From *k*-essence to generalized Galileons

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We determine the most general scalar field theories which have an action that depends on derivatives of order two or less, and have equations of motion that stay second order and lower on flat space-time. We show that those theories can all be obtained from linear combinations of Lagrangians made by multiplying a particular form of the Galileon Lagrangian by an arbitrary scalar function of the scalar field and its first derivatives. We also obtain curved space-time extensions of those theories which have second-order field equations for both the metric and the scalar field. This provides the most general extension, under the condition that field equations stay second order, of *k*-essence, Galileons, *k*-Mouflage as well as of the kinetically braided scalars. It also gives the most general action for a possible scalar classicalizer with second-order field equations. We discuss the relation between our construction and the Euler hierarchies of Fairlie *et al.* showing, in particular, that Euler hierarchies allow one to obtain the most general theory when the latter is shift symmetric. As a simple application of our formalism, we give the covariantized version of the conformal Galileon.

I. INTRODUCTION

Scalar field models with derivative self-interactions have attracted attention in various contexts. For instance, models of *k*-essence [1] provide an interesting framework in which to investigate important early- [2] as well as late-time [3] issues of modern cosmology, while models similar to *k*-essence have been proposed in the context of relativistic modified Newtonian dynamics [4]. All these models have the characteristic feature that their action depends solely on a scalar field π and its first derivative—clearly then, whatever the Lagrangian, the field equations stay second order.

More recently, scalar models with actions depending on second derivatives of the fields have been considered, mainly inspired by the decoupling limit of the Dvali-Gabadadze-Porrati (DGP) model and its cosmology [5,6] as well as the resulting modification of the gravitational interaction via the so-called Vainshtein mechanism [7,8]. Such models range from the "Galileon" [9], to "*k*-Mouflage" [10] or "kinetic gravity braided scalars" [11,12] and have different defining properties. A feature shared by the former and latter class of models, as well as some *k*-Mouflage models, is that they have an action which depends on second-order derivatives of the fields. Hence, it is not *a priori* obvious how the field equations can stay second order, a property necessary in order to avoid

propagating ghosts or extra degrees of freedom. This, however, can be achieved. The Galileon [9], for example, can be defined as the most general scalar theory which, in flat space-time, has field equations which are uniquely second-order in derivatives. We note that in fact, Galileons were introduced rather earlier than [9], by Fairlie *et al.* [13], as well as Horndeski [14], though in a different context. As we will also outline in Sec. V B, there the relevant Lagrangians were constructed through the successive application (called "Euler hierarchies") of the Euler-Lagrange operator to an arbitrary initial Lagrangian depending solely on the first derivatives of a scalar field (with also the possibility of introducing arbitrary functions of the field first derivatives at intermediate steps).

The curved space-time generalizations of those models are also interesting. As shown in Ref. [15] the simplest covariantization of the original, four-dimensional, Galileons led to field equations for the scalar and its stress tensor that contained third derivatives. However, [15] also showed how to eliminate these higher derivatives by introducing suitable nonminimal, curvature, couplings. Single scalar Galileons and their nonminimal covariantization were further generalized to the multifield (and *p*-forms) case, as well as to arbitrary dimensions, in Refs. [16–19]. More recently, [20,21] showed how to obtain the Galileons and covariant Galileons from models with extra dimensions. Finally, Ref. [11] pointed out that a family of models which have Lagrangians depending linearly on second derivatives of the fields, but also have second-order field equations, have interesting properties when considered on curved space-time, due to an essential mixing between the scalar and the metric dubbed in [11], kinetic gravity braiding.

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The DGP model also generated new interest in massive gravity and its Vainshtein mechanism. This mechanism was first discussed [7] in a simple nonlinear extension of the free theory for a massive graviton, the so-called Pauli-Fierz theory, as a way of getting rid of the bothering effects of the scalar polarization of the massive graviton, present for any nonvanishing graviton mass—what is known as the van Dam-Veltman-Zakharov discontinuity [22]. The Vainshtein mechanism, which was recently shown to work even in the simplest theories of massive gravity [23], can be attributed to the self-interactions of the scalar polarization of the graviton [8,24–26]. The latter interactions, which can be studied by taking an appropriate "decoupling limit" [24,27], take the form of derivative scalar self-couplings and the same is in fact true in the DGP model. For the Vainshtein mechanism to operate, however, there is no need to have field equations that are purely second order. This was shown explicitly in Ref. [10] in particular, which introduced a large family of scalar-tensor models called *k*-Mouflage, and which used the Vainshtein mechanism to screen the effect of a scalar field at small distances. Note also that the Vainshtein mechanism also serves as one of the bases of the recently introduced "classicalization [28]." Finally, we stress that a recent attempt to obtain a massive gravity devoided of the unwanted Boulware Deser ghost [29] has a decoupling limit sharing crucial properties with some of the above mentioned scalar field models [30].

Hence, it is clear that scalar models which have derivative self-interactions, possibly depending on second-order derivatives, have numerous interesting properties. However, to our knowledge, these theories have so far not been extensively classified nor even constructed. It is the purpose of this work to do so. Namely, here, we will construct all theories of a scalar field π in D dimension and on flat space-time, which have actions depending on first and twice-differentiated π's as well as on undifferentiated π's (hence without assuming necessarily a shift symmetry) but have field equations which stay of order two and lower. This will be carried out in Sec. III where our main result is first stated and summarized (Sec. III A) before being proven. We then show how to nonminimally complete those theories in curved space-time, maintaining second-order field equations for the scalar as well as for the metric (Sec. IV). Some examples are then discussed in relation with the Euler hierarchies construction, and we also illustrate our results giving the covariantization of the conformal Galileons (Sec. V). Finally, we comment on the relationship between our covariantized action and Horndeski's theory [14] in $D = 4$. An introductory Sec. II gathers some useful results.

II. GALILEON, KINETIC GRAVITY BRAIDING AND SOME USEFUL RESULTS AND NOTATIONS

In this section we work in flat space only, and introduce the models studied in the remainder of this paper as well as

some useful notation and results. We also revisit the Galileon model studied in [9]. Throughout we work in D space-time dimensions, with signature $(-, +, +, \dots)$.

A. Introduction and two useful lemma

All models we consider depend on a single scalar field π whose partial derivatives will be denoted by[1]

$$\pi_\mu \equiv \partial_\mu \pi, \quad \pi_{\mu\nu} \equiv \partial_\mu \partial_\nu \pi, \quad \pi_{\mu\nu\alpha} \equiv \partial_\mu \partial_\nu \partial_\alpha \pi \quad \text{etc.} \quad (1)$$

Since derivatives commute on flat space-time, these tensors are symmetric under interchange of any indices. The Lagrangians considered take the form

$$\mathcal{L} = \mathcal{T}_{(2n)}^{\mu_1 \dots \mu_n \nu_1 \dots \nu_n} \pi_{\mu_1 \nu_1} \dots \pi_{\mu_n \nu_n}, \quad (2)$$

where $\mathcal{T}_{(2n)}$ is a $2n$-contravariant tensor function of π and π_α only

$$\mathcal{T}_{(2n)} = \mathcal{T}_{(2n)}(\pi, \pi_\alpha). \quad (3)$$

Note that the integer n also denotes the number of twice-differentiated π's appearing in the Lagrangian (2). Thus $\mathcal{L} = \mathcal{L}(\pi, \pi_\alpha, \pi_{\alpha\beta})$ and the corresponding field equations are $\mathcal{E} = 0$, where

$$\mathcal{E} = \left[\frac{\partial}{\partial \pi} - \partial_\mu \left(\frac{\partial}{\partial \pi_\mu} \right) + \partial_\mu \partial_\nu \left(\frac{\partial}{\partial \pi_{\mu\nu}} \right) \right] \mathcal{L} \quad (4)$$

$$\equiv \hat{\mathcal{E}} \mathcal{L}. \quad (5)$$

For future use, we begin by giving the *sufficient* conditions such that these equations are of order 2 or lower (in derivatives) on flat space-time. To do so, first note the following lemma:

Lemma. The field equation derived from the Lagrangian $\mathcal{L} = \mathcal{T}_{(2)}^{\mu\nu} \pi_{\mu\nu}$ does not contain any derivative of order higher than 2.

This is straightforward to verify. The second term in (4) yields one contribution in third derivatives of π, namely $\pi_{\alpha\beta\mu} \partial \mathcal{T}^{\alpha\beta} / \partial \pi_\mu$. An identical contribution arises from the last term in (4). However, given the relative sign in (4), these terms in third derivatives cancel. This simple result is in fact at the basis of the model of kinetically braided scalar of Ref. [11], where $\mathcal{T}_{(2)}^{\mu\nu} = f(\pi, \pi_\alpha) g^{\mu\nu}$ and $g_{\mu\nu}$ is the metric. In fact the above lemma generalizes to curved space-time as shown in Appendix B and used in Sec. IV.

Now consider any $n > 1$. The second term in (4) again yields contributions to the equation of motion which are third order in derivatives. Those are cancelled by terms coming from the final term in Eq. (4) by virtue of the above lemma. However, the final term in Eq. (4) also yields "dangerous terms" (by which we mean terms of order

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[1]Note that, when considered on curved space-time, π_μ will denote the covariant derivative acting on π, $\nabla_\mu \pi$, and so on for $\pi_{\mu\nu} \dots$, i.e., partial derivatives are just to be replaced by covariant derivatives in the notation (1).

three or more in derivatives) which are of the form $\pi_{\mu_k \mu_l \nu_l}$, $\pi_{\nu_k \mu_l \nu_l}$ or $\pi_{\nu_k \mu_k \mu_l \nu_l}$ where all the indices are contracted with those of $\mathcal{T}_{(2n)}$. Since derivatives commute on flat space-times, we immediately have the following result:

Main lemma. A sufficient condition for the field equations derived from the Lagrangian (2) to stay of order less or equal to 2 is that the tensor $\mathcal{T}_{(2n)}^{\mu_1 \mu_2 \dots \mu_n \nu_1 \nu_2 \dots \nu_n}$ is totally antisymmetric in its first n indices μ_i as well as (separately) in its last n indices ν_i.

The main purpose of this paper is to study the converse of this simple result. Before doing so, and for future use, we first revisit the Galileon theory. As we will see, this provides a simple and fundamental example of the type of theory we will discuss.

B. The flat space-time Galileon revisited

The starting point of Galileon models [9,15] are Lagrangians $\mathcal{L}^{\text{Gal}}$ of the form (2) with the tensor $\mathcal{T}_{(2n)}$ satisfying (3) as well as the properties of the main lemma above. Furthermore, in flat space-time, they have equations of motion of order *strictly equal* to 2 (that is, they do not contain undifferentiated or once differentiated π, but only twice-differentiated π).

As we now outline, there exist several possible ways of writing the Galileon action: each differs from the other by a different choice of tensor $\mathcal{T}_{(2n)}$ and a total derivative. However they lead to the same equations of motion and are hence equivalent.

To begin with, define the $2m$-contravariant tensor $\mathcal{A}_{(2m)}$ by

$$\mathcal{A}_{(2m)}^{\mu_1 \mu_2 \dots \mu_m \nu_1 \nu_2 \dots \nu_m} \equiv \frac{1}{(D-m)!} \varepsilon^{\mu_1 \mu_2 \dots \mu_m \sigma_1 \sigma_2 \dots \sigma_{D-m}}$$
$$\times \varepsilon^{\nu_1 \nu_2 \dots \nu_m}{}_{\sigma_1 \sigma_2 \dots \sigma_{D-m}} \qquad (6)$$

where the totally antisymmetric Levi-Civita tensor is given by

$$\varepsilon^{\mu_1 \mu_2 \dots \mu_D} = -\frac{1}{\sqrt{-g}} \delta_1^{[\mu_1} \delta_2^{\mu_2} \dots \delta_D^{\mu_D]} \qquad (7)$$

with square brackets denoting unnormalized permutations. [The definitions (6) and (7) are also valid in arbitrary curved space-times with metric $g_{\mu\nu}$ and $D \geq m$.] Thus $\mathcal{A}_{(2m)}$ is antisymmetric in its first m indices as well as, separately, in its last m indices. Other useful properties of $\mathcal{A}_{(2m)}$ are given in Appendix A.

