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Einstein–Yang-Mills–Chern-Simons solutions in $D = 2n + 1$ dimensions

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We investigate finite energy solutions of the Einstein–Yang-Mills–Chern-Simons system in odd spacetime dimensions, $D = 2n + 1$, with $n > 1$. Our configurations are static and spherically symmetric, approaching at infinity a Minkowski spacetime background. In contrast with the Abelian case, the contribution of the Chern-Simons term is nontrivial already in the static, spherically symmetric limit. Both globally regular, particlelike solutions and black holes are constructed numerically for several values of D. These solutions carry a nonzero electric charge and have finite mass. For globally regular solutions, the value of the electric charge is fixed by the Chern-Simons coupling constant. The black holes can be thought of as nonlinear superpositions of Reissner-Nordström and non-Abelian configurations. A systematic discussion of the solutions is given for $D = 5$, in which case the Reissner-Nordström black hole becomes unstable and develops non-Abelian hair. We show that some of these non-Abelian configurations are stable under linear, spherically symmetric perturbations. A detailed discussion of an exact $D = 5$ solution describing extremal black holes and solitons is also provided.

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I. INTRODUCTION

In recent years the interest in the properties of gravity in more than $D = 4$ spacetime dimensions has increased significantly. This interest was enhanced by the development of string theory, which requires a ten-dimensional spacetime, to be consistent from a quantum theoretical viewpoint. Even in the absence of matter, solutions to the Einstein equations in dimensions higher than $3 + 1$ exhibit properties which are strikingly different. For example, in a $4 + 1$-dimensional asymptotically flat vacuum spacetime with a given Arnowitt-Deser-Misner mass and angular momentum, the geometry need not necessarily be that of the Myers-Perry [1] black hole. Notably, in this case there is the black ring [2] solution whose horizon topology is $S^2 \times S^1$, in contrast to S^3 of the former [1].

The rapid progress in the last decade has provided a rather extensive picture of the landscape of solutions for the five-dimensional case [3], including configurations with Abelian matter fields [4–9]. Although the situation for $D > 5$ is more patchy, analytical [10] and numerical [11] results suggest that the nonstandard solutions found in $D = 5$ have higher dimensional generalizations; moreover, even more complex configurations are likely to exist as the spacetime dimension increases.

In the case of higher dimensional gravitating systems of matter fields featuring nonlinear interactions, in particular, with non-Abelian (nA) gauge fields, black hole and regular solutions are still relatively scarcely explored. This is an important direction since the theory of gravitating nA gauge fields can be regarded as the most natural generalization of Einstein-Maxwell theory. Moreover, for the better known case of a $D = 3 + 1$-dimensional spacetime, the results in the literature show that various well-known, and rather intuitive, features of self-gravitating solutions with Maxwell fields are not shared by their counterparts with nA gauge fields. For example, the Einstein-Yang-Mills (EYM) equations admit black hole solutions that are not uniquely characterized by their mass, angular momentum, and YM charges, thus violating the no-hair conjecture [12]. Therefore the uniqueness theorem for electrovacuum black hole spacetimes ceases to apply for EYM systems. Also, in contrast with the Abelian situation, self-gravitating YM fields can form particlelike configurations [13]. Another surprising result is the existence of nA solutions which are static but not spherically symmetric [14]. However, since it turns out that all these asymptotically flat solutions are unstable, their physical relevance is obscure.[1] (A detailed review of $D = 4$ gravitating particlelike and black hole solutions with nA gauge fields can be found in Ref. [18].)

The study of $D > 4$ black hole solutions with non-Abelian matter fields is only in its beginnings. Based on the experience with Einstein-Maxwell solutions, it is natural to expect that higher dimensions $D > 4$ allow for a rich

[1]For the sake of completeness, one should mention that the picture is very different once one gives up the assumption of asymptotic flatness. For example, in anti–de Sitter (AdS) $3 + 1$-dimensional spacetime, stable nA solutions have been shown to exist [15]; there are also monopole and dyon solutions even in the absence of a Higgs field. As found in [16], some of the AdS nA solutions may provide a model of holographic superconductors. Also, the nonasymptotically flat nA solutions in [17] (with a dilaton field possessing a Liouville potential) have found interesting applications in providing gravity duals of $\mathcal{N} = 1$ super-Yang-Mills theory.

landscape of solutions that do not have four-dimensional counterparts. At the same time, considering such configurations is a legitimate task, since the gauged supersymmetric models generically contain non-Abelian fields.

Most of the solutions displayed so far in the literature are spherically symmetric (an exception being the results in [19]). As a new feature and in contrast with the situation in the $D = 4$ case, a generic property of the asymptotically flat higher dimensional EYM solutions is that their mass and action, as defined in the usual way, diverge [20–23]. This can be understood heuristically by noting that the Derrick scaling requirement [24] is not fulfilled in spacetimes for dimension five and higher. Finite energy solutions exist only when the usual YM system is augmented with higher derivative corrections in the nA action [25]. Such terms can occur in the low energy effective action of string theory and represent the gauge field counterparts of the Lovelock gravitational hierarchy [26] (for a review of these aspects, see [27]).

In a recent work [28], a different way of regularizing the mass of $D = 4 + 1$-dimensional asymptotically flat, gravitating nA solutions was proposed. This was done by introducing a Chern-Simons (CS) term in the action. The CS density is a higher order term in the YM curvature and connection, and as such can be viewed as an *alternative* to the higher order curvature terms of the YM hierarchy employed previously in [22,25]. It turns out that this prescription *does* result in finite mass globally regular and black hole solutions and leads to a variety of new features as compared to the well-known case of $D = 4$ EYM solutions. For example, these configurations carry an electric charge, emerging as perturbations of the Reissner-Nordström (RN) black holes. Moreover, in contrast to all other known asymptotically flat nA black holes without scalars, some of these solutions in [28] were found to be stable under linear, spherically symmetric perturbations. Also, for a particular value of the CS coupling constant, it was possible to construct both solitons and extremal black hole solutions, by exploiting the model in [29].

In this work we propose a general framework for the study of Einstein–Yang-Mills–Chern-Simons (EYMCS) solutions for an arbitrary $D = 2n + 1$ spacetime dimension. Our configurations are static and spherically symmetric, approaching at infinity a Minkowski spacetime background. Based on numerical results for $D = 5$ and $D = 7, 9$, we conjecture that the presence of a CS term in the action allows for finite energy solutions for any $D = 2n + 1$, with $n > 1$. (The case $D = 3$ is special, since it requires the presence of a negative cosmological constant.) Most of the numerical results in this paper are for the $D = 5$, in which case, we provide a systematic discussion of the solutions in [28]. We have also discussed some results for $D = 7, 9$, which reveal some new features of the solutions.

The paper is structured as follows: In Sec. II we present the general framework and analyze the field equations for an $SO(D + 1)$ gauge group. In Sec. III, the general features of a consistent truncation of the general model for an $SO(D - 1) \times SO(2)$ gauge group are discussed. Numerical results for $D = 5$ and $D > 5$, respectively, are presented in Secs. IV and V. We conclude with Sec. VI, where the significance of, and further consequences arising from, the solutions we have constructed are discussed.

II. THE GENERAL MODEL

A. The action and field equations

In odd spacetime dimensions, the usual gauge field action can be augmented by a (dynamical) CS term. Restriction to odd dimensions follows from the fact that Chern-Pontryagin (CP) densities are defined only in even dimensions, and the CS density is defined formally in one dimension lower than the CP density. The resulting odd-dimensional space is then interpreted as the spacetime on which the dynamical CS term appears in the Lagrangian.

Such terms appear in various supersymmetric theories, the $\mathcal{N} = 8$, $D = 5$ gauged supergravity model [30,31] being perhaps the best known case, due to its role in the conjectured AdS/CFT correspondence. However, in this work we shall restrict ourselves to a simple EYMCS model, which does not seem to correspond to a consistent truncation of any gauged supergravity model. Also, our solutions approach asymptotically the Minkowski spacetime background. In the case of an Abelian gauge group in $D = 5$, a CS term leads to some new features[2] only for rotating black holes [33]. However, we shall see that for a nA gauge group, the CS term can affect the properties of solutions even in the static, spherically symmetric case.

We consider the following action for the EYMCS model in $D = 2n + 1$ dimensions:

$$S = \int_{\mathcal{M}} d^D x \sqrt{-g}\left(\frac{R}{16\pi G} - \mathcal{L}_{\rm YM}\right) - \kappa \int_{\mathcal{M}} d^D x \mathcal{L}_{\rm CS}^{(D)}, \tag{2.1}$$

where

$$\mathcal{L}_{\rm YM} = \tfrac{1}{2}{\rm Tr}\{F_{\mu\nu}F^{\mu\nu}\} \tag{2.2}$$

is the usual Yang-Mills Lagrangian, with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu] \tag{2.3}$$

the gauge field strength tensor, while e and κ are the gauge and the CS coupling constant, respectively. (The value of κ is fixed in supersymmetric theories. However, in this work we shall treat κ as a free input parameter. This has also been motivated by the study in [34] of the Einstein-Maxwell-CS system, which revealed a nontrivial dependence of the properties of the solutions on the value of κ.)

[2]Note that the situation can be different for charged magnetic branes; see e.g. the asymptotically AdS_5 Abelian solutions with a CS term in [32].

The definition of the Chern-Simons density on D-dimensional spacetime follows from that of the corresponding Chern-Pontryagin density on $D+1$ (even) dimensions. The latter is, by definition, a total divergence

$$\boldsymbol{\nabla}\cdot\boldsymbol{\Omega} = \mathrm{Tr}\{F\wedge F\ldots\wedge F\}, \qquad n\,\text{times}$$

in $2n$ dimensions, and the CS density on a $D = 2n-1$-dimensional spacetime is formally defined as one of the $2n$ components of the density $\boldsymbol{\Omega}$.

The CS densities $\mathcal{L}_{\mathrm{CS}}^{(D)}$ thus defined are *gauge variant*. The explicit expressions of the first three, in $D = 3$-, 5-, and 7-dimensional spacetimes, are

$$\mathcal{L}_{\mathrm{CS}}^{(3)} = \varepsilon^{\lambda\mu\nu}\,\mathrm{Tr}\{A_\lambda[F_{\mu\nu} - \tfrac{2}{3}eA_\mu A_\nu]\}, \tag{2.4}$$

$$\mathcal{L}_{\mathrm{CS}}^{(5)} = \varepsilon^{\lambda\mu\nu\rho\sigma}\,\mathrm{Tr}\{A_\lambda[F_{\mu\nu}F_{\rho\sigma} - eF_{\mu\nu}A_\rho A_\sigma + \tfrac{2}{5}e^2A_\mu A_\nu A_\rho A_\sigma]\}, \tag{2.5}$$

$$\mathcal{L}_{\mathrm{CS}}^{(7)} = \varepsilon^{\lambda\mu\nu\rho\sigma\tau\kappa}\,\mathrm{Tr}\Big\{A_\lambda\Big[F_{\mu\nu}F_{\rho\sigma}F_{\tau\kappa} - \frac{4}{5}eF_{\mu\nu}F_{\rho\sigma}A_\tau A_\kappa - \frac{2}{5}e^2F_{\mu\nu}A_\rho F_{\sigma\tau}A_\kappa + \frac{4}{5}e^3F_{\mu\nu}A_\rho A_\sigma A_\tau A_\kappa - \frac{8}{35}e^4A_\mu A_\nu A_\rho A_\sigma A_\tau A_\kappa\Big]\Big\}, \tag{2.6}$$

which are all manifestly gauge variant. Remarkably however, the Euler-Lagrange variations of these densities are actually *gauge covariant*. Indeed, these variational terms are expressed in gauge covariant form for arbitrary $D = 2n+1$ as

$$(n+1)\varepsilon^{\mu_1\mu_2\mu_3\mu_4\ldots\mu_{2n-1}\mu_{2n}}F_{\mu_1\mu_2}F_{\mu_3\mu_4}\ldots F_{\mu_{2n-1}\mu_{2n}}.$$

Perhaps what is still more relevant in our case, where we restrict attention to *static* solutions only, is the fact that the CS densities (2.4), (2.5), and (2.6), etc., in that case reduce to a very useful form which can be expressed for the arbitrary $D = 2n+1$ case. Working in a gauge such that $\partial_t A_\mu = 0$, one can show that, up to a total divergence term (which we ignore here since we are only interested in the Euler-Lagrange equations), the effective arbitrary n CS Lagrangian is

$$\mathcal{L}_{\mathrm{CS}}^{(2n+1)} = (n+1)\varepsilon^{i_1i_2i_3i_4\ldots i_{2n-1}i_{2n}}\,\mathrm{Tr}\{A_0F_{i_1i_2}F_{i_3i_4}\ldots F_{i_{2n-1}i_{2n}}\}.$$

The field equations are obtained by varying the action (2.1) with respect to the field variables $g_{\mu\nu}$, A_μ,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu},$$
$$D_\mu(\sqrt{-g}F^{\mu\tau}) = \kappa\frac{(D+1)}{2\sqrt{-g}}\varepsilon^{\tau\lambda\mu\ldots\nu\rho}F_{\lambda\mu}\ldots F_{\nu\rho}, \tag{2.7}$$

where

$$T_{\mu\nu} = 2\,\mathrm{Tr}\{F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\} \tag{2.8}$$

is the energy momentum tensor. One can show that this tensor is covariantly conserved (i.e. $\nabla_\mu T^{\mu\nu} = 0$) for solutions of the YMCS equations.

