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# Black hole solutions in $f(R)$ gravity coupled with nonlinear Yang-Mills field

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It is shown that in the static, spherically symmetric spacetime, the problem of metric $f(R)$ gravity coupled with nonlinear Yang-Mills (YM) field constructed from the Wu-Yang ansatz as source can be solved in all dimensions. By nonlinearity it is meant that the YM Lagrangian depends arbitrarily on its invariant. A particular form is considered to be in the power-law form with limit of the standard YM theory. The formalism admits black hole solutions with single or double horizons in which $f(R)$ can be obtained, in general numerically. In 6-dimensional case we obtain an exact solution given by $f(R) = \sqrt{R}$ gravity that couples with the YM field in a consistent manner.

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## I. INTRODUCTION

With the hopes to explain a number of cosmological problems covering dark energy, accelerated expansion, quantum gravity and many related matters, extensions/modifications of general relativity theory gained momentum anew during the recent decade. Each extension adds new degrees of freedom and accommodates new parameters apt for the sake of better physics. Lovelock gravity [1], for instance, constitutes one such extension which abides by the ghost free combinations of higher order invariants resulting in second order equations alone. Next higher order to the Einstein-Hilbert extension in this hierarchy came to be known as the Gauss-Bonnet extension [2] which makes use of the quadratic invariants. Apart from this hierarchy, arbitrary dependence on the Ricci scalar $R$ which has been popular in recent times is known as the $f(R)$ gravity (see e.g. [3] and references therein and for a review paper see [4]). Compared to other theories which employ tensorial invariants this sounds simpler and the fact that the ghosts are eliminated makes it attractive [5]. The simplest form, namely $f(R) = R$, is the well-known Einstein-Hilbert Lagrangian, which constitutes the simplest theory of gravity. Given the simplest theory in hand, why to investigate complex versions of it? The idea is to add new degrees of freedom through nonlinearities, creating curvature sources that may be counterbalanced by the energy-momentum of some physical sources. Once $f(R) = R$, passes all the classical experimental tests any function $f(R)$ may be interpreted as a self-similar version of $f(R) = R$, creating no serious difficulties at the classical level. Introducing an effective Newtonian constant, as a matter of fact, plays a crucial role in this matter. At the quantum level, however, problems such as unitarity, renormalizability of the linearized theory are of vital importance to be tackled with.

As an example, we refer to the particular form $f(R) = R^N$ ($N =$ rational number) [6]. This admits, among others, an exact solution which simulates the geometry of a charged object i.e. the Reissner-Nordström geometry [6]. That is, $f(R) = R^N$ behaves geometrically as if we have $f(R) = R+$ (electrostatic field). Remarkably, the power $N$ plays the role of ‘‘charge’’ so that the geometry $f(R) = R^N$ becomes locally isometric to the geometry of Reissner-Nordström. In a similar manner various combinations of polynomial forms plus $ln(1 + R)$, $\sin R$, and other functions of $R$ can be considered as potential candidates for $f(R)$. Some of these have already appeared in the literature [7]. Beside the case of $f(R) = R+$ (electrostatic field), and more aptly, cases such as $f(R) = R+$ (nonminimal scalar field) cases also have been investigated [7].

Recently, the nonminimal Yang-Mills fields coupled with $f(R)$ gravity has also been studied [8]. In this paper we show that Yang-Mills (YM) field can be accommodated within the context of metric $f(R)$ gravity as well. To the best of our knowledge such a study, especially in higher dimensions which constitutes our main motivation, is absent in the literature. While numerical solutions to the problem of black holes in $f(R)$-YM theory [9] started to appear in the literature our interest is in finding exact solutions. It should be added also that the class of black holes in $f(R)$ gravity can be distinct from the well-known classes such as Myers-Perry (in [2]). Some classes of black holes in this paper also obey this rule since they are not asymptotically flat in the usual sense. We show that in most of our solutions asymptotically (i.e. $r \to \infty$) an effective cosmological constant can be identified which depends on the dimension of spacetime, YM charge $Q$ and the integration parameters. Herein, we do not propose an $f(R)$ Lagrangian *a priori*, instead we determine the $f(R)$ function in accordance with the YM sources. By using the Wu-Yang ansatz for the YM field [10,11] it is shown that a general class of solutions can be obtained in the $f(R)$ gravity where the geometric source matches with the energy-momentum of the YM field. Let us note that the Wu-Yang ansatz works miraculously in all higher dimensions which renders possible to solve $f(R)$ gravity not only in $d = 4$, but in all $d > 4$ as well. Further, the zero trace

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condition for the energy-momentum is imposed to obtain conformal invariant solutions which constitutes a particular class. Although our starting point is the nonlinear YM field in which the Lagrangian is an arbitrary function of the YM invariant our main concern is the special limit, namely, the linear (or standard) YM theory. Power-law type nonlinearity is a well-known class which we consider as an example and as we have shown elsewhere [12] the choice of the power plays a crucial role in the satisfaction of the energy conditions, i.e. Weak, Strong or Dominant. Essentially, conformal invariant property is one of the reasons that we consider Power-YM (PYM) field in higher dimensions. The implementation of zero trace condition for the energy-momentum tensor becomes relatively simpler in this PYM class. As in the example of Born-Infeld electrodynamics case which plays crucial role in resolution of point like singularities, in analogy, similar expectations can be associated with the nonlinear version of the YM theory. Such nonlinearities resemble the self-interacting scalar fields which serve to define different vacua in quantum field theory. More to that, in general relativity the nonlinear terms effect black hole formation significantly, it is therefore tempting to take such combinations seriously. It is our belief that with the nonlinear YM field we can establish effective cosmological parameters to contribute, in accordance with the energy conditions cited above, to the distinction between the phantom and quintessence data of our universe. This is a separate problem of utmost importance that should be considered separately. We show that, the choice $f(R) = \sqrt{R}$ in 6-dimensions yields an exact solution for $f(R)$ gravity coupled with YM fields which is nonasymptotically flat/non-de Sitter in the sense that it contains deficit angles at $r \to \infty$. Other classes of solutions that are asymptotically de Sitter, unfortunately cannot be expressed in a closed form as $f(R)$.

Organization of the paper is as follows. In Sec. II we introduce our theory of nonlinear YM field coupled to $f(R)$ gravity and give exact solutions. Section III specifies the nonlinearity of YM field to PYM case in all dimensions. The First Law of thermodynamics in our formalism is discussed briefly in Sec. IV. We complete the paper with Conclusion which appears in Sec. V.

## II. NON-LINEAR YM FIELD IN $f(R)$ GRAVITY

We start with an action given by

$$S = \int d^d x \sqrt{-g}\left[\frac{f(R)}{2\kappa} + L(F)\right] \tag{1}$$

in which $f(R)$ is a real function of Ricci scalar $R$, $L(F)$ is the nonlinear YM Lagrangian with $F = \frac{1}{4}\,\mathrm{tr}(F^{(a)}_{\mu\nu}F^{(a)\mu\nu})$. The particular choice $L(F) = -\frac{1}{4\pi}F$ will reduce to the case of standard YM theory. Here

$$\mathbf{F}^{(a)} = \frac{1}{2}F^{(a)}_{\mu\nu}dx^{\mu} \wedge dx^{\nu} \tag{2}$$

is the YM field 2-form with the internal index $(a)$ running over the degrees of freedom of the YM nonabelian gauge field. Our unit convention is chosen such that $c = G = 1$ so that $\kappa = 8\pi$.