A first possible Lagrangian for the Galileon is given by [16]

$$\mathcal{L}_N^{\text{Gal},1} = (\mathcal{A}_{(2n+2)}^{\mu_1 \dots \mu_{n+1} \nu_1 \dots \nu_{n+1}} \pi_{\mu_{n+1}} \pi_{\nu_{n+1}}) \pi_{\mu_1 \nu_1} \dots \pi_{\mu_n \nu_n}$$
$$\equiv \mathcal{T}_{(2n),\text{Gal},1}^{\mu_1 \dots \mu_n \nu_1 \dots \nu_n} \pi_{\mu_1 \nu_1} \dots \pi_{\mu_n \nu_n}, \qquad (8)$$

with

$$\mathcal{T}_{(2n),\text{Gal},1}^{\mu_1 \dots \mu_n \nu_1 \dots \nu_n} \equiv \mathcal{A}_{(2n+2)}^{\mu_1 \dots \mu_{n+1} \nu_1 \dots \nu_{n+1}} \pi_{\mu_{n+1}} \pi_{\nu_{n+1}}. \qquad (9)$$

Here and henceforth N indicates the number of π's appearing in the Lagrangian of a given Galileon model so that

$$N = n + 2.$$

As discussed in [16], the Lagrangian $\mathcal{L}_N^{\text{Gal},1}$ also reads:

$$\mathcal{L}_N^{\text{Gal},1} = -\sum_{\sigma \in S_{n+1}} \epsilon(\sigma) [\pi^{\mu_{\sigma(1)}} \pi_{\mu_1}]$$
$$\times [\pi^{\mu_{\sigma(2)}}{}_{\mu_2} \pi^{\mu_{\sigma(3)}}{}_{\mu_3} \dots \pi^{\mu_{\sigma(n+1)}}{}_{\mu_{n+1}}],$$
$$= -\sum_{\sigma \in S_{n+1}} \epsilon(\sigma) g^{\mu_{\sigma(1)} \nu_1} g^{\mu_{\sigma(2)} \nu_2} \dots g^{\mu_{\sigma(n+1)} \nu_{n+1}} (\pi_{\nu_1} \pi_{\mu_1})$$
$$\times (\pi_{\nu_2 \mu_2} \pi_{\nu_3 \mu_3} \dots \pi_{\nu_{n+1} \mu_{n+1}}), \qquad (10)$$

where σ denotes a permutation of signature $\epsilon(\sigma)$ of the permutation group S_{n+1}, and in order for the Lagrangian to be nonvanishing,

$$n + 1 \leq D \Leftrightarrow N \leq D + 1. \qquad (11)$$

This is the original form presented in [9], and the equality of (8) and (10) can be seen [16] using the identity

$$\sum_{\sigma \in S_D} \epsilon(\sigma) g^{\mu_{\sigma(1)} \nu_1} g^{\mu_{\sigma(2)} \nu_2} \dots g^{\mu_{\sigma(D)} \nu_D} = -\varepsilon^{\mu_1 \mu_2 \dots \mu_D} \varepsilon^{\nu_1 \nu_2 \dots \nu_D}. \qquad (12)$$

Using (4), the field equations derived from the Lagrangian $\mathcal{L}_N^{\text{Gal},1}$ read

$$\mathcal{E} = -N \times \mathcal{E}_N = 0, \qquad (13)$$

where

$$\mathcal{E}_N = -\sum_{\sigma \in S_{n+1}} \epsilon(\sigma) \prod_{i=1}^{n+1} \pi^{\mu_{\sigma(i)}}{}_{\mu_i},$$
$$= \mathcal{A}_{(2n+2)}^{\mu_1 \dots \mu_{n+1} \nu_1 \dots \nu_{n+1}} \pi_{\mu_1 \nu_1} \pi_{\mu_2 \nu_2} \dots \pi_{\nu_{n+1} \mu_{n+1}}. \qquad (14)$$

These are only second order, as advertised. Notice that the index N on $\mathcal{E}_N$ indicates that it is the equation of motion coming from $\mathcal{L}_N^{\text{Gal},1}$ (which contains N factors of π): thus $\mathcal{E}_N$ contains $N - 1$ factors of π. The Galileon model with the largest number of fields in D dimensions has $N = D + 1$. In this case, $\mathcal{E}_{D+1}$ is simply proportional to the determinant of the Hessian, the matrix of second derivatives $\pi_{\mu\nu}$. As such, the equation $\mathcal{E}_{D+1} = 0$ is known as the Monge-Ampère equation, and it has various interesting properties, in particular, in relation to integrability (see, e.g., [31]). At the same time

$$\mathcal{L}_{D+1}^{\text{Gal},1} \propto \det \begin{pmatrix} \pi_{\mu\nu} & \pi_\nu \\ \pi_\mu & 0 \end{pmatrix},$$

which is the left-hand side of the Bateman equation [31,32].

Finally, in D dimensions, the total Galileon Lagrangian is given by a linear combination of Lagrangians $\mathcal{L}_N^{\text{Gal},1}$ with $N = 2, \dots, D + 1$. In $D = 4$ dimensions, these are simply the 4 terms given in [9].

A second possible Lagrangian for the Galileon with, again, $N = n + 2$ fields is given by

$$\mathcal{L}_N^{\text{Gal},2} = (\mathcal{A}_{(2n)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n} \pi_{\mu_1} \pi_\lambda \pi^\lambda{}_{\nu_1}) \pi_{\mu_2\nu_2} \ldots \pi_{\mu_n\nu_n}, \quad (15)$$

$$\equiv \mathcal{T}_{(2n),\text{Gal},2}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n} \pi_{\mu_1\nu_1} \ldots \pi_{\mu_n\nu_n}, \quad (16)$$

where (see also Sec. VA)

$$\mathcal{T}_{(2n),\text{Gal},2}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n} = \frac{1}{n} \mathcal{A}_{(2n)}^{\alpha_1\ldots\alpha_n\nu_1\ldots\nu_n} [(\pi^{\mu_1}\pi_{\alpha_1})\delta^{\mu_2}{}_{\alpha_2} \ldots \delta^{\mu_n}{}_{\alpha_n}$$
$$+ \delta^{\mu_1}{}_{\alpha_1}(\pi^{\mu_2}\pi_{\alpha_2})\delta^{\mu_3}{}_{\alpha_3} \ldots \delta^{\mu_n}{}_{\alpha_n} + \ldots$$
$$+ \delta^{\mu_1}{}_{\alpha_1} \ldots \delta^{\mu_{n-1}}{}_{\alpha_{n-1}}(\pi^{\mu_n}\pi_{\alpha_n})]. \quad (17)$$

Finally, the third form of interest is given by

$$\mathcal{L}_N^{\text{Gal},3} = (\mathcal{A}_{(2n)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n} \pi_\lambda \pi^\lambda) \pi_{\mu_1\nu_1} \ldots \pi_{\mu_n\nu_n} \quad (18)$$

so that

$$\mathcal{T}_{(2n),\text{Gal},3}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n} = X \mathcal{A}_{(2n)}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n} \quad (19)$$

where[2]

$$X \equiv \pi_\mu \pi^\mu. \quad (20)$$

The three Lagrangians (8), (15), and (18) are in fact all equal up to a total derivative. Indeed, on defining J_N^μ by

$$J_N^\mu = X \mathcal{A}_{(2n)}^{\mu\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n} \pi_{\nu_1} \pi_{\mu_2\nu_2} \ldots \pi_{\mu_n\nu_n}, \quad (21)$$

it follows directly that

$$\mathcal{L}_N^{\text{Gal},2} = -\frac{1}{2} \mathcal{L}_N^{\text{Gal},3} + \frac{1}{2} \partial_\mu J_N^\mu. \quad (22)$$

Furthermore on using the properties of $\mathcal{A}_{(2n)}$ given in Appendix A, it follows that

$$(N - 2)\mathcal{L}_N^{\text{Gal},2} = \mathcal{L}_N^{\text{Gal},3} - \mathcal{L}_N^{\text{Gal},1}. \quad (23)$$

Thus we also have

$$\mathcal{L}_N^{\text{Gal},1} = \frac{N}{2} \mathcal{L}_N^{\text{Gal},3} - \frac{N-2}{2} \partial_\mu J_N^\mu, \quad (24)$$

$$\mathcal{L}_N^{\text{Gal},1} = -N \mathcal{L}_N^{\text{Gal},2} + \partial_\mu J_N^\mu. \quad (25)$$

From (22), (24), and (25) it therefore follows that the equations of motion of all three Galileon Lagrangians are identical, given by (14), and strictly of second order.

Finally, observe from (14) and (18) that $\mathcal{L}_N^{\text{Gal},3}$ can be rewritten as

$$\mathcal{L}_N^{\text{Gal},3} = X \mathcal{E}_{N-1} \quad (26)$$

where $\mathcal{E}_{N-1}$ are the equations of motion coming from $\mathcal{L}_{N-1}^{\text{Gal}}$ (where we drop the index 1, 2, 3). In this form, it

is manifest that Galileon models containing a given number N of π fields can be obtained from the field equations of the same models with one less field. This property, though implicit in [9], is very well explained by the hierarchical construction of [13] which preceeded by far Ref. [9] and discussed first, as far as we know, what are called here and elsewhere Galileons (see [33]). This hierarchical construction will be discussed in Sec. V.

C. Galileon Lagrangians in terms of cycles

The three equivalent Galileon Lagrangians presented above all satisfy the sufficient conditions of the main lemma of Sec. II A. In order to study the necessary conditions it will be useful to introduce a new notation consisting of the cycles $[i]$ and $\langle i \rangle$.

We define $[i]$ by

$$[i] \equiv \pi^{\mu_1}{}_{\mu_2} \pi^{\mu_2}{}_{\mu_3} \pi^{\mu_3}{}_{\mu_4} \cdots \pi^{\mu_i}{}_{\mu_1}, \quad (27)$$

so that for example

$$[1] = \Box\pi, \qquad [2] = \pi^\alpha{}_\beta \pi^\beta{}_\alpha. \quad (28)$$

Similarly,

$$\langle i \rangle \equiv \pi_{\mu_1} \pi^{\mu_1}{}_{\mu_2} \pi^{\mu_2}{}_{\mu_3} \pi^{\mu_3}{}_{\mu_4} \cdots \pi^{\mu_i}{}_{\mu_{i+1}} \pi^{\mu_{i+1}}, \quad (29)$$

so that

$$\langle 1 \rangle = \pi_\alpha \pi^\alpha{}_\beta \pi^\beta \quad (30)$$

Note that $[i]$ contains i factors of π as well as i twice-differentiated π's, whereas $\langle i \rangle$ contains $i + 2$ factors of π, but again i twice-differentiated π's.

