization condition [24], this is not necessary.

fundamental fields. For example, for the case of fundamental fermions we have

$$\begin{aligned}\check{S}_{\check{\psi}} &= i\int dx \bar{\check{\psi}} \star \gamma^{\mu}(\partial_{\mu} - ig\check{A}_{\mu}\star)\check{\psi} \\ &\equiv i\int dx \bar{\check{\psi}}\gamma^{\mu}\partial_{\mu}\check{\psi} + \check{S}_{jA}.\end{aligned} \tag{3}$$

The $U(1)_{\star}$ gauge transformations read

$$\begin{aligned}\check{A}_{\mu} &\to \check{A}'_{\mu} = U_{\check{\lambda}} \star \check{A}_{\mu} \star U_{\check{\lambda}}^{-1} + ig^{-1}(\partial_{\mu}U_{\check{\lambda}}) \star U_{\check{\lambda}}^{-1}, \\ \check{F}_{\mu\nu} &\to \check{F}'_{\mu\nu} = U_{\check{\lambda}} \star \check{F}_{\mu\nu} \star U_{\check{\lambda}}^{-1}, \\ U_{\check{\lambda}} &= e_{\star}^{i\check{\lambda}} = 1 + i\check{\lambda} - \frac{1}{2}\check{\lambda}\star\check{\lambda} + O(\check{\lambda}^3)\end{aligned} \tag{4}$$

with a local parameter $\check{\lambda}(x)$. The action $\check{S}_A$ is invariant under these gauge transformations (see e.g., [7,16,17]). The external current $\check{j}^{\mu}(x)$ transforms covariantly,

$$\check{j}^{\mu\prime} = U_{\check{\lambda}} \star \check{j}^{\mu} \star U_{\check{\lambda}}^{-1}, \tag{5}$$

so that the equations of motion, $\delta\check{S}/\delta\check{A}_{\mu} = 0$,

$$\check{D}_{\nu}\check{F}^{\nu\mu}(x) = \frac{4\pi}{c}\check{j}^{\mu}(x), \tag{6}$$

are gauge-covariant under the transformations (4) and (5). The covariant derivative is defined as

$$\check{D}_{\mu}\Phi := \partial_{\mu}\Phi + ig[\check{A}_{\mu} \overset{\star}{,} \Phi]. \tag{7}$$

The same transformation rule (5) for the current follows from (3) assuming standard transformation properties $\check{\psi} \to \check{\psi}' = U_{\check{\lambda}} \star \check{\psi}$ for the fermions.

The compatibility condition for the equations of motion (6) yields a covariant conservation law for the current,

$$\check{D}_{\mu}\check{D}_{\nu}\check{F}^{\nu\mu} = 0 \Rightarrow \check{D}_{\mu}\check{j}^{\mu} = 0. \tag{8}$$

Let us now turn to the actions. The sum $\check{S}_A + \check{S}_{\check{\psi}}$ is $U(1)_{\star}$ invariant. Let us check what happens to (2). By performing an infinitesimal gauge transformation (4) and (5), in the action for the currents we obtain,

$$\delta\check{S}_{jA} = -\frac{1}{gc^2}\int dx\{(\partial_{\mu}\check{j}^{\mu}) \star \check{\lambda}\}, \tag{9}$$

which means that noncommutative Maxwell theory in the presence of currents has $U(1)_{\star}$ symmetry, if the noncommutative currents j^{μ} are conserved $\partial_{\mu}j^{\mu} = 0$. However, this disagrees with the fact that the currents are covariantly conserved, which follows as an identity from the equations of motion (8). Hence, the total action $\check{S}$ (2) is not gauge-invariant. In fact, this same feature is already known for the Yang-Mills theory coupled with external currents. There is no consistent way to introduce currents in non-Abelian gauge theories already at the classical level [27] (see also [28]) without violating the gauge invariance, although a gauge-*covariant* theory can be formulated [29] both at classical and quantum levels. Therefore, it is not unexpected that the same problem is encountered here due to a close analogy between the $U(1)_{\star}$ noncommutative gauge theory and non-Abelian gauge theories [1,2].

B. The Seiberg-Witten map

Despite the presence of all desired covariance properties of the $U(1)_{\star}$ Maxwell equations, this is not what one needs to analyze phenomenological predictions of the NC theory. One likes to deal with gauge-*invariant* field strength and the currents rather than with gauge-*covariant* ones. In other words, one has to introduce ordinary commutative $U(1)$ fields in place of the $U(1)_{\star}$ ones.

This is done with the help of the Seiberg-Witten map [7], which is very well known for $\check{A}_{\mu}$ and $\check{F}_{\mu\nu}$. At the first order in θ it reads

$$\begin{aligned}\check{A}_{\mu} &= A_{\mu} + \frac{g}{2}\theta^{\alpha\beta}A_{\alpha}[\partial_{\beta}A_{\mu} + f_{\beta\mu}], \\ f_{\mu\nu} &= \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \\ \check{F}_{\mu\nu} &= f_{\mu\nu} - g\theta^{\alpha\beta}[f_{\alpha\mu}f_{\beta\nu} - A_{\alpha}\partial_{\beta}f_{\mu\nu}]. \\ \check{\lambda} &= \lambda - \frac{g}{2}\theta^{\alpha\beta}\partial_{\alpha}\lambda \cdot A_{\beta}.\end{aligned} \tag{10}$$

To find the SW map for currents $\check{j}^{\mu}$ one has to demand that under the SW map the $U(1)_{*}$ gauge transformation (5) of the current $\check{j}^{\mu}$ is induced by the $U(1)$ gauge transformations of A_{μ} and j^{μ} through a functional dependence of $\check{j}^{\mu}$ on the latter fields. For infinitesimal transformations this condition reads

$$\check{j}^{\mu}(A, j) + \check{\delta}_{\check{\lambda}}\check{j}^{\mu}(A, j) = \check{j}^{\mu}(A + \delta_{\lambda}A, j) \tag{11}$$

where $\check{\delta}_{\check{\lambda}}$ and δ_{λ} denote the corresponding gauge variations. Note, that $\delta_{\lambda}j = 0$, as it has to be in the commutative electrodynamics. By virtue of (5)

$$\check{\delta}_{\check{\lambda}}\check{j}^{\mu}(A, j) = i[\check{\lambda} \overset{\star}{,} \check{j}^{\mu}] = -\theta^{\alpha\beta}(\partial_{\alpha}\lambda)(\partial_{\beta}j^{\mu}) + O(\theta^2), \tag{12}$$

which coincides with the gauge transformation of $g\theta^{\alpha\beta}A_{\alpha}\partial_{\beta}j^{\mu}$ caused by $\delta_{\lambda}A_{\alpha} = -g^{-1}\partial_{\alpha}\lambda$. So, finally, the additional SW map for the current is

$$\check{j}^{\mu} = j^{\mu} + g\theta^{\alpha\beta}A_{\alpha}\partial_{\beta}j^{\mu}. \tag{13}$$

This result coincides with the one derived previously in Ref. [25]. The SW map is not unique. There is a three-parameter ambiguity in the solutions of the SW equations which will be discussed in Sec. V below.

As it was already mentioned above, the NC action for electromagnetic field interacting with an external current is *not* gauge-invariant, while the field equations are gauge-covariant. Therefore, it makes sense to apply the SW map to the equation of motion (6) and to the compatibility condition (8). To the first order in $\theta^{\mu\nu}$ one immediately gets

$$\partial_\nu f^{\nu\mu} - g\theta^{\alpha\beta}(\partial_\nu(f_\alpha{}^\nu f_\beta{}^\mu) - f_{\nu\alpha}\partial_\beta f^{\nu\mu} - A_\alpha \partial_\nu \partial_\beta f^{\nu\mu})$$
$$= \frac{4\pi}{c}(j^\mu + g\theta^{\alpha\beta}A_\alpha \partial_\beta j^\mu), \tag{14}$$

$$\partial_\mu j^\mu + g\theta^{\alpha\beta}(f_{\mu\alpha}\partial_\beta j^\mu + A_\alpha \partial_\beta \partial_\mu j^\mu) = 0. \tag{15}$$

One can directly check the compatibility of these two equations within the same θ-accuracy by acting with ∂_μ on (14) and using (15) to get an identity. Only the antisymmetricity of the tensors $\theta^{\alpha\beta}$ and $f^{\mu\nu}$ is referred to in the process, as well as the Bianki identity for the latter.