Variation of the action with respect to the metric gives the field equations as

$$f_R R^{\nu}_{\mu} + \left(\Box f_R - \frac{1}{2}f\right)\delta^{\nu}_{\mu} - \nabla^{\nu}\nabla_{\mu}f_R = \kappa T^{\nu}_{\mu} \tag{3}$$

where $f_R = \frac{df(R)}{dR}$ and $\Box f_R = \nabla_{\mu}\nabla^{\mu}f_R = \frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}\partial^{\mu})f_R$. Further, $\nabla^{\nu}\nabla_{\mu}f_R = g^{\alpha\nu}(f_R)_{,\mu;\alpha} = g^{\alpha\nu}[(f_R)_{,\mu,\alpha} - \Gamma^{m}_{\mu\alpha}(f_R)_{,m}]$. The trace of the field equation implies

$$f_R R + (d-1)\Box f_R - \frac{d}{2}f = \kappa T \tag{4}$$

in which $T = T^{\mu}_{\mu}$.

The energy-momentum tensor is chosen to be

$$T^{\nu}_{\mu} = L(F)\delta^{\nu}_{\mu} - \mathrm{tr}(F^{(a)}_{\mu\alpha}F^{(a)\nu\alpha})L_F(F) \tag{5}$$

in which $L_F(F) = \frac{dL(F)}{dF}$.

The YM ansatz, following the higher dimensional extension of Wu-Yang ansatz, is given by

$$\begin{aligned}
\mathbf{A}^{(a)} &= \frac{Q}{r^2}C^{(a)}_{(i)(j)}x^i dx^j,\\
Q &= \text{YM magnetic charge},\\
r^2 &= \sum_{i=1}^{d-1}x_i^2,
\end{aligned} \tag{6}$$

$$2 \le j+1 \le i \le d-1, \text{ and } 1 \le a \le (d-2)(d-1)/2,$$

$$\begin{aligned}
x_1 &= r\cos\theta_{d-3}\sin\theta_{d-4}\ldots\sin\theta_1,\\
x_2 &= r\sin\theta_{d-3}\sin\theta_{d-4}\ldots\sin\theta_1,\\
x_3 &= r\cos\theta_{d-4}\sin\theta_{d-5}\ldots\sin\theta_1,\\
x_4 &= r\sin\theta_{d-4}\sin\theta_{d-5}\ldots\sin\theta_1,\\
&\ldots\\
x_{d-2} &= r\cos\theta_1,
\end{aligned}$$

in which $C^{(a)}_{(b)(c)}$ is the nonzero structure constants [10]. The spherically symmetric metric is written as

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega^2_{d-2}, \tag{7}$$

where $A(r)$ is the only unknown function of $r$ and

$$d\Omega^2_{d-2} = d\theta_1^2 + \sum_{i=2}^{d-2}\prod_{j=1}^{i-1}\sin^2\theta_j d\theta_i^2, \tag{8}$$

with

$$0 \le \theta_{d-2} \le 2\pi, \qquad 0 \le \theta_i \le \pi, \qquad 1 \le i \le d-3.$$

The YM equations take the form

$$\mathbf{d}\left[{}^{\star}\mathbf{F}^{(a)}L_F(F)\right] + \frac{1}{\sigma}C^{(a)}_{(b)(c)}L_F(F)\mathbf{A}^{(b)} \wedge^{\star} \mathbf{F}^{(c)} = 0, \quad (9)$$

where $\star$ means duality. For our future use we add also that

$$F = \frac{1}{4}\ \mathrm{tr}(F^{(a)}_{\mu\nu}F^{(a)\mu\nu}) = \frac{(d-2)(d-3)Q^2}{4r^4} \quad (10)$$

and

$$\mathrm{tr}\,(F^{(a)}_{t\alpha}F^{(a)t\alpha}) = \mathrm{tr}(F^{(a)}_{r\alpha}F^{(a)r\alpha}) = 0, \quad (11)$$

while

$$\mathrm{tr}\,(F^{(a)}_{\theta_i\alpha}F^{(a)\theta_i\alpha}) = \frac{(d-3)Q^2}{r^4}. \quad (12)$$

From (5) the nonzero energy-momentum tensor components are

$$T^t_t = L = T^r_r, \quad (13)$$

$$T^{\theta_i}_{\theta_i} = L - \frac{(d-3)Q^2}{r^4}L_F. \quad (14)$$

The trace of Eq. (5) implies

$$T = d.L - 4FL_F \quad (15)$$

which yields from Eq. (3)

$$f = \frac{2}{d}[f_R R + (d-1)\Box f_R - \kappa(d.L - 4FL_F)]. \quad (16)$$

To write the exact form of the field equations we need the general form of Ricci scalar and Ricci tensor which are given by

$$R = -\frac{r^2A'' + 2(d-2)rA' + (d-2)(d-3)(A-1)}{r^2}, \quad (17)$$

$$R^t_t = R^r_r = -\frac{1}{2}\frac{rA'' + (d-2)A'}{r}, \quad (18)$$

$$R^{\theta_i}_{\theta_i} = -\frac{rA' + (d-3)(A-1)}{r^2}. \quad (19)$$

in which a prime denotes derivative with respect to $r$. Overall, the field equations read now

$$f_R\left(-\frac{1}{2}\frac{rA'' + (d-2)A'}{r}\right) + \left(\Box f_R - \frac{1}{2}f\right) - \nabla^t\nabla_t f_R = \kappa L, \quad (20)$$

$$f_R\left(-\frac{1}{2}\frac{rA'' + (d-2)A'}{r}\right) + \left(\Box f_R - \frac{1}{2}f\right) - \nabla^r\nabla_r f_R = \kappa L, \quad (21)$$

$$f_R\left(-\frac{rA' + (d-3)(A-1)}{r^2}\right) + \left(\Box f_R - \frac{1}{2}f\right) - \nabla^{\theta_i}\nabla_{\theta_i} f_R = \kappa\left(L - \frac{4}{(d-2)}FL_F\right). \quad (22)$$

