

**Strong interactions and exact solutions in nonlinear massive gravity**

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(Received 18 April 2011; published 21 September 2011)

We investigate strong coupling effects in a covariant massive gravity model, which is a candidate for a ghost-free nonlinear completion of Fierz-Pauli. We analyze the conditions to recover general relativity via the Vainshtein mechanism in the weak field limit, and find three main cases depending on the choice of parameters. In the first case, the potential is such that all nonlinearities disappear and the vDVZ discontinuity cannot be avoided. In the second case, the Vainshtein mechanism allows to recover general relativity within a macroscopic radius from a source. In the last case, the strong coupling of the scalar graviton completely shields the massless graviton, and weakens gravity when approaching the source. In the second part of the paper, we explore new exact vacuum solutions, that asymptote to de Sitter or anti de Sitter space depending on the choice of parameters. The curvature of the space is proportional to the mass of the graviton, thus providing a cosmological background which may explain the present-day acceleration in terms of the graviton mass. Moreover, by expressing the potential for nonlinear massive gravity in a convenient form, we also suggest possible connections with a higher-dimensional framework.

DOI: [10.1103/PhysRevD.84.064033](https://doi.org/10.1103/PhysRevD.84.064033)

PACS numbers: 04.50.Kd, 14.70.Kv

**I. INTRODUCTION**

Attempts to build a theory of massive gravity date back to the work by Fierz and Pauli (FP) in 1939 [1]. They considered a mass term for linear gravitational perturbations, which is uniquely determined by requiring the absence of ghost degrees of freedom. The mass term breaks the gauge invariance of general relativity (GR), leading to a graviton with 5 degrees of freedom instead of the 2 found in GR. There have been intensive studies in to what happens beyond the linearized theory of FP. In 1972, Boulware and Deser found a scalar ghost mode at the nonlinear level, the so-called sixth degree of freedom in the FP theory [2]. This issue has been reexamined using an effective field theory approach [3], where gauge invariance is restored by introducing Stückelberg fields. In this language, the Stückelberg fields acquire nonlinear interactions containing more than two time derivatives, signalling the existence of a ghost. In order to construct a consistent theory, nonlinear terms should be added to the FP model, which are tuned so that they remove the ghost order by order in perturbation theory.

Interestingly, this approach sheds light on another famous problem with FP massive gravity; due to contributions of the scalar degree of freedom, solutions in the FP model do not continuously connect to solutions in GR, even in the limit of zero graviton mass. This is known as the van Dam, Veltman, and Zakharov (vDVZ) discontinuity [4,5]. Observations such as light bending in the solar system would exclude the FP theory, no matter how small the graviton mass is. In 1972, Vainshtein [6] proposed a mechanism to avoid this conclusion; in the small mass limit, the scalar degree of freedom becomes strongly coupled and the linearized FP theory is no longer reliable.

In this regime, higher-order interactions, which are introduced to remove the ghost degree of freedom, should shield the scalar interaction and recover GR on sufficiently small scales.

Until recently, it was thought to be impossible to construct a ghost-free theory for massive gravity that is compatible with current observations [7,8]. A breakthrough came with a 5D braneworld model known as Dvali-Gabadadze-Porrati (DGP) model [9]. In this model there appears a continuous tower of massive gravitons from a four-dimensional perspective, and GR can be recovered for a given range of scales, due to strong coupling interactions [10,11]. In this paper, we explore the consequences of a promising new development along these lines that seems able to provide a consistent theory of massive gravity directly in four dimensions. In order to avoid the presence of a ghost, interactions have to be chosen in such a way that the equations of motion for the scalar degrees of freedom contain no more than two time derivatives. Recently, it was shown that there is a finite number of derivative interactions that gives rise to second-order differential equations. These are dubbed Galileon terms because of a symmetry under a constant shift of the scalar field derivative [12]. Therefore, one expects that any consistent nonlinear completion of FP contains these Galileon terms in the limit in which the scalar mode decouples from the tensor modes, the so-called *decoupling limit*. This turns out to be a powerful criterion for building higher-order interactions with the desired properties. Indeed, following this route, de Rham and Gabadadze constructed a family of ghost-free extensions to the FP theory, which reduce to the Galileon terms in the decoupling limit [13].

In this work, we investigate the consequences of strong coupling effects in this theory. We first reexpress the

potential for the most general version of nonlinear massive gravity, as developed by de Rham, Gabadadze and collaborators, in a particularly compact, easy-to-handle form. Among other things, this way of writing the potential suggests intriguing relations with a higher-dimensional setup, which might offer new perspectives for analyzing this theory. Moreover, we show that, for certain parameter choices, the potential for massive gravity coincides with the action describing nonperturbative brane objects, independently supporting connections with a higher-dimensional framework.

Armed with these tools, we then focus on the Vainshtein mechanism for this potential. We show that this theory is able to reproduce the behavior of linearized solutions in general relativity below the Vainshtein radius, but only in a specific region of parameter space. This result provides stringent constraints on nonlinear massive gravity. Moreover, we are able to physically reinterpret these findings in the decoupling limit in terms of an effective theory with Galileon interactions. We show that the condition to successfully implement the Vainshtein mechanism is associated with the sign of a direct coupling between the massless graviton and the scalar degrees of freedom.

We also present new exact solutions in the vacuum that asymptote de Sitter or anti-de Sitter space depending on the choice of the parameters. Asymptotically de Sitter configurations can be expressed in an explicit time-dependent form. These solutions may provide an interesting background for the observed Universe where the rate of the accelerated expansion of the Universe is set by the graviton mass. A small graviton mass, as required by the solar system constraints on deviations from standard general relativity, is then in agreement with the observed value of the cosmological constant. On the other hand, asymptotically anti-de Sitter configurations may have interesting applications to the AdS/CFT correspondence.

The paper is organized as follows: in Sec. II, we discuss how to construct a nonlinear potential for massive gravity and point out new connections with a higher-dimensional setup. In Sec. III, we show how linearized Einstein's gravity is recovered within a certain macroscopic radius from a mass source, via the Vainshtein mechanism. Only a subset of parameter space presents a successful Vainshtein effect in the weak field limit. For a better understanding of the theory and, in particular, of the ghost mode, it is important to find analytic nonlinear solutions; we present new exact solutions in Sec. IV. Finally, we conclude in Sec. V, leaving technical details of the calculations developed in the main text to the Appendixes.

## II. COVARIANT NONLINEAR MASSIVE GRAVITY

We start by introducing the covariant Fierz-Pauli mass term in four-dimensional space-time

$$\mathcal{L}_{\text{FP}} = m^2 \sqrt{-g} \mathcal{U}^{(2)}, \quad \mathcal{U}^{(2)} = (H_{\mu\nu} H^{\mu\nu} - H^2), \quad (2.1)$$

where the tensor  $H_{\mu\nu}$  is a covariantization of the metric perturbations, namely

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \equiv H_{\mu\nu} + \partial_\mu \phi^\alpha \partial_\nu \phi^\beta \eta_{\alpha\beta}. \quad (2.2)$$

The Stückelberg fields  $\phi^\alpha = (x^\alpha - \pi^\alpha)$  are introduced to restore reparametrization invariance, hence transforming as scalars [3]. The internal metric  $\eta_{\alpha\beta}$  corresponds to a nondynamical reference metric, usually assumed to be Minkowski space-time. Therefore, around flat space, we can rewrite  $H_{\mu\nu}$  as

$$\begin{aligned} H_{\mu\nu} &= h_{\mu\nu} + \eta_{\beta\nu} \partial_\mu \pi^\beta + \eta_{\alpha\mu} \partial_\nu \pi^\alpha - \eta_{\alpha\beta} \partial_\mu \pi^\alpha \partial_\nu \pi^\beta, \\ &\equiv h_{\mu\nu} - \mathcal{Q}_{\mu\nu}. \end{aligned} \quad (2.3)$$

From now on, indices are raised/lowered with the dynamical metric  $g_{\mu\nu}$ , unless otherwise stated. For example,  $H^\mu{}_\nu = g^{\mu\rho} H_{\rho\nu}$ . Moreover, the Lagrangian (2.1) is invariant under coordinate transformations  $x^\mu \rightarrow x^\mu + \xi^\mu$ , provided  $\pi^\mu$  transforms as

$$\pi^\mu \rightarrow \pi^\mu + \xi^\mu. \quad (2.4)$$

The scalar component  $\pi$  of the Stückelberg field can be extracted from the relation  $\pi^\mu = \eta^{\mu\nu} \partial_\nu \pi / \Lambda_3$ , with  $\Lambda_3^3 = m^2 M_{pl}$  (the meaning of the scale  $\Lambda_3$  will be explained in the following). The dynamics of  $\pi$  are the origin of the two problems discussed in the introduction: the DB ghost excitation and the vDVZ discontinuity. With respect to the first problem, as noticed by Fierz and Pauli, one can remove the ghost excitation, to linear order in perturbations, by choosing the quadratic structure  $h_{\mu\nu} h^{\mu\nu} - h^2$ . When expressed in the Stückelberg field language, by means of the scalar-graviton  $\pi$ , higher derivative terms in the action are arranged in a such way to form a total derivative, leading to second-order equations of motion. However, when going beyond linear order, the equation of  $\pi$  acquires higher time derivatives, signaling the presence of a ghost mode [3]. Remarkably, de Rham and Gabadadze were able to construct a potential, tuned at each order in perturbations, to give a total derivative for the dangerous terms, leading to equations of motion that are at most second order in time derivatives [13].