The modified Maxwell Eqs. (14) are nonlinear with respect to the field already when the external current is away, $j = 0$. The nonlinearity is restricted to the second power of the field, because we confined ourselves to the first order of the noncommutativity parameter θ when deducing them: expansion in powers of θ generates expansion in powers of the field.

Equations of motion (14) and (15), are $U(1)$-gauge-covariant by construction, even though they look noncovariant as containing the potentials together with the gauge-invariant field intensities and the current. It is important that the potentials are involved with the small factor of θ. For this reason the potential-containing terms can be omitted from the set of equations if taken on its solutions, determined within the accuracy accepted. To prove this statement, note first that if the factor $\partial_\mu j^\mu$, which is of the order of θ according to Eq. (15), is substituted into its potential-containing term $g\theta^{\alpha\beta}A_\alpha\partial_\beta\partial_\mu j^\mu$, the latter becomes of the order of θ^2 and is, hence, to be disregarded in Eq. (15). Analogously, the difference of the two potential-containing terms in the left- and right-hand sides of Eq. (14) $A_\alpha\theta^{\alpha\beta}\partial_\beta(\partial_\nu f^{\nu\mu} - \frac{4\pi}{c}j^\mu)$ is also $\sim\theta^2$, because the factor $(\partial_\nu f^{\nu\mu} - \frac{4\pi}{c}j^\mu)$ in it is of the order of θ according to Eq. (14). Hence, we are left with the explicitly gauge-invariant set of equations

$$\partial_\nu f^{\nu\mu} - g\theta^{\alpha\beta}(\partial_\nu(f_\alpha{}^\nu f_\beta{}^\mu) - f_{\nu\alpha}\partial_\beta f^{\nu\mu}) = \frac{4\pi}{c}j^\mu, \tag{16}$$

$$\partial_\mu j^\mu + g\theta^{\alpha\beta}f_{\mu\alpha}\partial_\beta j^\mu = 0. \tag{17}$$

To treat the nonlinear Eqs. (14) and (15) we shall need to give them a recurrence form by expanding A_μ and j^μ in them in the θ-series:

$$A_\mu = A_\mu^{(0)} + A_\mu^{(1)}(\theta) + O(\theta^2), \tag{18}$$

$$j^\mu = j^{(0)\mu} + j^{(1)\mu} + O(\theta^2), \tag{19}$$

where $A^{(0)}$, $j^{(0)}$ satisfy commutative Maxwell and current-conservation equations

$$\partial_\nu f^{(0)\nu\mu} = \frac{4\pi}{c}j^{(0)\mu}, \qquad \partial_\mu j^{(0)\mu} = 0, \tag{20}$$

and $A^{(j)}$ and $j^{(j)}$ are corrections of the jth order in θ. By using (20), we obtain to the first order in θ

$$\partial_\nu f^{(1)\nu\mu} - g\theta^{\alpha\beta}(\partial_\nu(f_\alpha^{(0)\nu} f_\beta^{(0)\mu}) - f_{\nu\alpha}^{(0)}\partial_\beta f^{(0)\nu\mu}) = \frac{4\pi}{c}j^{(1)\mu}, \tag{21}$$

$$\partial_\mu j^{(1)\mu} + g\theta^{\alpha\beta}f_{\mu\alpha}^{(0)}\partial_\beta j^{(0)\mu} = 0. \tag{22}$$

Solutions of (21) and (22) are not unique. One can add a current $\tilde{j}^\mu$, which satisfies $\partial_\mu \tilde{j}^\mu = 0$, to $j^{(1)\mu}$. This is the same as Eq. (20) for $j^{(0)\mu}$. This ambiguity is, therefore, not a physical one and $\tilde{j}^\mu$ can be absorbed in $j^{(0)\mu}$. In the examples considered in the next section, where the source $j_\mu^{(0)}$ is static and spherically symmetric, one can even take $j^{(1)\mu} = 0$. Similarly, the ambiguity for $f^{(1)\mu\nu}$ can be removed by imposing the falloff conditions at infinity on this field or fixing an external field.

Just for the sake of completeness, let us check what happens if one applies the SW map at the level of the action (2). One easily obtains to the first order in the NC parameter

$$S_{\mathrm{SW}} = -\frac{1}{16\pi c}\int dx\left\{\left(1 + \frac{g}{2}\theta^{\alpha\beta}f_{\alpha\beta}\right)f_{\mu\nu}f^{\mu\nu} - 2g\theta^{\alpha\beta}f^{\mu\nu}f_{\alpha\mu}f_{\beta\nu}\right\}$$
$$- \frac{1}{c^2}\int dx\left\{j^\mu A_\mu + \frac{g}{2}\theta^{\alpha\beta}j^\mu A_\alpha(\partial_\beta A_\mu + f_{\beta\mu}) + g\theta^{\alpha\beta}A_\mu A_\alpha(\partial_\beta j^\mu)\right\}. \tag{23}$$

Equating to zero the variation of this action with respect to A_μ yields the following equation

$$\partial_\nu\left[f^{\nu\mu}\left(1 + \frac{g}{2}\theta^{\alpha\beta}f_{\alpha\beta}\right)\right] = \frac{4\pi}{c}j^\mu\left(1 + \frac{g}{2}\theta^{\alpha\beta}f_{\alpha\beta}\right) + g\theta^{\alpha\beta}[\partial_\nu(f_\alpha{}^\nu f_\beta{}^\mu) + \partial_\alpha(f^{\mu\nu}f_{\beta\nu})]$$
$$+ g\theta^{\beta\mu}\left\{\partial_\alpha(f^{\alpha\nu}f_{\beta\nu}) - \frac{1}{4}\partial_\beta(f^{\alpha\nu}f_{\alpha\nu}) - \frac{4\pi}{c}\left[f_{\beta\alpha}j^\alpha + A_\alpha(\partial_\beta j^\alpha) - \frac{A_\beta}{2}(\partial_\alpha j^\alpha)\right]\right\}. \tag{24}$$

When the current is away, it can be directly checked that these equations coincide with (14) on solutions of the latter. On the contrary, when $j^\mu \neq 0$, Eqs. (24) exhibit trouble. The gauge invariance of (24) cannot be restored even by doing the expansions (18) and (19). In what follows, we discard the action approach, and use exclusively the NC Maxwell and current-conservation Eqs. (14) and (15) (together with their expanded versions (21) and (22)).

Some clarifying remarks on the nonequivalence of the SW maps in the action and in equations of motion are in order. By definition, for an $U(1)_\star$ theory, $\check{S}(\check{A}, \check{j}) = S_{\rm SW}(A, j)$. Therefore, the equation of motion (24) can be rewritten as

$$0 = \frac{\delta S_{\rm SW}}{\delta A_\mu(x)} = \int dy \left[\frac{\delta \check{S}}{\delta \check{A}_\nu(y)} \frac{\delta \check{A}_\nu(y)}{\delta A_\mu(x)} + \frac{\delta \check{S}}{\delta \check{j}^\nu(y)} \frac{\delta \check{j}^\nu(y)}{\delta A_\mu(x)} \right]. \tag{25}$$

The first term in the brackets above vanishes on the equations of motion (6), while the second one does not (since there is no equation of motion produced by variations of currents). Hence, the equations of motion obtained by varying $S_{\rm SW}$ are not equivalent to that obtained from the original action $\check{S}$. On the opposite, if the SW map is applied to the equations of motion of the $U(1)_\star$ gauge theory, such an equivalence is, of course, preserved (after truncation to the first order in θ). The reason for nonequivalence of the two procedures is in the nondynamical nature of the external current, which does not generate any equations of motion, but participates in the SW map.