Herein

$$\Box f_R = \frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}\partial^r)f_R = A'f'_R + Af''_R + \frac{(d-2)}{r}Af'_R, \quad (23)$$

$$\nabla^t\nabla_t f_R = g^{tt}f_{R,t;t} = g^{tt}(f_{R,t,t} - \Gamma^m_{tt}f_{R,m}) = \frac{1}{2}A'f'_R, \quad (24)$$

$$\nabla^r\nabla_r f_R = g^{rr}f_{R,r;r} = g^{rr}(f_{R,r,r} - \Gamma^m_{rr}f_{R,m}) = Af''_R + \frac{1}{2}A'f'_R, \quad (25)$$

and

$$\nabla^{\theta_i}\nabla_{\theta_i} f_R = g^{\theta_i\theta_i}f_{R,\theta_i;\theta_i} = g^{\theta_i\theta_i}(f_{R,\theta_i,\theta_i} - \Gamma^m_{\theta_i\theta_i}f_{R,m}) = \frac{A}{r}f'_R. \quad (26)$$

The $tt$ and $rr$ components of the field equations imply

$$\nabla^r\nabla_r f_R = \nabla^t\nabla_t f_R \quad (27)$$

or equivalently

$$f''_R = 0. \quad (28)$$

This leads to the solution

$$f_R = \xi + \eta r \quad (29)$$

where $\xi$ and $\eta$ are two integration constants and to avoid any nonphysical case we assume that $\eta, \xi > 0$. The other field equations become

$$(\xi + \eta r)\left(-\frac{1}{2}\frac{rA'' + (d-2)A'}{r}\right) + \frac{1}{2}\eta A' + \frac{(d-2)}{r}A\eta - \frac{1}{2}f = \kappa L, \quad (30)$$

$$(\xi + \eta r)\left(-\frac{rA' + (d-3)(A-1)}{r^2}\right) + A'\eta + \frac{(d-3)}{r}A\eta - \frac{1}{2}f = \kappa\left(L - \frac{4}{(d-2)}FL_F\right). \quad (31)$$

In a similar manner the $\theta_i\theta_i$ and $tt$ components yield

$$(\xi + \eta r)\left(\frac{2(d-3)(A-1) - r^2A'' - (d-4)rA'}{2r^2}\right) + \left(\frac{A}{r} - \frac{A'}{2}\right)\eta = \kappa\frac{4}{(d-2)}FL_F. \quad (32)$$

Equation (16) can equivalently be expressed by

$$f = \frac{2}{d}\left[f_R R + (d-1)\left(A'\eta + \frac{(d-2)}{r}A\eta\right) - \kappa(d.L - 4FL_F)\right]. \quad (33)$$

Here we comment that in the limit of linear Einstein-YM (EYM) theory one may set $L = -\frac{1}{4\pi}F$, $L_F = -\frac{1}{4\pi}$, $\eta = 0$, and $\xi = 1$ to get $f_R = 1$ or equivalently $f = R$ and consequently

$$2(d-3)(A-1) - r^2 A'' - (d-4)rA' = -\frac{4(d-3)Q^2}{r^2}. \tag{34}$$

This admits a solution as

$$A(r) = \begin{cases} 1 - \frac{m}{r^{d-3}} - \frac{d-3}{d-5}\frac{Q^2}{r^2}, & d > 5 \\ 1 - \frac{m}{r^2} - \frac{2Q^2 \ln r}{r^2}, & d = 5 \end{cases} \tag{35}$$

which was reported before [11]. Here we note that $m$ is an integration constant related to mass of the black hole.

## III. PYM FIELD COUPLED TO $f(R)$ GRAVITY

### A. General integral for the PYM field in $f(R)$ gravity

Our first approach to the solution of the field equations, concerns the PYM theory which is a particular nonlinearity given by the Lagrangian $L = -\frac{1}{4\pi}F^s$, in which $s$ is a real parameter [13]. The EYM limit is obtained by setting $s = 1$.

The metric function, then, reads as ($\xi = 0$)

$$A(r) = \begin{cases} \frac{d-3}{d-2} + \Lambda r^2 - \frac{m}{r^{d-2}} - \frac{(d-1)(d-2)^{d-1/2}(d-3)^{d-1/4}}{2^{d-5/2}\eta d}\frac{Q^{d-1/2}\ln r}{r^{d-2}}, & s = \frac{d-1}{4} \\ \frac{d-3}{d-2} + \Lambda r^2 - \frac{m}{r^{d-2}} - \frac{4^{2-s}s(d-2)^{s-1}(d-3)^s Q^{2s}}{(4s+1)(d-4s-1)\eta r^{4s-1}}, & s \neq \frac{d-1}{4} \end{cases} \tag{36}$$

in which $\Lambda$ and $m$ arise naturally as integration constants. Obviously $\Lambda$ is identified as the cosmological constant while $m$ is related to the mass. The fact that our metric is asymptotically de Sitter seems to be manifest only with deficit angles at $r \to \infty$. We add that in order to have an exact solution we had to set $\xi = 0$, which means that in this case we were unable to obtain the $f(R) = R$ gravity from the general solution.

Using the metric solution we also find $f(R(r))$ and $R(r)$ as

$$f(R(r)) = \begin{cases} 2\eta\frac{d-3}{r} - \frac{(d-3)^{d-1/4}(d-2)^{d-5/4}}{2^{d-5/2}}\frac{Q^{d-1/2}}{r^{d-1}}, & s = \frac{d-1}{4} \\ 2\eta\frac{d-3}{r} - \frac{(d-3)^s(d-2)^{s-1}(4s-d+2)}{4^{s-1}}\frac{Q^{2s}}{r^{4s}}, & s \neq \frac{d-1}{4} \end{cases} \tag{37}$$

and

$$R(r) = \begin{cases} \frac{d-3}{r^2} - \Lambda d(d-1) - \frac{(d-1)(d-3)^{d-1/4}(d-2)^{d-5/4}}{2^{d-5/2}\eta d}\frac{Q^{d-1/2}}{r^d}, & s = \frac{d-1}{4} \\ \frac{d-3}{r^2} - \Lambda d(d-1) - \frac{(4s-d+2)s(d-3)^s(d-2)^{s-1}}{4^{s-2}(4s+1)\eta}\frac{Q^{2s}}{r^{4s+1}}, & s \neq \frac{d-1}{4} \end{cases}. \tag{38}$$

We recall from Eq. (29) that

$$f_R = \frac{df}{dR} = \eta r \tag{39}$$

or equivalently

$$\frac{df/dr}{dR/dr} = \eta r. \tag{40}$$

As it is seen from the expressions of $R(r)$ and $f(r)$ it is not possible to eliminate $r$ to have the exact form of $f(R)$, instead we have a parametric form for $f(R)$.

Among all possible cases, we are interested in the condition $4s - d + 2 = 0$. Since this particular choice brings significant simplifications in (37) and (38). Table I shows for which values of $s$ and $d$ this is satisfied.