We now review their construction, introducing alternative ways to express the potential, which provide a new connection with a five-dimensional point of view. In terms of the helicity-zero mode, corresponding to the field  $\pi$ , we can write the tensor  $H_{\mu\nu}$  of Eq. (2.3) as

$$H_{\mu\nu} = h_{\mu\nu} + \frac{2}{M_{pl} m^2} \Pi_{\mu\nu} - \frac{1}{M_{pl}^2 m^4} \Pi_{\mu\nu}^2, \quad (2.5)$$

where  $\Pi_{\mu\nu} = \partial_\mu \partial_\nu \pi$  and  $\Pi_{\mu\nu}^2 = \Pi_{\mu\alpha} \Pi_\nu^\alpha$ . At a given order  $n$  in perturbations, the idea is to add terms of the form

$$m^2 \sqrt{-g} \mathcal{U}^{(n)} = m^2 \sqrt{-g} \sum_{i=0}^n c_i^n (H_{\mu\nu}^{n-i})(H^i), \quad (2.6)$$

to the FP action (2.1), and to choose the coefficients  $c_i^n$  in order to get a total derivative for the leading contributions of the scalar mode, namely  $(\Pi_{\mu\nu})^n$ . The key finding of [13] is that these total derivatives are unique at each order, and that the series stops at quintic order in perturbations. Let us describe in more detail the structure and origin of these terms. Following the notation of [13], the total derivatives are given by

$$\mathcal{L}_{\text{der}}^{(n)} = - \sum_{m=1}^n (-1)^m \frac{(n-1)!}{(n-m)!} (\text{tr} \Pi_{\mu\nu}^m) \mathcal{L}_{\text{der}}^{(n-m)}, \quad (2.7)$$

with  $\mathcal{L}_{\text{der}}^{(0)} = 1$  and

$$\begin{aligned} \mathcal{L}_{\text{der}}^{(1)} &= \text{tr} \Pi_{\mu\nu}, & \mathcal{L}_{\text{der}}^{(2)} &= (\text{tr} \Pi_{\mu\nu})^2 - \text{tr} \Pi_{\mu\nu}^2, \\ \mathcal{L}_{\text{der}}^{(3)} &= (\text{tr} \Pi_{\mu\nu})^3 - 3(\text{tr} \Pi_{\mu\nu})(\text{tr} \Pi_{\mu\nu}^2) + 2\text{tr} \Pi_{\mu\nu}^3, \\ \mathcal{L}_{\text{der}}^{(4)} &= (\text{tr} \Pi_{\mu\nu})^4 - 6(\text{tr} \Pi_{\mu\nu})^2(\text{tr} \Pi_{\mu\nu}^2) + 8(\text{tr} \Pi_{\mu\nu})(\text{tr} \Pi_{\mu\nu}^3) \\ &\quad + 3(\text{tr} \Pi_{\mu\nu}^2)^2 - 6\text{tr} \Pi_{\mu\nu}^4. \end{aligned}$$

and  $\mathcal{L}_{\text{der}}^{(n)}$  vanishes for  $n > 4$ . These expressions are related to a matrix determinant (see also Note Added at the end of this paper). To see this, consider a generic squared real matrix  $A$ , and a complex number  $z$ . Then, the following formula holds

$$\det(\mathbb{1} + zA) = 1 + \sum_{n=1}^{\infty} z^n \det_n(A) \quad (2.8)$$

where  $\det_n(A)$  can be written in terms of traces as

$$\begin{aligned} \det_1(A) &= \text{tr} A, \\ \det_2(A) &= \frac{1}{2} ((\text{tr} A)^2 - \text{tr} A^2), \\ \det_3(A) &= \frac{1}{6} ((\text{tr} A)^3 - 3(\text{tr} A)(\text{tr} A^2) + 2\text{tr} A^3), \\ \det_4(A) &= \frac{1}{24} ((\text{tr} A)^4 - 6(\text{tr} A)^2(\text{tr} A^2) \\ &\quad + 8(\text{tr} A)(\text{tr} A^3) + 3(\text{tr} A^2)^2 - 6\text{tr} A^4). \end{aligned} \quad (2.9)$$

Moreover, all terms  $\det_n(A)$  with  $n > 4$  vanish for a  $4 \times 4$  matrix. Therefore, for the choice  $A_{\mu}^{\nu} = \Pi_{\mu}^{\nu}$  we get the simple relation  $\mathcal{L}_{\text{der}}^{(n)} = n! \det_n(\Pi)$ , and the series indeed stops at  $n = 4$ . If one chooses a sum of determinants of the form

$$\sum_{i=1}^4 \det(\mathbb{1} + z_i \Pi) - 4, \quad (2.10)$$

one can generate each  $\det_n(\Pi)$  term with a separate coefficient  $\beta_n$ , provided a solution to  $\sum_{i=1}^4 z_i^n = \beta_n$  exists, which is guaranteed by the Newton identities. Then, the Lagrangian for the helicity-zero mode (that is, neglecting

for the moment the contributions of tensor modes, and of the vector components of the Stückelberg field) is

$$\mathcal{L}_{\pi} = \sum_{n=1}^4 \beta_n \det_n(\Pi). \quad (2.11)$$

We now briefly turn away from the present discussion, and show an interesting way to construct the Lagrangian (2.11) from a higher-dimensional point of view. Consider a five dimensional Minkowski space-time, and embed on it a test 3-brane (i.e. we do not include back-reaction from brane dynamics). Under this assumption, the five-dimensional Riemann tensor vanishes; using the Gauss equation, the intrinsic curvature on the brane is related to the extrinsic curvature as ( $R^{\alpha}_{\beta\gamma\delta}$  is constructed in terms of brane induced metric)

$$\begin{aligned} R^{\alpha}_{\beta\gamma\delta} &= K_{\gamma}^{\alpha} K_{\beta\delta} - K_{\delta}^{\alpha} K_{\beta\gamma}, \\ R_{\mu\nu} &= K K_{\mu\nu} - K_{\mu}^{\alpha} K_{\nu\alpha}, \\ R &= K^2 - K_{\mu\nu} K^{\mu\nu}. \end{aligned} \quad (2.12)$$

We then consider a four-dimensional Lagrangian given by

$$\mathcal{L}_{\text{brane}} = \sqrt{-g} [\alpha_1 K + \alpha_2 R + \alpha_3 \mathcal{K}_{\text{GB}} + \alpha_4 R_{\text{GB}}], \quad (2.13)$$

where  $R$  is the Ricci scalar,  $R_{\text{GB}}$  is the Gauss-Bonnet term,  $R_{\text{GB}} = R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\alpha\beta}^2$ ,  $K$  is trace of the extrinsic curvature, and the  $\mathcal{K}_{\text{GB}}$  is the boundary term associated with the five-dimensional Gauss-Bonnet term:  $\mathcal{K}_{\text{GB}} = K^3 - 3K K_{\mu\nu}^2 + 2K_{\mu\nu}^3$ . Using the expression for the intrinsic curvature in terms of the extrinsic curvature, Eq. (2.12), the Lagrangian (2.13) can be written as

$$\mathcal{L}_{\text{brane}} = - \sum_{n=1}^4 \beta_n \det_n(K), \quad (2.14)$$

where  $\beta_n = -n! \alpha_n$ , thus it has *exactly* the same structure as Eq. (2.11). We then denote the position modulus of the probe 3-brane as  $\pi$ . The induced metric on the brane is determined by  $\pi$  as

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_{\mu} \pi \partial_{\nu} \pi, \quad (2.15)$$

and the extrinsic curvature is given by

$$K_{\mu\nu} = \gamma \partial_{\mu} \partial_{\nu} \pi, \quad \gamma = \frac{1}{\sqrt{1 + (\partial \pi)^2}}. \quad (2.16)$$

If we take a limit  $\partial \pi \ll 1$ , i.e.  $\gamma \rightarrow 1$ , we find that the extrinsic curvature is simply  $K_{\mu\nu} = \Pi_{\mu\nu}$ . Then the Lagrangian (2.14) reduces to (2.11). This suggests that there may be a higher-dimensional interpretation behind the Lagrangian (2.11). Although we find these arguments very compelling, so far we have not been able to pursue these connections further, and for this reason we will not develop them in this work. But we should note that what we discussed follows the same construction as in the

so-called DBI Galileon [14]. If we do not take the limit  $\gamma \rightarrow 1$ , the Lagrangian (2.14) becomes nontrivial, and it reproduces the Galileon terms. Since the four-dimensional Gauss-Bonnet piece is a total derivative, there is no contribution from this term even away from the  $\gamma \rightarrow 1$  limit.

After this digression to five dimensions, we would like to return to Lagrangian (2.11), and discuss how to render it fully covariant. To do so, we need to understand how to go back from the field  $\pi$  to the original Stückelberg fields  $\phi^\mu$ , in order to restore the dependence of the vector mode and the full metric. Here we follow the approach discussed in Ref [15]. If only the scalar mode  $\pi$  is considered, then we can solve for  $\Pi_{\mu\nu}$  in terms of  $\phi^\mu$ , using (2.2) and (2.5). The result is a second-order algebraic equation for  $\Pi_{\mu\nu}$ , with solution  $\Pi_{\mu}{}^\nu = \Lambda_3[\delta_{\mu}{}^\nu - (\sqrt{(\partial\phi)\eta(\partial\phi)^T})_{\mu}{}^\nu]$ , where  $[(\partial\phi)\eta(\partial\phi)^T]_{\mu}{}^\nu \equiv \partial_{\mu}\phi^{\alpha}\partial^{\nu}\phi^{\beta}\eta_{\alpha\beta}$ , and just to remind the reader  $\Lambda_3^3 = m^2 M_{pl}$ . The previous expression is written in terms of the square root of a matrix, formally understood as  $\sqrt{M_{\mu}^{\alpha}}\sqrt{M_{\alpha}^{\nu}} = M_{\mu}{}^{\nu}$ .