Using this notation, and in the case of $N = 4$ fields, the three Galileon Lagrangians can be written as

$$\mathcal{L}_{N=4}^{\text{Gal},1} = -(\Box\pi)^2(\pi_\mu \pi^\mu) + 2(\Box\pi)(\pi_\mu \pi^{\mu\nu} \pi_\nu)$$
$$+ (\pi_{\mu\nu} \pi^{\mu\nu})(\pi_\rho \pi^\rho) - 2(\pi_\mu \pi^{\mu\nu} \pi_{\nu\rho} \pi^\rho)$$
$$= X([2] - [1]^2) + 2([1]\langle 1 \rangle - \langle 2 \rangle),$$

$$\mathcal{L}_{N=4}^{\text{Gal},2} = -(\Box\pi)\pi_\mu \pi^{\mu\nu} \pi_\nu + \pi_\mu \pi^{\mu\nu} \pi_{\nu\rho} \pi^\rho = \langle 2 \rangle - [1]\langle 1 \rangle,$$

$$\mathcal{L}_{N=4}^{\text{Gal},3} = (\pi_\lambda \pi^\lambda)(\pi_{\mu\nu} \pi^{\mu\nu} - (\Box\pi)^2) = X([2] - [1]^2).$$

Furthermore it will be useful to define

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_r \\ 1 & 2 & \cdots & r \end{bmatrix} = [1]^{p_1}[2]^{p_2} \cdots [r]^{p_r}, \quad (31)$$

as well as

$$\left\langle \begin{matrix} q_1 & q_2 & \cdots & q_s \\ 1 & 2 & \cdots & s \end{matrix} \right\rangle = \langle 1 \rangle^{q_1}\langle 2 \rangle^{q_2} \cdots \langle s \rangle^{q_s}, \quad (32)$$

where the p_i and q_i are positive (or vanishing) integers.[3]

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[2]Note that in the context of k-inflation and other models, X is often defined to be $-\pi_\mu \pi^\mu/2$. In order to simplify our equations, the factor of $-1/2$ is not included here.

[3]Note that the right-hand sides of Eqs. (31) and (32) are uniquely specified, respectively, by the ordered sets $(p_1, p_2, \cdots, p_r)$ as well as $(q_1, q_2, \cdots, q_s)$. Hence our notation on the left-hand side of Eqs. (31) and (32) is a bit redundant, but we feel this will ease the reading of some of the equations below.

Using this notation it follows from (17) that $\mathcal{L}_N^{\text{Gal},2}$ can be expressed as a sum of terms

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_r \\ 1 & 2 & \cdots & r \end{bmatrix}\left\langle \begin{matrix} q_1 & \cdots & q_s \\ 1 & \cdots & s \end{matrix} \right\rangle$$

with $\sum_{j=1}^s q_j = 1$. (See above in case $N = 4$.)

Thus, for example, the equations of motion coming $\mathcal{L}_{N=4}$ are

$$0 = \mathcal{E}_{N=4} = -(\Box\pi)^3 - 2(\pi_\mu{}^\nu \pi_\nu{}^\rho \pi_\rho{}^\mu) + 3(\Box\pi)(\pi_{\mu\nu}\pi^{\mu\nu})$$

$$= -[1]^3 - 2[3] + 3[1][2]$$

$$= -\begin{bmatrix} 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}. \tag{33}$$

More generally, for any N, $\mathcal{E}_N$ can be expressed as a specific linear combination of monomials of the form (31)

$$\mathcal{E}_N = \sum \mathcal{C}_{p_1,\cdots,p_r}^N \begin{bmatrix} p_1 & p_2 & \cdots & p_r \\ 1 & 2 & \cdots & r \end{bmatrix}, \tag{34}$$

where the sum runs over the $r-$ uplets $(p_1, \cdots, p_r)$ verifying the constraint

$$N = 1 + \sum_{i=1}^{i=r} i \times p_i, \tag{35}$$

and $\mathcal{C}_{p_1,\cdots,p_r}^N$ are real coefficients which do not depend on the dimension of space-time, but only depend on the number of fields N, explicitly

$$\mathcal{C}_{p_1,p_2,\cdots p_r}^N = (-1)^{N+p_1+p_2+\cdots+p_r}\frac{(N-1)!}{(p_1!p_2!\cdots p_r!)1^{p_1}2^{p_2}\cdots r^{p_r}}.$$

III. UNIQUENESS THEOREM IN FLAT SPACE-TIME

A. The result

We are now in the position to study the converse of the main lemma of Sec. II A. However, before presenting the proof in Sec. III B, we first state our result. Namely, in flat space-time, the most general theory satisfying the three conditions

(i) its Lagrangian contains derivatives of order 2 or less of the scalar field π;

(ii) its Lagrangian is polynomial in the second derivatives of π;

(iii) the corresponding field equations are of order 2 or lower in derivatives

has a Lagrangian which is given by an arbitrary linear combination of the Lagrangians $\mathcal{L}_n\{f\}$ (each containing n of twice-differentiated π) of the form

$$\mathcal{L}_n\{f\} = f(\pi, X) \times \mathcal{L}_{N=n+2}^{\text{Gal},3},$$

$$= f(\pi, X) \times (X \mathcal{A}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n} \pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}). \tag{36}$$

Here $f(\pi, X)$ is an arbitrary scalar function of π and X, generally different for each n, and the braces in $\mathcal{L}_n\{f\}$ denote that $\mathcal{L}_n$ is a functional of f. The equations of motion corresponding to each $\mathcal{L}_n\{f\}$ are[4]

$$0 = 2(f + Xf_X)\mathcal{E}_N + 4(2f_X + Xf_{XX})\mathcal{L}_{N+1}^{\text{Gal},2}$$

$$+ X[2Xf_{X\pi} - (n-1)f_\pi]\mathcal{E}_{N-1}$$

$$- n(4Xf_{X\pi} + 4f_\pi)\mathcal{L}_N^{\text{Gal},2} - nXf_{\pi\pi}\mathcal{L}_{N-1}^{\text{Gal},1}. \tag{37}$$

where $N = n + 2$. Notice the dependence of these equations on $\pi^{\mu\nu}$ as well as π^μ (when f is nonconstant).

B. Proof of uniqueness

In flat space-time, the only scalar quantities which are polynomial in second derivatives of π must be constructed from $[i]$ and $\langle i \rangle$, defined, respectively, in Eqs. (27) and (29). Recall that these both contain i times a twice-differentiated π. Hence, the most general scalar theory obeying conditions (i) and (ii) has a Lagrangian which is a linear combination of monomials, each of the form $\mathcal{L}_{q_1,q_2,\cdots,q_s}^{p_1,p_2,\cdots,p_r}$ defined by

$$\mathcal{L}_{q_1,q_2,\cdots,q_s}^{p_1,p_2,\cdots,p_r} = f(\pi, X) \times \begin{bmatrix} p_1 & p_2 & \cdots & p_r \\ 1 & 2 & \cdots & r \end{bmatrix}\left\langle \begin{matrix} q_1 & q_2 & \cdots & q_s \\ 1 & 2 & \cdots & s \end{matrix} \right\rangle,$$

$$\tag{38}$$

where f is an arbitrary scalar function of π and X (different for each monomial $\mathcal{L}_{q_1,q_2,\cdots,q_s}^{p_1,p_2,\cdots,p_r}$), and the other quantities have been defined in (31) and (32). Formally then

$$\mathcal{L} = \sum_{\{p_i\}\{q_j\}} C_{\{p_i\}\{q_j\}} \mathcal{L}_{q_1,q_2,\cdots,q_s}^{p_1,p_2,\cdots,p_r}, \tag{39}$$

where the sum is over the set of $\{p_i\} = (p_1, \cdots, p_r)$ and $\{q_j\} = (q_1, \cdots, q_s)$, and (the ratio of) the constant coefficients $C_{\{p_i\}\{q_i\}}$ will be determined below. Note also that the number N of fields, and the number n of twice-differentiated π which appear in a given product

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_r \\ 1 & 2 & \cdots & r \end{bmatrix}\left\langle \begin{matrix} q_1 & q_2 & \cdots & q_s \\ 1 & 2 & \cdots & s \end{matrix} \right\rangle$$

are given, respectively, by

$$N = \left(\sum_{i=1}^{i=r} p_i \times i\right) + \left(\sum_{j=1}^{j=s} q_j \times (j+2)\right), \tag{40}$$

$$n = N - 2\sum_{j=1}^{j=s} q_j. \tag{41}$$

We now look for the most general theory which obeys condition (iii) as well as (i) and (ii).

---

[4]We use the notation $f_X \equiv f_{,X}$, $f_\pi \equiv f_{,\pi}$ and so on.

1. Fourth-order derivatives containing $\Box\pi^\alpha{}_\beta$

Inspired by the proof given in [9], start from a particular monomial of the form $\mathcal{L}^{p_1,p_2,\cdots,p_r}_{q_1,q_2,\cdots,q_s}$, given in (38), with specific values of the p_i (for $1 \le i \le r$) and of the q_j (for $1 \le j \le s$). If the theory considered obeys condition (iii), *all* third- and fourth-order derivatives must disappear from the field equations. There are many fourth-order derivative terms, but we first focus on those containing $\Box\pi^\alpha{}_\beta$ (others will be discussed later). When varying $\mathcal{L}^{p_1,p_2,\cdots,p_r}_{q_1,q_2,\cdots,q_s}$, such terms can appear through

$$\delta_\pi[i]^{p_i} \supset \frac{2ip_i}{i-1}[\Box(i-1)][i]^{p_i-1}\delta\pi \qquad (i>1) \quad (42)$$

$$\delta_\pi\langle i\rangle^{q_i} \supset 2q_i\langle\Box(i-1)\rangle\langle i\rangle^{q_i-1}\delta\pi \qquad (i>1) \quad (43)$$

where

$$[\Box(j)] \equiv \sum_{k=1}^{j} \pi^{\mu_1}{}_{\mu_2}\pi^{\mu_2}{}_{\mu_3}\cdots\pi^{\mu_{k-1}}{}_{\mu_k}(\Box\pi^{\mu_k}{}_{\mu_{k+1}})\cdots$$
$$\pi^{\mu_{j-1}}{}_{\mu_j}\pi^{\mu_j}{}_{\mu_1} \quad (44)$$

$$= j \times \pi^{\mu_1}{}_{\mu_2}\pi^{\mu_2}{}_{\mu_3}\cdots\pi^{\mu_{j-1}}{}_{\mu_j}(\Box\pi^{\mu_j}{}_{\mu_1}), \quad (45)$$

$$\langle\Box(j)\rangle \equiv \sum_{k=1}^{j} \pi_{\mu_1}\pi^{\mu_1}{}_{\mu_2}\cdots\pi^{\mu_{k-1}}{}_{\mu_k}(\Box\pi^{\mu_k}{}_{\mu_{k+1}})\cdots$$
$$\pi^{\mu_j}{}_{\mu_{j+1}}\pi^{\mu_{j+1}}. \quad (46)$$

Now consider the contribution from (42). On varying $\mathcal{L}^{p_1,p_2,\cdots,p_r}_{q_1,q_2,\cdots,q_s}$ this yields the term

$$\frac{2ip_i}{i-1}f(\pi,X)\times[1]^{p_1}[2]^{p_2}\cdots[i-1]^{p_{i-1}}[\Box(i-1)]$$
$$\times[i]^{p_i-1}\cdots[r]^{p_r}\left\langle\begin{matrix}q_1 & q_2 & \cdots & q_s \\ 1 & 2 & \cdots & s\end{matrix}\right\rangle \quad (47)$$

in the equations of motion, and it can only be cancelled if one adds to the Lagrangian a term proportional to

$$f\times\begin{bmatrix}p_1+1 & p_2 & \cdots & p_{i-2} & p_{i-1}+1 & p_i-1 & p_{i+1} & \cdots & p_r \\ 1 & 2 & \cdots & i-2 & i-1 & i & i+1 & \cdots & r\end{bmatrix}$$
$$\times\left\langle\begin{matrix}q_1 & q_2 & \cdots & q_s \\ 1 & 2 & \cdots & s\end{matrix}\right\rangle. \quad (48)$$

Indeed, variation of the first term $[1]^{p_1+1} \equiv (\Box\pi)^{p_1+1}$ gives, after integrating by parts and shifting the $\Box$ operator onto one of the cycles $[i-1]$, a term proportional to (47). The same term is obtained from varying one of the twice-differentiated π inside a cycle $[i-1]$ and, on integrating by parts, acting with the derivatives on one $\Box\pi$. Thus a *necessary* condition for the theory considered to obey conditions (i), (ii) and (iii) is that it must contain in its action the specific linear combination

$$f\left\langle\begin{matrix}q_1 & q_2 & \cdots & q_s \\ 1 & 2 & \cdots & s\end{matrix}\right\rangle\times\left\{\begin{bmatrix}p_1 & p_2 & \cdots & p_r \\ 1 & 2 & \cdots & r\end{bmatrix}\right.$$
$$+\alpha_\Box\begin{bmatrix}p_1+1 & p_2 & \cdots & p_{i-2} & p_{i-1}+1 & p_i-1 & p_{i+1} & \cdots & p_r \\ 1 & 2 & \cdots & i-2 & i-1 & i & i+1 & \cdots & r\end{bmatrix}\right\}, \quad (49)$$

where $\alpha_\Box$, which is nothing other than the ratio of two of the C coefficients defined in (39), is given by

$$\alpha_\Box = -\frac{ip_i}{(p_1+1)(p_{i-1}+1)(i-1)}.$$

Notice that $\alpha_\Box$ is independent of $f(\pi,X)$. It is straightforward to check that the two terms in factor of f in (49) each have the same number N of fields, and number n of twice-differentiated π, as given in (40) and (41).