It is not difficult to observe that with these specific choices, plus $\Lambda = 0$, we obtain

$$f(R) = \mu_\circ \sqrt{R} \tag{41}$$

in which the constant $\mu_\circ$ is defined by

$$\mu_\circ = 2\eta\sqrt{d-3}. \tag{42}$$

Accordingly the metric function $A(r)$ takes the form

$$A(r) = \frac{d-3}{d-2} - \frac{m}{r^{d-2}} - \frac{(d-2)^{d-2/4}(d-3)^{d-2/4}Q^{d-2/2}}{4^{d-6/4}(d-1)\eta r^{d-3}} \tag{43}$$

with the scalar curvature

$$R(r) = \frac{d-3}{r^2}. \tag{44}$$

Since the constant term $\frac{d-3}{d-2} \neq 1$, in (43) our solution for $r \to \infty$ is given by

TABLE I. The table for $d$ versus $s$ that satisfies the condition $4s - d + 2 = 0$. The reason for making this choice is technical for it simplifies the expressions in (37) and (38) to great extent.

| $d =$ | 5 | 6 | 7 | 8 | 9 | 10 | $d$ |
|---|---|---|---|---|---|---|---|
| $s =$ | $\frac{3}{4}$ | 1 | $\frac{5}{4}$ | $\frac{3}{2}$ | $\frac{7}{4}$ | 2 | $\frac{d-2}{4}$ |

$$ds^2 \overset{r\to\infty}{=} -d\bar{t}^2 + d\bar{r}^2 + (\alpha\bar{r})^2 d\Omega_{d-2}^2 \tag{45}$$

where $\bar{t} = \sqrt{\alpha}t$, $r = \alpha\bar{r}$, $(\alpha = \frac{d-3}{d-2})$. This may be interpreted as a deficit angle at $r \to \infty$. It can also be seen easily from Table I that for the linear YM theory ($s = 1$) in $d = 6$, with $\eta = \frac{\sqrt{3}}{6}$, $f(R) = \sqrt{R}$ yields an exact solution.

### B. Thermodynamics of the black hole solution

The black hole solution given by (36) admits horizon(s) provided

$$A(r_h) = 0, \tag{46}$$

which implies

$$m = \begin{cases} \frac{d-3}{d-2} r_h^{d-2} + \Lambda r_h^d - \frac{(d-1)(d-2)^{d-1/2} Q^{d-1/2} (d-3)^{d-1/4}}{2^{d-5/2}\eta d} \ln r_h, & s = \frac{d-1}{4} \\ \frac{d-3}{d-2} r_h^{d-2} + \Lambda r_h^d - \frac{4^{2-s} s(d-2)^{s-1}(d-3)^s Q^{2s}}{(4s+1)(d-4s-1)\eta r_h^{4s-d+1}}, & s \neq \frac{d-1}{4} \end{cases}. \tag{47}$$

The standard definition of Hawking temperature

$$T_H = \frac{1}{4\pi} A'(r_h) \tag{48}$$

yields

$$T_H = \begin{cases} \frac{\frac{\eta}{4} d(d-3)^{1/4}((r_h^2\Lambda+1)d-3) - \sqrt{2}(d-2)^{d-5/4}(d-1) r_h^{1-d} Q^{d-1/2} (\frac{d-3}{4})^{d/4} r_h}{(d-3)^{1/4}\pi r_h \eta d}, & s = \frac{d-1}{4} \\ \frac{[\eta(4s+1)(d-2)(r^2\Lambda d + d - 3) - 4(\frac{d-2}{4})^s (d-3)^s Q^{2s} s r_h^{1-4s}]}{4\pi\eta(d-2)(4s+1) r_h}, & s \neq \frac{d-1}{4} \end{cases}. \tag{49}$$

It is known that the area formula $S = \frac{\mathcal{A}_H}{4G}$, in $f(R)$ gravity becomes [14–16]

$$S = \frac{\mathcal{A}_h}{4G} f_R|_{r=r_h} \tag{50}$$

in which

$$f_R = \eta r_h \tag{51}$$

and

$$\mathcal{A}_h = \frac{d-1}{\Gamma(\frac{d+1}{2})} \pi^{d-1/2} r_h^{d-2} \tag{52}$$

where $r_h$ is the radius of the event horizon or cosmological horizon of the black hole. Using $S$ with the definition of the heat capacity in constant charge we get

$$C_Q = T_H \left(\frac{\partial S}{\partial T_H}\right)_Q = \frac{\pi^{d-1/2}\eta(d-1)^2 r^{d-1}}{4\Gamma(\frac{d+1}{2})} \frac{\eta(d-2)(s+\frac{1}{4})(\Lambda r^2 d + d - 3) - 4s(\frac{d-2}{4})^s (d-3)^s Q^{2s} r^{-4s+1}}{\eta(d-2)(s+\frac{1}{4})(\Lambda r^2 d - d + 3) + 16 s^2 (\frac{d-2}{4})^s (d-3)^s Q^{2s} r^{-4s+1}}. \tag{53}$$

A thorough analysis of the zeros/infinities of this function reveal about local thermodynamic stability/phase transitions, which will be ignored here.

### C. A general approach with $s=1$

#### 1. $d \geq 6$

In this section, for the linear YM theory ($s = 1$), we let $\xi$ to get nonzero value and attempt to find the general solution. As one may notice the case of $d = 5$ is distinct so that we shall find a separate solution for it but for $d \geq 6$ the most general solution reads

$$A(r) = 1 - \frac{m}{r^{d-3}} - \frac{(d-3)Q^2}{(d-5)\xi r^2} + \eta\Delta_1 + \eta^2\Delta_2 + \eta^3\Delta_3 + \eta^4\Delta_4 + (-1)^d r^2 \gamma^{d-1}(d-1) m \ln\left|\frac{\xi + \eta r}{r}\right| + P_{d-7}(\gamma), \tag{54}$$

in which $\gamma = \frac{\eta}{\xi}$ with the following abbreviations

$$\begin{aligned} \Delta_1 &= \frac{4(d-3)}{3(d-5)} \frac{Q^2}{\xi^2 r} + \frac{d-1}{d-2} \frac{m}{\xi r^{d-4}} - \frac{2}{d-2} \frac{r}{\xi}, \\ \Delta_2 &= \frac{2}{(d-2)(d-5)} \frac{r^2}{\xi^2} \ln\left|\frac{\xi + \eta r}{r}\right| - \frac{d-1}{d-3} \frac{m}{\xi^2 r^{d-5}} - \frac{2(d-3)}{(d-5)} \frac{Q^2}{\xi^3}, \\ \Delta_3 &= \frac{4(d-3)}{d-5} \frac{Q^2}{\xi^4} r + \frac{d-1}{d-4} \frac{m}{\xi^3 r^{d-6}}, \\ \Delta_4 &= -\frac{4(d-3)}{d-5} \frac{Q^2}{\xi^5} r^2 \ln\left|\frac{\xi + \eta r}{r}\right| - \frac{d-1}{d-5} \frac{1}{r^{d-7}}, \end{aligned} \tag{55}$$

$$P_{d-7}(\gamma)=(-1)^{d-1}(d-1)m\gamma^5\Big[\overbrace{\gamma^{d-7}r-\frac{1}{2}\gamma^{d-8}+\frac{1}{3}\gamma^{d-9}r^{-1}-\ldots+\frac{1}{d-6}\gamma^{-(d-8)}}^{d-6\ \text{term(s)}}\Big],\qquad d\geq 7.$$

Using these results, we plot Fig. 1 which displays (for $d=6$), $A(r)$ and $f(R)$ for different values of $\eta$.