We now go beyond the pure scalar sector case, and define

$$\begin{aligned}\mathcal{K}_{\mu}{}^{\nu} &\equiv \delta_{\mu}{}^{\nu} - \left(\sqrt{(\partial\phi)\eta(\partial\phi)^T}\right)_{\mu}{}^{\nu} \\ &= \delta_{\mu}{}^{\nu} - \left(\sqrt{\mathbb{1} - g^{-1}H}\right)_{\mu}{}^{\nu} \\ &= \delta_{\mu}{}^{\nu} - \left(\sqrt{g^{-1}[\eta + \mathcal{Q}]}\right)_{\mu}{}^{\nu}.\end{aligned}\quad (2.17)$$

Notice that the previous quantity contains also contributions from vector and tensor degrees of freedom. On the other hand, by construction  $\mathcal{K}_{\mu}{}^{\nu}$  becomes  $\Pi_{\mu}{}^{\nu}/\Lambda_3$  when only the scalar mode is considered. The full nonlinear Lagrangian for massive gravity is then constructed by substituting  $\Pi$  by  $\mathcal{K}$  in (2.11). Namely

$$\begin{aligned}\mathcal{L}_{\mathcal{K}} &= -[\alpha_1 \det_1(\mathcal{K}) + 2\alpha_2 \det_2(\mathcal{K}) + 6\alpha_3 \det_3(\mathcal{K}) \\ &\quad + 24\alpha_4 \det_4(\mathcal{K})]\end{aligned}\quad (2.18)$$

where  $\alpha_n = -n!\beta_n$ , and the determinants  $\det_n(\mathcal{K})$  are defined in Eqs. (2.9) with  $A = \mathcal{K}$ . The second term, for positive  $\alpha_2$ , reduces to the Fierz-Pauli term (2.1) when expanding the Lagrangian in terms of  $H_{\mu\nu}$  around the Minkowski metric. Therefore, since we would like to have the Fierz-Pauli as the first correction to Einstein's gravity at leading order, we are not interested on the contributions from the first term,  $\det_1 \mathcal{K}$ . Then, from now on we set  $\alpha_1 = 0$ . As a result, a family of nonlinear massive gravity Lagrangians can be written as

$$\mathcal{L} = \frac{M_{pl}^2}{2} \sqrt{-g}(R - 2\Lambda - m^2 \mathcal{U}),\quad (2.19)$$

where  $\mathcal{U} = \mathcal{L}_{\mathcal{K}}$  with  $\alpha_1 = 0$  and  $\alpha_2 = 1$ . These Lagrangians are parametrized by  $m$ ,  $\alpha_3$  and  $\alpha_4$ ; moreover we added a bare cosmological constant  $\Lambda$ .

In this family of Lagrangians, there is a special choice of parameters:  $\alpha_3 = -1/3$  and  $\alpha_4 = 1/12$ . It corresponds to the choice  $z = -1$  in the expansion (2.8) with  $A = \mathcal{K}$ . The Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{NG}} &= 2m^2 \sqrt{-g}(\det(\mathbb{1} - \mathcal{K}) - \text{tr}\mathcal{K}) \\ &= 2m^2(\sqrt{-\det(\partial_{\mu}\phi^{\alpha}\partial_{\nu}\phi^{\beta}\eta_{\alpha\beta})} - \text{tr}\mathcal{K}),\end{aligned}\quad (2.20)$$

where the first term is the Nambu-Goto type of action for a bosonic 3-brane [16]. However, we will discuss in what follows that this particular choice, even though has a striking physical interpretation, does not allow to recover the GR solutions via the Vainshtein mechanism.

The  $\alpha_2$  term in the potential (2.18) was first suggested in [15] as nonlinear completion of FP theory. It was shown that this term reduces to a particular choice of Galileon terms in the decoupling limit, without any coupling between the scalar mode and the massless graviton. Formally, the decoupling limit corresponds to a limit in which the scale  $\Lambda_3 = m^2 M_{pl}$  is kept fixed, while sending  $M_{pl} \rightarrow \infty$  and  $m \rightarrow 0$ . Once  $\alpha_3$  and  $\alpha_4$  are included, the picture changes in an interesting way, and couplings between  $\pi$  and the massless graviton appear, even in the decoupling limit. These couplings have important theoretical and observational consequences; as we will discuss in detail in the next section. Interestingly, these mixing terms are finite in number, and do not spoil the fundamental property that the equations of motion for  $\pi$  are second order [13]. It is still an open question whether this remains true away from the decoupling limit, ensuring the absence of ghost degrees of freedom. A full Hamiltonian analysis should be carried out to set a final word on the subject. However, there are hints that the theory is ghost-free perturbatively, and for the particular choice of  $\alpha_3 = \alpha_4 = 0$  this has been shown up to and including quartic order in perturbations [15].

We have now the necessary ingredients to discuss the second problem addressed in the Introduction, namely, the vDVZ discontinuity.

### III. VAINSHTEIN MECHANISM AT WORK

In [17], we showed that Vainshtein mechanism applies for Lagrangians as (2.19), setting to zero the coefficients  $\alpha_3$  and  $\alpha_4$ , and the bare cosmological constant. (See Ref. [18] for spherical symmetric solutions in the FP theory). Here we extend the analysis to arbitrary coefficients. We determine stringent constraints on the parameter space of these theories, in order to ensure that the Vainshtein mechanism works. As we are going to discuss, our results find a natural interpretation in terms of the dynamics of helicity-zero mode, in the decoupling limit.

In order to discuss solutions associated with Lagrangian (2.19), it is convenient to express  $\mathcal{K}$ , given in Eq. (2.17), in terms of matrices as

$$\mathcal{K} = \mathbb{1} - \sqrt{\mathcal{M}},\quad (3.1)$$

where  $\mathbb{1}$  denotes the identity matrix and  $\mathcal{M} = g^{-1}[\eta + \mathcal{Q}]$ . The task is to calculate the trace of  $\mathcal{M}^n$ . Given that  $\mathcal{M}$  is a square matrix, it is diagonalizable, and can be expressed as  $\mathcal{M} = \mathcal{U}D\mathcal{U}^{-1}$ , for some invertible matrix  $\mathcal{U}$ , where  $D$  is a diagonal matrix containing the eigenvalues of  $\mathcal{M}$ . We shall call the eigenvalues  $\lambda_1, \dots, \lambda_4$ . Then, since  $\mathcal{M}^n = \mathcal{U}D^n\mathcal{U}^{-1}$ , the traces in the formulas above can be expressed in terms of eigenvalues

$$\text{tr } \mathcal{M}^n = \sum_i \lambda_i^n, \quad (3.2)$$

and the traces of  $\mathcal{K}^n$  result

$$\begin{aligned} \text{tr } \mathcal{K} &= 4 - \text{tr} \sqrt{\mathcal{M}}, \\ \text{tr } \mathcal{K}^2 &= 4 - 2\text{tr} \sqrt{\mathcal{M}} + \text{tr} \mathcal{M}, \\ \text{tr } \mathcal{K}^3 &= 4 - 3\text{tr} \sqrt{\mathcal{M}} + 3\text{tr} \mathcal{M} - \text{tr} \mathcal{M}^{3/2}, \\ \text{tr } \mathcal{K}^4 &= 4 - 4\text{tr} \sqrt{\mathcal{M}} + 6\text{tr} \mathcal{M} - 4\text{tr} \mathcal{M}^{3/2} + \text{tr} \mathcal{M}^2. \end{aligned} \quad (3.3)$$

Using the formulas for expressing the determinants in terms of traces (2.9), we can easily construct the potential.

We now discuss the conditions to recover GR results in the small graviton mass limit, within a certain radius from a mass source. In particular, we are interested to determine under which circumstances the Vainshtein mechanism applies. In order to do so, we study spherically symmetric perturbations around flat space, expressed in spherical coordinates as  $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$ , with  $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ . We start our discussion using the unitary gauge,  $\pi^\mu = x^\mu - \phi^\mu = 0$ . Consider the following Ansatz for the metric

$$ds^2 = -N(r)^2 dt^2 + F(r)^{-1} dr^2 + r^2 H(r)^{-2} d\Omega^2, \quad (3.4)$$

that reduces the potential in (2.19) to

$$\begin{aligned} \sqrt{-g}\mathcal{U} &= -\frac{r^2 \sin(\theta)}{\sqrt{F}H^2} \{2[\sqrt{F}((2H-3)N+1) \\ &+ H^2N + H(2-6N) + 6N-3] - 6\alpha_3(H-1) \\ &\times [\sqrt{F}((H-3)N+2) - 2HN + H + 4N-3] \\ &- 24\alpha_4(1-\sqrt{F})(1-H)^2(1-N)\}. \end{aligned} \quad (3.5)$$

Notice that in GR one can set  $H(r) = 1$  by a coordinate transformation, but this is not possible here, since we have already chosen a gauge. The field equations are obtained by varying the action (2.19) with respect to  $N$ ,  $H$  and  $F$ . The  $N$ -equation is the Hamiltonian constraint, so it only depends on  $F$  and  $H$ . Since the equation for  $H$  is quite complicated, we instead consider a combination of the three equations which gives  $\nabla^\mu G_{\mu\nu} = 0$ , where  $G_{\mu\nu}$  is the Einstein tensor; it corresponds to the Bianchi identity. Therefore, we work with the Hamiltonian constraint, the Bianchi identity and the equation for  $F$ . The corresponding expressions are lengthy, so we relegate them to Appendix A.