In what follows, we shall be seeking to construct finite mass/energy solutions of the above equations. This is made possible by the fact that the energy density functional arising from the Lagrangian (2.1) actually satisfies Derrick scaling[3] by virtue of the presence of the CS term in it. This is because the CS term, which in $D = 2n+1$-dimensional spacetime scales as $L^{-(2n+1)}$, balances the Yang-Mills term which scales as L^{-4}. Specifically, in the $2n = D-1$ space dimensions, $2n+1 \geq 2n \geq 4$, for the cases of interest here, namely, for $n \geq 2$.

Insofar as the CS term here plays the role of regularizing the energy by providing the required Derrick balance, this makes it an alternative to employing YM higher order curvature terms [22,25] for this purpose. The latter is of course more versatile since its use is not restricted, as in the CS cases, to $2n+1$-dimensional spacetimes.

B. A spherically symmetric ansatz

1. The metric

In D-dimensional spacetime, we restrict ourselves to static fields that are spherically symmetric in the $D-1$ spacelike dimensions with a general metric Ansatz

$$ds^2 = f_1(r)dr^2 + f_2(r)d\Omega_{D-2}^2 - f_0(r)dt^2, \tag{2.9}$$

where r and t are the radial and time coordinates, while $d\Omega_{D-2}^2$ is the metric on the round $(D-2)$ sphere (note that this Ansatz still has some freedom in the choice of the radial coordinate).

The numerical work has been done for a metric gauge choice $f_2(r) = r^2$ and

$$f_1(r) = N(r),$$
$$f_0(r) = N(r)\sigma^2(r), \quad \text{where } N(r) = 1 - \frac{m(r)}{r^{D-3}}, \tag{2.10}$$

the function $m(r)$ being related to the local mass-energy density (as defined in the standard way) up to some D-dependent factor.

Another convenient metric gauge choice used in the literature is

$$f_1(r) = \frac{f_2(r)}{r^2} = \frac{m(r)}{f(r)}, \qquad f_0(r) = f(r), \tag{2.11}$$

corresponding to an isotropic coordinate system (the $D = 5$ exact solution discussed in Sec. IV D is found for this choice of coordinates).

[3]Strictly speaking, Derrick scaling [24] applies only in flat space background. However, in practice it works also for gravitating configurations in asymptotically flat spaces [35] (at least in the spherically symmetric case).

2. The YM fields

The construction of a static, spherically symmetric YM Ansatz in $D = 5$ spacetime dimensions leading to a nonzero CS term has been discussed in [36]. In what follows, we present an extension of that result for a generic $D = 2n + 1$ case.

There is some arbitrariness in the choice of the gauge group. The only restriction is that it should be large enough to accommodate for a static spherically symmetric Ansatz, with a nonvanishing electric potential.[4] In $D = 2n + 1$ dimensions, the smallest simple gauge group supporting a nonvanishing CS term is $SO(2n + 2)$. Here we shall take the $SO(2n + 2)$ YM fields in one or the other chiral representation of $SO_{\pm}(2n + 2)$. Our spherically symmetric Ansatz is expressed in terms of the representation matrices

$$\Sigma^{(\pm)}_{\alpha\beta} = -\frac{1}{4}\left(\frac{1 \pm \Gamma_{2n+3}}{2}\right)[\Gamma_\alpha, \Gamma_\beta], \qquad \alpha, \beta = 1, 2, \ldots, 2n + 2, \tag{2.12}$$

$\Gamma_\alpha = (\Gamma_i, \Gamma_M)$, with the index $M = (2n + 1, 2n + 2)$, being the gamma matrices in $2n + 2$ dimensions and Γ_{2n+3}, the corresponding chiral matrix.[5] We shall adopt a normalization convention such that

$$2\,\mathrm{Tr}\{\Sigma^{(\pm)}_{\alpha\beta}, \Sigma^{(\pm)}_{\alpha'\beta'}\} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}. \tag{2.13}$$

Our construction of a spherically symmetric gauge field Ansatz is based on the formalism of Schwarz [37]. An alternative formalism, in [38], is familiar in the literature, but the calculus of [37] was found to be more convenient for the purposes of this work.

Our spherically symmetric Ansatz for the YM connection $A_\mu = (A_0, A_i)$ is

$$A_0 = \frac{1}{e}\{-(\varepsilon\chi)^M \hat{x}_j \Sigma^{(\pm)}_{jM} - \chi^{2n+3}\Sigma^{(\pm)}_{2n+1,2n+2}\}, \tag{2.14}$$

$$A_i = \frac{1}{e}\left\{\left(\frac{\phi^{2n+3} + 1}{r}\right)\Sigma^{(\pm)}_{ij}\hat{x}_j + \left[\left(\frac{\phi^M}{r}\right)(\delta_{ij} - \hat{x}_i\hat{x}_j) + (\varepsilon A_r)^M \hat{x}_i\hat{x}_j\right]\Sigma^{(\pm)}_{jM} + A_r^{2n+3}\hat{x}_i\Sigma^{(\pm)}_{2n+1,2n+2}\right\} \tag{2.15}$$

(with x^i the usual Cartesian coordinates on R^{D-1} and $x^0 = t$) in which the summed over indices $M, N = 2n+1, 2n + 2$ run over two values such that we can label the functions $(\phi^M, \phi^{2n+3}) \equiv \vec{\phi}$, $(\chi^M, \chi^{2n+3}) \equiv \vec{\chi}$, and $(A_r^M, A_r^{2n+3}) \equiv \vec{A}_r$ like three isotriplets $\vec{\phi}$, $\vec{\chi}$, and $\vec{A}_r$, all depending on the $2n$-dimensional spacelike radial variable r and time t. ε is the two-dimensional Levi-Civita symbol, while $\hat{x}_i = x_i/r$ (with $x_i x_i = r^2$).

In what follows we are interested in configurations without a dependence on time. Then the parametrization used in the Ansatz (2.14) and (2.15) results in a gauge covariant expression for the YM curvature $F_{\mu\nu} = (F_{ij}, F_{i0})$,

$$F_{ij} = \frac{1}{e}\left\{\frac{1}{r^2}(|\vec{\phi}|^2 - 1)\Sigma^{(\pm)}_{ij} + \frac{1}{r}\left[D_r\phi^{2n+3} + \frac{1}{r}(|\vec{\phi}|^2 - 1)\right] \times \hat{x}_{[i}\Sigma^{(\pm)}_{j]k}\hat{x}_k + \frac{1}{r}D_r\phi^M \hat{x}_{[i}\Sigma^{(\pm)}_{j]M}\right\}, \tag{2.16}$$

$$\begin{aligned} F_{i0} = \frac{1}{e}\Big\{ & -\frac{1}{r}\phi^M(\varepsilon\chi)^M\Sigma^{(\pm)}_{ij}\hat{x}_j \\ & + \frac{1}{r}[\phi^{2n+3}(\varepsilon\chi)^M - \chi^{2n+3}(\varepsilon\phi)^M]\Sigma^{(\pm)}_{iM} \\ & - \left[(\varepsilon D_r\chi)^M + \frac{1}{r}[\phi^{2n+3}(\varepsilon\chi)^M - \chi^{2n+3}(\varepsilon\phi)^M]\right] \\ & \times \hat{x}_i\hat{x}_j\Sigma^{(\pm)}_{jM} - D_r\chi^{2n+3}\hat{x}_i\Sigma^{(\pm)}_{2n+1,2n+2}\Big\}, \end{aligned} \tag{2.17}$$

in which we have used the notation

$$\begin{aligned} D_r\phi^a &= \partial_r\phi^a + \varepsilon^{abc}A_r^b\phi^c, \\ D_r\chi^a &= \partial_r\chi^a + \varepsilon^{abc}A_r^b\chi^c, \end{aligned} \tag{2.18}$$

as the $SO(3)$ covariant derivatives of the two triplets $\vec{\phi} \equiv \phi^a = (\vec{\phi}^M, \phi^{2n+3})$, $\vec{\chi} \equiv \chi^a = (\vec{\chi}^M, \chi^{2n+3})$, with respect to the $SO(3)$ gauge connection $\vec{A}_r \equiv A_r^a$.

After taking the traces over the spin matrices, it is convenient to relabel the triplets of radial function as

$$\vec{\phi} \equiv (\phi^M, \phi^3), \quad \vec{\chi} \equiv (\chi^M, \chi^3), \quad \text{and} \quad \vec{A}_r \equiv (A_r^M, A_r^3), \tag{2.19}$$

with $M = 1, 2$ now.

The triplet $\vec{A}_r(r)$ plays the role of a connection in the residual one-dimensional system after the imposition of symmetry, and encodes the $SO(D - 1)$ arbitrariness of this Ansatz. In one dimension there is no curvature; hence it can be gauged away in practice [36].

However, finding solutions within the YM Ansatz (2.14) and (2.15) (which after setting $\vec{A}_r = 0$ still features six independent functions) is technically a difficult task. A further consistent truncation of the general Ansatz is $\phi^2 = \chi^2 = 0$, leading to an EYMCS system with six unknown functions, four of them being gauge potentials parametrizing the gauge field and two being metric functions. Indeed, the two gauge functions suppressed are redundant and would only be excited in an eventual stability analysis of our solutions. To make connection with previous results on EYM solutions, we shall note

[4] For the most interesting case $D = 5$, this condition rules out the possibility of using the minimal non-Abelian gauge group $SO(3)$ [21].

[5] Thus, the fact that we are using an anti-Hermitian representation for the $SO(D + 1)$ algebra matrices leads to a factor of i in front of $\mathcal{L}^{(D)}_{\mathrm{CS}}$.

$$\phi^1(r) = \tilde{w}(r), \qquad \phi^3(r) = w(r), \qquad \chi^1(r) = \tilde{V}(r),$$
$$\chi^3(r) = V(r), \quad \text{with} \quad \vec{A}_r = 0. \tag{2.20}$$

Reference [36] gave numerical evidence for the existence of solutions within the above Ansatz for the case of $D = 5$ [i.e. an $SO(6)$ gauge group] and AdS asymptotics. Some of the features discussed there are generic. For example, the resulting system has some residual symmetry under a rotation of the "doublets" $w(r)$, $\tilde{w}(r)$ and $V(r)$, $\tilde{V}(r)$ with the same constant angle u (e.g. $w \to w\cos u + \tilde{w}\sin u$, etc.). Note that for configurations with $\tilde{w}(r) = \tilde{V}(r) = 0$ the gauge potentials are invariant under the "chiral" transformations generated by $\Sigma^{(\pm)}_{2n+1,2n+2}$. The configurations with $w(r) = V(r) = 0$ instead change just by a sign under the same transformations. Also, this Ansatz is invariant under the parity transformation $\phi^a \to -\phi^a$, $\chi^a \to -\chi^a$.