It can easily be seen that in the limit $\eta\to 0$ and $\xi=1$, the metric function reduces to

$$A(r)=1-\frac{m}{r^{d-3}}-\frac{(d-3)Q^2}{(d-5)r^2},\qquad (56)$$

which is nothing but the well-known black hole solution in $f(R)=R$, EYM theory [11]. On the other hand for $\eta\neq 0\neq\xi$, it is observed that asymptotical flatness does not hold.

To complete our solution we find the asymptotic behavior of the metric function $A(r)$. As one observes from (54), at $r\to\infty$ $A(r)$ becomes

$$A(r)\simeq 1+\frac{\Lambda_{\text{eff}}}{3}r^2,\qquad (57)$$

in which

$$\Lambda_{\text{eff}}=3\Big[\frac{2}{(d-2)(d-5)}-\frac{4(d-3)}{d-5}Q^2\eta^4+(-1)^d\gamma^{d-1}(d-1)\xi^2 m\Big]\gamma^2\ln|\eta|.\qquad (58)$$

As one may see in Fig. 1, we add here that $\eta$ plays a crucial role in making the metric function asymptotically de Sitter, anti-de Sitter and asymptotically flat ($|\eta|<1$, $|\eta|>1$ and $|\eta|=1$ respectively).

*a. Thermodynamics of the BH solution in 6-dimensions* In this part we would like to study the thermodynamics of the solution (54) and compare the result with the case of linear gravity $f(R)=R$. As one can see from the form of the solution (54), we are not able to study analytically in any arbitrary dimensions $d$, and therefore we only consider $d=6$. The metric solution in $d=6$ dimensions is given by

$$\begin{aligned}A(r)=1&-\frac{m}{r^3}-\frac{3Q^2}{\xi r^2}+\eta\Big(\frac{4Q^2}{\xi^2 r}+\frac{5}{3}\frac{m}{\xi r^2}-\frac{1}{2}\frac{r}{\xi}\Big)\\&+\eta^2\Big(\frac{1}{2}\frac{r^2}{\xi^2}\ln\Big|\frac{\xi+\eta r}{r}\Big|-\frac{5}{2}\frac{m}{\xi^2 r}-6\frac{Q^2}{\xi^3}\Big)\\&+\eta^3\Big(12\frac{Q^2}{\xi^4}r+\frac{5}{2}\frac{m}{\xi^3}\Big)\\&+\eta^4\Big(-12\frac{Q^2}{\xi^5}r^2\ln\Big|\frac{\xi+\eta r}{r}\Big|-5r\Big),\end{aligned}\qquad (59)$$

and therefore the Hawking temperature is given by

$$T_H=\Big(\frac{3}{4\pi r_h}-\frac{3}{4}\frac{Q^2}{\pi r_h^3}\Big)+\Big(\frac{17Q^2}{16\pi r_h^2}-\frac{3}{16}\frac{1}{\pi}\Big)\eta-\Big(\frac{123Q^2}{64\pi r_h}+\frac{120r_h\ln r_h+139r_h}{192\pi}\Big)\eta^2+\mathcal{O}(\eta^3)\qquad (60)$$

and the specific heat capacity reads

$$C_Q=\Big(\frac{8\pi^2 r_h^4(Q^2-r_h^2)}{(3r_h^2-9Q^2)}\Big)+\Big(-\frac{8}{9}\frac{\pi^2 r_h^5(3r_h^4-17Q^2r_h^2+7Q^4)}{(r_h^2-3Q^2)^2}\Big)\eta+\mathcal{O}(\eta^2).\qquad (61)$$

First we comment herein that, to get the above result we considered $\xi=1$. Second we add that, by $\eta=0$ we get the case of EYM black hole in $R$-gravity. In the case of pure $R$-gravity we put $\eta=0$ and $Q=0$ which leads to

$$T_H=\frac{3}{4\pi r_h},\quad\text{and}\quad C_Q=-\frac{8\pi^2}{3}r_h^4\qquad (62)$$

which are the Hawking temperature and Heat capacity of the 6-dimensional Schwarzschild black hole. Divergence in the Heat capacity for particular YM charge and therefore a thermodynamic instability is evident from this expression.

### 2. *d*=5

As we stated before, dimension $d=5$ behaves different from the other dimensions. The metric function is given in this case by

$$A(r)=1-\frac{m}{r^2}-\frac{2Q^2}{\xi r^2}\ln r+\eta\Delta_1+\eta^2\Delta_2+\eta^3\Delta_3+\eta^4\Delta_4\qquad (63)$$

in which

$$\begin{aligned}\Delta_1&=\frac{1}{9\xi^2 r}[24Q^2\ln r+12m+2Q^2-6\xi r^2],\\ \Delta_2&=\frac{1}{3\xi^3}\Big[2\xi r^2\ln\frac{\xi+\eta r}{r}-12Q^2\ln r-3Q^2-6m\xi\Big],\\ \Delta_3&=\frac{2r}{\xi^4}[2m\xi-3Q^2+4Q^2\ln r],\\ \Delta_4&=-\frac{2r^2}{\xi^5}\Big[4Q^2\ln r\ln\frac{\xi+\eta r}{\xi\sqrt{r}}+4Q^2\text{dilog}\Big(\frac{\xi+\eta r}{\xi}\Big)+(2m\xi-Q^2)\ln\Big(\frac{\xi+\eta r}{r}\Big)\Big].\end{aligned}\qquad (64)$$

Herein $m$ is an integration constant and

$$\text{dilog}(x) = \int_1^x \frac{\ln t}{1-t} dt \tag{65}$$

is the dilogarithm function. Here also the EYM limit with $\eta \to 0$ and $\xi = 1$ is obvious.

Similar to the higher than 6-dimensional case we give here also the asymptotic behavior of the metric solution (65) as $r \to \infty$,

$$A(r) \simeq 1 + \frac{\Lambda_{\text{eff}}}{3} r^2, \tag{66}$$

in which

$$\Lambda_{\text{eff}} = -2\frac{\eta^4}{\xi^5}\left\{6\left[3m\xi^5 \ln\eta - Q^2 \ln\left(\frac{\eta}{\xi}\right)\ln(\xi\eta)\right] - \frac{\xi^3}{\eta^2}\ln\eta + 2\pi^2 Q^2\right\} \tag{67}$$

is the effective cosmological constant.