A similar reasoning can be applied to the "$\langle\rangle$"piece in (38). Indeed, using (43) it follows that any theory obeying conditions (i), (ii) and (iii), and which has a term in its action given by $\mathcal{L}^{p_1,p_2,\cdots,p_r}_{q_1,q_2,\cdots,q_s}$, must also contain a term

$$\alpha_\langle\rangle\times f\times\begin{bmatrix}p_1+1 & p_2 & \cdots & p_r \\ 1 & 2 & \cdots & r\end{bmatrix}$$
$$\times\left\langle\begin{matrix}q_1 & \cdots & q_{j-1}+1 & q_j-1 & q_{j+1} & \cdots & q_s \\ 1 & \cdots & j-1 & j & j+1 & \cdots & s\end{matrix}\right\rangle, \quad (50)$$

where

$$\alpha_\langle\rangle = -\frac{q_j}{(p_1+1)(q_{j-1}+1)}.$$

Once again observe that the above terms (38) and (50) have the same number N of fields, and same number n of twice-differentiated π's.

We can thus define mappings F and G on the set of monomials (38) which appear in the Lagrangian of any theory obeying conditions (i), (ii) and (iii), such that F maps any term $\mathcal{L}^{p_1,p_2,\cdots,p_r}_{q_1,q_2,\cdots,q_s}$, i.e., (38), to the monomial (47) and similarly G maps the term (38) to (50). Then, any two terms related by those mappings (or their inverse F^{-1} and G^{-1}) must have coefficients which are equal (up to combinatorial factors) in order to eliminate fourth-order derivatives containing $\Box\pi^\alpha{}_\beta$. These mappings can easily be pictured by a graph whose nodes are labeled by the set $\{p_1,p_2,\cdots,p_r,q_1,q_2,\cdots,q_s\}$ and represent the monomial, and such that two nodes are connected if and only if they are image of each other by the mappings F or G or their inverse (or successive applications of F, G, F^{-1} or G^{-1}).

Now observe that in going from (38) to (48), the power of the cycle $[i]$ is lowered by 1, whereas the power of the cycles $[i-1]$ and $[1]$ is increased by 1. Similarly, in going from the term (38) to (50) the power of the cycle $\langle j\rangle$ is lowered by 1 whereas that of the cycles $\langle j-1\rangle$ and $[1]$ increases by 1. Hence, by acting recursively with the

mappings F and G, starting from the cycles $[r]$ and $\langle s \rangle$ which have the *largest* length r and s respectively, one ends up with the conclusion that any term $\mathcal{L}^{p_1,p_2,\cdots,p_r}_{q_1,q_2,\cdots,q_s}$ is connected (*after several applications of the maps F and G*) to a term which has all but p_1 and q_1 vanishing. That is, a term of the form

$$\mathcal{L}^p_q = f \times [1]^p \langle 1 \rangle^q = f \times (\Box \pi)^p (\pi^\mu \pi_{\mu\nu} \pi^\nu)^q, \quad (51)$$

where

$$q = \sum_{j=1}^s q_j, \quad (52)$$

$$p = \sum_{i=1}^r (i p_i) + \sum_{j=1}^s q_j (j - 1). \quad (53)$$

Alternatively, on using (40) and (41), p and q are determined by the number of fields and twice-differentiated π in the part of the monomial containing second derivatives, through

$$p = \frac{1}{2}(3n - N), \quad (54)$$

$$q = \frac{1}{2}(N - n). \quad (55)$$

Since the term (51) is the *same* for all $\mathcal{L}^{p_1,p_2,\cdots,p_r}_{q_1,q_2,\cdots,q_s}$ which has fixed values of N and n, one can conclude that the graph introduced above, and representing monomials with fixed values of N and n, is *connected*. In other words, it means that any term $\mathcal{L}^{p_1,p_2,\cdots,p_r}_{q_1,q_2,\cdots,q_s}$ with fixed N and n must appear in the action, with a common function f and fixed coefficients.

Hence, to conclude the first part of our reasoning, we get a family of (possibly trivial but each uniquely determined) theories indexed by the values of N and n. More specifically, for given values of N and n the theory is uniquely specified by the coefficient in front of (51) (as well as the function f). The coefficients of all the other monomials with the same number of N and n will be proportional to the coefficient of (51) and the proportionality factor will be independent of the specific function f (since this factor is uniquely determined by requiring the vanishing of terms with four derivatives $\Box \pi^\alpha{}_\beta$, and this procedure is blind to the chosen form for f).

2. Other fourth-order derivative terms

Now consider other fourth-order derivative terms which may appear in the equations of motion. In particular, if $q \geq 2$, the field equation derived from (51) will contain a fourth derivative term proportional to

$$f \times (\Box \pi)^p (\pi^\lambda \pi^\rho \pi^\sigma \pi^\tau \pi_{\lambda\rho\sigma\tau})(\pi^\mu \pi_{\mu\nu} \pi^\nu)^{q-2} \quad (56)$$

(obtained by varying one of the twice-differentiated π appearing in one cycle $\langle 1 \rangle$ and integrating by parts on the

other cycle $\langle 1 \rangle$). However, it is impossible to cancel this term by the variation of any of the other terms in the Lagrangian (connected to $\mathcal{L}^p_q$ through F and G). Thus we conclude that there are two possibilities—the power q appearing in (51) must take the value 0 or 1, so that on using (55),

$$q = 0 \Leftrightarrow N = n \quad (57)$$

$$q = 1 \Leftrightarrow N = n + 2. \quad (58)$$

3. Second-order equations of motion

We first focus on $q = 0$, namely, one particular theory which obeys (i), (ii), (iii), has a fixed value of n, and $N = n$. From (52), it follows that all the q_j must vanish so that the Lagrangian is a sum of monomials made of

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_r \\ 1 & 2 & \cdots & r \end{bmatrix}$$

only. As we have just seen, these have relative coefficients which are fixed and independent of the choice of the function f. One can conclude that those coefficients must be the ones, $C^{N+1}_{p_1,\cdots,p_r}$, appearing in the expansion of $\mathcal{E}_{N+1}$ (see Eq. (34)). Indeed, we know that if we consider $\mathcal{E}_{N+1}$ as an action, this action has a vanishing equation of motion since $\mathcal{E}_{N+1}$ is a total derivative. Hence it obeys the hypotheses (i), (ii) and (iii) and it has also the correct power, $n = N$, of twice-differentiated π appearing in the expansion in terms of monomials

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_r \\ 1 & 2 & \cdots & r \end{bmatrix}.$$

Hence, one is led to the conclusion that any theory obeying (i), (ii) and (iii), with a fixed value of n and $N = n$ must have a Lagrangian of the form $g \times \mathcal{E}_{N+1}$, where g is some function of π and X. Thus, on using (26), it can be rewritten as (defining $f \equiv g X^{-1}$)

$$\mathcal{L}^{(3)}_n\{f\} \equiv f(\pi, X) \times \mathcal{L}^{\text{Gal},3}_{n+2} = f(\pi, X) \times \mathcal{L}^{\text{Gal},3}_N. \quad (59)$$

A similar argument applies to the other family of models. Here $q = 1 = \sum_{j=1}^s q_j$, and $N = n + 2$ for a given value of n: these models must have a Lagrangian of the form

$$\mathcal{L}^{(2)}_n\{f\} \equiv f(\pi, X) \times \mathcal{L}^{\text{Gal},2}_{n+2}, = f(\pi, X) \times \mathcal{L}^{\text{Gal},2}_N. \quad (60)$$

Indeed $\mathcal{L}^{\text{Gal},2}_N$ obeys hypotheses (i), (ii) and (iii), and, as recalled after Eq. (32), has an expansion in terms of monomials

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_r \\ 1 & 2 & \cdots & r \end{bmatrix} \begin{pmatrix} q_1 & \cdots & q_s \\ 1 & \cdots & s \end{pmatrix}$$

with $\sum_{j=1}^{j=s} q_j = 1$.

Finally, it is straightforward to show that the field equations derived from Lagrangians (59) and (60) are indeed second order (so that third-order derivative terms also vanish). This is in fact a direct consequence of the "main lemma" of the previous section, since using the expressions (15) and (18) one can see that Lagrangians (59) and (60) are of the form (2). In the case of $\mathcal{L}_n^{(3)}$, the explicit equations of motion are given in (37).

4. Unique family of models satisfying the "main lemma"

Finally, we are in a position to show that (59) and (60) are in fact equivalent up to total derivatives, so that the unique theory satisfying (i), (ii) and (iii) is indeed given by an arbitrary linear combination of the Lagrangians $\mathcal{L}_n^{(3)}\{f\}$ as advertised in Sec. III A.

Define $\mathcal{L}_n^{(1)}\{f\} \equiv f(\pi, X)\mathcal{L}_{n+2}^{\text{Gal},1}$. Then the identity (23) amongst the three Galileon models is now generalized to

$$n\mathcal{L}_n^{(2)}\{f\} = \mathcal{L}_n^{(3)}\{f\} - \mathcal{L}_n^{(1)}\{f\}, \tag{61}$$

and similarly (22) becomes

$$2\mathcal{L}_n^{(2)}\{f + Xf_X\} = -\mathcal{L}_{n-1}^{(1)}\{Xf_\pi\} - \mathcal{L}_n^{(3)}\{f\} + \partial_\mu(f(\pi, X)J_{(n)}^\mu). \tag{62}$$

$J_{(n)}^\mu \equiv J_N^\mu$ is given in Eq. (21). Elimination of $\mathcal{L}_n^{(1)}$ between these two equations yields a recurrence relation between $\mathcal{L}_n^{(2)}$ and $\mathcal{L}_n^{(3)}$, namely,

$$\mathcal{L}_n^{(2)}\{f\} = -(n-1)\mathcal{L}_{n-1}^{(2)}\left\{\frac{\partial g_1}{\partial \pi}\right\} + \mathcal{L}_n^{(3)}\left\{\frac{g_1}{X}\right\} + \mathcal{L}_{n-1}^{(3)}\left\{\frac{\partial g_1}{\partial \pi}\right\}$$
$$+ \text{tot.div.}, \tag{63}$$

with

$$g_1\{f\} = -\frac{1}{2}\int_0^X dY f(\pi, Y).$$

On using (63) repeatedly, it follows that (up to a total derivative) $\mathcal{L}_n^{(2)}\{f\}$ can be expressed as a linear combination of $\mathcal{L}_i^{(3)}\{f\}$. Specifically

$$\mathcal{L}_n^{(2)}\{f\} = \mathcal{L}_0^{(3)}\left\{\frac{\partial g_{n,1}}{\partial \pi}\right\} + \sum_{i=1}^{n-1}\mathcal{L}_i^{(3)}\left\{\frac{g_{n,i}}{X} + \frac{\partial g_{n,i+1}}{\partial \pi}\right\}$$
$$+ \mathcal{L}_n^{(3)}\left\{\frac{g_{n,n}}{X}\right\} + \text{tot.div.}, \tag{64}$$

where $\mathcal{L}_0^{(3)}\{f\} = Xf$ for consistency, and

$$g_{n,i}\{f\} \equiv \frac{(n-1)!}{(i-1)!}g_{n-i+1}\{f\},$$

$$g_i\{f\} \equiv -\frac{1}{2^i}\left(\frac{\partial}{\partial \pi}\right)^{i-1}\int_{X_0}^X dX_1\int_{X_0}^{X_1} dX_2\cdots\int_{X_0}^{X_{i-1}} dX_i f(\pi, X_i).$$

Thus Eq. (64) shows the equivalence of the $q = 0$ and $q = 1$ Lagrangians given in (59) and (60) respectively.