A further simplification of the YM Ansatz consists in taking

$$\tilde{w}(r) = \tilde{V}(r) = 0, \tag{2.21}$$

which is a consistent truncation, $SO(D-1) \times SO(2)$, of the general Ansatz.

C. The equations

The truncated YM Ansatz (2.20) together with the generic metric element (2.9) leads to the following set of EYMCS equations:

$$f_0'' - \frac{f_0'^2}{2f_0} - \frac{f_0' f_1'}{2f_1} + \frac{D-3}{2f_2}\left(f_0' f_2' + 2f_0 f_1 - \frac{f_0 f_2'^2}{2f_2}\right) - \frac{\alpha^2}{4f_2}\Bigg((D-2)(2D-7)f_1(\tilde{V}w - V\tilde{w})^2 + 2(2D-5)f_2(V'^2 + \tilde{V}'^2)$$
$$+ \frac{3(D-2)(D-3)f_0 f_1}{f_2}(w^2 + \tilde{w}^2 - 1)^2 + 2(D-2)f_0(w'^2 + \tilde{w}'^2)\Bigg) = 0, \tag{2.22}$$

$$\frac{f_0' f_2'}{f_0} + \frac{(D-3)f_2'^2}{2f_2} - (D-3)f_1 + \alpha^2\Bigg(\frac{f_2}{f_0}(V'^2 + \tilde{V}'^2) - (D-2)(w'^2 + \tilde{w}'^2) + \frac{(D-2)(D-3)f_1}{2f_2}(w^2 + \tilde{w}^2 - 1)^2$$
$$- \frac{(D-2)f_1}{2f_0}(\tilde{V}w - V\tilde{w})^2\Bigg) = 0, \tag{2.23}$$

$$f_2'' - \frac{f_1' f_2'}{2f_1} + \frac{(D-5)f_2'^2}{4f_2} - (D-3)f_1 + \frac{\alpha^2}{2}\Bigg((D-2)(w'^2 + \tilde{w}'^2) + \frac{f_2}{f_0}(V'^2 + \tilde{V}'^2)\frac{(D-2)f_1}{2f_0}(\tilde{V}w - V\tilde{w})^2$$
$$+ \frac{(D-2)(D-3)f_1}{2f_2}(w^2 + \tilde{w}^2 - 1)^2\Bigg) = 0, \tag{2.24}$$

$$w'' + \frac{1}{2}\left(\frac{f_0'}{f_0} - \frac{f_1'}{f_1} + \frac{(D-4)f_2'}{f_2}\right)w' + \frac{f_1}{2f_0}\tilde{V}(\tilde{V}w - V\tilde{w}) - \frac{(D-3)f_1}{f_2}w(w^2 + \tilde{w}^2 - 1)$$
$$- \kappa\frac{(D^2-1)f_1}{(D-2)f_2^{(D-4)/2}\sqrt{f_1 f_0}}(w^2 + \tilde{w}^2 - 1)^{(D-5)/2}((w^2 + \tilde{w}^2 - 1)V' + (D-3)(V\tilde{w} - \tilde{V}w)\tilde{w}') = 0, \tag{2.25}$$

$$\tilde{w}'' + \frac{1}{2}\left(\frac{f_0'}{f_0} - \frac{f_1'}{f_1} + \frac{(D-4)f_2'}{f_2}\right)\tilde{w}' + \frac{f_1}{2f_0}V(V\tilde{w} - \tilde{V}w) - \frac{(D-3)f_1}{f_2}\tilde{w}(w^2 + \tilde{w}^2 - 1)$$
$$- \kappa\frac{(D^2-1)f_1}{(D-2)f_2^{(D-4)/2}\sqrt{f_1 f_0}}(w^2 + \tilde{w}^2 - 1)^{(D-5)/2}((w^2 + \tilde{w}^2 - 1)\tilde{V}' + (D-3)(\tilde{V}w - V\tilde{w})w') = 0, \tag{2.26}$$

$$V'' + \left(\frac{(D-2)f_2'}{2f_2} - \frac{f_0'}{2f_0} - \frac{f_1'}{2f_1}\right)V' - \frac{(D-2)f_1\tilde{w}}{2f_2}(V\tilde{w} - \tilde{V}w) - \kappa(D^2-1)\sqrt{f_0 f_1}f_2^{1-D/2}(w^2 + \tilde{w}^2 - 1)^{(D-3)/2}w' = 0, \tag{2.27}$$

$$\tilde{V}'' + \left(\frac{(D-2)f_2'}{2f_2} - \frac{f_0'}{2f_0} - \frac{f_1'}{2f_1}\right)\tilde{V}' - \frac{(D-2)f_1\tilde{w}}{2f_2}(\tilde{V}w - V\tilde{w}) - \kappa(D^2-1)\sqrt{f_0 f_1}f_2^{1-D/2}(w^2 + \tilde{w}^2 - 1)^{(D-3)/2}\tilde{w}' = 0, \tag{2.28}$$

where a prime denotes a derivative with respect to r. Also, to simplify the expression of the above relations, we note

$$\alpha^2 = \frac{16\pi G}{(D-2)e^2} \tag{2.29}$$

and absorb a factor of $1/e^{(D-3)/2}$ in the expression of κ.

After fixing a metric gauge, Eq. (2.23) becomes a Hamiltonian constraint. There is also a constraint equation for the gauge fields,

$$\frac{f_2^{(D-2)/2}}{\sqrt{f_0 f_1}}(\tilde{V}V' - V\tilde{V}') + (D-2)f_2^{(D-4)/2}\sqrt{\frac{f_0}{f_1}}(w\tilde{w}' - \tilde{w}w') - \kappa(D^2-1)(\tilde{V}w - V\tilde{w})(w^2+\tilde{w}^2-1)^{(D-3)/2} = 0, \tag{2.30}$$

which originates from the variational equation for $\vec{A}_r$ [one can show that (2.30) is a first integral of Eqs. (2.25), (2.26), (2.27), and (2.28)].

Also, in what follows, we shall restrict to a dimension $D \geq 5$. The case $D = 3$ should be discussed separately, since the existence of physically interesting solutions requires the presence of a cosmological constant.

D. The asymptotics and a truncation

Numerical evidence for the existence of asymptotically AdS_5 solutions within the full Ansatz (2.20) was given in Ref. [36]. However, it seems that the presence in that case of a negative cosmological constant was crucial in arriving at that result.[6] In the asymptotically flat case, we could not find such solutions (with four essential functions) but only configurations within the restricted $SO(D-1)\times SO(2)$ Ansatz (2.21).

Although we do not have a rigorous proof of that, some analytical indications in this direction come from the study of the asymptotics of the general solutions close to the horizon and at infinity. Here it is convenient to use the metric Ansatz (2.10), with two functions $N(r)$ and $\sigma(r)$. For black hole solutions, the horizon is located at $r = r_h > 0$, with $N(r_h) = 0$ and $\sigma(r_h) > 0$, while $N'(r_h) > 0$ in the nonextremal case.

Then we suppose that all functions admit the following behavior as $r \to r_h$:

$$\begin{aligned}
w(r) &= \cos U_1 \sum_{k=0}^{\infty} w_k (r-r_h)^k,\\
\tilde{w}(r) &= \sin U_1 \sum_{k=0}^{\infty} \tilde{w}_k (r-r_h)^k,\\
V(r) &= \cos U_1 \sum_{k=0}^{\infty} V_k (r-r_h)^k,\\
\tilde{V}(r) &= \sin U_1 \sum_{k=0}^{\infty} \tilde{V}_k (r-r_h)^k,\\
N(r) &= \sum_{k=1}^{\infty} \bar{N}_k (r-r_h)^k,\\
\sigma(r) &= \sum_{k=0}^{\infty} \bar{\sigma}_k (r-r_h)^k,
\end{aligned} \tag{2.31}$$

which satisfy the regularity condition $w\tilde{V} - \tilde{w}V \to 0$ as $r \to r_h$. The coefficients w_k, $\tilde{w}_k$, $\tilde{V}_k$, V_k, and σ_k are computed order by order by substituting this expansion in the field equations. It turns out that the only free parameters are $\sigma(r_h)$, $w(r_h)$, and V_1. Moreover, we have verified that, at least up to order four,[7] $w_k/\tilde{w}_k = V_k/\tilde{V}_k = 1$.

Interestingly, a similar analysis for large values of r leads to the same conclusions. Here we suppose the solutions admit a power series expansion with

$$\begin{aligned}
w(r) &= \cos U \sum_{k=0}^{\infty} \frac{W_k}{r^k}, \qquad \tilde{w}(r) = \sin U \sum_{k=0}^{\infty} \frac{\tilde{W}_k}{r^k},\\
V(r) &= \cos U \sum_{k=0}^{\infty} \frac{V_k}{r^k}, \qquad \tilde{V}(r) = \sin U \sum_{k=0}^{\infty} \frac{\tilde{V}_k}{r^k},\\
N(r) &= 1 + \sum_{k=D-3}^{\infty} \frac{M_k}{r^k}, \qquad \sigma(r) = 1 + \sum_{k=1}^{\infty} \frac{\sigma_k}{r^k}.
\end{aligned} \tag{2.32}$$

After plugging this expansion in the field equations, we have found that $W_k = \tilde{W}_k$ and $V_k = \tilde{V}_k$, at least up to order $D+5$. Moreover, the only free parameters in the above expressions are W_2 and V_2.

This result, together with the corresponding one for the near horizon expansion (2.31), strongly suggests that the functions $w(r)$, $\tilde{w}(r)$ and $V(r)$, $\tilde{V}(r)$ have a constant ratio for any $r > r_h$ for any physical solution of (2.22), (2.23), (2.24), (2.25), (2.26), (2.27), and (2.28). Although we do not have a rigorous proof, this conjecture has been confirmed by our numerics, and all black hole solutions we have found have in fact only two essential gauge functions.[8] This applies also for asymptotically flat solitons (i.e. without an event horizon), in which case we have

[6]This is not an entirely surprising result. We recall that already in $D = 4$ dimensions and a gauge group $SO(3)$, the presence of a negative cosmological constant Λ leads to some new qualitative features [15]. In particular, $\Lambda < 0$ allows for EYM static solutions with a nonvanishing electric potential, which is not the case for asymptotically flat configurations [39].

[7]Beyond this order, the involved relations were too complicated to deal with.

[8]Although we could construct $D = 5$ black hole solutions within the general ansatz (2.20), it turns out that, within the numerical accuracy, the ratios $w(r)/\tilde{w}(r)$ and $V(r)/\tilde{V}(r)$ were in fact always constant.

also failed to find asymptotically solutions within the general Ansatz (2.20). In asymptotically AdS_5 spacetime, the asymmetry between w, $\tilde{w}$ and V, $\tilde{V}$ explicitly appears in the large-r behavior, being introduced by the cosmological term; see the results in Sec. 2 of Ref. [36].

Then, for the remainder of this work we shall deal with the case of solutions within the restricted $SO(D-1)\times SO(2)$ Ansatz with two essential functions: a magnetic potential, $w(r)$, and an electric one, $V(r)$.