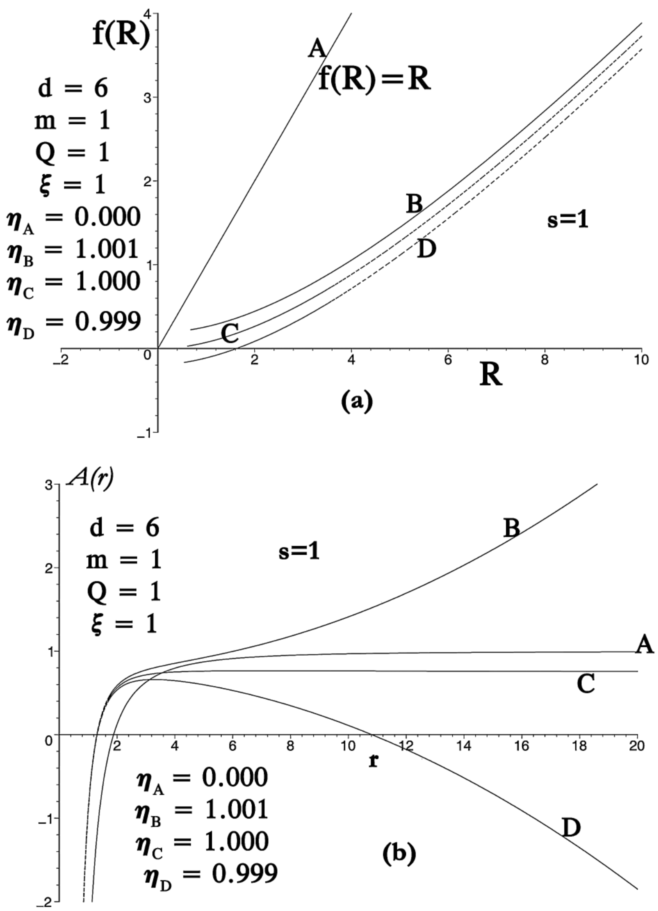


FIG. 1. The plot of 6-dimensional $f(R)$ [Fig. 1(a)] and $A(r)$ [Fig. 1(b)]. We choose $\Lambda = 0$, and four different values of $\eta$ ($\eta_A$, $\eta_B$, $\eta_c$, and $\eta_D$) are depicted as plots A, B, C and D. From Fig. 1(b) it can be seen that in A, B, and C we have single, while in D double horizons.

### D. Black holes with a conformally invariant YM source

One of the interesting choice for $s$ in Einstein-Power-Maxwell theory-which has been considered first by Hassaine and Martinez [13]-is given by $s = \frac{d}{4}$ (for all $d \geq 4$) which is conformally invariant. This choice yields a zero trace for the energy-momentum tensor in any dimensions, i.e., $T = T^\mu_\mu = 0$. In EPYM case also $s = \frac{d}{4}$ leads to a traceless energy-momentum tensor and a metric solution for the field equations with arbitrary values of $\xi$ and $\eta$ is given by

$$A(r) = 1 - \frac{m}{r^{d-3}} + 4(d-2)^{d/4}\left(\frac{d-3}{4}\right)^{d/4}\frac{Q^{d/2}}{\xi r^{d-2}} + r^2\left[\frac{2\gamma^2}{d-2} + (-1)^d(d-1)m\gamma^{d-1}\right] \times \ln\left|\frac{\xi(1+\gamma r)}{r}\right| - \frac{2\gamma}{d-2}r + q(\gamma) \tag{68}$$

in which

$$q(\gamma) = (-1)^d(d-1)m\gamma^{d-2}\sum_{k=1}^{d-2}\frac{(-1)^k r^{2-k}}{\gamma^{k-1}k}, \tag{69}$$

and $\gamma = \frac{\eta}{\xi}$. Figure 2 ($d = 5$, $s = \frac{5}{4}$) and Fig. 3 ($d = 6$, $s = \frac{3}{2}$) depict $A(r)$, $R(r)$ and $f(R)$ which relate the conformally invariant ($s = \frac{d}{4}$) cases for different $\eta$ values. For $r \to \infty$, it can be seen easily from Eq. (68) that we have an effective cosmological constant term, given by

$$\Lambda_{\text{eff}} = 3\left[\frac{2\gamma^2}{d-2} + (-1)^d(d-1)m\gamma^{d-1}\right]\ln|\eta| \tag{70}$$

We note that as a limit, once $\eta \to 0$ (or equivalently $\gamma \to 0$) and $\xi \to 1$ the solution reduces to

$$A(r) = 1 - \frac{m}{r^{d-3}} + 4(d-2)^{d/4}\left(\frac{d-3}{4}\right)^{d/4}\frac{Q^{d/2}}{r^{d-2}} \tag{71}$$

which is the metric function in Einstein-PYM theory in $f(R) = R$ gravity. Determination of horizons and thermodynamical properties in this limit is much more feasible in comparison with the intricate expression (68). To complete this section we give the Hawking temperature and specific heat capacity for $d = 5$ which read

$$T_H = \frac{2r_h^3 - \sqrt[4]{24Q^{10}}}{4\pi r_h^4} + \frac{2\sqrt[4]{24Q^{10}} - r_h^3}{6\pi r_h^3}\eta - \frac{17r_h^3 + 10\sqrt[4]{24Q^{10}} + 12r_h^3\ln r_h}{18\pi r_h^2}\eta^2 + \mathcal{O}(\eta^3) \tag{72}$$