Finally, observe that $\mathcal{L}_D^{(3)}\{f\}$ is in fact a linear combination of $\mathcal{L}_k^{(3)}$ with $k = 0, \ldots, D - 1$. This follows from (64) together with the fact that for $n = D$ Eq. (61) reduces to $D\mathcal{L}_D^{(2)}\{f\} = \mathcal{L}_D^{(3)}\{f\}$. Thus to conclude, the most general Lagrangian in D dimensions obeying conditions (i), (ii) and (iii) is given by

$$\mathcal{L} = \sum_{n=0}^{D-1}\mathcal{L}_n\{f_n\}, \tag{65}$$

where f_n are arbitrary functions of π and X and $\mathcal{L}_n\{f\} \equiv \mathcal{L}_n^{(3)}\{f\}$.

IV. COVARIANTIZATION

We now turn our discussion from flat space-time to curved space-time. Let us start from the general Lagrangian $\mathcal{L}_n\{f\}$ in our family (36), writing it in the form of Eq. (2) with $\mathcal{T}_{(2n)}$ defined by

$$\mathcal{T}_{(2n)} = \mathcal{T}_{(2n)}(\pi, X) = f(\pi, X)\times X\mathcal{A}_{(2n)}, \tag{66}$$

and where n is the number of second derivatives of the field in the Lagrangian. On replacing all partial derivatives appearing in this Lagrangian by covariant derivatives, we obtain a minimally covariantized theory. As we will now see, following [16], the field equations of this covariantized model contain derivatives of order higher than two (varying the action with respect to π or with respect to the metric): however, we will also see that there exists a nonminimal covariantization removing all such higher-order derivatives.

Let us then consider variation with respect to the scalar π of the minimally covariantized version of (36). It reads

$$\delta\mathcal{L}_n\{f\} = \delta\mathcal{T}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_n\nu_n}$$
$$+ n\mathcal{T}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\delta\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\cdots\pi_{\mu_n\nu_n}. \tag{67}$$

Above, the only "dangerous terms" (recall that these are terms leading to expressions in the field equations depending on derivatives of order higher than two) come only from the second term on the right-hand side. Indeed by virtue of a straightforward generalization to curved space-time of the first lemma in Sec. II A (see the proof in Appendix B), dangerous terms coming from the first piece on the right-hand side of (67) are exactly compensated by terms coming from the second piece. After these compensations, first derivatives of the Riemann tensor (i.e., third derivatives of the metric) remain, and are therefore troublesome. Indeed we have

$$\delta\mathcal{L}_n\{f\} \sim -\frac{n(n-1)}{4}\mathcal{T}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\pi^\lambda$$
$$\times R_{\mu_1\mu_2\nu_1\nu_2;\lambda}\pi_{\mu_3\nu_3}\cdots\pi_{\mu_n\nu_n}\delta\pi, \tag{68}$$

where here and below we use the same notation as in [16] so that the symbol $\sim$ means that we write only the dangerous terms, up to total derivatives.

In order for the equations of motion to be second order, it is enough to add a finite number of terms whose

variations exactly cancel the third derivatives of the metric. Starting with the above dangerous term, we can add a term proportional to $(\int_{X_0}^{X} \mathcal{T}_{(2n)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n}(\pi, X_1) dX_1) \times R_{\mu_1 \mu_2 \nu_1 \nu_2} \pi_{\mu_3 \nu_3} \cdots \pi_{\mu_n \nu_n}$ to our Lagrangian. Then we need to add another term in order to compensate for the dangerous terms arising from our correction, and so on. The general term in this series of Lagrangians is

$$\mathcal{L}_{n,p}\{f\} = \mathcal{P}_{(p)}^{\mu_1 \mu_2 \cdots \mu_n \nu_1 \nu_2 \cdots \nu_n} \mathcal{R}_{(p)} \mathcal{S}_{(q \equiv n - 2p)} \qquad (69)$$

where we again follow the notation of [16] and

$$\mathcal{R}_{(p)} \equiv \prod_{i=1}^{p} R_{\mu_{2i-1} \mu_{2i} \nu_{2i-1} \nu_{2i}},$$

$$\mathcal{S}_{(q \equiv n - 2p)} \equiv \prod_{i=0}^{q-1} \pi_{\mu_{n-i} \nu_{n-i}},$$

and $\mathcal{P}_{(p)}$ is the (p times) repeated integral of $\mathcal{T}_{(2n)}$ with respect to X defined by[5]

$$\mathcal{P}_{(p)}^{\mu_1 \mu_2 \cdots \mu_n \nu_1 \nu_2 \cdots \nu_n}$$
$$\equiv \int_{X_0}^{X} dX_1 \int_{X_0}^{X_1} dX_2 \cdots \int_{X_0}^{X_{p-1}} dX_p \mathcal{T}_{(2n)}^{\mu_1 \mu_2 \cdots \mu_n \nu_1 \nu_2 \cdots \nu_n}(\pi, X_1).$$

Notice that we use the conventions $\mathcal{L}_{n,0}\{f\} = \mathcal{L}_n\{f\}$, $\mathcal{S}_1 = \pi_{\mu_n \nu_n}$ and $\mathcal{S}_{q \leq 0} = 1$. It follows from the lemma of Appendix B that $\mathcal{L}_{n,p}\{f\}$ does not yield any more dangerous terms for $p \geq \lfloor n/2 \rfloor$ (where $\lfloor n/2 \rfloor$ denotes the integer part of $n/2$). Thus, as we will confirm, only a finite number of terms $\mathcal{L}_{n,p}\{f\}$ is necessary. In order to make sure that adding these terms to the initial Lagrangian exactly cancels all higher derivatives in the equations of motion, and to determine the coefficients that make this possible, we now compute their variation $\delta \mathcal{L}_{n,p}$ with respect to the field π, only paying attention to dangerous terms. We get (after suitable integrations by parts)

$$\begin{aligned}
\delta \mathcal{L}_{n,p} &= \delta \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p)} \mathcal{S}_{(q)} + (n - 2p) \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p)} \mathcal{S}_{(q-1)} \delta \pi_{\mu_{2p+1} \nu_{2p+1}} \\
&= 2 \mathcal{P}_{(p-1)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \pi^\lambda \delta \pi_\lambda \mathcal{R}_{(p)} \mathcal{S}_{(q)} + (n - 2p) \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p)} \mathcal{S}_{(q-1)} \delta \pi_{\mu_{2p+1} \nu_{2p+1}} \\
&\sim -2p \mathcal{P}_{(p-1)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p-1)} \mathcal{S}_{(q)} \pi^\lambda R_{\mu_{2p-1} \mu_{2p} \nu_{2p-1} \nu_{2p}; \lambda} \delta \pi \\
&\quad - \frac{(n-2p)(n-2p-1)}{4} \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p)} \mathcal{S}_{(q-2)} \pi^\lambda R_{\mu_{2p+1} \mu_{2p+2} \nu_{2p+1} \nu_{2p+2}; \lambda} \delta \pi.
\end{aligned}$$

Notice that, once again, the third derivatives of π coming from the variation of $\mathcal{P}_{(p)}$ are exactly canceled by those coming from the variation of $\mathcal{S}_{(q)}$. The second Bianchi identity *ensures* that no other derivatives of the Riemann tensor appear. To summarize, in order that the equations of motion contain no more than second-order derivatives, the Lagrangian must be given by the linear combination

$$\mathcal{L}_n^{\text{cov}}\{f\} = \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{C}_{n,p} \mathcal{L}_{n,p}\{f\}, \qquad (70)$$

where the coefficients obey the specific recurrence relation $\mathcal{C}_{n,p+1} = -\frac{(n-2p)(n-2p-1)}{8(p+1)} \mathcal{C}_{n,p}$ for $p \geq 0$ and $\mathcal{C}_{n,0} = 1$. This gives

$$\mathcal{C}_{n,p} = \left(-\frac{1}{8}\right)^p \frac{n!}{(n-2p)! p!}. \qquad (71)$$

Note that similar expressions have been obtained in [16]. However, there, the covariantization was derived for the Lagrangians $\mathcal{L}_N^{\text{Gal},1}$. It was also observed that the covariantized action could be written in different forms using various total derivatives. The same is true here and hence,

taking also into account (24), it is not straightforward to compare the covariantized form (70) and (71) to the one given in [16].

Remarkably the Lagrangian $\mathcal{L}_n^{\text{cov}}\{f\}$ also yields second-order equations for the metric. Indeed by computing the variation of each term appearing in the above linear combination, but this time with respect to the metric $g_{\mu\nu}$ (denoting the metric variation by $\delta g_{\mu\nu}$), we obtain (given that the tensor $\mathcal{P}_{(p)}$ does not depend on derivatives of the metric)

$$\begin{aligned}
\delta \mathcal{L}_{n,p} &\sim p \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p-1)} \delta R_{\mu_{2p-1} \mu_{2p} \nu_{2p-1} \nu_{2p}} \mathcal{S}_{(q)} \\
&\quad + (n - 2p) \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p)} \mathcal{S}_{(q-1)} \delta \pi_{\mu_{2p+1} \nu_{2p+1}}
\end{aligned}$$

with (see equations (31) and (32) in Ref. [16])

$$\begin{aligned}
&\delta \pi_{\mu_{2p+1} \nu_{2p+1}} \\
&= -\frac{1}{2} \pi^\lambda (\delta g_{\lambda \mu_{2p+1}; \nu_{2p+1}} + \delta g_{\lambda \nu_{2p+1}; \mu_{2p+1}} - \delta g_{\mu_{2p+1} \nu_{2p+1}; \lambda}), \\
&\mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \delta R_{\mu_{2p-1} \mu_{2p} \nu_{2p-1} \nu_{2p}} \\
&= 2 \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \delta g_{\mu_{2p-1} \nu_{2p}; \mu_{2p} \nu_{2p-1}} \\
&\quad + \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \delta g^\sigma_{\mu_{2p-1}} R_{\sigma \mu_{2p} \nu_{2p-1} \nu_{2p}}.
\end{aligned}$$

This yields

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[5]X_0 being an arbitrary constant whose presence is related to the possibility of adding terms—all of which vanish in flat space—that avoid higher derivatives. See, e.g., (91) and (92).

$$\delta \mathcal{L}_{n,p} \sim 2p \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p-1)} \mathcal{S}_{(q)} \delta g_{\mu_{2p-1}\nu_{2p};\mu_{2p}\nu_{2p-1}} + \frac{(n-2p)}{2} \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p)} \mathcal{S}_{(q-1)} \pi^\lambda \delta g_{\mu_{2p+1}\nu_{2p+1};\lambda}.$$

After two integrations by parts, we are led to

$$\delta \mathcal{L}_{n,p} \sim 4p \mathcal{P}_{(p-1)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p-1)} \mathcal{S}_{(q)} \pi^\lambda \pi_{\nu_{2p-1}\mu_{2p}\lambda} \delta g_{\mu_{2p-1}\nu_{2p}}$$

$$- \frac{p(n-2p)}{2} \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p-1)} \mathcal{S}_{(q-1)} \pi^\lambda R_{\mu_{2p}\mu_{2p+1}\nu_{2p-1}\nu_{2p+1};\lambda} \delta g_{\mu_{2p-1}\nu_{2p}}$$

$$- \frac{p(n-2p)}{2} \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p-1)} \mathcal{S}_{(q-1)} \pi^\lambda R_{\mu_{2p-1}\mu_{2p}\nu_{2p-1}\nu_{2p};\lambda} \delta g_{\mu_{2p+1}\nu_{2p+1}}$$

$$- \frac{(n-2p)(n-2p-1)}{2} \mathcal{P}_{(p)}^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \mathcal{R}_{(p)} \mathcal{S}_{(q-2)} \pi^\lambda \pi_{\lambda\mu_{2p+2}\nu_{2p+2}} \delta g_{\mu_{2p+1}\nu_{2p+1}}.$$

On relabeling $\mu_{2p-1} \leftrightarrow \mu_{2p+1}$ and $\nu_{2p} \leftrightarrow \nu_{2p+1}$ in the second term, the second and third terms are seen to cancel. Also, by changing $\nu_{2p} \leftrightarrow \nu_{2p-1}$ in the first term, it is clear that all the other dangerous terms vanish if the coefficients $\mathcal{C}_{n,p}$ are given by Eq. (71).