III. SOLUTIONS FOR THE POTENTIAL PRODUCED BY A STATIC CHARGE IN THE PRESENCE OF A MAGNETIC BACKGROUND

A. Regular solutions

In this section we are studying the linear in θ NC corrections to the 4-vector potential of a static spherically symmetric charge. The corrections to be found are both magnetostatic and electrostatic (the latter will occur only when an external magnetic field is present).

We impose the stationarity conditions

$$\partial_0 A^{(1)\mu}(x) = 0, \tag{26}$$

on these corrections, bearing in mind that the unperturbed solutions, i.e. those to the equations

$$\partial_\nu f^{(0)\nu\mu} = \frac{4\pi}{c} j^{(0)\mu}, \qquad \partial_\mu j^{(0)\mu} = 0,$$
$$f^{(0)}_{\mu\nu} = \partial_\mu A^{(0)}_\nu - \partial_\nu A^{(0)}_\mu, \tag{27}$$

are also subjected to the stationarity conditions. More precisely, we take the total external charge Ze distributed with a constant density throughout a spherical region of the space with the radius a. The current density $j^{(0)\mu}$ is defined in the inner part of the sphere $r < a$, called region I, and in its outer part $r > a$, called region II, as follows

$$j^\mu = (c\rho, 0), \qquad \rho(\mathbf{x}) = \begin{cases} \frac{3}{4\pi}\frac{Ze}{a^3}, & r \in \mathrm{I} \\ 0, & r \in \mathrm{II} \end{cases}, \qquad r = |\mathbf{x}|. \tag{28}$$

In what follows we shall equip designations of potentials, relating to these regions, with the indices I or II. The homogeneous—inside the sphere—charge distribution is taken for simplicity. Extensions to arbitrary spherical symmetric distributions, continuous ones included, may be also considered when necessary. The charge density (28) tends to the Dirac delta-function in the point-charge limit: $\rho(\mathbf{x}) = Ze\delta^3(\mathbf{x})$, as $a \to 0$. However, owing to the coordinate noncommutativity, different coordinate components cannot be simultaneously given definite values, hence no spherical *physical object* should be taken with its radius smaller than the elementary length. For this reason we will restrict our consideration to the values $a > \sqrt{\theta}$. On the other hand, after the SW map is accomplished, we deal with a commutative plain and must take care of providing consistency to the resulting theory defined everywhere on it, both for $r < a$ and $r > a$. Therefore, when considering the coordinate values down to the origin $r = 0$, we must seriously treat possible singularities in this point and their consequences. This will lead us to imposing regularity boundary conditions in the origin. This is a must at least as long as the θ-expansion is relied on. The point is that higher orders of θ are accompanied by higher orders of the electromagnetic potential and its derivatives. Therefore, if a singularity is admitted in the lowest order, it would strengthen with every next order of the θ-expansion, hence the latter would not exist.

Note, that in Ref. [30] it was even suggested that the smearing of charges replaces the use of noncommutative products in equations of motion. We do not share this point of view.

Equation (28), certainly, satisfies the current-conservation equation in (27). Equation (27) is satisfied by the following electromagnetic potential $A^{(0)\mu}$,

$$A^{(0)\mu} = (A^{(0)0}, A^{(0)i}),$$
$$A^{(0)0}(r) = \begin{cases} \frac{Ze}{2a^3} - r^2 + \frac{3}{2}\frac{Ze}{a}, & r \in \mathrm{I} \\ \frac{Ze}{r}, & r \in \mathrm{II} \end{cases}, \tag{29}$$
$$A^{(0)i} = -\frac{1}{2} f^{(0)}_{ik} x^k, \qquad f^{(0)}_{ij} = \text{const},$$

where we have included a solution of the homogeneous counterpart of Eq. (27) $A^{(0)i} = -\frac{1}{2} f^{(0)}_{ik} x^k$ corresponding to a constant external magnetic field $B_i = \frac{1}{2}\varepsilon_{ijk} f^{(0)}_{jk}$. The case $A^{(0)i} = 0$ has been considered previously [26].

The zeroth component of Eq. (29) satisfies the boundary

$$A_{\mathrm{I}}^{(0)0}(0) \neq \infty, \qquad A_{\mathrm{II}}^{(0)0}(r)|_{r\to+\infty} = 0, \qquad (30)$$

and smoothness $A_{\mathrm{I}}^{(0)0}(r)|_{r=a} = A_{\mathrm{II}}^{(0)0}(r)|_{r=a}$, $\partial_r A_{\mathrm{I}}^{(0)0}(r)|_{r=a} = \partial_r A_{\mathrm{II}}^{(0)0}(r)|_{r=a}$ conditions. The boundary conditions (30) completely determine the solution (29) for $A^{(0)0}$ of the Laplace equation. The second condition in (30) excludes the linear function $E^i x^i$ corresponding to a homogeneous electric field of arbitrary strength and of arbitrary direction as a possible solution for $f^{(0)0i}$.

We restrict ourselves to a space-space noncommutativity ($\theta^{0\mu} = 0$). From the spherical symmetry of the current $j^{(0)\mu}$ (28) and of the solution $A^{(0)0}$ (29) it follows that Eq. (22) is satisfied by $j^{(1)\mu} = 0$, no correction to the current is required. This implies that the current remains dynamically intact, $j^\mu = j^{(0)\mu}$, so we may refer to it as a fixed external current, as this is customary in an $U(1)$-theory. The Maxwell equation in SW approach (21) reads,

$$\partial_i f^{(1)i\mu} + g\theta^{ij}[(\partial_i A^{(0)0})(\partial_j \partial^\mu A^{(0)0}) + 2f_{ik}^{(0)}(\partial_k \partial_j A^{(0)\mu})] = 0. \qquad (31)$$

Taking into account the stationarity condition, for the zeroth component ($\mu = 0$), Eq. (31) reduces to

$$\nabla^2 A^{(1)0} + 2gB^i\theta^j(\delta_{ij}\nabla^2 A^{(0)0} - \partial_i\partial_j A^{(0)0}) = 0, \qquad (32)$$

where we have introduced the vector $\boldsymbol{\theta}$ with components $\theta^i = \frac{1}{2}\varepsilon_{ijk}\theta^{jk}$. The spacial part of Eq. (31) reads

$$\partial_i f^{(1)ik} + g\theta^{ij}(\partial_i A^{(0)0})(\partial^k \partial_j A^{(0)0}) = 0 \quad \text{or}$$
$$\nabla^2 A^{(1)k} - \partial_i\partial_k A^{(1)i} + g\theta^{ij}(\partial_i A^{(0)0})(\partial^k \partial_j A^{(0)0}) = 0, \qquad (33)$$

because the second space derivative of the external constant magnetic field potential $A^{(0)k}$ is zero.

Once the starting Eqs. (21) and (22) contain only field strengths and not potentials we may impose, for instance, the Coulomb gauge condition $\partial_k A^k = 0$ both on $A^{(0)k}$ and on $A^{(1)k}$. It is worthwhile making explicitly sure that the resulting equations are noncontradictory. To this end, let us act with the differential operator ∂_k on (33) and see that we do not come to a contradiction with the equality $\partial_k A^{(1)k} = 0$. This implies the requirement $\theta^{ij}\partial_k[(\partial_i A^{(0)0})(\partial_k\partial_j A^{(0)0})] = 0$. This expression is equal to $\theta^{ij}[(\partial_k\partial_i A^{(0)0})(\partial_k\partial_j A^{(0)0}) + (\partial_i A^{(0)0})(\partial_k^2\partial_j A^{(0)0})]$. The first term here disappears due to the antisymmetricity of θ^{ij}. The second one is zero thanks to the spherical symmetry of $A_{(0)0}$, already exploited above, because the latter implies that this scalar function contains only one 3-vector, which is the radius-vector $\mathbf{x}$. Then the product of the two factors in the brackets is proportional to the product of different components of this same vector $x_i x_j$. This tensor disappears when multiplied by the antisymmetric tensor θ^{ij}. In the Coulomb gauge we obtain finally for spacial components ($\mu = k = 1, 2, 3$)

$$\nabla^2 A^{(1)k} + g\theta^{ij}(\partial_i A^{(0)0})(\partial_k\partial_j A^{(0)0}) = 0. \qquad (34)$$

One can observe that unlike Eq. (32), the latter Eq. (34) does not contain a contribution from the external magnetic field. These facts show that the introduction of a constant and homogeneous magnetic background will not modify the magnetic field produced by a static charge, but instead, provides a correction to its electric field, which otherwise ($\mathbf{B} = \mathbf{0}$) does not gain a first-order θ-correction, since Eq. (32) for it becomes a source-free Laplace equation with the regular boundary conditions (30).