First, let us study solutions in the weak field limit, by expanding  $N$ ,  $F$  and  $H$  as

$$N = 1 + n, \quad F = 1 + f, \quad H = 1 + h, \quad (3.6)$$

and truncating the field equations to first order in these perturbations. As we will see in what follows, this linearization procedure is not completely consistent for all values of the radial coordinate  $r$ , and we will need to improve it. In order to analyze the system, it is convenient to introduce a new radial coordinate

$$\rho = \frac{r}{H(r)}, \quad (3.7)$$

so that the linearized metric is expressed as

$$ds^2 = -(1+2n)dt^2 + (1-\tilde{f})d\rho^2 + \rho^2 d\Omega^2, \quad (3.8)$$

where  $\tilde{f} = f - 2h - 2\rho h'$  and a prime denotes a derivative with respect to  $\rho$ . As discussed above, one should be careful with this change of coordinates, since, after fixing a gauge, a change of frame in the metric modifies the Stückelberg field  $\pi^\mu$  as well. However, for the moment, let us focus on the change of the metric part; later we will discuss what happens to  $\pi^\mu$ . At linear order, the equations for the functions  $n(\rho)$ ,  $\tilde{f}(\rho)$  and  $h(\rho)$  in the new variable  $\rho$  are

$$0 = (m^2\rho^2 + 2)\tilde{f} + 2\rho(\tilde{f}' + m^2\rho^2 h' + 3m^2\rho h), \quad (3.9)$$

$$0 = m^2\rho^2(n - 2h) - 2\rho n' - \tilde{f}, \quad (3.10)$$

$$0 = \tilde{f} + \rho n'. \quad (3.11)$$

In this linear expansion, the solutions for  $n$  and  $\tilde{f}$  are

$$2n = -\frac{8GM}{3\rho} e^{-m\rho}, \quad \tilde{f} = -\frac{4GM}{3\rho} (1+m\rho) e^{-m\rho}, \quad (3.12)$$

where we fix the integration constant so that  $M$  is the mass of a point particle at the origin, and  $8\pi G = M_{pl}^{-2}$ . These solutions exhibit the vDVZ discontinuity, since the post-Newtonian parameter  $\gamma = f/2n$  is  $\gamma = 1/2(1+m\rho)$ , which in the massless limit reduces to  $\gamma = 1/2$ , in disagreement with GR, and with Solar System observations ( $\gamma = 1$  in GR, while observations provide  $1 - \gamma \approx 10^{-5}$  [19]).

However, in order to understand what really happens in this limit, we must also analyze the behavior of  $h$  as  $m \rightarrow 0$ . For doing this, we consider scales below the Compton wavelength  $m\rho \ll 1$ , and at the same time ignore higher-order terms in  $GM$ . Under these approximations, the equations of motion can still be truncated to linear order in  $\tilde{f}$  and  $n$ , but since  $h$  is not necessarily small, we have to keep all nonlinear terms in  $h$ . The resulting equations are then (see Appendix A for their derivation)

$$0 = 2\tilde{f} + 2\rho\tilde{f}' + m^2\rho^2[(1 - 2(3\alpha_3 + 1)h + 3(\alpha_3 + 4\alpha_4)h^2][(2 + \tilde{f})\rho h' + (1 + h)\tilde{f}] + 6h[1 - (3\alpha_3 + 1)h + (\alpha_3 + 4\alpha_4)h^2]], \quad (3.13)$$

$$0 = -\tilde{f} - 2\rho n' + m^2\rho^2(n - 2[1 + n + (3\alpha_3 + 1)n]h + [(3\alpha_3 + 1)(n + 1) + 3(\alpha_3 + 4\alpha_4)n]h^2), \quad (3.14)$$

$$0 = \rho n'[-1 + 2(3\alpha_3 + 1)h - 3(\alpha_3 + 4\alpha_4)h^2] - \tilde{f}[1 - (3\alpha_3 + 1)h]. \quad (3.15)$$

We start with the  $N$ -Eq. (3.13). Since it only depends on  $\tilde{f}$  and  $h$ , one can solve for  $\tilde{f}$  in terms of  $h$ , including all nonlinear terms in  $h$ . The solution, dropping higher-order terms proportional to  $m^4$ ,  $(GM)^2$  and  $m^2GM$ , is

$$\tilde{f} = -2\frac{GM}{\rho} - (m\rho)^2[h - (1 + 3\alpha_3)h^2 + (\alpha_3 + 4\alpha_4)h^3]. \quad (3.16)$$

Then we take the second Eq. (3.14), obtained by varying the action with respect to  $F$ , and use the solution (3.16) for  $\tilde{f}$ . We find an expression for  $n$  as a nonlinear function of  $h$ . It turns out to be simpler to work with  $n'$ , given by

$$2\rho n' = \frac{2GM}{\rho} - (m\rho)^2[h - (\alpha_3 + 4\alpha_4)h^3], \quad (3.17)$$

where again we have dropped terms with  $m^4$ ,  $(GM)^2$  and  $m^2GM$ . Finally, the constraint Eq. (3.15) gives an equation for  $h$ , after substituting the solutions for  $n$  and  $\tilde{f}$  given, respectively, by (3.17) and (3.16). We should stress that this equation for  $h$  is *exact*, so there are no higher-order corrections. It is given by

$$\begin{aligned} & \frac{GM}{\rho}[1 - 3(\alpha_3 + 4\alpha_4)h^2] \\ &= -(m\rho)^2\left\{\frac{3}{2}h - 3(1 + 3\alpha_3)h^2 + [(1 + 3\alpha_3)^2 + 2(\alpha_3 + 4\alpha_4)]h^3 - \frac{3}{2}(\alpha_3 + 4\alpha_4)^2h^5\right\}. \end{aligned} \quad (3.18)$$

If we linearize the Eqs. (3.16), (3.17), and (3.18) with respect to  $h$ , we recover the solution (3.12); on the other hand, below the so-called Vainshtein radius

$$\rho_V = \left(\frac{GM}{m^2}\right)^{1/3}, \quad (3.19)$$

$h$  becomes larger than 1. Therefore, in this regime, we have to include higher-order contributions due to  $h$  to Eqs. (3.16) and (3.17). There are three qualitatively different cases, depending on the values of the parameters  $\alpha_3$  and  $\alpha_4$ :

- (i) *Case with  $\alpha_3 = -1/3$ ,  $\alpha_4 = 1/12$ .* This is a special situation, since all higher-order contributions in  $h$  vanish from Eqs. (3.16) and (3.17). Therefore, there

is no Vainshtein effect and the solutions to the equations are those given in (3.12) for  $\rho < 1/m$ . This model, corresponding to the bosonic 3-brane Lagrangian (2.20), is ruled out by Solar System observations.

- (ii) *Case with  $\alpha_3 = -4\alpha_4 \neq -1/3$ .* For  $\rho \ll \rho_V$  we can solve for  $h$  from the last equation (3.18) keeping only the highest order terms in  $h$ . Then, the solution for  $h$  is

$$h = -\frac{1}{(1 - 12\alpha_4)^{2/3}}\frac{\rho_V}{\rho}, \quad (3.20)$$

which implies  $|h| \gg 1$  for  $\rho \ll \rho_V$ , as expected. We can then use this solution and Eqs. (3.16) and (3.17) to give the expressions for  $n$  and  $\tilde{f}$  within the Vainshtein radius, namely

$$\begin{aligned} 2n &= -\frac{2GM}{\rho}\left[1 - \frac{1}{2(1 - 12\alpha_4)^{2/3}}\left(\frac{\rho}{\rho_V}\right)^2\right], \\ \tilde{f} &= -\frac{2GM}{\rho}\left[1 - \frac{1}{2(1 - 12\alpha_4)^{1/3}}\left(\frac{\rho}{\rho_V}\right)\right]. \end{aligned} \quad (3.21)$$

Therefore, the corrections to GR solutions are indeed small for  $\rho$  smaller than the Vainshtein radius  $\rho_V$ .