Our covariantized Lagrangian (70) depends on D functions of π and X as well as $\sum_{n=0}^{D-1} \lfloor n/2 \rfloor$ functions of π (the integration constants). However, this is due to our parametrization: these "constants" can be reabsorbed in a redefinition of the functional coefficients f_n. That is, we can parametrize our theories by starting from the term with the highest number of Riemann tensors, leaving a total of D arbitrary functions only.

To conclude, we have started from the most general theory in flat space-time (36) and obtained an extended family of Lagrangians (70) which yield second-order equations of motion in curved space-time. As we will see below, when $D = 4$, this can be shown to be the most general such theory in curved space-time.

V. EXAMPLES

Although in Sec. III we have completely classified all models of the form (2) (and more generally all models obeying conditions (i), (ii) and (iii) of Sec. III), it is of interest to discuss some specific models in which the tensor $\mathcal{T}_{(2n)}$ of Eqs. (2) and (3) takes a simple form. We will in turn give a straightforward method with which to construct a suitable tensor $\mathcal{T}_{(2n)}$—namely satisfying the properties given in the main lemma of Sec. II A. We then discuss the hierarchical constructions of [13] (see also [33] for a recent summary) in our notation. Finally, we give the covariantization of the conformal Galileon in four dimensions (Sec. V C), and explain the link between our work and that of Horndeski [14] (Sec. V D).

A. Construction with antisymmetric tensors

One way to obtain a tensor $\mathcal{T}_{(2n)}$ is to consider a set of twice contravariant tensors $F_i^{\mu\nu}$, $i = 1 \ldots n$, (and $n \leq D$) depending on π and π^λ (where μ and ν label space-time

indices, whereas i labels the tensors) and then to define the D-contravariant tensor $\mathcal{E}_{\{F_i\}}$ by

$$\mathcal{E}_{\{F_i\}}{}^{\mu_1 \ldots \mu_n \sigma_1 \ldots \sigma_{D-n}} \equiv \varepsilon_{\nu_1 \ldots \nu_n}{}^{\sigma_1 \ldots \sigma_{D-n}} F_{(1}^{\mu_1 \nu_1} F_2^{\mu_2 \nu_2} \ldots F_{n)}^{\mu_n \nu_n}, \tag{72}$$

where the brackets denote symmetrization over the tensors $F_i^{\mu\nu}$. The tensor $\mathcal{E}_{\{F_i\}}$ is easily seen to be completely antisymmetric over its first n indices as well as (separately) over its last $D - n$ indices.

The special choice

$$F_i^{\mu\nu} = g^{\mu\nu}, \quad \text{for all } i \tag{73}$$

gives a tensor $\mathcal{E}$ equal to the Levi-Civita ε tensor. Another interesting case arises by choosing, for example, $F_1^{\mu\nu} = \pi^\mu \pi^\nu$, and all the other $F_i^{\mu\nu}$ given by the metric as in (73). In that case the tensor $\mathcal{E}$ will be denoted by $\tilde{\varepsilon}_{(n)}$ and is given by

$$\tilde{\varepsilon}_{(n)}{}^{\mu_1 \ldots \mu_n \sigma_1 \ldots \sigma_{D-n}} = \frac{1}{n} \varepsilon_{\nu_1 \ldots \nu_n}{}^{\sigma_1 \ldots \sigma_{D-n}} (\pi^{\mu_1} \pi^{\nu_1} g^{\mu_2 \nu_2} \ldots g^{\mu_n \nu_n}$$

$$+ g^{\mu_1 \nu_1} \pi^{\mu_2} \pi^{\nu_2} g^{\mu_3 \nu_3} \ldots g^{\mu_n \nu_n} + \ldots$$

$$+ g^{\mu_1 \nu_1} \ldots g^{\mu_{n-1} \nu_{n-1}} \pi^{\mu_n} \pi^{\nu_n}). \tag{74}$$

This tensor enters, in particular, in the form (15)–(17) of the action for the Galileon, as we will see below. We can also construct a tensor $\mathcal{E}$ given by a linear combination of ε and $\tilde{\varepsilon}_{(n)}$ by choosing

$$F_i^{\mu\nu} = \frac{\partial^2 F_i}{\partial \pi_\mu \partial \pi_\nu}, \tag{75}$$

where F_i is some scalar function of X and π: such $F_i^{\mu\nu}$ arise in the hierarchical construction of [13] (see Sec. V A). In this case

$$\frac{\partial^2 F_i}{\partial \pi_\mu \partial \pi_\nu} = 4 \frac{\partial^2 F_i}{\partial X^2} \pi^\mu \pi^\nu + 2 \frac{\partial F_i}{\partial X} g^{\mu\nu}. \tag{76}$$

Note that if there is more than one tensor $F_i^{\mu\nu}$ proportional to π^μ (or π^ν) in Eq. (72), then the contraction with the Levi-Civita tensor on the right-hand side of this equation implies a vanishing $\mathcal{E}$.

Having at hand the tensors $\mathcal{E}$, it is easy to build tensors $\mathcal{T}$ with the required properties: namely, simply replace the Levi-Civita ε tensor on the right-hand side of Eq. (6) by the $\mathcal{E}$ tensor, and define, in analogy with (9) a tensor $\mathcal{T}$ given by

$$\mathcal{T}_{(2n)}^{\mu_1\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n} \equiv f \times \mathcal{E}_{\{F_i\}}{}^{\mu_1\mu_2\ldots\mu_n\sigma_1\sigma_2\ldots\sigma_{D-n}}$$
$$\times \mathcal{E}_{\{G_i\}}{}^{\nu_1\nu_2\ldots\nu_n}{}_{\sigma_1\sigma_2\ldots\sigma_{D-n}}. \qquad (77)$$

Here the tensors $\mathcal{E}$ are built with (possibly different) given sets of functions F_i and G_i, and the function f is an arbitrary scalar function of π_μ and π. For example, if we choose $f = 1$, one of the $\mathcal{E}$ tensors to be the tensor $\tilde{\varepsilon}_{(n)}$, and the other to be the Levi-Civita tensor ε, then $\mathcal{T}$ is simply the tensor $\mathcal{T}_{(2n),\text{Gal},2}$ of Eq. (17), namely

$$\mathcal{T}_{(2n)}^{\mu_1\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n} \equiv \tilde{\varepsilon}_{(n)}{}^{\mu_1\mu_2\ldots\mu_n\sigma_1\sigma_2\ldots\sigma_{D-n}}\varepsilon^{\nu_1\nu_2\ldots\nu_n}{}_{\sigma_1\sigma_2\ldots\sigma_{D-n}}. \qquad (78)$$

B. Euler hierarchies

Another way to generate tensors of the form (72) is provided by the *Euler hierarchies* of [13], which we now revisit using the notations and results of the present paper. We start from a set of arbitrary scalar functions $F_\ell = F_\ell(\pi^\mu)$ which depend *only* on first derivatives of the scalar field π, and work in flat D-dimensional space-time. Then, denoting $W_0 = 1$, we define the recursion relation

$$W_{\ell+1} = -\hat{\mathcal{E}}F_{\ell+1}W_\ell, \qquad (79)$$

where $\hat{\mathcal{E}}$ is the Euler-Lagrange operator defined in Eq. (5). Hence W_ℓ is the field equation of a Lagrangian $\mathcal{L}_\ell$ defined by

$$\mathcal{L}_\ell = F_\ell W_{\ell-1}, \qquad (80)$$

and we build that way an "Euler hierarchy" of equations of motion and Lagrangians for each ℓ. An interesting aspect of this hierarchy is that one can show that W_ℓ is given by

$$W_\ell = \mathcal{A}_{\mu_1\mu_2\cdots\mu_\ell}{}^{\nu_1\nu_2\cdots\nu_\ell}\frac{\partial^2 F_1}{\partial\pi_{\mu_1}\partial\pi_{\rho_1}}\frac{\partial^2 F_2}{\partial\pi_{\mu_2}\partial\pi_{\rho_2}}\cdots\frac{\partial^2 F_\ell}{\partial\pi_{\mu_\ell}\partial\pi_{\rho_\ell}}$$
$$\times \pi_{\rho_1\nu_1}\pi_{\rho_2\nu_2}\cdots\pi_{\rho_\ell\nu_\ell}, \qquad (81)$$

and

$$\mathcal{L}_\ell = \mathcal{A}_{\mu_1\cdots\mu_{\ell-1}}{}^{\nu_1\cdots\nu_{\ell-1}}\frac{\partial^2 F_1}{\partial\pi_{\mu_1}\partial\pi_{\rho_1}}\cdots\frac{\partial^2 F_{\ell-1}}{\partial\pi_{\mu_{\ell-1}}\partial\pi_{\rho_{\ell-1}}}$$
$$\times F_\ell\pi_{\rho_1\nu_1}\cdots\pi_{\rho_{\ell-1}\nu_{\ell-1}}. \qquad (82)$$

The proof proceeds by induction [13], and is also given in Appendix C. Notice that $\mathcal{L}_\ell$ can be obtained from (77) by setting $f \equiv \frac{1}{(\ell-1)!}F_\ell$ and contracting the tensor $\mathcal{E}_{\{\partial^2 F_k/\partial\pi_\mu\partial\pi_\nu\}}$ with ε. The above generic hierarchy stops after *at most* D steps (it may stop earlier depending on the properties of the functions F_k) and one finds a vanishing

W_{D+1}. As discussed in [13], and as we will see in a simple example below, the last nontrivial equation of motion W_D is simply given (whatever the choice made for the functions F_k) by the equation of motion of the maximal Galileon in D dimensions, namely, the one for which $N = D + 1$, see (11).