We see that equations for $A^{(1)0}$ and $A^{(1)i}$ decouple and can be analyzed separately. We start with the equation for $A^{(1)i}$. In region I, Eq. (34) reads,

$$\nabla^2 A_{\mathrm{I}}^{(1)k}(\mathbf{x}) = -g\left(\frac{Ze}{a^3}\right)^2 \theta^{ik}x^i, \qquad (35)$$

and for region II we have

$$\nabla^2 A_{\mathrm{II}}^{(1)k}(\mathbf{x}) = -g\left(\frac{Ze}{r^3}\right)^2 \theta^{ik}x^i. \qquad (36)$$

The general solutions are

$$A_{\mathrm{I}}^{(1)k}(\mathbf{x}) = -\frac{g}{10}\left(\frac{Ze}{a^3}\right)^2 r^2\theta^{ik}x^i + a_{\mathrm{I}}^{(1)k}(r, \vartheta, \varphi), \qquad (37)$$

$$A_{\mathrm{II}}^{(1)k}(\mathbf{x}) = -\frac{g}{4}\left(\frac{Ze}{r^2}\right)^2 \theta^{ik}x^i + a_{\mathrm{II}}^{(1)k}(r, \vartheta, \varphi), \qquad (38)$$

where ϑ and φ are the azimuthal and polar angles of the coordinate radius-vector $\mathbf{x}$, respectively. The functions $a_\lambda^{(1)k}$ ($\lambda = \mathrm{I}$), II are solutions of the homogeneous Laplace equation $\nabla^2 a_\lambda^{(1)k} = 0$,

$$a_\lambda^{(1)k}(r, \vartheta, \varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}[\alpha_{(\lambda)l,m}^k r^l + \beta_{(\lambda)l,m}^k r^{-(l+1)}]Y_{l,m}(\vartheta, \varphi),$$

where the functions $Y_{l,m}(\vartheta, \varphi)$ are spherical harmonics [31]. The constants $\alpha_{(\lambda)l,m}^k$ and $\beta_{(\lambda)l.m}^k$ are fixed by the same type of boundary and smoothness conditions as before,

$$\begin{aligned}
&\mathbf{A}_{\mathrm{I}}^{(1)}(\mathbf{x})|_{r=a} = \mathbf{A}_{\mathrm{II}}^{(1)}(\mathbf{x})|_{r=a},\\
&\frac{\partial}{\partial r}\mathbf{A}_{\mathrm{I}}^{(1)}(\mathbf{x})|_{r=a} = \frac{\partial}{\partial r}\mathbf{A}_{\mathrm{II}}^{(1)}(\mathbf{x})|_{r=a},\\
&\mathbf{A}_{\mathrm{I}}^{(1)}(\mathbf{x})|_{r\to 0} \neq \infty,\\
&\mathbf{A}_{\mathrm{II}}^{(1)}(\mathbf{x})|_{r\to\infty} = 0.
\end{aligned} \qquad (39)$$

The right-hand sides in Eqs. (35) and (36) can be expressed in terms of spherical harmonics with $l = 1$ using the relations

$$Y_{1,\pm 1} = \mp \frac{1}{r}\sqrt{\frac{3}{8\pi}}(x^1 \pm ix^2), \qquad Y_{1,0} = \frac{1}{r}\sqrt{\frac{3}{4\pi}}x^3. \quad (40)$$

Then the coefficients $\alpha^k_{(\lambda)l,m}$ and $\beta^k_{(\lambda)l,m}$ are found to be ($\alpha^k_{(\mathrm{I})l,m} = \alpha^k_{l,m}$, $\beta^k_{(\mathrm{II})l,m} = \beta^k_{l,m}$)

$$\begin{aligned}
\alpha^k_{(\mathrm{II})l,m} &= \beta^k_{(\mathrm{I})l,m} = 0,\\
\alpha^k_{1,\pm 1} &= \frac{1}{4}g\left(\frac{Ze}{a^2}\right)^2\sqrt{\frac{2\pi}{3}}[\mp\theta^{1k} + i\theta^{2k}],\\
\beta^k_{1,\pm 1} &= \frac{2}{5}g\frac{(Ze)^2}{a}\sqrt{\frac{2\pi}{3}}[\mp\theta^{1k} + i\theta^{2k}],\\
\alpha^k_{1,0} &= \frac{1}{4}g\left(\frac{Ze}{a^2}\right)^2\sqrt{\frac{4\pi}{3}}\theta^{3k},\\
\beta^k_{1,0} &= \frac{2}{5}g\frac{(Ze)^2}{a}\sqrt{\frac{4\pi}{3}}\theta^{3k},
\end{aligned} \quad (41)$$

while all the rest of the coefficients $\alpha^k_{l,m}$ and $\beta^k_{l.m}$ with $l \neq 1$ are identically zero. Finally we have

$$\begin{aligned}
A^{(1)k}_{\mathrm{I}}(\mathbf{x}) &= -\frac{g}{4}\left(\frac{Ze}{a^2}\right)^2\left(\frac{2}{5}\frac{r^2}{a^2} - 1\right)\theta^{ik}x^i, \qquad r < a,\\
A^{(1)k}_{\mathrm{II}}(\mathbf{x}) &= \frac{g}{4}\left(\frac{Ze}{r^2}\right)^2\left(\frac{8}{5}\frac{r}{a} - 1\right)\theta^{ik}x^i, \qquad r > a.
\end{aligned} \quad (42)$$

As a matter of fact, this solution disappears in the origin $\mathbf{A}^{(1)}_{\mathrm{I}}(0) = 0$, although such a boundary condition was not imposed. There is no finite limit of (42) if $a \to 0$.

Now we proceed to evaluating the solutions of (32). For region I we obtain

$$\nabla^2 A^{(1)0}_{\mathrm{I}}(\mathbf{x}) = 4g\left(\frac{Ze}{a^3}\right)(\mathbf{B}\cdot\boldsymbol{\theta}), \quad (43)$$

and for region II we have

$$\nabla^2 A^{(1)0}_{\mathrm{II}}(\mathbf{x}) = 2g\left(\frac{Ze}{r^5}\right)[3(\mathbf{x}\cdot\mathbf{B})(\mathbf{x}\cdot\boldsymbol{\theta}) - r^2(\mathbf{B}\cdot\boldsymbol{\theta})], \quad (44)$$

we remind that the vector $\boldsymbol{\theta}$ has components $\theta^i = \frac{1}{2}\varepsilon_{ijk}\theta^{jk}$.