and

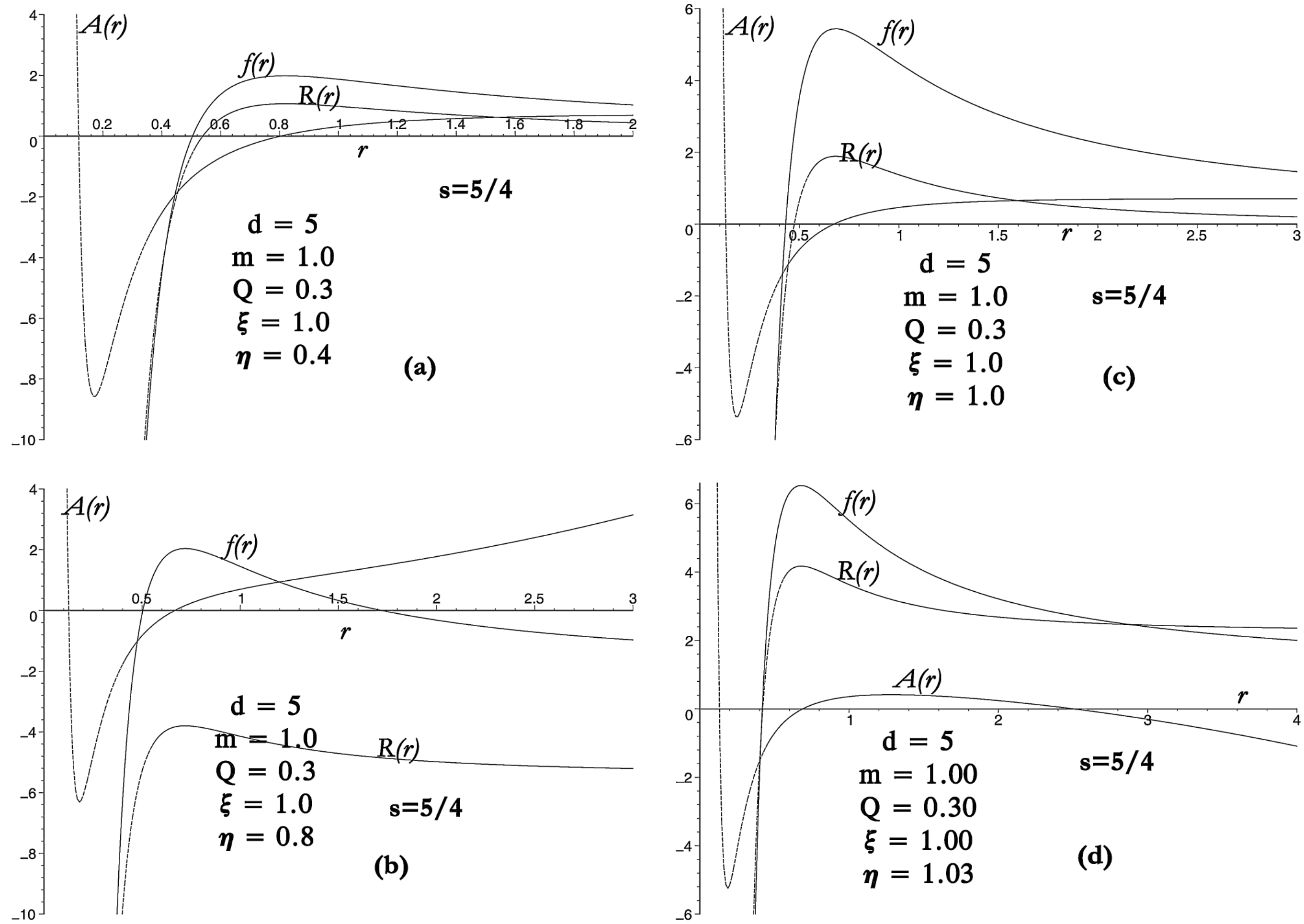


FIG. 2. The 5-dimensional plots of $A(r)$, $f(r)$ and $R(r)$ from Eq. (68), for a variety of parameters given in Figs. 2(a)–2(d). Since $s = \frac{5}{4}$ in this particular case, the source is the PYM field with Lagrangian $L \sim F^{5/4}$. These are all black hole solutions with inner and outer horizons. A general analytic expression for $f(R)$ seems out of our reach.

$$C_Q = -\frac{3}{4}\frac{\pi^2 r_h^3(\sqrt[4]{24Q^{10}} - 2r_h^3)}{2\sqrt[4]{24Q^{10}} - r_h^3} + \frac{3\pi^2 r_h^4(4\sqrt[4]{24Q^{10}}r_h^3 - r_h^6 - 2\sqrt{6}Q^5)}{(2\sqrt[4]{24Q^{10}} - r_h^3)}\eta + O(\eta^2). \tag{73}$$

### 1. Constant $d$-dimensional curvature $R = R_0$

G. Cognola, *et al.* in Ref. [15] have considered the constant four-dimensional curvature $R = R_0$ in pure $f(R)$ gravity, which implies a de-Sitter universe. Here we wish to follow the same procedure in higher dimensions in $f(R)$ gravity coupled with the nonminimal PYM field. As stated before, in order to have a traceless energy-momentum in $d$-dimensions we need to consider the case of conformally invariant YM source which is given by $s = \frac{d}{4}$ in the PYM source. In 4-dimensions $s = 1$ is satisfied automatically for the zero trace condition.

We start with the trace of the Eq. (4) which leads to

$$f'(R_0) = \frac{d}{2R_0}f(R_0) \tag{74}$$

and therefore the field Eqs. (3) become

$$G_\mu^\nu + \Lambda_{\text{eff}}\delta_\mu^\nu = \kappa\tilde{T}_\mu^\nu \tag{75}$$

with the effective cosmological constant and energy-momentum tensor as

$$\Lambda_{\text{eff}} = \frac{(d-2)R_0}{2d}, \qquad \tilde{T}_\mu^\nu = \frac{2R_0}{f(R_0)d}T_\mu^\nu. \tag{76}$$

Now, we follow [15,16] to give the form of the entropy akin to the possible BH solution. As we indicated in Eq. (50) the entropy of the modified gravity with constant curvature is given by

$$S = \frac{\mathcal{A}_h}{4G}f_{R_0} \tag{77}$$

which after considering (78) it becomes

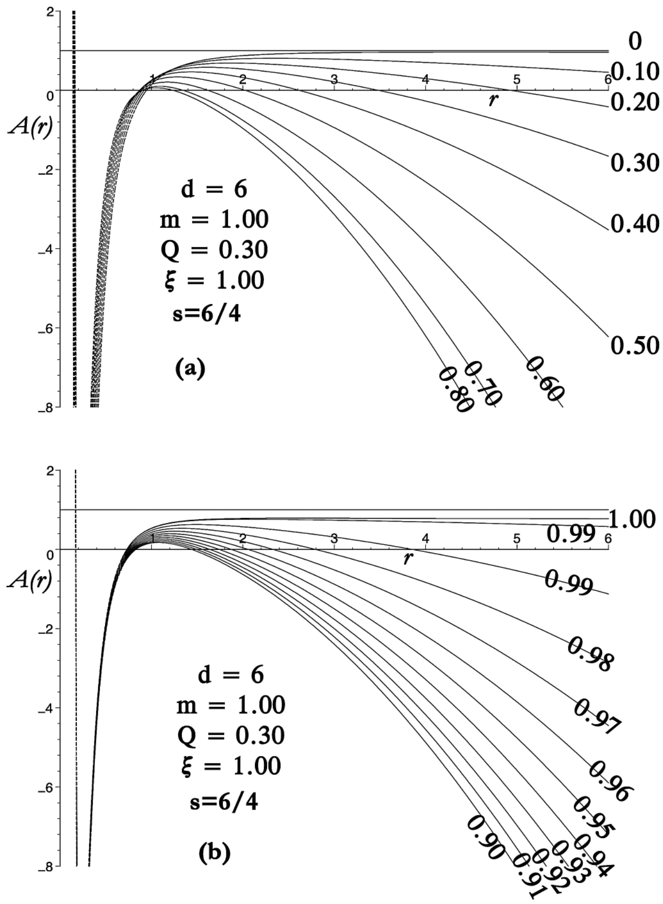


FIG. 3. The plot of the metric function $A(r)$ corresponding to the conformally invariant case from Eq. (68) in $d = 6$ and $s = \frac{3}{2}$, for a set of $\eta$ parameters. Black hole formations with single/double horizons are explicitly seen. Specifically, Fig. 3(a) for $0 \le \eta \le 0.80$ and Fig. 3(b) for $0.9 \le \eta \le 1.0$.

$$S = \frac{\mathcal{A}_h d}{8GR_0} f(R_0). \tag{78}$$

Since our main concern in this paper is not the particular class of $R = R_0 =$ constant curvature space time we shall not extend our discussion here any further.