- (iii) *Case with  $\alpha_3 \neq -4\alpha_4$ .* This case can be divided in two:  $\alpha_3 + 4\alpha_4 > 0$  and  $\alpha_3 + 4\alpha_4 < 0$ . The latter is the most intriguing case, and we focus on it. In the limit  $\rho \ll \rho_V$ , the solution for  $h$  is given by

$$\begin{aligned} h &= -\left(\frac{2}{\alpha_3 + 4\alpha_4}\right)^{1/3}\frac{\rho_V}{\rho} \\ &\quad - \frac{[2(1 + 3\alpha_3)^2 + 3(\alpha_3 + 4\alpha_4)]}{9[2(\alpha_3 + 4\alpha_4)^5]^{1/3}}\frac{\rho}{\rho_V}, \end{aligned} \quad (3.22)$$

so that  $|h| \gg 1$ . Notice that, in solving Eq. (3.18), we include also the next-to-leading order, that results to be linear in  $\rho$ . It turns out that this expression for  $h$  provides a correction of the same order as  $GM/\rho$  in the  $n$  and  $\tilde{f}$  equations. Indeed, plugging this expression in the equations for  $n$  and  $\tilde{f}$ , one gets

$$2n = \mathcal{O}\left(\left(\frac{\rho}{\rho_V}\right)^2\right)\frac{GM}{\rho}, \quad \tilde{f} = \mathcal{O}\left(\frac{\rho}{\rho_V}\right)\frac{GM}{\rho}. \quad (3.23)$$

Surprisingly, the contribution from the scalar mode  $h$  exactly cancels the usual  $1/\rho$  potential at leading order. So, gravity becomes *weaker* approaching the source, for distances smaller than Vainshtein radius, and larger than Schwarzschild radius. This implies that the strong coupling of the scalar graviton not only shields interactions of the scalar mode  $h$ , but also those of the massless graviton. As we will see later while analyzing the decoupling limit, this is related to a coupling between the scalar mode and

graviton that cannot be removed by a local field transformation if  $\alpha_3 \neq -4\alpha_4$  [13]. In this case, local tests of gravity would completely rule out the predictions of the theory at leading order in  $h_{\mu\nu}$ . However, when  $\alpha_3 + 4\alpha_4 > 0$ , apart from the previous solution, a second solution exists in the limit  $\rho \ll \rho_V$  [20], where  $h = 1/\sqrt{3(\alpha_3 + 4\alpha_4)} + \mathcal{O}(\rho^3)$ . In this case, the metric fields  $n$  and  $f$  present small corrections to the GR solution within the Vainshtein radius, and the solution cannot be ruled out by Solar System tests.

In order to fully understand the system with  $\alpha_3 \neq -4\alpha_4$ , it is imperative to study how the two possible solutions extend beyond the Vainshtein radius, both in vacuum and in the presence of matter. This is beyond the scope of the present analysis and will be studied in a future publication [21].

To summarize, only the choices  $\alpha_3 + 4\alpha_4 > 0$  and  $\alpha_3 = -4\alpha_4 \neq -1/3$  allow to recover standard GR in the weak field limit, below the Vainshtein radius  $\rho_V$ . This fact imposes a stringent constraint in parameter space for the theory described by Lagrangian (2.19).

Now, we would like to go back to the issue of the coordinate transformation introduced in Eq. (3.7), and discuss another way to interpret the previous results. As mentioned earlier, a coordinate transformation introduces a change in  $\pi$  of the form (2.4). When changing  $r/H = \rho$ , we excite the radial component of the Stückelberg field, as  $\pi^\rho = -\rho h$ . Thus the strong coupling nature of  $h$  is encoded in  $\pi^\rho = \eta^{\rho\rho} \partial_\rho \pi / \Lambda_3^3$ , when working with the coordinate  $\rho$ . As a result, the nonlinear analysis previously done is more transparent in the decoupling limit, as developed in [13]. As previously mentioned, this limit is achieved by taking  $m \rightarrow 0$  and  $M_{pl} \rightarrow \infty$ , while keeping  $\Lambda_3 = (m^2 M_{pl})^{1/3}$  fixed. This implies that, when substituting  $H_{\mu\nu}$  back into the full Lagrangian (2.19), we do not consider the vector mode, and expand the potential at linear order in  $h_{\mu\nu}$ . The resulting expression describes the theory in the decoupling limit, and contains the quadratic Hilbert-Einstein piece, total derivatives in  $\pi$  given by (2.11), and finally mixing terms between  $h_{\mu\nu}$  and  $\pi$ . The mixing terms are

$$h^{\mu\nu} \sum_i X_{\mu\nu}^{(i)}, \quad \text{with} \quad \sum_i X_{\mu\nu}^{(i)} = \left. \frac{\partial \mathcal{L}_{\mathcal{K}}}{\partial h^{\mu\nu}} \right|_{h_{\mu\nu}=0}, \quad (3.24)$$

and each  $X^{(i)}$  is of order  $\mathcal{O}(\Pi^i)$ . There are only three mixing contributions, which can be absorbed (except for  $X^{(3)}$ ) in the remaining terms by the nonlinear field redefinition

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{\pi}{M_{pl}} \eta_{\mu\nu} - \frac{1 + 3\alpha_3}{\Lambda_3^3 M_{pl}} \partial_\mu \pi \partial_\nu \pi. \quad (3.25)$$

After this field redefinition, the resulting Lagrangian is  $([\Pi^n] \equiv \text{tr}\Pi^n)$  [13]

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_{\text{GR}}(\hat{h}_{\mu\nu}) + \frac{3}{2} \pi \square \pi - \frac{3(1 + 3\alpha_3)}{2\Lambda_3^3} \\ & \times (\partial\pi)^2 \square \pi + \frac{(\partial\pi)^2}{2\Lambda_3^6} [(1 + 3\alpha_3)^2 + 2(\alpha_3 + 4\alpha_4)] \\ & \times ([\Pi^2] - [\Pi]^2) - \frac{5}{4\Lambda_3^9} (1 + 3\alpha_3)(\alpha_3 + 4\alpha_4) \\ & \times (\partial\pi)^2 ([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3]) + \frac{M_{pl}}{\Lambda_3^6} \hat{h}^{\mu\nu} X_{\mu\nu}^{(3)}, \end{aligned} \quad (3.26)$$

where  $\mathcal{L}_{\text{GR}}$  is the quadratic Einstein-Hilbert action for  $\hat{h}_{\mu\nu}$ , and

$$\begin{aligned} X_{\mu\nu}^{(3)} = & -\frac{1}{2} (\alpha_3 + 4\alpha_4) \{6\Pi_{\mu\nu}^3 - 6(\text{tr}\Pi)\Pi_{\mu\nu}^2 \\ & + 3[(\text{tr}\Pi)^2 - (\text{tr}\Pi^2)]\Pi_{\mu\nu} - [(\text{tr}\Pi)^3 - 3(\text{tr}\Pi^2) \\ & \times (\text{tr}\Pi) + 2(\text{tr}\Pi^3)]\eta_{\mu\nu}\}. \end{aligned} \quad (3.27)$$

Interestingly, the Lagrangian obtained so far in the decoupling limit allows us to reinterpret the constraints on  $\alpha_3$  and  $\alpha_4$  obtained earlier in this section. One can observe that the remaining direct coupling between the scalar  $\pi$  and graviton  $\hat{h}_{\mu\nu}$  either vanishes or is negative for the two cases which successfully implement Vainshtein mechanism.

Observe also that the kinetic terms for  $\pi$  in (3.26) are precisely the Galileon terms, which give rise to second-order differential equations for  $\pi$  [12]. The nonlinear structure of these terms involving  $\pi$  is essential to recover GR in some range of scales distances; for the choice  $\alpha_3 = -4\alpha_4 = -1/3$ , we obtain an Einstein frame Lagrangian for Brans-Dicke gravity with a vanishing Brans-Dicke parameter and no potential. However, this particular choice is not compatible with observations, as we have already mentioned during our general analysis, in the first part of this section.

We would like now to further analyze the equations of motion for this system in the decoupling limit. The Lagrangian (3.26) is exact in the decoupling limit, as there are no higher-order terms besides those shown, so the linearized Einstein equation for  $\hat{h}_{\mu\nu}$  is

$$\delta G_{\mu\nu}(\hat{h}) - \frac{1}{M_{pl}\Lambda_3^6} X_{\mu\nu}^{(3)} = 0. \quad (3.28)$$

If we assume spherical symmetry, then it can shown that  $X_{\mu\nu}^{(3)}$  is simply

$$X_{tt}^{(3)} = -\frac{\alpha_3 + 4\alpha_4}{\rho^2} (\pi'^3)', \quad X_{\rho\rho}^{(3)} = 0. \quad (3.29)$$

Therefore, the solutions for the linearized metric,  $\hat{h}_{tt} = -2\hat{n}$  and  $\hat{h}_{\rho\rho} = -\hat{f}$ , reduce to

$$\begin{aligned}\hat{f} &= -\frac{2GM}{\rho} + \frac{\rho^2}{\Lambda_3^6 M_{pl}} (\alpha_3 + 4\alpha_4) \left(\frac{\pi'}{\rho}\right)^3, \\ 2\rho\hat{n}' &= -\hat{f}.\end{aligned}\quad (3.30)$$

On the other hand, the equation of motion for  $\pi$  derived from the action (3.26), is given by (see [12])

$$\begin{aligned}3\left(\frac{\pi'}{\rho}\right) + \frac{6}{\Lambda_3^3}(1+3\alpha_3)\left(\frac{\pi'}{\rho}\right)^2 + \frac{2}{\Lambda_3^6}[(1+3\alpha_3)^2 + 2(\alpha_3 + 4\alpha_4)] \\ \times \left(\frac{\pi'}{\rho}\right)^3 + \frac{6M_{pl}(\alpha_3 + 4\alpha_4)}{\rho^2\Lambda_3^6}(\rho\hat{n}')\left(\frac{\pi'}{\rho}\right)^2 = \frac{M}{4\pi\rho^3 M_{pl}},\end{aligned}\quad (3.31)$$

where the integration constant is again chosen so that  $M$  is a mass of a particle at the origin. Using the relation between  $\pi$  and  $h$ ,  $h = -\pi'/(m^2 M_{pl}\rho)$ , it is simple to check that the solutions for  $\tilde{f}$ ,  $n$  and  $h$  given by Eqs. (3.16), (3.17), and (3.18) agree with the expressions in Eqs. (3.25), (3.30), and (3.31). This confirms that the results obtained earlier in this section are in perfect agreement with what is found in terms of the dynamics of the scalar field  $\pi$ , in the decoupling limit.