The general solutions can be expressed as

$$A^{(1)0}_{\mathrm{I}}(\mathbf{x}) = \frac{2}{3}g\left(\frac{Ze}{a^3}\right)r^2(\mathbf{B}\cdot\boldsymbol{\theta}) + a^{(1)0}_{\mathrm{I}}(r,\vartheta,\varphi), \quad (45)$$

$$A^{(1)0}_{\mathrm{II}}(\mathbf{x}) = -g(Ze)\frac{(\mathbf{x}\cdot\mathbf{B})(\mathbf{x}\cdot\boldsymbol{\theta})}{r^3} + a^{(1)0}_{\mathrm{II}}(r,\vartheta,\varphi), \quad (46)$$

where $a^{(1)0}_{\mathrm{I}}$, $a^{(1)0}_{\mathrm{II}}$ are homogeneous solutions, which are fixed through the boundary conditions (39) yielding

$$\begin{aligned}
A^{(1)0}_{\mathrm{I}}(\mathbf{x}) &= 2g\left(\frac{Ze}{a}\right)\left\{\left(\frac{2}{5}\frac{r^2}{a^2} - 1\right)(\mathbf{B}\cdot\boldsymbol{\theta}) - \frac{1}{5a^2}(\mathbf{x}\cdot\mathbf{B})(\mathbf{x}\cdot\boldsymbol{\theta})\right\},\\
&\quad r < a,\\
A^{(1)0}_{\mathrm{II}}(\mathbf{x}) &= g\left(\frac{Ze}{r}\right)\left\{\frac{1}{r^2}\left(\frac{3}{5}\frac{a^2}{r^2} - 1\right)(\mathbf{x}\cdot\mathbf{B})(\mathbf{x}\cdot\boldsymbol{\theta})\right.\\
&\quad \left. - \left(\frac{1}{5}\frac{a^2}{r^2} + 1\right)(\mathbf{B}\cdot\boldsymbol{\theta})\right\} \quad r > a.
\end{aligned} \quad (47)$$

By using (10) and (13) together with the solutions obtained above one can obtain the noncommutative fields $\check{A}^\mu$ and currents $\check{j}^\mu$ at the same (linear) order in θ. As expected, noncommutative fields differ from their SW counterparts already at this order.

It might make sense to confront the result (47) with the fact [32] that within certain models the large magnetic field regime mimics the noncommutivity: the (effective) action for some composite or gauge fields includes their Moyal-like product, so these fields may be imagined as defined on the coordinates, out of which the ones, orthogonal to the magnetic field, do not mutually commute. To attribute the origin of noncommutativity dealt with in the present context to this effect we have to identify the elementary length with the Larmour radius, $\sqrt{\theta} = 1/\sqrt{eB}$, taking the magnetic field large and coinciding with θ in direction. Then, $B = \infty$ is the commutative limit, while the dimensionality of space, $d = 4$, is reduced by two—the number of coordinates, orthogonal to $\mathbf{B}$: the orthogonal subspace merely does not exist. On the contrary, while B is large, but still finite, the orthogonal subspace is noncommutative, while the total space is "almost" $d - 2 = 2$ dimensional. (The reservation "almost" means that in some domains of the space, say, near a charge, where its field dominates over the external magnetic field, the dimensionality of space is again [33] $d = 4$). Unfortunately, the result (47) cannot cover this case, because the condition $B\theta = 1$ lies beyond the applicability of the expansion in powers of θ and B, used in deriving Eq. (47).

B. Alternative magnetic solutions

The solutions that we obtained above depend crucially on the regularity condition imposed at $r = 0$. Relaxing this condition, one can obtain another solution for the vector-potential

$$\begin{aligned}
A^{(1)k}_{\mathrm{I}}(\mathbf{x}) &= -\frac{g}{4}\left(\frac{Ze}{a^2}\right)^2\left(\frac{2}{5}\frac{r^2}{a^2} + \frac{8}{5}\frac{a^3}{r^3} - 1\right)\theta^{ik}x^i,\\
A^{(1)k}_{\mathrm{II}}(\mathbf{x}) &= -\frac{g}{4}\left(\frac{Ze}{r^2}\right)^2\theta^{ik}x^i
\end{aligned} \quad (48)$$

that does not obey the finiteness boundary condition in the origin, but decreases at large distance from the source faster than (42), in other words it is short-range. Its outer part, $A^{(1)k}_{\mathrm{II}}(\mathbf{x})$, which does not depend on the size a of the charge, coincides with the magnetic solution found in [34] for the field produced by a pointlike static charge (the limit of $a = 0$ in (28)); it is highly singular in the origin $r = 0$ in that case. Unlike Eq. (42) this solution is not the field of a magnetic dipole, since it decreases at large distances faster than that.

Solution (42) differs from (48) by adding the function

$$2g\frac{(Ze)^2}{5a}\frac{\theta^{ik}x^i}{r^3}, \tag{49}$$

which is a solution for the vector potential to the homogeneous Laplace equation with the constant coefficient $2g\frac{(Ze)^2}{5a}$ chosen in such a way as to cancel the singularity in the origin of (the inner part of) the solution (48). There are three more homogeneous solutions that do not include any vectors or tensors, other than the ones inherent to the problem: the radius-vector $\mathbf{x}$ and the noncommutativity tensor θ^{ik}:

$$\frac{\theta^k}{r}, \qquad \frac{x_k}{r^3}, \qquad \theta^{ki}x_i. \tag{50}$$

The first one does not satisfy the Coulomb gauge condition and should not, therefore, be included into consideration. The second one is a pure gauge; it does not contain any field strength in it. Its appearance is due to the fact that the Coulomb gauge fixes the gauge degree of freedom only up to a gauge transformation caused by a function $\lambda(r)$ obeying the free Laplace equation. So, with $\lambda = 1/r$ the discussed solution is $-\partial_k\lambda$. Thus, only the third solution $\theta^{ki}x_i$ remains yet to be considered. It is linear in $\mathbf{x}$. Once the smoothness conditions in (39) include the first and the second derivatives, the linear solution can be matched only with itself on the internal boundary. The third homogeneous solution cannot be associated with the source, a constant coefficient to be put in front of it remaining arbitrary. We conclude that the two solutions (48) and (42) exhaust all magnetic solutions produced by the static charge. There are, certainly, many more, homogeneous solutions not associated with the charge. One of them is $\theta^{ki}x_i$. It corresponds to a constant and homogeneous magnetic field of arbitrary strength, but of fixed direction: it is directed along the noncommutativity vector $\boldsymbol{\theta}$. This field should be absorbed into the (more general) external magnetic field $\mathbf{B}$, already included in (29) at the zero-order level. Note that the total magnetic energy of the static charge $\simeq \int |\mathbf{rot A}|^2 d^3x$ is finite for (42) and infinite for (48) when a is finite.

Which of the two solutions (48) or (42) should be selected? Let us first seek an answer to this question beyond the intrinsic context of the NC theory, taking into consideration a possible future use of the solution. According to [34] the magnetic field carried by solutions for the 3-vector potential interacts with the orbital momentum and the spin of the electron in a NC hydrogen atom problem elaborated in [35], wherein the noncommutative charge is taken for the nucleus and placed in the origin $r = 0$. This interaction energy, computed using our solution (42) is finite in the origin. On the contrary, it behaves as r^{-3} for solution (48) (and even worse, as r^{-4}, if the outer part of (48) is continued to the origin to form the point charge solution of [34]). The contribution of this interaction energy causes the fall-down onto the center and thus makes the problem inconsistent. Nevertheless, the similar situation is not considered (although not quite righteously) as a real trouble in quantum mechanics, because the finite size of the nucleus provides a sufficient cutoff. So, purely pragmatically, we cannot completely justify the exclusive necessity of selecting solution (42), but we shall come back to this discussion later in this section after considering also solutions alternative to the electric solution (47).

C. Alternative electrostatic solutions

A clue observation that has helped to solve Eqs. (43) and (44) and may be used to check its solution (47) is that a linear combination $a(r)(\mathbf{B}\cdot\boldsymbol{\theta}) + b(r)(\mathbf{x}\cdot\boldsymbol{B})(\mathbf{x}\cdot\boldsymbol{\theta})$ reproduces itself with different coefficients after being acted on by the Laplace operator according to the formulas:

$$\begin{aligned}\nabla^2 a(r)(\boldsymbol{B}\cdot\boldsymbol{\theta}) &= (\boldsymbol{B}\cdot\boldsymbol{\theta})\left(a'' + \frac{2a'}{r}\right),\\ \nabla^2 b(r)(\mathbf{x}\cdot\mathbf{B})(\mathbf{x}\cdot\boldsymbol{\theta}) &= \left(b'' + \frac{6b'}{r}\right)(\mathbf{x}\cdot\mathbf{B})(\mathbf{x}\cdot\boldsymbol{\theta})\\ &\quad + 2b(r)(\boldsymbol{B}\cdot\boldsymbol{\theta}),\end{aligned} \tag{51}$$

where the primes denote differentiations over r. Using Eq. (51) and the general solution [36]