III. THE $SO(D-1)\times SO(2)$ MODEL. GENERAL PROPERTIES

A. The equations and scaling properties

The equations of the model simplify drastically for the truncation (2.21). Their form within the metric parametrization (2.10) reads

$$\begin{aligned} & w'' + \left(\frac{D-4}{r} + \frac{N'}{N} + \frac{\sigma'}{\sigma}\right)w' - \kappa\frac{(D^2-1)}{(D-2)} \\ & \quad \times \frac{(w^2-1)^{(D-3)/2}}{\sigma N r^{D-4}}V' + \frac{(D-3)w(1-w^2)}{r^2 N} = 0, \\ & V'' + \left(\frac{D-2}{r} - \frac{\sigma'}{\sigma}\right)V' - \kappa\frac{(D^2-1)}{r^{D-2}} \\ & \quad \times \sigma(w^2-1)^{(1/2)(D-3)}w' = 0, \\ & m' = \frac{\alpha^2}{2}r^{D-2} \\ & \quad \times \left(\frac{V'^2}{\sigma^2} + \frac{(D-2)Nw'^2}{r^2} + (D-2)(D-3)\frac{(1-w^2)^2}{2r^4}\right), \\ & \sigma' = \alpha^2(D-2)\frac{\sigma w'^2}{2r}, \end{aligned} \tag{3.1}$$

with the gauge constraint (2.30) vanishing identically. These equations can also be derived from the effective action

$$\begin{aligned} S_{\mathrm{eff}} = \int dt dr\Big\{\sigma m' - \frac{1}{2}\alpha^2\Big[& r^{D-2}\Big((D-2)\frac{N\sigma w'^2}{r^2} - \frac{V'^2}{\sigma} \\ & + \frac{(D-2)(D-3)}{2r^4}\sigma(1-w^2)^2\Big) \\ & - 2\kappa(D^2-1)V(w^2-1)^{(D-3)/2}w'\Big]\Big\} \end{aligned} \tag{3.2}$$

(note that, as required, there is no coupling with the geometry for the term proportional to κ).

A generic feature of the YMCS model within the $SO(D-1)\times SO(2)$ truncation is the existence of a first integral for the electric potential $V(r)$,

$$V' = \frac{\sigma}{r^{D-2}}\left(\frac{P}{\alpha^2} + (D^2-1)\kappa F(w)\right), \tag{3.3}$$

where P is an integration constant (we shall see that, for globally regular solutions, its value is fixed by κ). The function $F(w)$ has the following general expression in terms of the hypergeometric function ${}_2F_1$:

$$F(w) = (-1)^{(1/2)(D+1)}_2 F_1\left(\frac{1}{2}, \frac{3-D}{2}, \frac{3}{2}; w^2\right)w, \tag{3.4}$$

its explicit form for several dimensions being

$$\begin{aligned} & F(w) = w, \quad \text{for } D = 3, \\ & F(w) = -w + \frac{1}{3}w^2, \quad \text{for } D = 5, \\ & F(w) = w - \frac{2}{3}w^3 + \frac{w^5}{5}, \quad \text{for } D = 7, \end{aligned}$$

and

$$F(w) = -w + w^3 - \tfrac{3}{5}w^5 + \tfrac{1}{7}w^7, \quad \text{for } D = 9.$$

One should also note that Eqs. (3.1) together with the first integral (3.3) are invariant under the scaling

$$\begin{aligned} & r \to \lambda r, \quad m \to \lambda^{D-3}m, \quad \sigma \to \sigma, \quad w \to w, \quad V \to V/\lambda, \\ & \quad P \to \lambda^{D-2}P, \ \text{ and } \ \alpha \to \lambda\alpha, \quad \kappa \to \lambda^{D-4}\kappa, \end{aligned} \tag{3.5}$$

with λ an arbitrary positive parameter. There is also a second scaling symmetry of Eqs. (3.1),

$$V \to \tilde{\lambda}V, \qquad \sigma \to \tilde{\lambda}\sigma, \tag{3.6}$$

together with $t \to t/\tilde{\lambda}$, all other variables remaining unchanged. This symmetry is lost after setting $\sigma(\infty) = 1$ as a boundary condition.

The last symmetry of the equations of the model consists in simultaneously changing the sign of the CS coupling constant together with the electric or magnetic potential,

$$\kappa \to -\kappa, \qquad V \to -V, \quad \text{or} \quad \kappa \to -\kappa, \qquad w \to -w \tag{3.7}$$

[the first integral (3.3) implies also $P \to -P$ in the first case]. In what follows, we shall use this symmetry to study solutions with a positive κ only.

B. The behavior at infinity

Unfortunately, it is not possible to find an exact solution of Eqs. (3.1) with a nontrivial magnetic gauge potential $w(r)$, except for a special value of κ in $D = 5$ dimensions. However, one can write an approximate form of the solutions as a power series with a finite number of undetermined constants, both at infinity and at the horizon/origin of the coordinate system. This analysis allows us to obtain some information on the possible global behavior of solutions.

In deriving the asymptotic form of solutions as $r \to \infty$, we assume that the spacetime approaches the Minkowski background at infinity, while the configurations have a finite Arnowitt-Deser-Misner mass. Then we arrive at the following expansion of the solutions at $r \to \infty$:

$$
\begin{aligned}
m(r) &= M_0 - \frac{1}{2}\alpha^2(D-3)\frac{Q^2}{r^{D-3}} + \ldots, \\
\sigma(r) &= 1 - \frac{\alpha^2(D-3)^2J^2}{4r^{2(D-2)}} + \ldots, \\
w(r) &= \pm 1 - \frac{J}{r^{D-3}} + \ldots, \\
V(r) &= V_0 - \frac{Q}{r^{D-3}} + \ldots.
\end{aligned} \tag{3.8}
$$

In the above relations, J, M_0, V_0 are parameters given by numerics which fix all higher order terms, while Q is a constant fixing the electric charge of the solutions,

$$Q = \frac{1}{D-3}\left(\frac{P}{\alpha^2} + (D^2-1)\kappa F(\pm 1)\right), \tag{3.9}$$

where

$$F(-1) = -F(1) = (-1)^{(D-1)/2}\sqrt{\pi}\frac{\Gamma(\frac{D-1}{2})}{2\Gamma(\frac{D}{2})}.$$

The set (3.8) of boundary conditions is shared by both globally regular and black hole solutions.

C. Soliton solutions: The expansion at $r = 0$

These are perhaps the simplest possible solutions of the system (3.1) and can be viewed as higher dimensional generalizations of the Bartnik-Mckinnon solutions [13], though dressed with an electric charge. The most striking feature here is that the electric charge of the solitons is fixed by the value of the CS coupling constant. Technically, this results from the fact that the term $P/\alpha^2 + (D^2-1)\kappa F(w)$ in the first integral (3.3) should vanish as $r \to 0$. Since $w(0) = 1$ for regular solutions, the parameter P is fixed to be

$$P = 2\alpha^2(-1)^{(D-1)/2}\kappa\sqrt{\pi}\frac{\Gamma(\frac{D+3}{2})}{\Gamma(\frac{D}{2})}. \tag{3.10}$$

One finds, e.g., $P = 16\alpha^2\kappa$, $-128\alpha^2\kappa/5$, and $256\alpha^2\kappa/7$ for $D = 5$, 7, and 9, respectively.

Then, for globally regular soliton-type solutions, the electric charge parameter which enters the far field expression (3.8) is fixed by

$$Q = Q^{(c)} = \kappa\frac{4(-1)^{(D-1)/2}\sqrt{\pi}}{(D-3)}\frac{\Gamma(\frac{D+3}{2})}{\Gamma(\frac{D}{2})}. \tag{3.11}$$

Its expression for the values of D considered in numerics is $Q^{(c)} = 16\kappa$, $-64\kappa/5$, and $256\kappa/21$ for $D = 5$, 7, and 9, respectively.

Also, one finds that the globally regular configurations have the following expansions near the origin $r = 0$:

$$
\begin{aligned}
w(r) &= 1 - br^2 + O(r^4), \\
V(r) &= (-1)^{(D-1)/2}2^{(D-3)/2}b^{(D-1)/2}(D+1)\kappa\sigma_0 r^2 + O(r^4), \\
m(r) &= (D-2)\alpha^2b^2r^{D-1} + O(r^D), \\
\sigma(r) &= \sigma_0 + (D-2)\alpha^2b^2\sigma_0 r^2 + O(r^4).
\end{aligned} \tag{3.12}
$$

The free parameters here are $b = -\frac{1}{2}w''(0)$ and $\sigma_0 = \sigma(0)$. The coefficients of all higher order terms in the $r \to 0$ expansion are fixed by these parameters.

D. Black holes: The near horizon solution

For the solutions in this work, the event horizon is located at a constant value of the radial coordinate $r = r_h$, with $N(r_h) = 0$. In this case, one can also write an approximate form of the solutions near the horizon, as a power series in $r - r_h$. In the nonextremal case, the first terms in this expansion are

$$
\begin{aligned}
m(r) &= r_h^{D-3} + m_1(r - r_h) + \ldots, \\
\sigma(r) &= \sigma_h + \frac{3\sigma_h w_1^2}{2r_h}(r - r_h) + \ldots, \\
w(r) &= w_h + w_1(r - r_h) + \ldots, \\
V(r) &= v_1(r - r_h) + \ldots,
\end{aligned} \tag{3.13}
$$

where

$$
\begin{aligned}
v_1 &= \frac{\sigma_h}{\alpha^2 r_h^{D-2}}(P + \alpha^2(D^2-1)\kappa F(w_h)), \\
m_1 &= \frac{\alpha^2}{2}r_h^{D-2}\left(\frac{v_1^2}{\sigma_h^2} + (D-2)(D-3)\frac{(1-w_h^2)^2}{2r_h^4}\right), \\
w_1 &= \frac{(D-3)w_h(w_h^2-1)}{r_h(D-3-\frac{m_1}{r_h^{D-4}})} \\
&\quad + \frac{(D^2-1)\kappa r_h v_1(w_h^2-1)^{(D-3)/2}}{(D-2)\sigma_h(-m_1 + (D-3)r_h^{D-4})}, \\
\sigma_1 &= \frac{\alpha^2(D-2)}{2r_h}\sigma_h w_1^2,
\end{aligned} \tag{3.14}
$$

while $N(r) = N'(r_h)(r - r_h) + \ldots$, with $N'(r_h) = (D - 3 - m_1/r_h^{D-4})/r_h$. The obvious condition $N'(r_h) > 0$ implies the existence of a lower bound on the event horizon radius $r_{\min}$, for given values of Q and κ. The only free parameters in the expansion above are σ_h and w_h. In a numerical approach, their values are found by matching the near horizon form of the solutions (3.13) with the asymptotic expansion (3.8).

As $r \to r_{\min}$, the function $N(r)$ develops a double zero at the horizon, i.e. $N(r) = N_2(r - r_h)^2 + \ldots$, and the black holes become extremal. We shall see that such solutions exist indeed, emerging as limiting configurations of a branch of nonextremal black holes. The near horizon expansion is more constrained in this case. Supposing $w_h^2 \neq 1$, one finds that the event horizon radius r_h is the largest positive solution of the equation

$$1-\frac{\alpha^2(D-2}{4}(1-w_h^2)^2\left(\frac{1}{r_h^2}+\frac{2(D-2)(D-3)r_h^{2(D-5)}w_h^2}{(D^2-1)^2\kappa^2(w_h^2-1)^{D-3}}\right)=0, \tag{3.15}$$

the value of the electric charge parameter also being fixed [with $w(\infty)=\pm 1$]:

$$Q=\frac{(D-2)r_h^{2(D-4)}w_h}{(D^2-1)\kappa(w_h^2-1)^{(D-5)/2}}\times\left(\frac{(D^2-1)^2\kappa^2(w_h^2-1)^{(D-5)/2}}{(D-2)(D-3)w_h r_h^{2(D-4)}}(F(\pm 1)-F(w_h))-1\right). \tag{3.16}$$

Also, the coefficient N_2 in the leading order expansion of the metric function $N(r)$ is

$$N_2=\frac{(D-3)(D-4)}{2r_h^2}\left[1+\frac{\alpha^2(D-2)(D-6)(w_h^2-1)^2}{4(D-4)r_h^2}\times\left(\frac{2(D-2)^2(D-3)r_h^{2(D-4)}w_h^2}{(D-6)(D^2-1)\kappa^2(w_h^2-1)^{D-3}}-1\right)\right]. \tag{3.17}$$

The parameters w_1, v_1, and σ_1 have a similar expression as those found in the nonextremal case [there, one should replace the expressions (3.15) and (3.16) for r_h and Q, respectively].