## IV. FIRST LAW OF THERMODYNAMICS

In this section we follow Ref. [17] to find a higher dimensional form of the Misner-Sharp energy [18] inside the horizon of the static spherically symmetric black hole in $f(R)$ gravity. The corresponding metric is given by (7) and the horizon is found from $A(r_h) = 0$. The field Eqs. (3) may be written as

$$G_\mu^\nu = \kappa\left[\frac{1}{f_R} T_\mu^\nu + \frac{1}{\kappa}\hat{T}_\mu^\nu\right] \tag{79}$$

where $G_\mu^\nu$ is the Einstein tensor and $\hat{T}_\mu^\nu$ is a stress-energy tensor for the effective curvature which reads

$$\hat{T}_\mu^\nu = \frac{1}{f_R}\left[\nabla^\nu\nabla_\mu f_R - \left(\Box f_R - \frac{1}{2}f + \frac{1}{2}R\right)\delta_\mu^\nu\right]. \tag{80}$$

At the horizon $tt$ and $rr$ parts of (79) imply

$$\frac{d-2}{2r_h}A' f_R - \frac{(d-2)(d-3)}{2r_h^2} f_R = \kappa\left(T_0^0 + \frac{1}{2\kappa}[(f - Rf_R) - A' f_R']\right) \tag{81}$$

which upon multiplying by an infinitesimal displacement $dr_h$ on both sides can be reexpressed in the form

$$\frac{A'}{4\pi} d\left(\frac{2\pi\mathcal{A}_h}{\kappa} f_R\right) - \frac{1}{2\kappa}\left[\frac{(d-2)(d-3)}{r_h^2} f_R + (f - Rf_R)\right]\mathcal{A}_h dr_h = \mathcal{A}_h T_0^0 dr_h. \tag{82}$$

We add here that all functions are calculated at the horizon, for instance $A' = \frac{dA(r)}{dr}|_{r=r_h}$. The latter equation suggests that

$$dE = \frac{1}{2\kappa}\left[\frac{(d-2)(d-3)}{r_h^2} f_R + (f - Rf_R)\right]\mathcal{A}_h dr_h \tag{83}$$

in which $E$ is the Misner-Sharp energy in our case. Therefore (82) becomes

$$TdS - dE = PdV \tag{84}$$

where we set Hawking temperature $T = \frac{A'}{4\pi}$, entropy of the black hole $S = \frac{2\pi\mathcal{A}_h}{\kappa} f_R$, radial pressure of matter fields at the horizon $P = T_r^r = T_0^0$ and finally the change of volume of the black hole at the horizon is given by $dV = \mathcal{A}_h dr_h$. The exact form of the Misner-Sharp energy stored inside the horizon may be found as

$$E = \frac{1}{2\kappa}\int\left[\frac{(d-2)(d-3)}{r_h^2} f_R + (f - Rf_R)\right]\mathcal{A}_h dr_h \tag{85}$$

in which the integration constant is set to zero (to read more see Refs. [17,19]).

As an example we study the case of PYM field in $f(R)$ gravity in Sec. III A. Also to have an exact form for $f(R)$ we employ the metric (43) which corresponds to $f(R) = \mu_\circ\sqrt{R}$. Equation (82) yields,

$$\frac{A'}{4\pi}d\left(\frac{2\pi\mathcal{A}_h}{\kappa}\eta r_h\right)-\frac{1}{2\kappa}\left[\frac{(d-2)(d-3)}{r_h}\eta+\frac{\eta(d-3)}{r_h}\right]\mathcal{A}_h dr_h=\mathcal{A}_h\left(\frac{-1}{4\pi}\left[\frac{(d-2)(d-3)Q^2}{4r_h^4}\right]^{d-2/4}\right)dr_h, \quad (86)$$

in which $R=\frac{d-3}{r_h^2}$ has been used. Now, this equation leads to

$$A'=\frac{(d-3)}{r_h}-\frac{1}{\eta(d-1)}\left(4\left[\frac{(d-2)(d-3)Q^2}{4r_h^4}\right]^{d-2/4}\right). \quad (87)$$

By taking derivative of (43) and substituting for $m$ in terms of $r_h$ the foregoing equation easily follows.

Finally, one can see that the total energy is expressed by

$$E=\frac{\eta(d-3)(d-1)}{2\kappa(d-2)}\mathcal{A}_h. \quad (88)$$

## V. CONCLUSION

An arbitrary dependence on the Ricci scalar in the form of $f(R)$ as Lagrangian yields naturally an arbitrary geometrical curvature. The challenge is to find a suitable energy-momentum that will match this curvature by solving the highly nonlinear set of equations. For a number of reasons it has been suggested that $f(R)$ gravity may solve the long-standing problems such as, accelerated expansion and dark energy problems of cosmology. Richer theoretical structure naturally provides more parameters to fit recent observational data. We have shown that in analogy with the electromagnetic (both linear and nonlinear) field, the Yang-Mills field also can be employed and solved within the context of $f(R)$ gravity. So far, $f(R)$ as a modified theory of gravity has been considered mainly in $d=4$, whereas we have been able in the presence of YM fields to extend it to $d>4$. In addition to the parameters of the theory the dimension of space time also contribute asymptotically to the effective cosmological constant created in $f(R)$ gravity. Admittedly, out of the general numerical solution technically it is not possible to invert scalar curvature $R(r)$ as $r=r(R)$ and obtain $f(R)$ in a closed form. This happens only in very special cases. In particular dimensions and nonlinearities we obtained black holes with single/multi horizons. From the obtained solutions for PYM field coupled $f(R)$ gravity we can discriminate three broad classes as follows:

i) the asymptotically flat class in which $\eta=0$, $\xi=1$. This class was already known [11].

ii) the asymptotically de Sitter/anti-de Sitter class corresponding to $\eta\neq 0$, $\xi=1$ ($s=1$).

iii) the nonasymptotically flat/nonasymptotically de Sitter class for $\eta\neq 0\neq\xi$, $s=\frac{d}{4}$.

Our solutions admit black hole solutions with single/multi horizons. In the proper limits we recover all the well-known metrics to date. The case (ii) at large distance limit exhibits deficit angle as shown in Eq. (45).

Conformally invariant class with zero trace of the energy-momentum tensor, is obtained with the PYM Lagrangian $L(F)=-\frac{1}{4\pi}(F)^{5/4}$ in $d=5$. In general, the power of $F$ becomes meaningful within the context of energy conditions and causality. By introducing effective pressure $P_{\mathrm{eff}}$ and energy density $\rho_{\mathrm{eff}}$ through $P_{\mathrm{eff}}=\omega\rho_{\mathrm{eff}}$ and using the PYM fields in energy conditions $\omega$ factor (i.e. whether $\omega<-1$, or $\omega>-1$) can be determined as a cosmological factor [4]. This will be our next project in this line of study. It may happen that, certain set of powers eliminate nonphysical fields such as phantoms and alikes. As far as exact solutions are concerned a remarkable solution is obtained in the case of standard YM Lagrangian $L(F)=-\frac{1}{4\pi}F$ with $d=6$ in $f(R)=\sqrt{R}$ gravity which automatically restricts the curvature to $R>0$.

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