$$y(x) = \left(\frac{\xi}{x}\right)^{\gamma}\left(\eta + \int_{\xi}^{x} g(x')\left(\frac{x'}{\gamma}\right)^{\gamma} dx'\right) \tag{52}$$

to the linear first-order differential equation

$$y' + \frac{\gamma}{x}y = g(x), \tag{53}$$

where ξ and η are arbitrary constants, and γ is either 2 or 6 in our case, one can find all solutions to Eqs. (43) and (44) of the form $a(r)(\boldsymbol{B}\cdot\boldsymbol{\theta}) + b(r)(\mathbf{x}\cdot\mathbf{B})(\mathbf{x}\cdot\boldsymbol{\theta})$, extending, when necessary, beyond the homogeneous charge distribution (28). (The reason to confine ourselves to this class of solutions is that, once the inhomogeneity in (43) and (44) is linear both in $\mathbf{B}$ and θ, only solutions with the same property may be referred to as produced by the source under consideration). The homogeneous solutions of Eqs. (43) and (44) so found are

$$a)\,(\boldsymbol{B}\cdot\boldsymbol{\theta}),\quad b)\,\frac{(\boldsymbol{B}\cdot\boldsymbol{\theta})}{r},\quad c)\,\frac{(\boldsymbol{B}\cdot\boldsymbol{\theta})}{r^3} - \frac{3(\mathbf{x}\cdot\mathbf{B})(\mathbf{x}\cdot\boldsymbol{\theta})}{r^5},\quad d)\,r^2(\boldsymbol{B}\cdot\boldsymbol{\theta}) - 3(\mathbf{x}\cdot\mathbf{B})(\mathbf{x}\cdot\boldsymbol{\theta}). \tag{54}$$

Again, the same as before, the condition that the field strength should decrease for large distances r from the charge is not sufficient to fix the solution: more boundary conditions are needed as an effect of gauge invariance violation by external current in the NC theory as discussed in Sec. II. We may discard solution (d) as giving rise to (anisotropic) electric field (linearly) growing in the remote region, but we must note that such a solution makes an interesting option of an external field admitted by sourceless equations of motion. The constant solution (a) should be disregarded as a pure gauge, left still unfixed after the gauge conditions used above were imposed. (Recall that in the $U(1)$ gauge theory the Lorentz gauge imposed to reduce the Maxwell equations for the field strengths to the Laplace equations for potentials in a stationary problem, where fields and potentials do not depend on time t, turns into the Coulomb gauge for the 3-vector potential. However, there remains a residual gauge freedom determined by the gauge parameter $\lambda = \lambda_1 t + \lambda_2(\mathbf{x})$ with λ_1 being a constant and $\boldsymbol{\nabla}^2\lambda_2(\mathbf{x}) = 0$. Therefore, the scalar potential A_0 remains fixed only up to this constant λ_1 and up to a function λ_2 subject to the homogenous Laplace equation). By linearly combining the remaining two solutions (b) and (c) (54) with the solution (47) one can form all solutions, satisfying boundary conditions, other than (39), but still, perhaps, also physically reasonable. Let us discuss such possibilities. First, solution (b) multiplied by the constant factor gZe may be added to solution (47) to exclude from it the Coulomb (rightmost in (47)) term. Note, however, that the other long-range correction to the Coulomb potential $gZe(\mathbf{x}\cdot\mathbf{B})(\mathbf{x}\cdot\boldsymbol{\theta})/r^3$ still cannot be excluded. Second, solution (c) multiplied by $gZea^2/5$, when added to (47), leads to a solution of (43) and (44) free of electric quadrupole term (see Eq. (58) in he next section). Both new solutions, as well as any other solution formed by combining (b) and (c) with (47), are singular in the origin, although (b) does not produce the fall-down onto the center. Quite the opposite, combining (b) with (47) would lead to a solution with the $1/r$ behavior of the potential that is considered as admissible in the standard theory.

We have now to answer the question set in Sec. III B, bearing in mind that an applicability for use cannot serve a sufficient criterion for fixing a physical solution of the field-and-current equations of motion in the NC theory. To finally ground why we are keeping to the choice of (47) and (42) as the only appropriate solution is the following. Let us remember that the principal motivation for creating an NC theory was the complete ultraviolet regularization by the elementary length of everything, including the Coulomb potential itself, because many ultraviolet troubles are already due to this weakest singularity, to say nothing of stronger singularities in the origin that might appear in the theory. Solutions (47) and (42) are the only ones among other possibilities discussed in the present section that are totally free of any singularities in the origin. In other words, these are the only solutions, *regularizable* by the charge size. Naturally, these should not necessarily survive the limiting transition to a point charge, since the latter notion is away from the NC theory. Indeed, Eq. (42) does not.

Another, more technical, reason to stick to these solutions lies in the validity of the approximation considered. The effective parameters used in the expansion here are $f^2\theta$ and $fj\theta$. Both of them remain small on the nonsingular solutions chosen, (42) and (47), even when the size a is taken as its minimum $a = \sqrt{\theta}$. Namely, $f\theta = g(Ze)^2$ for (42) and $f\theta = gZeB\theta$ for (47). So only the values of the charge Z and the external magnetic field b are restricted. A use of any singular solution would lead us out the applicability domain for sufficiently small r.

IV. PROPERTIES OF REGULAR SOLUTIONS

Solutions (42) and (47) provide stationary long-distance corrections to the zeroth order potential (29) induced by a static spherical charge. These corrections may be understood as higher order form factors of a finite-size spherical charge induced by the noncommutativity, because they can be interpreted in terms of appearance of an effective charge density surrounding that charge, as well as of the dipole magnetic and quadrupole electric moments.

Irrespective of whether the external magnetic field $\mathbf{B}$ is present, a magnetic field carried by $A^{(1)k}(\mathbf{x})$ proportional to θ and independent of the external magnetic field is induced. On the contrary, the electric field remains unchanged within the first order of θ if $\mathbf{B} = \mathbf{0}$, but gains first-order corrections if the external magnetic field is present. The spherically-symmetric source does not undergo corrections.

A. Magnetic dipole

The leading long-distance part of the vector-potential (42) behaves like that of a magnetic dipole, the static charge (28) being thus a carrier of an equivalent magnetic moment $\mathcal{M}$

$$\boldsymbol{A} = \frac{\mathcal{M}\times\mathbf{x}}{r^3}, \qquad \mathcal{M} = \boldsymbol{\theta}(Ze)^2\frac{2g}{5a} \tag{55}$$

(we used the designation $\times$ for the vector product). Though the magnetic moment grows infinitely in the limit of a pointlike charge, $a \to 0$, this fact should not be thought of as a problem, since the size of the charge is restricted from below, $a > l$, by the elementary length $l = \sqrt{\theta}$.

Taking expression (55) for the magnetic moment, in Ref. [26] we studied an efficient tool of getting stronger bounds on the noncommutativity parameter based on the fact that in all scattering processes leptons do not show any size at all. Once theoretical calculations of the lepton anomalous magnetic moments, based on standard commutative models, do not contradict their observable values within the existing experimental and theoretical accuracy, we admitted that, at the worst, all clearance in their values may be attributed to the effect of noncommutativity described above. Then, as long as we relied on the existing experimental bounds of the lepton size, we obtained that the noncommutativity parameter is bounded by the values following already from other present-day estimates. But admitting the point-likeness of the electron we got the hitherto strongest bound of 10^4 TeV among the ones based on particle physics experiments.

The magnetic moment of a proton should contribute to the hyperfine splitting of the $1S_{1/2}$-states in the hydrogen atom. When calculated with the help of the outer part of (42), the spitting is proportional to $(1/a)\overline{r^{-3}}$, where the bar means averaging over $r > a$ in the S-state outside of the proton, with a now taken as its size. On the other hand with the outer part of solution (48) the corresponding contribution [34] makes $\overline{r^{-4}}$. The two expressions are of the same order of magnitude, because the averaging effectively results in the substitution $r = a$, owing to the singular character of the averaged function and the fact that the proton size a is much less than the Bohr radius $a_0 = \hbar^2/m_e\alpha$, where m_e is the electron mass and $\alpha = 1/137$ is the fine-structure constant. So, taking into account the noncommutative magnetic moment of the proton does not change the existing bound on the noncommutativity parameter.