For $D=5$ and $Q>Q^{(c)}$, we have found numerical evidence for the existence of a different type of extremal black hole, with $w(r_h)=1$. As $r\rightarrow r_h$, these solutions have basically the same leading order expression as an extremal RN black hole. However, the next to leading order terms in the near horizon expansion exhibit noninteger powers of $r-r_h$. These special $D=5$ configurations will be discussed in Sec. IV C below.

E. An $AdS_2\times S^{D-2}$ solution

As expected, the near horizon structure of the extremal solutions with $w_h^2\neq 1$ can be extended to a full $AdS_2\times S^{D-2}$ solution of the field equations. This is a new, exact, essentially nA solution to the EYMCS field equations, with

$$ds^2=v_1\left(\frac{dr^2}{r^2}-r^2dt^2\right)+v_2 d\Omega_{D-2}^2,\qquad w(r)=w_0,\quad V(r)=qr, \tag{3.18}$$

where

$$q=-\frac{(D-2)(D-3)v_1v_2^{(D-6)/2}}{(D^2-1)\kappa}\frac{w_0}{(w_0^2-1)^{(D-5)/2}}. \tag{3.19}$$

The AdS radius satisfies the relation

$$v_1=\frac{8v_2^2}{(D-3)}\left(4(D-4)v_2+\alpha^2(D-2)(w_0^2-1)^2\times\left(6-D+\frac{2(D-2)^2(D-3)w_0^2v_2^{D-4}}{(D^2-1)^2\kappa^2(w_0^2-1)^{D-3}}\right)\right)^{-1}, \tag{3.20}$$

where the size of the S^{D-2} part of the metric results as a solution of the equation

$$v_2=\frac{1}{4}\alpha^2(D-2)(w_0^2-1)^2\times\left(1+\frac{2(D-2)(D-3)}{(D^2-1)\kappa^2}\frac{w_0^2}{(w_0^2-1)^{D-3}}v_2^{D-4}\right), \tag{3.21}$$

being a function of w_0 only. Unfortunately, one cannot write a simple solution of the above equation, except for $D=5$. In this case, the general solution reads

$$v_1=\frac{1536\alpha^2\kappa^4(w_0^2-1)^2}{(64\kappa^2-\alpha^2w_0^2)(64\kappa^2+\alpha^2w_0^2)},\qquad v_2=\frac{48\alpha^2\kappa^2(w_0^2-1)^2}{64\kappa^2-\alpha^2w_0^2},\qquad q=\frac{32\sqrt{3}\alpha\kappa^2w_0(w_0^2-1)}{\sqrt{64\kappa^2-\alpha^2w_0^2}(64\kappa^2+\alpha^2w_0^2)}. \tag{3.22}$$

One can see that, for any D, the properties of the $AdS_2\times S^{D-2}$ solution are uniquely specified by the constant w_0. Note that for $w_0^2\neq 1$, the nA magnetic gauge field is nonvanishing; the case $w_0=\pm 1$ is special and describes the near horizon geometry of the extremal RN solutions.

Although finding local solutions in the vicinity of the horizon does not guarantee the existence of the global solutions, the above result provides an argument that the EYMCS system is likely to present asymptotically flat, extremal black hole solutions. In $D=5$ and $D=7$, 9, this is confirmed by our numerical and analytic results. If there are extremal black holes in the bulk, the parameter q in (3.18) is related to the bulk charge parameter Q via

$$q=\frac{v_1}{v_2^{(D-2)/2}}((D-3)Q-(D^2-1)\kappa(F(w_0)+F(\pm 1)), \tag{3.23}$$

with $w(\infty)=\pm 1$, the allowed values at infinity of the bulk magnetic gauge potential.

It would be interesting to consider these solutions in the context of the attractor mechanism and to compute their entropy functions (for a discussion of Sen's entropy function for $D=5$ supergravity models containing Abelian Chern-Simons terms, see e.g. [40]).

F. Relevant parameters and global charges

The only global charges associated with the solutions are the mass $\mathcal{M}$ and the electric charge $\mathcal{Q}$,

$$\mathcal{M} = \frac{(D-2)V_{D-2}M_0}{16\pi G}, \quad \mathcal{Q} = \frac{(D-3)V_{D-2}Q}{g}, \quad (3.24)$$

where $V_{D-2} = 2\pi^{(D-1)/2}/\Gamma((D-1)/2)$ is the area of the unit $D-2$ sphere. The mass $\mathcal{M}$ is the charge associated with the Killing vector $\partial/\partial t$; the electric charge $\mathcal{Q}$ is associated with the $U(1)$ gauge symmetry generated by $\Sigma^{(\pm)}_{2n+1,2n+2}$. For the boundary conditions in this work, the electrostatic potential difference Φ between the horizon and infinity is fixed by the value at infinity of the electric potential $V(r)$, as read from (3.8). In a thermodynamical description of the system, Φ corresponds to the chemical potential,

$$\Phi = \frac{V_0}{e}. \quad (3.25)$$

The Hawking temperature and the entropy of the black holes are given by

$$T_H = \frac{1}{4\pi}\sigma(r_h)N'(r_h),$$
$$S = \frac{A_H}{4G}, \quad \text{with} \quad A_H = V_{D-2}r_h^{D-2}. \quad (3.26)$$

The solutions should also satisfy the first law of thermodynamics,

$$d\mathcal{M} = T_H dS + \Phi d\mathcal{Q}. \quad (3.27)$$

In the discussion of the black hole thermodynamics, we shall restrict ourselves to configurations in a canonical ensemble, the relevant potential being the Helmholz free energy

$$F[T_H, \mathcal{Q}] = \mathcal{M} - T_H S. \quad (3.28)$$

As usual, in practice, it is convenient to work with quantities which are invariant under the rescaling (3.5) (note that in the numerics we set $\alpha = 1$). For solutions in a canonical ensemble, we normalize the global quantities with respect to the charge parameter Q and define, e.g., the dimensionless quantities

$$f = \frac{F}{Q/g^2}, \qquad t_H = T_H Q^{1/(D-3)},$$
$$a_H = \frac{A_H}{Q^{(D-2)/(D-3)}}, \quad \text{and} \quad j = \frac{J}{Q}. \quad (3.29)$$

One should note that there is no conserved quantity associated with the parameter J which appears in the large-r asymptotics (3.8). Also, J does not enter the first law (3.27).

In addition to the mass and the electric charge, there is another global charge which has a topological origin. This is the volume integral $\mathcal{P}$ of the topological density calculated from the "magnetic" components, F_{ij}, namely, the CP density defined on the $(D-1)$-dimensional space dimensions with Euclidean signature. The correct expression for this CP density must take account of the fact that the gauge group for F_{ij} is $SO(D-1)$, and not[9] one or the other of the two chiral algebras $SO_{\pm}(D-1)$. Using the spherically symmetric components of the curvature F_{ij} given by (2.16), this charge density is calculated to be

$$\varepsilon_{i_1i_2i_3i_4\ldots i_{D-2}i_{D-1}} \mathrm{Tr}\{\Sigma_{D,D+1}F^{i_1i_2}F^{i_3i_4}\ldots F^{i_{D-2}i_{D-1}}\}$$
$$\simeq -\frac{1}{r^{D-2}}(w^2-1)^{(D-3)/2}w'. \quad (3.30)$$

Usually, for spherically symmetric soliton solutions, the integral of the above quantity is suitably normalized such that it yields *unit* magnetic CP charge, $\mathcal{P} = 1$. However, noticing that $(w^2-1)^{(D-3)/2}w'$ is just the derivative of the function $F(w)$ which enters the first integral (3.3), it is more interesting to use a nonstandard normalization and to define the magnetic CP charge $\mathcal{P}$ directly as the integral of (3.30). Then, taking into account the boundary conditions[10] (3.8) and (3.12), it follows that the soliton solutions exhibit an interesting connection between the electric charge and the magnetic CP charge

$$\mathcal{Q} = \kappa\mathcal{P}. \quad (3.31)$$

G. The Reissner-Nordström solution

The RN solution plays an important role in what follows. Thus, for completeness, we briefly discuss here its basic thermodynamical properties. This solution is recovered for $w(r) \equiv \pm 1$, in which case the magnetic components of the field strength vanish identically. Then the configuration becomes essentially Abelian and one finds the following exact solution:

$$m(r) = M_0 - \frac{\alpha^2(D-3)Q^2}{2r^{D-3}}, \qquad \sigma(r) = 1,$$
$$w(r) = \pm 1, \qquad V(r) = V_0 - \frac{Q}{r^{D-3}}, \quad (3.32)$$

with V_0 a constant which is usually chosen such that the electric potential vanishes on the horizon, i.e., $V_0 = Q/r_h^{D-3}$. The mass $\mathcal{M}$, electric charge $\mathcal{Q}$, and chemical potential Φ follow automatically from (3.24) and (3.25). The Schwarzschild-Tangherlini vacuum black hole corresponds to the case $Q = 0$.

The RN solution has an outer event horizon at $r = r_h$, with

[9] Note that the $SO(D-1)$ curvature consists of two $SO_{\pm}(D-1)$ curvatures, each contributing ± 1 CP charge in the spherically symmetric case. If the appropriate factor of $\Sigma_{D,D+1} = \gamma_D$ in (3.30) is not accounted for in the CP density, these two charges will cancel each other out. This is discussed in detail in the context of the $SO(D)$ monopole on IR^D in [41].

[10] Note that we take $w(\infty) = -1$, which was the case for all solutions we could find numerically.

$$r_h = \left(\frac{1}{2}(M_0 + \sqrt{M_0^2 - 2\alpha^2(D-3)Q^2})\right)^{1/(D-3)}, \quad (3.33)$$

whose existence imposes an upper bound for the electric charge for a given mass. The Hawking temperature and the entropy of this solution can be written in terms of r_h, Q (which are the parameters used in numerics) as

$$T_H = \frac{D-3}{4\pi r_h}\left(1 - \frac{(D-3)}{2}\frac{\alpha^2 Q^2}{r_h^{2(D-3)}}\right), \quad S = \frac{V_{D-2} r_h^{D-2}}{4G}. \quad (3.34)$$

With these definitions, one can easily verify that the first law (3.27) is indeed fulfilled. In addition, the RN black holes satisfy the Smarr law

$$\mathcal{M} = \frac{D-2}{D-3} T_H S + \Phi Q. \quad (3.35)$$

The relation (3.34) shows that, for a given Q, there is a minimal value of the event horizon radius at which an extremal black hole is approached,

$$r_h \geq r_h^{(\min)} = (\tfrac{1}{2}(D-3)\alpha^2 Q^2)^{1/(2(D-3))}. \quad (3.36)$$

Thus the Hawking temperature vanishes in this limit, while the event horizon area approaches a minimal value, with

$$a_H^{(\min)} = \tfrac{1}{2}(D-3)\alpha^2 V_{D-2}. \quad (3.37)$$

After expressing r_h as a function of A_H according to (3.26), one gets the following relation between the reduced variables t_H and a_H:

$$t_H = \frac{(D-3)}{4\pi}\left(\frac{V_{D-2}}{a_H}\right)^{1/(D-2)} \times \left(1 - \frac{\alpha^2(D-3)}{2}\left(\frac{V_{D-2}}{a_H}\right)^{(2(D-3))/(D-2)}\right). \quad (3.38)$$

Unfortunately, one cannot invert this relation to get $S(T_H, Q)$. However, one can see that, for a given Q, the solutions exist only for $0 \leq T_H \leq T_H^{(\max)}$, the maximal value of the Hawking temperature being

$$T_H^{(\max)} = \frac{1}{2^{(2D-7)/(2D-6)}\pi}(D-3)^{(4D-13)/(2(D-3))} \times (2D-5)^{-(((2D-5))(2(D-3)))}\frac{1}{(\alpha Q)^{1/(D-3)}}, \quad (3.39)$$

and the entropy at this turning point being

$$S(T_H^{(\max)}) = \frac{V_{D-2}}{G}\left(\frac{2^{(5D-14)/(2(D-2))}}{\alpha\sqrt{(D-3)(2D-5)}Q}\right)^{(D-2)/(D-3)}. \quad (3.40)$$

It is also of interest to express the free energy of the RN solution as a function of T_H, Q. The only relation we could find in this case reads

$$f = \frac{1}{t_H^{D-3}}\frac{(D-3)^{D-3}}{(D-2)2^{2D-7}\pi^{D-3}}\frac{V_{D-2}}{\alpha^2}\left(1 + \sqrt{1 - \frac{2(D-3)(2D-5)V_{D-2}^2}{\alpha^2(D-2)^2}\frac{1}{f^2}}\right)^{-1} \times \left[1 - f^2\frac{\alpha^2(D-2)^2}{2(2D-5)^2(D-3)V_{D-2}^2}\left(1 - \sqrt{1 - \frac{2(D-3)(2D-5)V_{D-2}^2}{\alpha^2(D-2)^2}\frac{1}{f^2}}\right)^2\right]^{D-3}. \quad (3.41)$$

A study of the above relations shows the existence of two branches of Abelian solutions, each with different thermal properties. (The generic picture here is dimension independent; then the well-known $D = 4$ result in [42] applies also in higher dimensions.) For a fixed Q, there is first a branch of large black holes whose entropy decreases with T_H, and are therefore unstable. This branch stops in a critical configuration with a maximal value of T_H given by (3.39), where a secondary branch of small black holes emerges. This branch has a positive specific heat and ends in an extremal configuration with an event horizon area given by (3.37). These features are exhibited later in Figs. 7 and 8 (see the RN curves there). The picture for $D > 5$ is qualitatively the same.