The maximal and nonmaximal Galileons can also be obtained by choosing $F_k = \pi_\mu\pi^\mu/2$ for all $k = 0,\ldots,\ell$. In this case, one has

$$W_\ell = \mathcal{A}_{(2\ell)}^{\mu_1\cdots\mu_\ell\nu_1\cdots\nu_\ell}\pi_{\mu_1\nu_1}\cdots\pi_{\mu_\ell\nu_\ell} = \mathcal{E}_{\ell+1} \qquad (83)$$

$$\mathcal{L}_\ell = \frac{1}{2}XW_{\ell-1} = \frac{1}{2}\mathcal{L}_{\ell+1}^{\text{Gal},3}. \qquad (84)$$

More generally, when the F_k are (possibly different) functions of X, that is $F_k = f_k(\pi^\lambda\pi_\lambda)$, one can show that

$$W_\ell = \alpha_\ell\mathcal{E}_{\ell+1} - \beta_\ell\mathcal{L}_{\ell+2}^{\text{Gal},1}, \qquad (85)$$

$$\mathcal{L}_\ell = f_\ell(X)(\alpha_{\ell-1}\mathcal{E}_\ell - \beta_{\ell-1}\mathcal{L}_{\ell+1}^{\text{Gal},1}), \qquad (86)$$

where α and β are given by

$$\alpha_\ell(\pi^\lambda\pi_\lambda) = \frac{2^{\ell-1}}{\ell}\sum_{k=1}^{\ell}\left((2f_k' + 4(\pi^\lambda\pi_\lambda)f_k'')\prod_{j\neq k}f_j'\right)$$

$$\beta_\ell(\pi^\lambda\pi_\lambda) = \frac{2^{\ell+1}}{\ell}\sum_{k=1}^{\ell}\left(f_k''\prod_{j\neq k}f_j'\right).$$

To conclude, since both $\mathcal{L}_{\ell+1}^{\text{Gal},1}$ and $\mathcal{E}_\ell$ are directly related to $\mathcal{L}_{\ell+1}^{\text{Gal},3}$ (see Eqs. (24) and (26) respectively), the Lagrangian (86) which we have constructed using the Euler hierarchies belongs to our family of general Lagrangians given in (36).

C. Conformal covariant Galileons in four dimensions

The conformal Galileons [9] provide another simple example of theories of the kind obtained in this work (other nontrivial examples in flat and curved space-times have been obtained in [21]). The Lagrangians for the conformal Galileons in four dimensions and flat space-time have been given explicitly in [20]. They read, in our notation,

$$\mathcal{L}_4^{C.\text{Gal}} = \frac{1}{20}e^{2\pi}X[10((\Box\pi)^2 - \pi_{\mu\nu}\pi^{\mu\nu})$$
$$+ 4(X\Box\pi - \pi^\mu\pi^\nu\pi_{\mu\nu}) + 3X^2],$$

$$\mathcal{L}_5^{C.\text{Gal}} = e^{4\pi}X\left[\frac{1}{3}((\Box\pi)^3 + 2\pi_{\mu_1}^{\mu_2}\pi_{\mu_2}^{\mu_3}\pi_{\mu_3}^{\mu_1} - 3\Box\pi\pi_{\mu\nu}\pi^{\mu\nu})\right.$$
$$+ X((\Box\pi)^2 - \pi_{\mu\nu}\pi^{\mu\nu})$$
$$\left. + \frac{10}{7}X(X\Box\pi - \pi^\mu\pi^\nu\pi_{\mu\nu}) - \frac{1}{28}X^3\right].$$

These can be rewritten in terms of $\mathcal{L}_N^{\text{Gal},i}$ and $\mathcal{L}_n^{(i)}\{f\}$, as

$$\mathcal{L}_4^{C.Gal} = -\frac{1}{2}e^{2\pi}\mathcal{L}_4^{Gal,3} - \frac{1}{5}e^{2\pi}X\mathcal{L}_3^{Gal,1} + \frac{3}{20}e^{2\pi}X^3,$$

$$= -\frac{1}{2}\mathcal{L}_2^{(3)}\{e^{2\pi}\} - \frac{1}{5}\mathcal{L}_1^{(1)}\{e^{2\pi}X\} + \frac{3}{20}e^{2\pi}X^3, \quad (87)$$

$$\mathcal{L}_5^{C.Gal} = -\frac{1}{3}e^{4\pi}\mathcal{L}_5^{Gal,3} - e^{4\pi}X\mathcal{L}_4^{Gal,3} - \frac{10}{7}e^{4\pi}X^2\mathcal{L}_3^{Gal,1}$$

$$- \frac{1}{28}e^{4\pi}X^4,$$

$$= -\frac{1}{3}\mathcal{L}_3^{(3)}\{e^{4\pi}\} - \mathcal{L}_2^{(3)}\{e^{4\pi}X\} - \frac{10}{7}\mathcal{L}_1^{(3)}\{e^{4\pi}X^2\}$$

$$- \frac{1}{28}e^{4\pi}X^4. \quad (88)$$

As expected, one sees that those Lagrangians are indeed in the family obtained by our proof of uniqueness. The co-variantization of theories (87) and (88) is straightforward. Only the first term in (87) and the first two terms in (88) need compensating factors. The relevant covariantization formulas are

$$\mathcal{L}_2^{cov}\{f\} = \mathcal{L}_2^{(3)}\{f\} + \frac{1}{2}R\int_{X_0}^{X}dY f(\pi, Y)Y, \quad (89)$$

$$\mathcal{L}_3^{cov}\{f\} = \mathcal{L}_3^{(3)}\{f\} - 3G_{\mu\nu}\pi^{\mu\nu}\int_{X_0}^{X}dY f(\pi, Y)Y. \quad (90)$$

To summarize, (87) must be completed with

$$\mathcal{L}_4^{C.Gal,c} = -\frac{1}{8}e^{2\pi}R(X^2 - X_0^2), \quad (91)$$

and (88) must be completed with

$$\mathcal{L}_5^{C.Gal,c} = \frac{1}{2}e^{4\pi}\left[G_{\mu\nu}\pi^{\mu\nu}(X^2 - X_0^2) - \frac{1}{3}R(X^3 - X_0^3)\right], \quad (92)$$

where X_0 is an arbitrary integration constant that can be taken to vanish. Note that the coefficients of the terms proportional to $(X^2 - X_0^2)$ in (91) and (92) can be directly read from the expressions given in [15,16]. However, one should take into account that one has to integrate once $e^{4\pi}X\mathcal{T}_{(4),Gal,3}$ to obtain the term proportional to $(X^3 - X_0^3)$ in (92).

D. Relation with Horndeski theories

In [14], Horndeski obtained the most general scalar-tensor theory in four dimensions which has field equations of second (and lower) order both for the scalar field and the metric. Using the notation of the present paper, the family of theories he obtained are written as

$$\mathcal{L}_H = -\mathcal{A}_{(3)}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\left(\kappa_1 R_{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_3\nu_3}\right.$$

$$\left. - \frac{4}{3}\kappa_{1,X}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3\nu_3}\right) \quad (93)$$

$$- \mathcal{A}_{(3)}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(\kappa_3 R_{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_3}\pi_{\nu_3}$$

$$- 4\kappa_{3,X}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}\pi_{\mu_3}\pi_{\nu_3}) \quad (94)$$

$$- \mathcal{A}_{(2)}^{\mu_1\mu_2\nu_1\nu_2}(FR_{\mu_1\mu_2\nu_1\nu_2} - 4F_{,X}\pi_{\mu_1\nu_1}\pi_{\mu_2\nu_2}) \quad (95)$$

$$- 2\kappa_8\mathcal{A}_{(2)}^{\mu_1\mu_2\nu_1\nu_2}\pi_{\mu_1}\pi_{\nu_1}\pi_{\mu_2\nu_2} \quad (96)$$

$$- 3(2F_{,\pi} + X\kappa_8)X + \kappa_9, \quad (97)$$

where κ_1, κ_3, κ_8, κ_9 and F are functions of π and X which are related by the constraint

$$F_{,X} = \kappa_{1,\pi} - \kappa_3 - 2X\kappa_{3,X}. \quad (98)$$

We now comment on the relation between the above theories and the one discussed in this paper.

First it is clear that the flat space restriction of Horndeski theories must be a subset of the most general flat space theories obtained in this work in Sec. III (and presented in III A) on setting $D = 4$.[6] Second, the set of theories presented in Sec. IV and obtained by covariantizing the most general 4D flat space theory with second-order field equations must be included in the set of theories discussed by Horndeski. In fact one can also show that the converse is true: the set of theories discussed by Horndeski and the set of theories presented in Sec. IV are identical for $D = 4$. To see this, it is enough to show that Horndeski theories are included in the set of theories discussed in Sec. IV. That this is the case[7] can be seen from the following rewriting of (93)–(97):

$$\mathcal{L}_H = \sum_{n=0}^{3}\mathcal{L}_n^{cov}\{f_n\} \quad (99)$$

with

$$Xf_0(\pi, X) = -\kappa_9(\pi, X) - \frac{X}{2}\int dX(2\kappa_8 - 4\kappa_{3,\pi})_{,\pi},$$

$$Xf_1(\pi, X) = X(4\kappa_{3,\pi} + \kappa_8) - \frac{1}{2}\int dX(2\kappa_8 - 4\kappa_{3,\pi}) + 6F_{,\pi},$$

$$Xf_2(\pi, X) = 4(F + X\kappa_3)_{,X}, \quad Xf_3(\pi, X) = \frac{4}{3}\kappa_{1,X}.$$

Hence it is quite interesting to note that starting from different hypotheses—namely in our case the (possibly nonunique) covariantization of a scalar theory in flat space-time with second-order field equations (and action polynomial in second derivatives), and in Horndeski's

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[6]Note that the flat space restriction of Horndeski theories is not obviously the most general second-order theory for a scalar field in flat space. The reason is that Horndeski construction relies on a condition on the metric field equations which is inapplicable in flat space, and which could reduce the set of theories obtained.

[7]This equivalence has also been obtained in the last version of the preprint [34].

case, a curved space-time scalar-tensor theory with second-order field equations both for the scalar and the metric—one ends up with *identical* sets of theories.

Notice that the parametrization given by Horndeski can be misleading because one might conclude that, e.g., the terms (93), (95), and (97) are not independent, since they all depend on the function κ_1 (the same can be said for (94), (95), and (97) which depend on κ_3). This is however not the case, and one can see easily from the calculations done in Sec. IV as well as in [16] that each of the terms (93)–(97) lead *separately* to field equations of second order both for the scalar and the metric. This is transparent in our rewriting (99). Moreover, Horndeski's parametrization does not elucidate the relation between the two families of Lagrangians $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(3)}$ which appear, respectively, in Eqs. (94) and (96) and Eqs. (93), (95), and (97). Besides, in contrast to Horndeski's our theorem is valid for an arbitrary number of dimensions.

VI. CONCLUSIONS

In this work, we have obtained the most general scalar theory which has an action depending on derivatives of order up to second and has second-order (and lower) field equations. Those theories were shown to have Lagrangians made by taking the product of an arbitrary function of the scalar field and its first derivatives with a special form of the Galileon Lagrangian, or any linear combinations of those Lagrangians. We have also shown how to covariantize those models, while maintaining the key property that field equations are second order. Finally, we have also discussed the relation between our construction and the Euler hierarchies of Fairlie et al. We have shown, in particular, that the latter construction allows one to obtain all theories which are shift symmetric.

Several questions are left for future work. On the formal side, it would be interesting to see how the above proof can be generalized to the case of p forms and/or multifields. One could also investigate the possibility of having actions which depend on derivatives of order higher than two, and yet give rise to field equations of second order. On the phenomenological side, work is needed to see which subsets of the theories introduced here retain the several interesting aspects recalled in the introduction. Finally, it would be interesting to study the cosmology of, as well as cosmological perturbation theory in, phenomenologically interesting models.