There is a different context [37], wherein the noncommutativity of coordinates is introduced associated with a charged particle spin. Then, most naturally, it also carries a magnetic moment.

A production of magnetic field by a static electric charge—the magneto-electric effect—was reported in quantum electrodynamics with a constant and homogeneous external (magnetic plus electric) field of the most general form [5]. The inhomogeneous magnetic field produced by a static charge in that problem exists in an approximation linear in the charge, when the charge is small. Contrary to that situation, in the present problem we have found a solution of nonlinear Maxwell Eqs. (14) and (15) within the first order of θ, and this solution is, for its magnetic component, quadratic in the charge eZ, as it is seen from (42). (The same statement holds true for the solution (48).) The absence of a linear-in-the-charge part of the magnetic field is in agreement with the statement in [3] that in NC electrodynamics without a background field the photon polarization tensor, responsible for the linear response, is zero in spite of the presence of the noncommutativity tensor θ^{ij}.

B. Enhancement of the Coulomb law and electric quadrupole

Let us now turn to the combined effects of noncommutativity and external homogeneous magnetic field, which is the correction (47) to the electrostatic potential. It is worth noting that this correction is linear in the charge eZ. This corresponds to the fact that now that there is a homogeneous magnetic-field background the linear response tensor is no longer trivial [3], although yet unable to provide the magneto-electric effect, so the magnetic correction remains quadratic in the charge eZ. As for the leading behavior of (47) in the remote region $r \gg a$, it follows the Coulomb law $\sim 1/r$. When united with (29), it gives the anisotropic, NC-corrected Coulomb potential

$$A^0_{\mathrm{Coulomb}}(\mathbf{x}) = \left(\frac{Ze}{r}\right)\left(1 - g\left\{\frac{1}{r^2}(\mathbf{x}\cdot\mathbf{B})(\mathbf{x}\cdot\boldsymbol{\theta}) + (\mathbf{B}\cdot\boldsymbol{\theta})\right\}\right), \quad r \gg a. \tag{56}$$

The correction may be attributed to the $1/r^3$ behavior of the right-hand side of Eq. (44)—the ''dark charge density'' distribution. (We use this term in analogy with the notion of the dark matter introduced to resume the responsibility for the observed gravitational field deviation from the Newtonian law). In the special case where the external field is oriented in parallel (antiparallel) to the noncommutativity vector, $\mathbf{B} \parallel \pm\boldsymbol{\theta}$, the overall multiplier of the standard Coulomb potential eZ/r in Eq. (56) becomes $1 \mp g|B||\theta|(\cos^2\vartheta + 1)$, where ϑ is the angle between $\mathbf{x}$ and $\mathbf{B}$. For the antiparallel configuration (lower sign) with positive charge $g > 0$ (as well as for parallel configuration with negative charge) the correction to the unity in this formula is positive for every direction of the radius-vector. The latter result allows us to estimate the maximum value of the long-range NC correction to the Coulomb field, which reads in either of the above cases

$$A^0_{\mathrm{LR}}(r) = \frac{Ze}{r}[1 + gB\theta)] \simeq \frac{Ze}{r}(1 + \delta). \tag{57}$$

Assuming a magnetic field of magnitude 10 Tesla (which is a very strong laboratory field) and the NC parameter of $(100\ \mathrm{TeV})^{-2}$ (which does not satisfy the strongest bounds on the NC scale, see e.g. [26]), we obtain $\delta = 6 \cdot 10^{-26}$ which is far beyond possibilities of any experimental verification. Even for the magnetic field on the surface of Soft Gamma Repeater which reaches 10^{11} T [38] and the same θ as above the correction $\delta = 6 \cdot 10^{-16}$ is very small.

On the other hand, for the same special cases when the magnetic field $\mathbf{B}$ and the vector $\boldsymbol{\theta}$ are parallel or antiparallel, the angular dependence of solution (47) leads to splitting between levels in an atom with different angular momentum projections onto the common direction of these vectors to compete with the Zeeman splitting (at a much lower level, of course).

We saw that the Coulomb field produced by a charge in external magnetic field far away from the former is enhanced as compared to Maxwell electrodynamics. This unprecedented property is absent from QED, where the linear in the charge correction to (the long-range part of) the Coulomb potential only makes it anisotropic without enhancing it [33,39]: the potential decreases as $1/r$ along the magnetic field following the same Coulomb law as in empty space, and it decreases along any other direction

$\vartheta \neq 0$ also following the Coulomb law, but taken with the coefficient $(\cos^2\vartheta + \beta\sin^2\vartheta)^{-1/2}$ smaller than unity. Here $\beta = (1 + \frac{\alpha}{3\pi}\frac{eB}{m_e^2})^{1/2} > 1$.

Note that the leading (Coulomb) part (56) survives the limiting transition $a \to 0$ to the pointlike charge. Also, when there is no external magnetic field, the (cubic in the charge) correction [4] to electrostatic potential in QED does not affect its long-ranged Coulomb part.

The long-distance next-to-leading part in $A_{\rm II}^{(1)0}$ corresponds to an equivalent electric quadrupole moment D_{ij}

$$A^0 = \frac{D_{ij}x_i x_j}{r^5}, \qquad D_{ij} = 2gZea^2(3B_i\theta_j - \delta_{ij}(\boldsymbol{B}\cdot\boldsymbol{\theta})) \tag{58}$$

that may be attributed to the finite-size charge (28), although it is spherically symmetric. The NC quadrupole moment vanishes in the limit $a \to 0$.

There is no electric-dipole part in $A_{\rm II}^{(1)}$. In this respect the situation is similar to QED, where the post-Coulomb long-range tail in the potential produced by a spherically symmetric charge in a magnetic field does not contain the dipole $1/r^2$ term either, but starts, according to [39], with $1/r^3$, the same as (47).

C. Powers of the charge and generalized Furry theorem

A remark is in order, which provides one with a tool—that may be referred to as a generalized Furry theorem—to judge, prior to calculations, about the powers of the charge eZ on general grounds within and beyond the leading approximation of the first order of θ. As long as the space-space NC theory conserves parity, the vector $\boldsymbol{\theta}$ is a pseudovector, the same is a magnetic field. Referring to the QFT language, we can say that this implies that the total number of legs in a many-photon diagram, characteristic of the nonlinear theory under study, connected to a magnetic field and to the "field" $\boldsymbol{\theta}$ must be even. Because of the Furry theorem that states that the overall number of photon legs should be even, we conclude that the number of photon legs connected to electrostatic field should be even separately. So, the magnetic field, produced by the static charge in the approximation, linear in θ, irrespective of whether the external magnetic field is present or not, must be even in the charge: the corresponding diagram contains one θ-leg, an even number or none of legs joining it to the external magnetic field (B-legs), one leg corresponding to the produced magnetic field (b-leg) and even number of legs joining to the external charge (Z-legs). This is in accordance with the result (42), wherein the latter number is two. On the other hand the electric field cannot undergo a correction linear in θ if external magnetic field is absent. In that case an impossible configuration with one θ-leg would be required. Such correction might be only of even order in θ, that fell beyond our consideration. The situation changes when external magnetic field is turned on. Now there is an admissible configuration of one θ-leg and an odd number of B-legs, so the first-order correction to the electric field should include one leg going to the produced electric field (e-leg) and an odd number of Z-legs (hence, odd powers of the charge Z) to keep the total numbers of legs connecting with the electrostatic field even. Out of these odd powers of the charge we got only one, because the keeping of only the first power of θ used when deriving the field Eqs. (14) and (15) has essentially reduced the otherwise unlimited extent of nonlinearity in the field.

When applied to the standard QED with no NC parameter, the generalized Furry theorem explains why the electric field produced by a static charge, besides being proportional to the value of the latter, has also contributions odd in it (the cubic contribution [4] was mentioned above). It also predicts the existence in QED of a magnetic field, quadratic in the static charge value, produced by such charge, when it is placed into external strong magnetic field—yet another manifestation of the magneto-electric effect.