H. The issue of perturbative static solutions around Reissner-Nordström black holes

Since the RN black hole is a solution of the model for any D, one might expect the existence of a branch of nA solutions connected with it. Such solutions, if they exist, would emerge as *static* perturbations around the RN background. However, as we shall argue, this is the case for $D = 5$ only.

Suppose that there is a perturbative solution of Eqs. (3.1) around the RN black hole,

$$\begin{aligned} m(r) &= m_0(r) + \epsilon m_1(r) + \ldots, \\ \sigma(r) &= 1 + \epsilon\sigma_1(r) + \ldots, \\ w(r) &= \pm 1 + \epsilon W_1(r) + \ldots, \\ V(r) &= V_0(r) + \epsilon V_1(r) + \ldots, \end{aligned} \quad (3.42)$$

with ϵ a small parameter. In the above relations, $m_0(r)$, $V_0(r)$ are the functions which enter the RN solution (3.32).

After substituting (3.42) in Eqs. (3.1), one finds that to lowest order, the equation for $W_1(r)$ decouples. This equation is the only relevant one, its general-D expression being

$$(r^{D-4}NW_1')' = 2((D-3)r^{D-6} \pm 8\kappa V_0'\delta_{D,5})W_1 \quad (3.43)$$

[with $N = 1 - m_0(r)/r^{D-3}$]. Then it turns out that the case $D = 5$ is special, since the CS term gives a nonzero contribution to the W_1 equation only for this dimension.

The perturbation $W_1(r)$ starts from some (arbitrary) nonzero value at the horizon and vanishes at infinity, in order to be consistent with the asymptotic behavior (3.13) and (3.8). However, one can show that for $D > 5$, there is no solution of (4.14) that satisfies this asymptotic behavior. To prove that, we rewrite Eq. (3.43) in the equivalent form

$$\frac{1}{2}(r^{D-4}N(W_1^2)')' = r^{D-4}NW_1'^2 + 2(D-3)r^{D-6}W_1^2, \tag{3.44}$$

recalling that $D > 5$ now. Then, after integrating from r_h to infinity, one finds that W_1 necessarily vanishes identically.[11] The same argument applies when considering higher order terms in the expansion (3.42). Therefore we conclude that, for $D > 5$, the RN solution is stable with respect to nA perturbations within the considered EYMCS model.[12]

For $D = 5$, a similar reasoning implies that the "mass" term $1 \pm \frac{8\kappa Q}{r^2}$ in Eq. (3.43) should necessarily be negative in the vicinity of the horizon. Then, one finds $\kappa Q > 0$ for $w(\infty) = -1$ and $\kappa Q < 0$ for $w(\infty) = 1$.

IV. THE RESULTS IN $D = 5$

A. Numerical methods

The scaling transformation (3.5) can be used to fix an arbitrary value[13] for α. The usual choice is $\alpha = 1$, which is what we employ for all solutions in this work. This fixes the EYM length scale $L = \sqrt{16\pi G/((D-2)e^2)}$, while the mass scale is fixed by $\mu = L^{(D-3)/2}/G$. All other quantities get multiplied with suitable factors of L.

To control the quality of the numerical results, we have performed some of the calculations with two different methods, finding excellent agreement. First, Eqs. (3.1) were solved with suitable boundary conditions which result from (2.31) and (2.32) using a standard solver [44]. This solver involves a Newton-Raphson method for boundary-value ordinary differential equations, equipped with an adaptive mesh selection procedure. Typical mesh sizes include 10^3–10^4 points. The solutions in this work have a typical relative accuracy of 10^{-7}. In this approach, the value of the electric potential at infinity V_0 is fixed, the electric charge resulting from numerics; i.e., the configurations are in a grand canonical ensemble. [The first integral (3.3) has been used to verify the accuracy of the solutions.]

In addition to employing this algorithm, families of solutions with a fixed electric charge were constructed by using a standard Runge-Kutta ordinary differential equation solver. In this approach we evaluate the initial conditions at $r = r_h + 10^{-5}$, for global tolerance 10^{-12}, adjusting for shooting parameters and integrating towards $r \to \infty$. In this case the electric charge Q was fixed via Eq. (3.3), the electrostatic potential V_0 resulting from numerics. We have confirmed that there is good agreement between the results obtained with these two different methods.

Also, for both approaches, we have restricted our integration to the (physically more relevant) region outside of the horizon, $r \geq r_h$.

B. Perturbative solutions: An instability of the RN_5 black hole

The branch of $D = 5$ nA solutions emerges as a perturbation of the RN black hole. (In what follows, we shall suppose without any loss of generality that the RN black hole has a positive electric charge, $Q > 0$.)

The perturbative solutions are found by solving Eq. (3.43), which for $D = 5$ reads

$$r(rNW_1')' - 4\left(1 \pm \frac{8\kappa Q}{r^2}\right)W_1 = 0, \tag{4.1}$$

where $N = 1 - \frac{Q^2 + r_h^4}{r_h^2 r^2} + \frac{Q^2}{r^4}$. Although this linear equation does not appear to be solvable in terms of known functions, one can construct an approximate solution near the horizon and at infinity. For a vacuum choice $w \equiv -1$, one finds that, as $r \to r_h$,

$$W_1(r) = W_h + \frac{2W_h r_h(-8\kappa Q + r_h^2)}{r_h^4 - Q^2}(r - r_h) + \ldots, \tag{4.2}$$

with all higher order coefficients fixed by W_h. Because (4.2) is linear, one can take $W_h = 1$, without any loss of generality.

At infinity, the only reasonable asymptotics reads

$$W_1(r) = \frac{J}{r^2} + \frac{2J(Q^2 - 4\kappa Q r_h^2 + r_h^4)}{3r_h^2 r^4} + \ldots, \tag{4.3}$$

in terms of a free parameter J. The solutions interpolating between (4.2) and (4.3) are constructed numerically, with typical results being shown in Fig. 1.

It turns out that such perturbative solutions do not exist for arbitrary values of (r_h, κ). Restricting to solutions of (4.1) with monotonic behavior[14] of $W_1(r)$, we find that for given (r_h, κ) for which such solutions exist, these pertain to a fixed value of Q. This value results from the numerics.

The existence of these configurations can be understood as follows. For $w(\infty) = -1$, the second term in (3.43) shows the existence of an effective mass term μ^2 for W_1 near the horizon, with $\mu^2 \sim 1 - 8\kappa Q/r_h^2$; all solutions we

[11] Basically, the right-hand side of (3.44) is greater than or equal to zero, while $N(r_h) = 0$ and $W_1(\infty) = 0$.

[12] At the cost of replacing the usual Yang-Mills term F^2 by F^{2p} [43], one can expect that a branch of the static nA solution emerges as a perturbation around the corresponding RN-type background, in the $D = 4p + 1$ case, also.

[13] In principle, one can use (3.5) to fix instead the value of the CS coupling constant κ. However, this choice is less interesting.

[14] Note that there are also solutions of (4.1) where $W_1(r)$ has nodes.

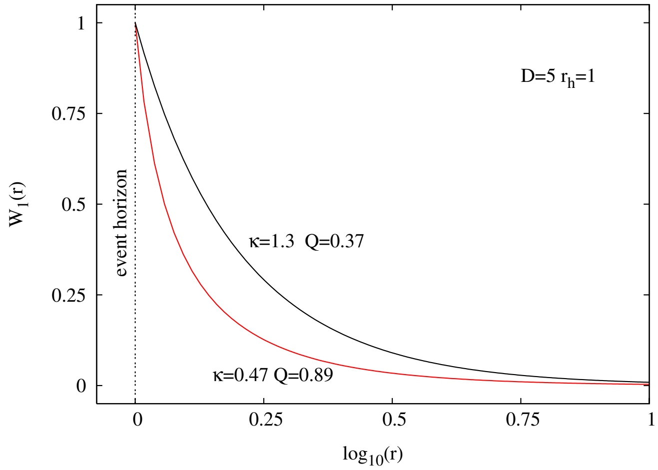

FIG. 1 (color online). The profiles of typical solutions of the $D = 5$ perturbation equation (4.1) are presented as a function of the radial coordinate r for two different values of κ. The values of the charge parameter Q for the corresponding critical Reissner-Nordström solutions with $r_h = 1$ are also shown.

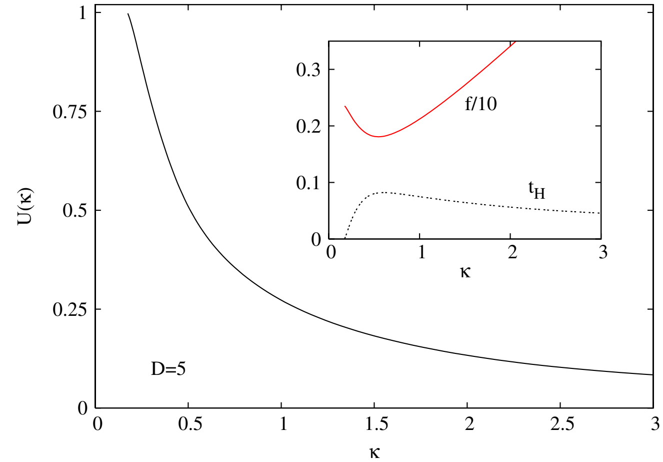

FIG. 2 (color online). The shape function $U(\kappa) = Q/r_h^2$ which gives the unstable Reissner-Nordström solution where a branch of non-Abelian configurations emerges is shown as a function of the CS coupling constant κ. The inset shows the scaled free energy and Hawking temperature of critical Abelian solutions.

could find have $\mu^2 < 0$. Then it is also convenient to introduce the dimensionless function

$$U(\kappa) = \frac{Q}{r_h^2}, \tag{4.4}$$

which uniquely fixes the parameters of the critical RN solution. One finds e.g.

$$t_H = \frac{\sqrt{U}(1 - U^2)}{2\pi}, \qquad f = \frac{\pi}{8U}(1 + 5U^2), \quad \text{while} \quad \frac{\mathcal{M}}{Q} = \frac{3\pi}{8U}(1 + U^2). \tag{4.5}$$

The shape of the function $U(\kappa)$ for the fundamental set of solutions of (4.1), for which $W_1(r)$ are nodeless, is shown in Fig. 2. The inset there shows the scaled free energy and temperature of the critical RN solutions where static linear nA perturbations arise, as a function of κ.

The function $U(\kappa)$ depends monotonically on the CS coupling constant κ. As $\kappa \to 1/8$, one finds $U = 1$; i.e., the corresponding RN solution becomes extremal, while the $D = 5$ RN black hole with the maximal value of the temperature (3.39) is unstable for $\kappa \simeq 0.55$. Also, one finds that the function U decreases along the branch of large black holes, with $U \simeq 1/4\kappa$ for large κ.