In summary, the Vainshtein mechanism applies for  $\alpha_3 + 4\alpha_4 > 0$  and  $\alpha_3 = -4\alpha_4 \neq -1/3$ . Only for these choices the weak field GR results are fully recovered at distances smaller than the Vainshtein radius. If these solutions match the asymptotic configurations (3.12) beyond the Vainshtein radius, then there would be three phases: on the largest scales beyond Compton wavelength,  $m^{-1} \ll \rho$ , the gravitational interactions decay exponentially due to the graviton mass, see Eq. (3.12). In the intermediate region  $\rho_V < \rho < m^{-1}$ , we obtain the  $1/r$  gravitational potential but Newton's constant is rescaled,  $G \rightarrow 4G/3$ . Moreover, the post-Newtonian parameter reduces to  $\gamma = 1/2$  in the  $m \rightarrow 0$  limit, instead of  $\gamma = 1$  of GR, showing the vDVZ discontinuity. Finally, below the Vainshtein radius  $\rho < \rho_V$ , the GR solution is recovered due to the strong coupling of the  $\pi$  mode (see, for example, Eq. (3.21) for the case  $\alpha_3 = -4\alpha_4 \neq -1/3$ ); in this regime the theory can then be rendered compatible with observations.

The solution discussed here provides a testing arena for studying the Boulware-Deser ghost. Instead of expanding the action in  $H_{\mu\nu}$  around Minkowski space-time up to higher orders in perturbations, we have the possibility to study *linear* perturbations around this nonperturbative solution, using the complete potential in Eq. (2.18). In order to obtain the full nonlinear solution, matching the three phases we have described, a numerical approach is necessary. In the next section, we consider a different family of vacuum solutions for this theory, which can be obtained analytically, and can lead to interesting candidates for realistic backgrounds.

## IV. EXACT SOLUTIONS

As we learned in the previous section, an essential property of this theory of massive gravity is the strong coupling phenomenon occurring in the proximity of a source. This allows, for certain regions of parameter space, to recover linearized general relativity at sufficiently small distances by means of the Vainshtein mechanism. This behavior, accompanied by the fact that Birkhoff theorem does not apply in this context, suggests that exact solutions for this theory, even imposing spherical symmetry, might be very different from the GR ones.

In this section, we will exhibit new spherically symmetric exact solutions in the vacuum for massive gravity, that generalize the ones of [17,22]. In an appropriate gauge, the solutions are asymptotically de Sitter or anti-de Sitter, depending on the choice of parameters.

While in [17] we focused on the case  $\alpha_3 = \alpha_4 = 0$ , we now generalize the analysis to arbitrary couplings in the Lagrangian (2.18). We adopt the unitary gauge and allow for arbitrary couplings  $\alpha_i$ ,  $i = 2, \dots, 4$  (as explained earlier, we set to zero the coefficient  $\alpha_1$ ). We start with the following form for the metric (for convenience, we implement slightly different conventions with respect to the previous section)

$$ds^2 = -C(r)dt^2 + 2D(r)dt dr + A(r)dr^2 + B(r)d\Omega^2 \quad (4.1)$$

so that, even though the space-time is spherically symmetric, the metric contains a cross term  $dt dr$ . We choose the following Ansatz for the metric functions [17,22],

$$\begin{aligned}B(r) &= b_0 r^2, \\ C(r) &= c_0 + \frac{c_1}{r} + c_2 r^2, \\ A(r) + C(r) &= Q_0, \\ D^2(r) + A(r)C(r) &= \Delta_0,\end{aligned}\quad (4.2)$$

and use the equations of motion to fix the constant parameters  $b_0, c_0, c_1, c_2, Q_0, \Delta_0$ . Einstein equations read

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (4.3)$$

with energy momentum tensor  $T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}_K}{\delta g^{\mu\nu}}$ , and  $\mathcal{L}_K$  given in Eq. (2.18).

In general relativity, diffeomorphism invariance allows one to choose the function  $B(r)$  to be  $B(r) = r^2$ , so that  $b_0 = 1$ . In this theory of massive gravity, after having fixed the gauge, this choice is no longer possible and the equations of motion determine  $b_0$ . In order to do this, one observes that the metric Ansatz (4.1) leads to the following identity between components of the Einstein tensor:  $C(r)G_{rr} + A(r)G_{tt} = 0$ . This combination on the energy momentum tensor provides the following value for  $b_0$ ,



$$b_0 = \left( \frac{1 + 6\alpha_3 + 12\alpha_4 \pm \sqrt{1 + 3\alpha_3 + 9\alpha_3^2 - 12\alpha_4}}{3(1 + 3\alpha_3 + 4\alpha_4)} \right)^2. \quad (4.4)$$

The upper branch generalizes the result of [17], while the lower branch is specifically associated with theories in which  $\alpha_3$  and/or  $\alpha_4$  are nonvanishing. After plugging the metric components (4.2) in the remaining Einstein equations, one can find the values for the other parameters. The corresponding general expressions are quite lengthy, and for this reason we relegate them to Appendix B. As a concrete, simple example, in the main text we work out the special case we focussed on the previous section,  $\alpha_3 = -4\alpha_4$ . In this case, a solution is given by the following values for the parameters

$$\begin{aligned} b_0 &= \frac{4}{9} \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right)^2, \\ c_0 &= \frac{\Delta_0}{b_0}, \\ c_2 &= \frac{m\Delta_0}{4(12\alpha_4 - 1)}, \\ Q_0 &= \frac{16(1 - 12\alpha_4)^4 + 81(1 - 8\alpha_4)^4 \Delta_0}{36[1 + 4\alpha_4(-5 + 24\alpha_4)]^2}. \end{aligned} \quad (4.5)$$

The previous solution is valid for  $\alpha_4$  in the ranges  $\alpha_4 < 1/12$  and  $\alpha_4 > 1/8$ . Notice that the case  $\alpha_4 = 1/12$  corresponds exactly to the Lagrangian (2.20), discussed in Sec. II. We find that  $c_1$  and  $\Delta_0$  are arbitrary; this vacuum solution is then characterized by two integration constants. The resulting metric coefficients can be rewritten in the following, easier-to-handle form:

$$\begin{aligned} A(r) &= \frac{9}{4} \Delta_0 \left( \frac{1 - 8\alpha_4}{1 - 12\alpha_4} \right)^2 [p(r) + \gamma + 1], \\ B(r) &= \frac{4}{9} \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right)^2 r^2 \\ C(r) &= \frac{9}{4} \Delta_0 \left( \frac{1 - 8\alpha_4}{1 - 12\alpha_4} \right)^2 [1 - p(r)], \\ D(r) &= \frac{9\Delta_0}{4} \left( \frac{1 - 8\alpha_4}{1 - 12\alpha_4} \right)^2 \sqrt{p(r)(p(r) + \gamma)} \end{aligned} \quad (4.6)$$

with  $(\mu = -c_1/c_0)$

$$\begin{aligned} p(r) &\equiv \frac{\mu}{r} + \frac{(1 - 12\alpha_4)m^2 r^2}{9(1 - 8\alpha_4)^2}, \\ \gamma &\equiv \frac{16}{81\Delta_0} \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right)^4 - 1. \end{aligned} \quad (4.7)$$

In order to have a consistent solution, we must demand that the argument of the square root appearing in the expression for  $D(r)$ , Eq. (4.6), is positive. A sufficient condition to ensure this is that  $\mu \geq 0$ , and

$$0 < \sqrt{\Delta_0} < \frac{4}{9} \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right)^2. \quad (4.8)$$

The metric then could be rewritten in a more transparent diagonal form, by means of a coordinate transformation. However, a coordinate transformation of the time coordinate is not permitted, since until this point we have adopted the unitary gauge. Therefore, we now renounce to this gauge choice, and allow for a nonzero vector  $\pi^\mu$  of the form  $\pi^\mu = (\pi_0(r), 0, 0, 0)$ . One finds that then the metric can be rewritten in a diagonal form, as

$$ds^2 = -C(r)dt^2 + \tilde{A}(r)dr^2 + B(r)d\Omega^2, \quad (4.9)$$

while the equations of motion for the fields involved are solved by

$$\begin{aligned} \tilde{A}(r) &= \frac{4}{9} \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right)^2 \frac{1}{1 - p(r)}, \\ \pi'_0(r) &= -\frac{\sqrt{p(r)(p(r) + \gamma)}}{1 - p(r)}, \end{aligned} \quad (4.10)$$

with  $C(r)$  and  $B(r)$  being the same as in Eq. (4.6). If one makes a time rescaling

$$t \rightarrow \frac{4(1 - 12\alpha_4)^2}{9\Delta_0^{1/2}(1 - 8\alpha_4)^2} t,$$

the resulting metric has then a manifestly de Sitter-Schwarzschild, or anti-de Sitter-Schwarzschild form. This depends on whether  $\alpha_4$  is smaller or larger than  $1/12$ , as can be seen inspecting the function  $p(r)$  in Eq. (4.7). On the other hand, we should point out that this time-rescaling cannot be performed, without further introducing a time-dependent contribution to  $\pi_0$ . As expected, the metric in Eq. (4.9) can also be obtained by making the following transformation of the time coordinate  $d\tilde{t} = dt + \pi'_0 dr$  to the original metric (4.1). This produces a nonzero time component for  $\pi^\mu$ , that does not vanish even in the limit  $m \rightarrow 0$ .