V. AMBIGUITIES IN THE SW MAP

It is very well known that the SW map is not uniquely defined. There are additional terms which can be interpreted as redefinition of the gauge fields [15]. Such terms have been discussed in the context of renormalization of the noncommutative Maxwell theory [22,40], noncommutative Dirac fields coupled with the Yang-Mills [21] and of the noncommutative chiral electrodynamics [23]. In the case of noncommutative $U(1)$ gauge theories, it was shown (e.g. [40]) that the SW map to the potentials admits, in first order in θ, the following extension,

$$\check{A}_\mu(x) = A_\mu(x) + \frac{g}{2}\theta^{\alpha\beta}A_\alpha(x)[\partial_\beta A_\mu(x) + f_{\beta\mu}(x)] + \mathbb{A}_\mu(x),$$
$$\mathbb{A}_\mu(x) = g\kappa_1\theta_{\mu\alpha}\partial_\beta f^{\alpha\beta}(x), \quad \kappa_1 = \text{const}, \tag{59}$$

that keeps the Euler-Lagrange equations the same as they are in the pure noncommutative $U(1)$ theory with the action $\check{S}_A$ (2). To see this it is sufficient to form $f^{\nu\mu}$ out of $\mathbb{A}_\mu(x)$ and make sure that it gives vanishing contribution into the first term of Eq. (14). Therefore κ_1 does not appear in equations of motion to the first-order accuracy. Once $\mathbb{A}_\mu(x)$ satisfies the homogeneous part of the equation of motion, the transformation (59) reduces to adding such a solution of the source-free equations to any other solution of Eq. (14).

A very similar ambiguity in the SW equations for currents (11) was observed in [25]. One can add two extra terms $\mathbb{J}^\mu(x)$ to the solution (13):

$$\check{j}^{\mu}(x) = j^{\mu}(x) + g\theta^{\alpha\beta}A_{\alpha}(x)\partial_{\beta}j^{\mu}(x) + \mathbb{J}^{\mu}(x),$$
$$\mathbb{J}^{\mu}(x) = g(\kappa_2\theta^{\alpha\beta}f_{\alpha\beta}j^{\mu} + \kappa_3\theta^{\mu\alpha}f_{\alpha\beta}j^{\beta}), \quad \kappa_2, \kappa_3 = \text{const.} \tag{60}$$

It is easy to check that the current (60) satisfies (11) for arbitrary values of κ_2 and κ_3 already because $\mathbb{J}^{\mu}(x)$ does not undergo gauge transformation.

Let us check how the ambiguities described above influence the solutions of NC Maxwell equations. At the zeroth order in θ, the equations remain the same, see (20). At the first order,

$$\mathbb{A}_{\mu}^{(1)} = g\kappa_1\theta_{\mu\alpha}\partial_{\beta}f^{(0)\alpha\beta} = -\frac{4\pi}{c}g\kappa_1\theta_{\mu\alpha}j^{(0)\alpha}. \tag{61}$$

Obviously, $\mathbb{A}_{\mu}^{(1)}$ vanishes for a static charge distribution in the case of a space-space noncommutativity. So, in our special problem the ambiguity $\mathbb{A}_{\mu}^{(1)}$ does not work at all. The Maxwell equation and the compatibility conditions at the order θ read

$$\partial_{\nu}f^{(1)\nu\mu} - g\theta^{\alpha\beta}(\partial_{\nu}(f_{\alpha}^{(0)\nu}f_{\beta}^{(0)\mu}) - f_{\nu\alpha}^{(0)}\partial_{\beta}f^{(0)\nu\mu}) = \frac{4\pi}{c}(j^{(1)\mu} + \mathbb{J}^{(1)\mu}), \tag{62}$$

$$\partial_{\mu}(j^{(1)\mu} + \mathbb{J}^{(1)\mu}) + g\theta^{\alpha\beta}f_{\mu\alpha}^{(0)}\partial_{\beta}j^{(0)\mu} = 0, \tag{63}$$

where $\mathbb{J}^{(1)}$ is a given function of the zeroth order electromagnetic potential and the current

$$\mathbb{J}^{(1)\mu} = g(\kappa_2\theta^{\alpha\beta}f_{\alpha\beta}^{(0)}j^{(0)\mu} + \kappa_3\theta^{\mu\alpha}f_{\alpha\beta}^{(0)}j^{(0)\beta}). \tag{64}$$

The Eqs. (62) and (63) do not depend on κ_1, while κ_2 and κ_3 enter both equations only through the combination $j^{(1)\mu} + \mathbb{J}^{(1)\mu}$. Besides, this combination is defined by exactly the same equation as in the case $\kappa_2 = \kappa_3 = 0$, see Eq. (22). Therefore, the whole ambiguity in the first-order corrections to the electromagnetic potentials and to the "acting" current $j^{(1)\mu} + \mathbb{J}^{(1)\mu}$ is no ampler than the natural arbitrariness of adding homogeneous solutions to solutions of Eqs. (22) and (63). (This ambiguity has been already discussed above, see the paragraph below Eq. (22).) On the other hand separate parts in the combination $j^{(1)\mu} + \mathbb{J}^{(1)\mu}$ remain ambiguous, the physical results being independent of any separation of the acting current into parts. The same conclusions can be drawn dealing directly with Eqs. (14) and (15) without appealing to the special case of space-space noncommutativity and stationarity.

VI. CONCLUSIONS

In this paper we have studied how one can introduce external currents (sources) in the classical NC Maxwell theory without violating the gauge covariance. We started with a $U(1)_{\star}$ gauge theory and found that this can be self-consistently done at the level of the equations of motion. Note, that in this case the currents transform under the gauge transformations, as well as the field strength. Further, we argued that to facilitate the comparison with predictions of commutative electrodynamics one needs the fields with the same gauge-transformation properties as in the commutative case. A transition to such fields is done by means of the SW map, and we extended this map to include the currents. Again, a consistent result is obtained if one works at the level of the Maxwell equations rather than at the level of the action. We wrote weakly nonlinear anisotropic equations of motion, wherein the fields and the currents are involved, that are valid up to the first power of the NC parameter θ. Although these equations contain potentials along with the field strengths they are gauge-covariant in the sense that the gauge-transformed potentials satisfy the same equations, moreover the potentials can be on-shell eliminated from the equations, i.e. the equations with the potential-containing terms omitted have common solutions with the primary equations. For the case of space-space noncommutativity we considered an example, where the external source is a homogeneously charged sphere of finite radius and solved the equations of motion in the presence of an external constant and homogeneous magnetic field. No first-order in θ correction to the source appears in the spherically symmetric problem under consideration. To select solutions we impose the boundary conditions that require that these be finite in the point where the source is centered. The magnetic solution fixed in this way *does not* admit the limiting transition to a point source, which is an admissible property, since in the noncommutative theory a size of a physical body cannot be smaller than the elementary length.

We studied the contents of the solutions obtained. We found angle-dependent correction to the electric field produced by a static charge that implies an enhancement of the Coulomb law in the remote region of space in the presence of a constant magnetic field—a remarkable macroscopic consequence of the microscopic elementary length inherent to the NC electrodynamics under study. We found also that the next-to-leading behavior of the electric field far from the charge in the presence of external magnetic field is the one of an electric quadrupole. We noted an uncustomary possibility contained in item (d) of Eq. (54) that an anisotropic electric field linearly growing with the distance from the charge and nonsingular in the origin satisfies the field equations and can be therefore thought of as another admissible external field added to the constant magnetic field already present or a constant electric field. Irrespective of whether the external magnetic field is present or not, according to the chosen solution, the static charge, apart of giving rise—as usual—to an electrostatic field, also behaves itself as a magnetic dipole, with the magnetic moment depending on its size and proportional to the second power of the charge.

Finally, we studied the ambiguities in the definition of the SW map in the presence of currents, and found that at the first order in θ this is precisely the ambiguity of adding a homogeneous solution in the current conservation equation.

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