No physically reasonable solutions of Eq. (3.43) are found for $\kappa < 1/8$, or for perturbations of the form $w(r) = +1 + \epsilon W(r)$, in which cases the effective mass for W is always real (we recall that we take $Q > 0$, $\kappa > 0$).

C. Nonperturbative black hole solutions

1. General properties

The instability discussed above signals the presence of a symmetry breaking branch of nA solutions bifurcating from the RN black hole.

Indeed, our numerical results provide evidence for the existence of finite mass black hole solutions of the EYMCS system with nontrivial magnetic gauge fields outside the horizon. These solutions smoothly interpolate between the asymptotics (3.13) and (3.8) [all $D = 5$ configurations have $w(\infty) = -1$].

On the basis of analytical and numerical results we have a pretty clear picture of the behavior of the $D = 5$ EYMCS black holes, this case being studied in a systematic way. The properties of the solutions depend on the value of the CS coupling constant κ. For a fixed r_h, the value of the electric charge parameter Q is also important, some basic features of the solutions depending on whether Q is less or greater than $Q^{(c)} = 16\kappa$.

As a general feature, we could not find configurations with multinodes of the function $w(r)$. Heuristically, this can be understood as follows: The existence of such configurations would imply $w(r) = 0$ as the limit of multinodes. However, such solutions would have infinite energy, with $m(r) = 3/2 \log r + M_0$, $\sigma(r) = 1$, and $V(r) = 0$, which is not compatible with the boundary conditions in the present work.

More importantly, since our solutions emerge as perturbations of RN black holes, we notice the existence of configurations without nodes of the magnetic potential $w(r)$. We shall see that some of these solutions are stable. Also, different from the case of other asymptotically flat hairy black holes with nA fields [12], $w(r)$ may take values outside the interval $[-1, +1]$.

Moreover, it is possible to find more than one solution for the same value of (κ, Q, r_h). In this case, apart from configurations with a monotonic behavior of $w(r)$, there are solutions with local extrema of the magnetic gauge potential; see Fig. 3 for such an example. However, it is likely that the solutions with local extrema are always

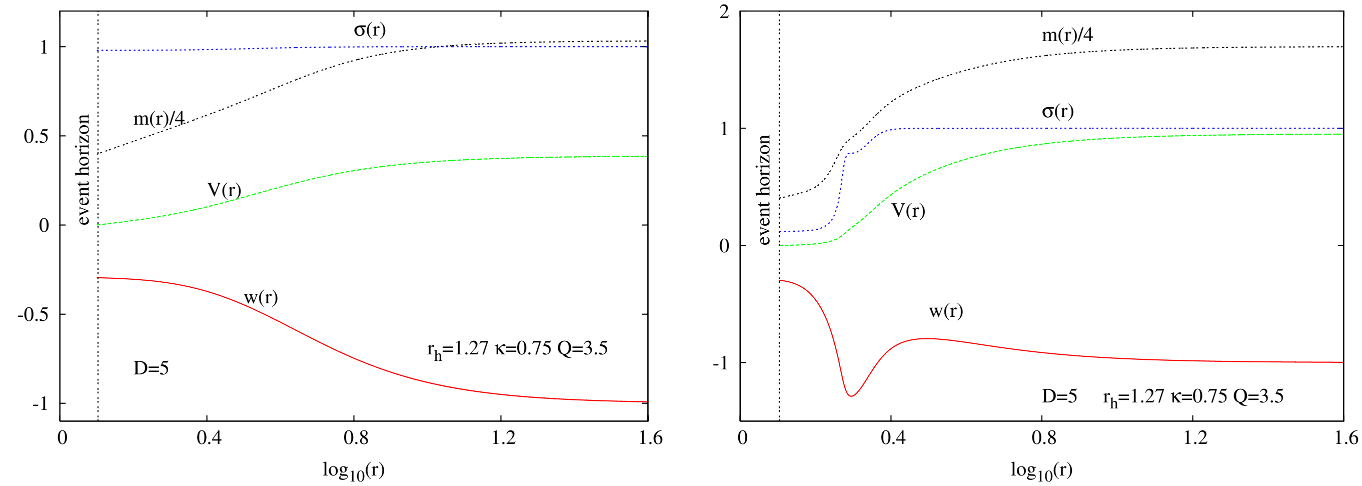

FIG. 3 (color online). The profiles of two EYMCS black hole solutions are presented as a function of the radial coordinate r. $m(r)$ and $\sigma(r)$ are metric functions, while $V(r)$ and $w(r)$ are electric and magnetic gauge potentials, respectively.

thermodynamically disfavored because spatial oscillations in w increase the total mass. Thus, in what follows we shall restrict ourselves to the study of the fundamental branch of solutions with a monotonic behavior of the nA gauge function, i.e., $w'(r) < 0$ everywhere.

Some of the general features of the solutions are shown in Fig. 4, where we plot the values w_h of the magnetic gauge potential at the horizon and V_0 of the electric potential at infinity as functions of r_H for a fixed κ and several values of the charge parameter Q. For example, one can see that the solutions exist for $w_h^{(\max)} \leq w_h \leq -1$, where the value $w_h^{(\max)}$ increases with Q. For $Q < Q^{(c)}$ one finds $w_h^{(\max)} < 1$, while $w_h^{(\max)} = 1$ for $Q \geq Q^{(c)}$.

The behavior of solutions as a function of the electric charge for a fixed event horizon radius (i.e., at fixed entropy) is displayed in Fig. 5. The value of the CS coupling constant is also fixed there. One can notice there the existence of both a maximal and a minimal value for Q. The nA solution emerges as a perturbation of a RN black hole for a minimal value of Q given by $r_h^2 U(\kappa)$. It exists up to a maximal value of $Q = r_h^2 + 16\kappa$, where an extremal black hole is approached with $w(r_h) = 1$. Also, above some value of Q close to $Q^{(c)}$, $w(r_h)$ becomes very close (but not equal) to 1.

In Fig. 6 we plot the order parameter J which enters the first relevant term in the large-r expansion of the magnetic gauge potential as a function of the Hawking temperature (i.e. a varying r_h) and several values of the electric charge parameter. Again, the behavior of J depends crucially on the value of Q. For $Q < Q^{(c)}$, J approaches a constant

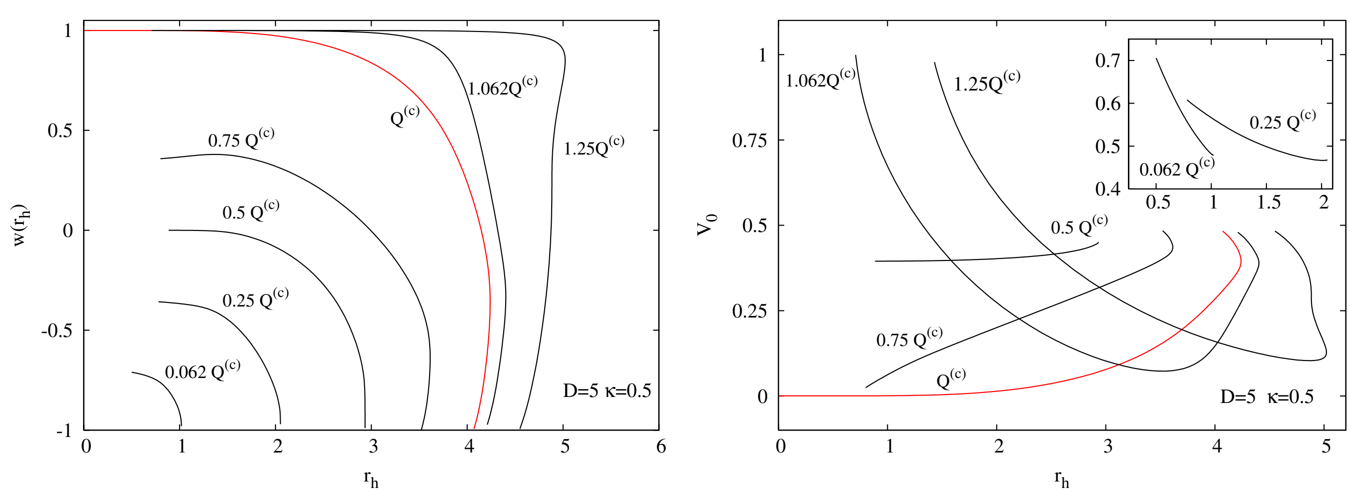

FIG. 4 (color online). The values at the event horizon of the magnetic gauge potential $w(r_h)$ (left panel) and of the electric potential at infinity V_0 (right panel) are shown as functions of the event horizon radius r_h for a fixed value of the Chern-Simons coupling constant κ and several values of the electric charge. Here and in Figs. 5–8, $Q^{(c)} = 16\kappa$ is the critical value of the electric charge parameter.

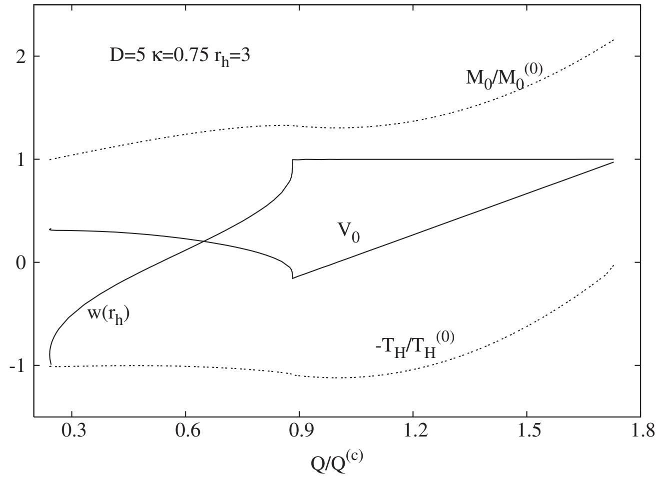

FIG. 5. Several relevant parameters are plotted as a function of the electric charge parameter Q for black hole solutions with a fixed value of the Chern-Simons coupling constant κ and a given event horizon radius. The values of the mass parameter M_0 and of the Hawking temperature T_H are normalized with respect to the critical Reissner-Nordström solution, where a branch of non-Abelian solutions emerges as a perturbation.

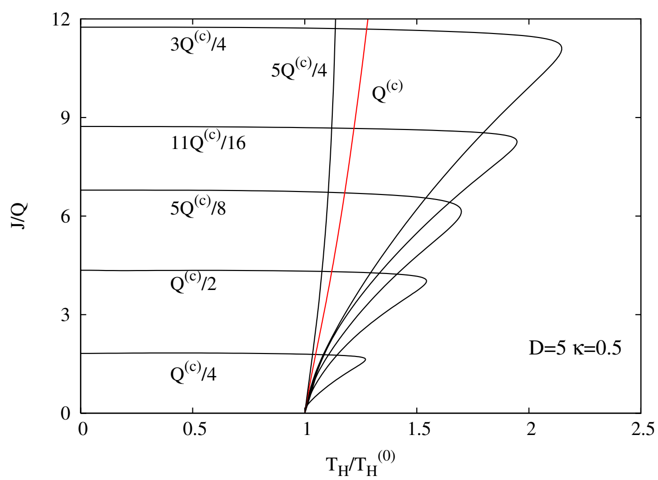

FIG. 6 (color online). The order parameter J which enters the asymptotics of the magnetic gauge potential at infinity is shown as a function of the Hawing temperature T_H for a fixed value of the Chern-Simons coupling constant κ and several values of the electric charge. Note that T_H is normalized with respect to the temperature of the critical Abelian solution, where a branch of non-Abelian solutions emerges as a perturbation.

value as $T_H \to 0$. The behavior is different for $Q \geq Q^{(c)}$, with J in that case increasing strongly with T_H and taking always large values, which makes its accurate computation difficult.