To summarize so far, we found vacuum solutions in this theory that are asymptotically de Sitter or anti-de Sitter, depending on the choice of the parameters. (Another family of solutions with similar behavior, but obtained for different choices of parameters, has been recently discussed in [23]). Let us point out that it is also possible to include a bare cosmological constant term  $\sqrt{-g}\Lambda$  to the Lagrangian (2.19). Our solutions to the Einstein equations, with our metric Ansatz, remain formally identical. The only difference is that the function  $p(r)$  in Eq. (4.7) becomes

$$p(r) = \frac{\mu}{r} + \frac{(1 - 12\alpha_4)}{9(1 - 8\alpha_4)^2} \left[ m^2 + \frac{4}{3}(1 - 12\alpha_4)\Lambda \right] r^2. \quad (4.11)$$

Notice that the additional integration constant  $\Delta_0$  cannot be used to “compensate” the contribution of the bare

cosmological constant  $\Lambda$  via a self-tuning mechanism, since  $\Delta_0$  does not explicitly appear in the previous formula. For asymptotically de Sitter solutions,  $\alpha_4 < 1/12$ , choosing  $\mu = 0$ , the metric can also be written in a time-dependent form, at the price of switching on additional components of  $\pi^\mu$ , as mentioned earlier. After dubbing

$$\tilde{m}^2 \equiv \frac{1}{(1 - 12\alpha_4)} \left[ m^2 + \frac{4}{3}(1 - 12\alpha_4)\Lambda \right],$$

we can make the following coordinate transformation  $t = F_t(\tau, \rho)$  and  $r = F_r(\tau, \rho)$  with

$$F_t(\tau, \rho) = \frac{4}{3\Delta_0^{1/2}\tilde{m}} \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right) \times \operatorname{arctanh} \left( \frac{\sinh(\frac{\tilde{m}\tau}{2}) + \frac{\tilde{m}^2\rho^2}{8} e^{\tilde{m}\tau/2}}{\cosh(\frac{\tilde{m}\tau}{2}) - \frac{\tilde{m}^2\rho^2}{8} e^{\tilde{m}\tau/2}} \right), \quad (4.12)$$

$$F_r(\tau, \rho) = \frac{3}{2} \left( \frac{1 - 8\alpha_4}{1 - 12\alpha_4} \right) \rho e^{\tilde{m}\tau/2}. \quad (4.13)$$

The metric becomes that of flat slicing of de Sitter

$$ds^2 = -d\tau^2 + e^{\tilde{m}\tau}(d\rho^2 + \rho^2 d\Omega^2), \quad (4.14)$$

where the Hubble parameter is given by

$$H = \frac{\tilde{m}}{2} = \frac{1}{2(1 - 12\alpha_4)^{1/2}} \left[ m^2 + \frac{4}{3}(1 - 12\alpha_4)\Lambda \right]^{1/2}. \quad (4.15)$$

The Stückelberg fields  $\pi^\mu$  are now given by  $\pi^\mu = (\pi^\tau(\tau, \rho), \pi^\rho(\tau, \rho), 0, 0)$ , with  $\pi^\tau = \pi_0 + F_t - \tau$ ,  $\pi^\rho = F_r - \rho$ . Interestingly, the value of the Hubble parameter is ruled by the mass of the graviton; in the case of vanishing bare cosmological constant, we have a self-accelerating solution, in which the smallness of the observed cosmological constant can be associated with the smallness of the graviton mass.

This self-accelerating solution, which reduces to that found perturbatively in Ref. [24], appears as an ideal background to explain present-day acceleration. Notice that this configuration is remarkably similar to that in the DGP braneworld model [25], though there are important differences. In order to study the viability of our nonperturbative solution, it is necessary to study the behavior of fluctuations, to confirm that there is no ghost. On the other hand, in the DGP model, the self-accelerating solution suffers from a ghost instability [10,11,26], which is related to the ghost in the FP theory on a de Sitter background.

## V. DISCUSSION

In this work, we investigated the consequences of strong coupling effects for a theory of nonlinear massive gravity, developed by de Rham, Gabadadze and collaborators. We first reexpressed the complete potential for this theory in a particularly compact and easy-to-handle form. Among

other things, this way of writing the potential suggested intriguing relations with a higher-dimensional setup, that might offer new perspectives. Moreover, we showed that, for certain parameter choices, the potential for massive gravity coincides with the action describing nonperturbative brane objects, independently supporting connections with a higher-dimensional framework.

We then studied the conditions to implement the Vainshtein mechanism in this context. The theory is able to reproduce the behavior of linearized general relativity below a certain scale, but only in a specific region of parameter space. This result provides stringent constraints on this nonlinear massive gravity models. Moreover, we were able to physically reinterpret our findings in the decoupling limit, in terms of an effective theory with Galileon interactions for the scalar-graviton. We showed that the condition to successfully implement Vainshtein mechanism is associated with the sign of a direct coupling between the massless graviton and scalar degree of freedom which cannot be removed by a local field transformation.

We also presented new exact solutions in the vacuum for this theory that asymptote either de Sitter or anti-de Sitter space, depending on the choice of the parameters. Asymptotically de Sitter configurations can be expressed in an explicit time-dependent form, providing an interesting background for the observed Universe, where the rate of the accelerated expansion of the Universe is set by the graviton mass. A small graviton mass, as required by the solar system constraints on deviations from standard general relativity, is then in agreement with the observed value of the cosmological constant. On the other hand, asymptotically anti-de Sitter configurations may have interesting AdS/CFT applications.

Our results naturally lead to various important questions to be examined in the future. It would be interesting to put in a firmer basis the connection between this theory of nonlinear massive gravity and the higher-dimensional setup described in Sec. II. Exploiting this relation might also shed light on the absence of ghost excitations at all orders in perturbations about Minkowski. In Sec. III, we found the intriguing result that, in some region of parameter space, the coupling between the scalar degrees of freedom and graviton shields the interactions of linearized graviton so that gravity appears to become *weaker* as the source is approached. To understand in full detail the system in this case, it is however necessary to analyze the dynamics of higher-order metric fluctuations and their couplings with the scalar sector. Although it seems unlikely that strong coupling dynamics of gravitational modes behave such to mimic general relativity in relevant regimes (by producing a sort of Vainshtein effect at higher-order in perturbations), we cannot exclude this possibility for all ranges of parameters. This paper provides all the necessary equations to study this deep issue, away from the

decoupling limit. It would also be interesting to study in more detail the exact vacuum configurations discussed in Sec. IV, in particular, to understand whether de Sitter solution controlled by the graviton mass can be rendered stable under perturbations. If so, it can be considered as a serious candidate for the observed universe where the present-day acceleration of the Universe is only due to gravitational degrees of freedom.

### ACKNOWLEDGMENTS

We would like to thank David Pirtskhalava for discussions. K. K. and G. N. are supported by the ERC. K. K. is also supported by STFC grant ST/H002774/1, while G. T. is supported by STFC Advanced Fellowship ST/H005498/1. K. K. acknowledges support from the Leverhulme trust.

*Note Added*—While this work was in the last stages of preparation, two papers [23,27] appeared, with some overlap with our Sec. II with respect to the formulation of the potential for nonlinear massive gravity in terms of determinants.

### APPENDIX A: EQUATIONS OF MOTION FOR SPHERICALLY SYMMETRIC AND ASYMPTOTICALLY FLAT BACKGROUNDS

Here we give details of the equations used in Sec. II to describe spherically symmetric solutions with the theory defined by (2.19). Using the Ansatz

$$ds^2 = -N(r)^2 dt^2 + F(r)^{-1} dr^2 + r^2 H(r)^{-2} d\Omega^2 \quad (\text{A1})$$

in the action (2.19), we obtain the potential

$$\begin{aligned} \sqrt{-g}\mathcal{U} = & -\frac{r^2 \sin(\theta)}{\sqrt{F}H^2} \{2[\sqrt{F}((2H-3)N+1) + H^2N \\ & + H(2-6N) + 6N-3] - 6\alpha_3(H-1) \\ & \times [\sqrt{F}((H-3)N+2) - 2HN + H + 4N-3] \\ & - 24\alpha_4(1-\sqrt{F})(1-H)^2(1-N)\}. \end{aligned} \quad (\text{A2})$$

Varying it with respect to  $N$  gives the equation of motion

$$\begin{aligned} 0 = & 3\alpha_3 m^2 r^2 (1-H)H^2 [\sqrt{F}(H-3) - 2H + 4] \\ & - 12\alpha_4 m^2 r^2 (\sqrt{F}-1)(1-H)^2 H^2 \\ & - H^2 [r(\dot{F} - 6m^2 r) + 3m^2 r^2 \sqrt{F} + F] \\ & + rH(r\dot{F}\dot{H} + 2F(r\ddot{H} + 3\dot{H})) - 5r^2 F(\dot{H})^2 \\ & + 2m^2 r^2 (\sqrt{F}-3)H^3 + H^4(m^2 r^2 + 1), \end{aligned} \quad (\text{A3})$$

where “dot” denotes derivatives with respect to  $r$ . Varying with respect to  $F$  gives

$$\begin{aligned} 0 = & -3\alpha_3 m^2 r^2 (1-H)H^2 [H(2N-1) - 4N + 3] \\ & - 12\alpha_4 m^2 r^2 (1-H)^2 H^2 (1-N) \\ & + 2rFH\dot{H}(r\dot{N} + N) - r^2 FN(\dot{H})^2 \\ & + H^2 [N(6m^2 r^2 - F) - r(2F\dot{N} + 3m^2 r)] \\ & + H^4 N(m^2 r^2 + 1) + 2m^2 r^2 H^3 (1-3N), \end{aligned} \quad (\text{A4})$$

and finally by varying with respect to  $H$ , one gets

$$\begin{aligned} 0 = & 6\alpha_3 m^2 r H^2 [\sqrt{F}(H(2N-1) - 3N + 2) + H(2-3N) \\ & + 4N - 3] - H^2 [r\dot{F}\dot{N} + N\dot{F} + 2F(r\ddot{N} + \dot{N}) \\ & + 2m^2 r \sqrt{F}(3N-1) - 12m^2 r N + 6m^2 r] \\ & + H(rN\dot{F}\dot{H} + 2F(rN\ddot{H} + r\dot{H}\dot{N} + 2N\dot{H})) \\ & - 4rFN(\dot{H})^2 + 2m^2 r H^3 [(\sqrt{F}-3)N + 1] \\ & + 24\alpha_4 m^2 r (\sqrt{F}-1)(H-1)H^2(N-1). \end{aligned} \quad (\text{A5})$$