APPENDIX A: USEFUL IDENTITIES

The definition of $\mathcal{A}_{(2n)}$ given in (6) is

$$\mathcal{A}_{(2n)}^{\mu_1\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n}$$
$$\equiv \frac{1}{(D-n)!}\varepsilon^{\mu_1\mu_2\ldots\mu_n\sigma_1\sigma_2\ldots\sigma_{D-n}}\varepsilon^{\nu_1\nu_2\ldots\nu_n}{}_{\sigma_1\sigma_2\ldots\sigma_{D-n}}, \quad \text{(A1)}$$

where [see (7)]

$$\varepsilon^{\mu_1\mu_2\ldots\mu_D} = -\frac{1}{\sqrt{-g}}\delta_1^{[\mu_1}\delta_2^{\mu_2}\ldots\delta_D^{\mu_D]} \equiv -\frac{1}{\sqrt{-g}}\delta_{12\ldots D}^{\mu_1\mu_2\ldots\mu_D},$$

$$\varepsilon_{\mu_1\mu_2\ldots\mu_D} = \sqrt{-g}\,\delta_{[\mu_1}^1\delta_{\mu_2}^2\ldots\delta_{\mu_D}^D \equiv \sqrt{-g}\,\delta_{\mu_1\mu_2\ldots\mu_D}^{12\ldots D},$$

so that (A1) can be rewritten as

$$\mathcal{A}_{(2n)}^{\mu_1\mu_2\cdots\mu_n}{}_{\nu_1\nu_2\cdots\nu_n} = -\frac{1}{(D-n)!}\delta_{\nu_1\nu_2\ldots\nu_n\sigma_1\sigma_2\ldots\sigma_{D-n}}^{\mu_1\mu_2\ldots\mu_n\sigma_1\sigma_2\ldots\sigma_{D-n}}$$
$$= -\delta_{\nu_1\cdots\nu_n}^{\mu_1\cdots\mu_n}. \quad \text{(A2)}$$

The identity between the three Galileon Lagrangians (8), (15), and (18) given in (23) then follows by using

$$\delta_{\nu_1\nu_2\cdots\nu_{n+1}}^{\mu_1\mu_2\cdots\mu_{n+1}} = \sum_{i=1}^{n+1}(-1)^{i-1}\delta_{\nu_i}^{\mu_1}\delta_{\nu_1\nu_2\cdots\nu_{i-1}\nu_{i+1}\ldots\nu_{n+1}}^{\mu_2\mu_3\cdots\mu_{n+1}}$$
$$= \delta_{\nu_1}^{\mu_1}\delta_{\nu_2\cdots\ldots\nu_{n+1}}^{\mu_2\cdots\mu_{n+1}}$$
$$+ \sum_{i=2}^{n+1}(-1)^{i-1}\delta_{\nu_i}^{\mu_1}\delta_{\nu_1\nu_2\cdots\nu_{i-1}\nu_{i+1}\cdots\nu_{n+1}}^{\mu_2\mu_3\cdots\mu_{i-1}\mu_i\mu_{i+1}\cdots\mu_{n+1}}.$$
$$\text{(A3)}$$

APPENDIX B: A FORMAL PROOF OF THE FIRST LEMMA A OF SEC. II A

Here we provide a formal proof of the lemma of Sec. II A. Since we also use this lemma in Sec. IV we discuss the proof in curved space-time with the understanding that we replace everywhere partial derivatives ∂ by covariant derivatives ∇ (i.e., $\pi_\alpha \equiv \nabla_\alpha\pi$, $\pi_{\alpha\beta} \equiv \nabla_\alpha\nabla_\beta\pi$ and so on). We consider $\mathcal{T}_{(2)}$, a twice contravariant tensor depending on π and its first derivatives π_μ. Varying the Lagrangian $\mathcal{L} = \mathcal{T}_{(2)}^{\mu\nu}\pi_{\mu\nu}$, and writing only the potentially dangerous terms (i.e., those terms which can lead to third and higher derivatives, which, following the notations of [16], we denote by using the symbol $\sim$) in the variations we find (using suitable integrations by part)

$$\delta\mathcal{L} = \frac{\partial\mathcal{T}_{(2)}^{\mu\nu}}{\partial\pi_\rho}\delta\pi_\rho\pi_{\mu\nu} + \mathcal{T}_{(2)}^{\mu\nu}\delta\pi_{\mu\nu}$$

$$\sim -\frac{\partial\mathcal{T}_{(2)}^{\mu\nu}}{\partial\pi_\rho}\pi_{\rho\mu\nu}\delta\pi + (\nabla_\nu\nabla_\mu\mathcal{T}_{(2)}^{\mu\nu})\delta\pi$$

$$\sim -\frac{\partial\mathcal{T}_{(2)}^{\prime\mu\nu}}{\partial\pi_\rho}\pi_{\rho\mu\nu}\delta\pi + (\partial_\nu\partial_\mu\mathcal{T}_{(2)}^{\mu\nu})\delta\pi$$

$$\sim -\frac{\partial\mathcal{T}_{(2)}^{\mu\nu}}{\partial\pi_\rho}\pi_{\rho\mu\nu}\delta\pi + \partial_\nu\left(\frac{\partial\mathcal{T}_{(2)}^{\mu\nu}}{\partial\pi_\rho}\partial_\mu\pi_\rho\right)\delta\pi$$

$$\sim -\frac{\partial\mathcal{T}_{(2)}^{\mu\nu}}{\partial\pi_\rho}\pi_{\rho\mu\nu}\delta\pi + \frac{\partial\mathcal{T}_{(2)}^{\mu\nu}}{\partial\pi_\rho}\partial_\nu\partial_\mu\pi_\rho\delta\pi$$

$$\sim -\frac{\partial\mathcal{T}_{(2)}^{\mu\nu}}{\partial\pi_\rho}(\pi_{\rho\mu\nu} - \pi_{\nu\mu\rho})\delta\pi$$

Since the commutation of the covariant derivatives only involves a contraction of the Riemann tensor with a first derivative of the field, the dangerous terms exactly cancel each other and this concludes the proof.

APPENDIX C: EULER HIERARCHY

In this appendix, following [13], we show how to obtain (81). The proof proceeds by induction. First notice that

$$W_1 = -\hat{\mathcal{E}}F_1 W_0 = \partial_\mu\frac{\partial F_1}{\partial\pi_\mu} = \frac{\partial^2 F_1}{\partial\pi_\mu\partial\pi_\nu}\pi_{\mu\nu},$$

so the proposition is true for $n = 1$. Suppose now that it is true for arbitrary fixed n. In order to show that it is still valid for $n + 1$ we will need the following identity

$$\hat{\mathcal{E}}F(\pi_\mu)W(\pi_\mu,\pi_{\mu\nu}) = (\hat{\mathcal{E}}W)F - 2\left[\frac{\partial W}{\partial\pi_\mu} - \frac{1}{2}\partial_\nu\left(\frac{\partial W}{\partial\pi_{\mu\nu}} + \frac{\partial W}{\partial\pi_{\nu\mu}}\right)\right]\pi_{\mu\rho}\frac{\partial F}{\partial\pi_\rho} - \left(W\frac{\partial^2 F}{\partial\pi_\mu\partial\pi_\nu}\pi_{\mu\nu} - \frac{\partial W}{\partial\pi_{\mu\nu}}\frac{\partial^2 F}{\partial\pi_\rho\partial\pi_\sigma}\pi_{\mu\rho}\pi_{\nu\sigma}\right).$$

We can now use this with $W = W_n$ (which depends only on first and second derivatives of π) and $F = F_{n+1}$. But since W_n is a divergence (it is the equation of motion of some lagrangian which does not depend explicitly on the field itself), its equation of motion vanishes $\hat{\mathcal{E}}W_n = 0$ (alternatively we could say that the operator $\hat{\mathcal{E}}$ is nilpotent). Also because of the antisymmetry of the Levi-Civita tensor, we can show that

$$\frac{\partial W_n}{\partial\pi_\mu} - \frac{1}{2}\partial_\nu\left(\frac{\partial W_n}{\partial\pi_{\mu\nu}} + \frac{\partial W_n}{\partial\pi_{\nu\mu}}\right) = \frac{\partial W_n}{\partial\pi_\mu} - \frac{1}{2}\underbrace{\left(\frac{\partial^2 W_n}{\partial\pi_{\mu\nu}\partial\pi_\rho} + \frac{\partial^2 W_n}{\partial\pi_{\nu\mu}\partial\pi_\rho}\right)\pi_{\nu\rho}}_{=2(\partial W_n/\partial\pi_\mu)} + \frac{1}{2}\underbrace{\left(\frac{\partial^2 W_n}{\partial\pi_{\mu\nu}\partial\pi_{\rho\sigma}} + \frac{\partial^2 W_n}{\partial\pi_{\nu\mu}\partial\pi_{\rho\sigma}}\right)\pi_{\nu\rho\sigma}}_{=0}.$$

So in the end this term vanishes too and we have

$$-\hat{\mathcal{E}}F_{n+1}W_n = W_n\frac{\partial^2 F_{n+1}}{\partial\pi_\mu\partial\pi_\nu}\pi_{\mu\nu} - \frac{\partial W_n}{\partial\pi_{\mu\nu}}\frac{\partial^2 F_{n+1}}{\partial\pi_\rho\partial\pi_\sigma}\pi_{\mu\rho}\pi_{\nu\sigma}$$

$$= \mathcal{A}_{\mu_1\cdots\mu_n}{}^{\nu_1\cdots\nu_n}\frac{\partial^2 F_1}{\partial\pi_{\mu_1}\partial\pi_{\rho_1}}\cdots\frac{\partial^2 F_n}{\partial\pi_{\mu_n}\partial\pi_{\rho_n}}\frac{\partial^2 F_{n+1}}{\partial\pi_\mu\partial\pi_\nu}\pi_{\rho_1\nu_1}\cdots\pi_{\rho_n\nu_n}\pi_{\mu\nu}$$

$$- \sum_{p=1}^n\mathcal{A}_{\mu_1\cdots\mu_n}{}^{\nu_1\cdots\nu_{p-1}\nu\nu_p\cdots\nu_n}\frac{\partial^2 F_1}{\partial\pi_{\mu_1}\partial\pi_{\rho_1}}\cdots\frac{\partial^2 F_p}{\partial\pi_{\mu_p}\partial\pi_\mu}\cdots\frac{\partial^2 F_n}{\partial\pi_{\mu_n}\partial\pi_{\rho_n}}$$

$$\times\frac{\partial^2 F_{n+1}}{\partial\pi_\rho\partial\pi_\sigma}\pi_{\rho_1\nu_1}\cdots\pi_{\rho_{p-1}\nu_{p-1}}\pi_{\mu\rho}\pi_{\nu\sigma}\pi_{\rho_{p+1}\nu_{p+1}}\cdots\pi_{\rho_n\nu_n}$$

$$= \left(\mathcal{A}_{\mu_1\cdots\mu_n}{}^{\nu_1\cdots\nu_n}\delta_{\mu_{n+1}}{}^{\nu_{n+1}} - \sum_{p=1}^n\mathcal{A}_{\mu_1\cdots\mu_n}{}^{\nu_1\cdots\nu_{p-1}\nu_{n+1}\nu_{p+1}\cdots\nu_n}\delta_{\mu_{n+1}}{}^{\nu_p}\right)$$

$$\times\frac{\partial^2 F_1}{\partial\pi_{\mu_1}\partial\pi_{\rho_1}}\cdots\frac{\partial^2 F_{n+1}}{\partial\pi_{\mu_{n+1}}\partial\pi_{\rho_{n+1}}}\pi_{\rho_1\nu_1}\cdots\pi_{\rho_{n+1}\nu_{n+1}}.$$

But we also have (with (A3))

$$\mathcal{A}_{\mu_1\cdots\mu_n}{}^{\nu_1\cdots\nu_n}\delta_{\mu_{n+1}}{}^{\nu_{n+1}} - \sum_{p=1}^n\mathcal{A}_{\mu_1\cdots\mu_n}{}^{\nu_1\cdots\nu_{p-1}\nu_{n+1}\nu_{p+1}\cdots\nu_n}\delta_{\mu_{n+1}}{}^{\nu_p} = \mathcal{A}_{\mu_1\cdots\mu_{n+1}}{}^{\nu_1\cdots\nu_{n+1}},$$

which proves our result for $n + 1$. Notice that in this proof, we often used the fact that partial derivatives commute over flat space-time.

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