Instead of using the last equation, it is simpler to use a constraint based on the (second) Bianchi identity. This can be achieved by taking a combination of the previous field equations which leads to  $\nabla^\mu G_{\mu\nu} = 0$  for the Einstein piece, where  $G_{\mu\nu}$  is the Einstein tensor. The constraint constructed in this way is

$$\begin{aligned} 0 = & \frac{1}{rHN} \{-3\alpha_3 [\sqrt{F}[H(N(4r\dot{H} + 6) - 2r\dot{H} + 3r\dot{N} - 4) \\ & + 2r(2-3N)\dot{H} + rH^3\dot{N} + H^2(-4r\dot{N} - 4N + 2)] \\ & + 2H^2(H(2N-1) - 3N + 2)] + 12\alpha_4(1-H) \\ & \times [\sqrt{F}[2r(N-1)\dot{H} + rH^2\dot{N} - H(r\dot{N} + 2N-2)] \\ & + 2H^2(N-1)] + \sqrt{F}[-H[2N(r\dot{H} + 3) + 3r\dot{N} - 2] \\ & + 2(3N-1)r\dot{H} + 2H^2(r\dot{N} + N)] - 2H^2[(H-3)N + 1]\}. \end{aligned} \quad (\text{A6})$$

We would like to study perturbations about flat space, hence

$$N = 1 + n, \quad F = 1 + f, \quad H = 1 + h, \quad (\text{A7})$$

and study linear perturbations. However,  $n$  and  $f$  are small, like in GR, but  $h$  could be large since this is where the scalar graviton has an influence. Therefore, we will keep higher orders in  $h$  and truncate the equations to first order in  $n$  and  $f$ . It is more convenient to introduce a new radial coordinate  $\rho = r/H$ , so that the linearized metric is expressed as

$$ds^2 = -(1 + 2\tilde{n})dt^2 + (1 - \tilde{f})d\rho^2 + \rho^2 d\Omega^2. \quad (\text{A8})$$

The change of coordinates fixes  $\tilde{f}$  in terms of  $f$ , as  $\tilde{f}(\rho) = f(r(\rho))\{\partial_\rho[\rho H(\rho)]\}^2$ , while we have the freedom to choose

$\tilde{n}$  and  $\tilde{h}$  in terms of  $n$  and  $h$ . For simplicity, we pick  $\tilde{n}(\rho) = n(r(\rho))$  and  $\tilde{h}(\rho) = h(r(\rho))$ . Therefore, we can drop the tildes of  $n$  and  $h$  from now on, and  $\tilde{f}$  simplifies to

$$1 - \tilde{f}(\rho) = [1 - f(r(\rho))][1 + h(\rho) + \rho h']^2, \quad (\text{A9})$$

where a prime denotes a derivative with respect to  $\rho$ . Then, Eq. (A3), in the new variable  $\rho$  and to leading order in  $\tilde{f}(\rho)$  but keeping all orders in  $h(\rho)$ , reads

$$0 = 2\tilde{f} + 2\rho\tilde{f}' + m^2\rho^2[1 - 2(3\alpha_3 + 1)h + 3(\alpha_3 + 4\alpha_4)h^2][(2 + \tilde{f})\rho h' + (1 + h)\tilde{f}] + 6h[1 - (3\alpha_3 + 1)h + (\alpha_3 + 4\alpha_4)h^2]. \quad (\text{A10})$$

Equivalently for Eq. (A4), one gets

$$0 = -\tilde{f} - 2\rho n' + m^2\rho^2(n - 2[1 + n + (3\alpha_3 + 1)n]h + [(3\alpha_3 + 1)(n + 1) + 3(\alpha_3 + 4\alpha_4)n]h^2). \quad (\text{A11})$$

Finally, for the constraint equation (A6), the expression to first order in  $\tilde{f}$  and  $n$ , but to all orders in  $h$  is

$$0 = \rho n'[-1 + 2(3\alpha_3 + 1)h - 3(\alpha_3 + 4\alpha_4)h^2] - \tilde{f}[1 - (3\alpha_3 + 1)h]. \quad (\text{A12})$$

These last three equations are the ones used in Sec. II to derived the Vainshtein mechanism.

$$c_2 = \frac{\Delta_0 m^2}{9b_0(1 + 3\alpha_3 + 4\alpha_4)^2} [1 - 2\Gamma_{\pm} + 4\alpha_4(2\Gamma_{\pm} - 7) + 2\alpha_3(1 - 18\alpha_4 - 2\Gamma_{\pm}) + \alpha_3^2(15 - 6\Gamma_{\pm}) + 18\alpha_3^3]$$

$$Q_0 = \frac{1}{81b_0(1 + 3\alpha_3 + 4\alpha_4)^4} [8 + 8\Gamma_{\pm} + 81\Delta_0 + \alpha_3[10 + 81\Delta_0 + 2^6 3^3 \alpha_4^3(2 + 3\Delta_0) + 9\Gamma_{\pm} + 216\alpha_4^2(1 + 18\Delta_0 + 4\Gamma_{\pm}) + 6\alpha_4(17 + 162\Delta_0 + 22\Gamma_{\pm})]\Gamma_{\pm} + 27\alpha_3^2[27 + 162\Delta_0 + 288\alpha_4^2(5 + 9\Delta_0) + 20\Gamma_{\pm} + 8\alpha_4(29 + 162\Delta_0 + 26\Gamma_{\pm})] + 2^7 3^3 \alpha_4^3(-1 + 6\Delta_0 + 2\Gamma_{\pm}) + 2^8 3^4 \alpha_4^4(1 + \Delta_0) + 81\alpha_3^4(41 + 81\Delta_0) + 54\alpha_3^3(41 + 162\Delta_0 + 24\alpha_4(14 + 27\Delta_0) + 20\Gamma_{\pm})\Gamma_{\pm} + 144\alpha_4^2(1 + 54\Delta_0 + 8\Gamma_{\pm})\Gamma_{\pm} + 48\alpha_4(2 + 27\Delta_0 + 3\Gamma_{\pm})]. \quad (\text{B7})$$

Notice that there are two branches, depending on the sign choice in  $b_0$  (see Eqs. (B4) and (B5)). For  $\alpha_3 = -4\alpha_4$ , the upper (lower) branch solution must be taken for  $\alpha_4 < 1/12$  ( $\alpha_4 > 1/12$ ). The solution for  $Q_0$  exists only if  $Q_0$  satisfies the condition  $4\sqrt{\Delta_0} + 2Q_0 > 0$ . Because of this condition, for  $\alpha_3 = -4\alpha_4$ , the solution is valid only for  $\alpha_4$  in the ranges  $\alpha_4 < 1/12$  and  $\alpha_4 > 1/8$ .

## APPENDIX B: GENERAL EXACT SOLUTION

In order to construct exact solutions to the Lagrangian (2.19), we use the following Ansatz

$$ds^2 = -C(r)dt^2 + 2D(r)dtdr + A(r)dr^2 + B(r)d\Omega^2, \quad (\text{B1})$$

with

$$B(r) = b_0 r^2, \\ C(r) = c_0 + \frac{c_1}{r} + c_2 r^2, \quad (\text{B2})$$

$$A(r) + C(r) = Q_0,$$

$$D^2(r) + A(r)C(r) = \Delta_0.$$

The constant parameters  $b_0, c_0, c_1, c_2, Q_0, \Delta_0$ , can then be fixed using Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (\text{B3})$$

with  $T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta\mathcal{L}_K}{\delta g^{\mu\nu}}$ , and  $\mathcal{L}_K$  given in Eq. (2.18). The combination  $C(r)G_{rr} + A(r)G_{tt} = 0$  fixes uniquely  $b_0$  to be

$$b_0 = \left(\frac{1 + 6\alpha_3 + 12\alpha_4 + \Gamma_{\pm}}{3(1 + 3\alpha_3 + 4\alpha_4)}\right)^2, \quad (\text{B4})$$

where

$$\Gamma_{\pm} \equiv \pm\sqrt{1 + 3\alpha_3 + 9\alpha_3^2 - 12\alpha_4}. \quad (\text{B5})$$

By requiring that the  $1/r^3$  term in the  $G_{tt}$  equation vanishes, we obtain the condition

$$c_0 = \frac{\Delta_0}{b_0}, \quad (\text{B6})$$

and using the rest of Einstein's equations leads to a solution for the remaining coefficients. These are

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