2. The thermodynamics of solutions

The nA solutions appear to exist for values of the CS coupling constant $\kappa \geq 1/8$. Similar to the RN case, the nA black holes with given κ, Q are found only for a finite interval of r_h (i.e., of the entropy). The detailed picture depends, however, on the ratio $Q/Q^{(c)}$. (We recall that for $D = 5$, $Q^{(c)} = 16\kappa$.)

For fixed $Q \neq Q^{(c)}$, the behavior of the solutions is rather similar to the Abelian case, and the temperature reaches its maximum at some intermediate value of the event horizon radius, an extremal black hole being approached for a minimal value of r_h. A plot of the horizon area as a function of the temperature reveals the existence of several branches of nA solutions. The typical picture can be summarized as follows. For a given $\kappa > 1/8$ and any value of Q, a branch of non-Abelian solutions emerges as a perturbation of a critical RN configuration with $r_h = \sqrt{Q/U(\kappa)}$. The entropy increases with temperature along this branch, which has, however, a small extension in both a_H and t_H. This branch continues in a secondary one, where the temperature still increases, while the event horizon area decreases. Thus, these solutions have negative specific heat. For $Q \neq Q^{(c)}$, this branch ends for a maximal value of t_H (whose value depends on Q), where a third branch of solutions emerges. This branch extends backward in (t_H, a_H) and has a positive specific heat. The Hawking temperature vanishes there for a minimal value $r_h^{(\min)}$ of the event horizon radius.

As $r \to r_h^{(\min)}$, an extremal nA black hole solution with an $AdS_2 \times S^3$ near horizon geometry is approached. For $Q < Q^{(c)}$, the parameters of this extremal black hole can be read from (3.15) and (3.16). For example, the near horizon expansion of the solutions implies

$$r_h^{(\min)} = 4\sqrt{3}\kappa \frac{(1 - w_h^2)}{\sqrt{64\kappa^2 - w_h^2}}, \tag{4.6}$$

where w_h satisfies the cubic equation

$$(64\kappa^2 - w_h^2)Q^2 + 2\kappa(1 + w_h)^2 \times (128\kappa^2(w_h - 2) + w_h(w_h^2 - 2w_h + 3)) = 0. \tag{4.7}$$

For $Q > Q^{(c)}$, the limiting extremal solution has

$$r_h = r_h^{(\min)} = \sqrt{Q - Q^{(c)}}, \tag{4.8}$$

and

$$\begin{aligned} H(r) &= \frac{4}{r_h^2}(r - r_h)^2 + \ldots, \\ \sigma(r) &= \sigma_h\left(1 + \frac{3}{2r_h}\frac{w_1^2 k^2}{2k - 1}(r - r_h)^{2k-1}\right) + \ldots, \\ V(r) &= \frac{2}{r_h \sigma_h}(r - r_h) + \ldots,\ w(r) \\ &= 1 + w_1(r - r_h)^k + \ldots, \end{aligned} \tag{4.9}$$

where w_1 and σ_h are parameters fixed by numerics and

$$k = \frac{1}{2}(-1 + \sqrt{5 + 32\kappa}) > 1. \tag{4.10}$$

Although close to the horizon, due to the scaling relation (3.6), this extremal solution is essentially similar to the RN one, but their bulk form is different. The nA solution presents a magnetic hair (outside the horizon) and a metric function $\sigma(r) \neq 1$. Another interesting feature of the nA configurations with $Q > Q^{(c)}$ is that the magnetic flux lines are "expelled" from the black holes as extremality is approached. That is, one finds that the nA magnetic field is vanishing on the horizon of the extremal black holes admitting the approximate expansion (4.9), $F_{ij} = 0$. Thus, these solutions seem to exhibit a sort of nA "Meissner effect" which is characteristic of superconducting media. This should be contrasted with the $Q < Q^{(c)}$ case, which possesses a nontrivial nA magnetic field on the horizon.

Some features of the nA black holes are shown in Figs. 7 and 8 (left panels), where we plot the reduced area of the horizon $a_H = 2\pi^2 r_h^3/Q^{3/2}$ as a function of the dimensionless temperature t_H for a fixed CS coupling constant and several values of the charge parameter Q. The branch of RN solutions as given by (3.38) and (3.41) is also shown there.

Furthermore, it turns out that the free energy $F = \mathcal{M} - T_H S$ of a RN solution is larger than the free energy of a lower branch nA solution with the same temperature and electric charge, except for configurations with κ close to 1/8 and small enough values of the charge, $Q \lesssim Q^{(c)}/3$. Therefore, the nA black holes are generically preferred. These aspects are exhibited in Figs. 7 and 8 (right panels), where the dimensionless free energy f is plotted as a function of the dimensionless temperature t_H. Moreover,

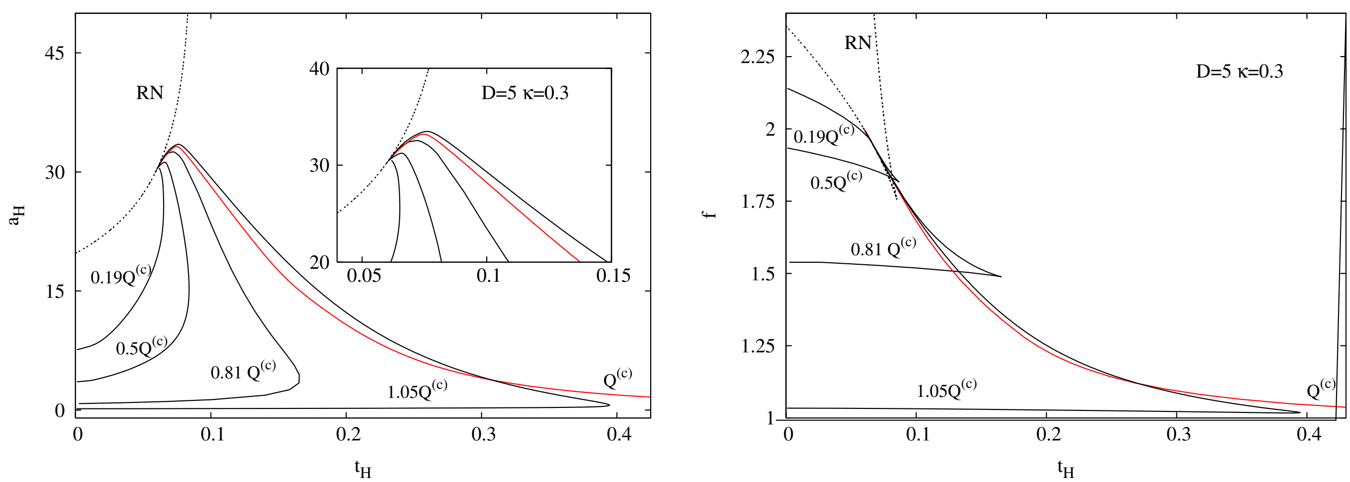

FIG. 7 (color online). The scaled event horizon area a_H (left panel) and free energy f (right panel) are plotted vs the scaled temperature t_H for the non-Abelian solutions with several values of the electric charge and a given value of κ. The branch of RN solutions is also shown.

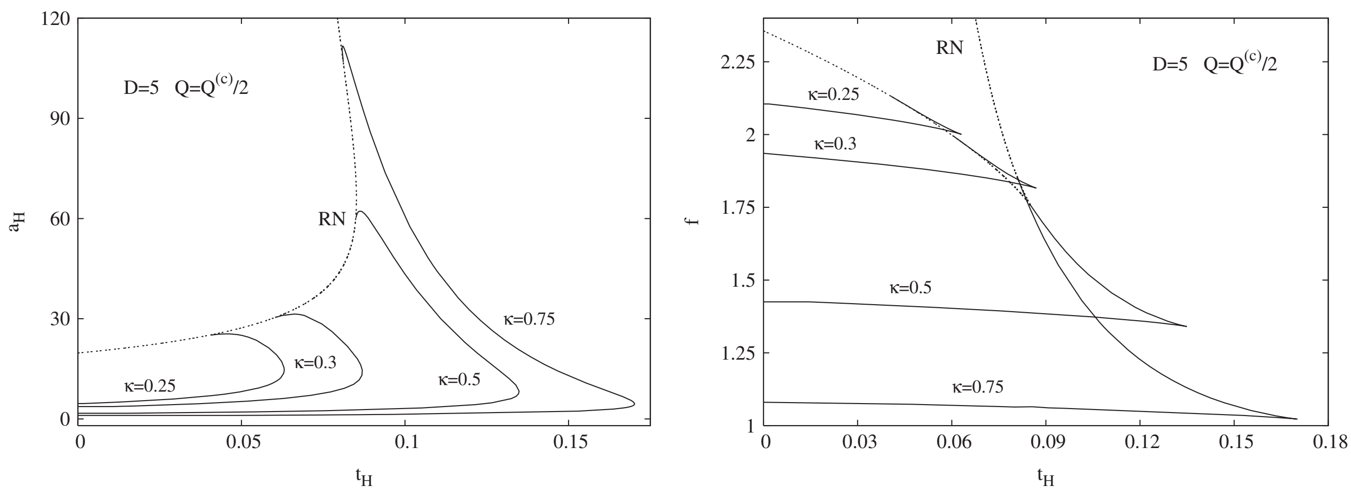

FIG. 8. Same as Fig. 7 for a fixed value of the electric charge parameter $Q = Q^{(c)}/2$ and several different values of the Chern-Simons coupling constant κ.

for the same values of the mass and electric charge, the RN solution typically has a smaller event horizon radius (and thus a smaller entropy) than the nA black hole [28]. Note, however, that most of the nA configurations have no RN counterparts; see Figs. 7 and 8.

Also, one can see that the interval of the scaled temperature t_H covered by the set of nA solutions shrinks to zero as κ approaches the minimal value $1/8$. As $\kappa \to 1/8$, all three branches described above collapse to a single point, which is the extremal black hole solution. This limiting solution admits a closed form expression and will be discussed separately.

The overall picture is somehow different for $Q = Q^{(c)}$, in which case, despite the presence of an electric charge, the nA black holes behave in a similar way to the vacuum Schwarzschild-Tangherlini solution, with a single branch of thermally unstable configurations; see Fig. 7. These solutions emerge again from a critical RN solution with $r_h = \sqrt{Q^{(c)}/U(\kappa)}$ and can be continued for an arbitrarily small value of the event horizon radius. As $r_h \to 0$, the black holes with $Q = Q^{(c)}$ approach a set of globally regular particlelike solutions, with t_H diverging in that limit, as expected.

D. On the existence of $D = 5$ stable black hole solutions

Typically, the existence of an unstable mode of a nA configuration is associated with the zeros of the magnetic gauge potential $w(r)$. Thus the fact that we have found nodeless solutions suggests the existence of configurations which are stable against spherically symmetric perturbations. Moreover, the nA solutions are arbitrarily close to the Abelian RN configuration, which is known to be stable.

Thus in this subsection we address the issue of linear stability of the black holes discussed above. For that purpose, we have to study the evolution of linear perturbations around the equilibrium configuration. For the pure EYM case and with a gauge group $SU(2)$, this has been studied by Okuyama and Maeda in [21], who derived the corresponding pulsation equation. Below we repeat their derivation with a slight modification due to the presence of a CS term and hence also an electric potential. Also, the purpose here is to show the existence of stable black hole solutions rather than study the unstable modes, which for our purposes here is of secondary importance.

In examining such time-dependent fluctuations, we consider the following metric Ansatz generalizing (2.10):

$$ds^2 = \frac{dr^2}{N(r,t)} + r^2 d\Omega_3^2 - N(r,t)\sigma^2(r,t)dt^2, \quad \text{with}$$
$$N(r,t) = 1 - \frac{m(r,t)}{r^2}. \tag{4.11}$$

On the gauge field sector, we shall restrict our study to perturbations within the considered $SO(4)\times SO(2)$ model. For the $U(1)$ part, one can take, without any loss of generality, an Ansatz with a single nonvanishing component $V(r,t)$.

The construction of a general time-dependent Ansatz for the $SO(3)$ gauge group in $D = 5$ dimensions has been extensively discussed in [21]. Interestingly, it turns out that the corresponding YM Ansatz is much more restricted in this case than in $D = 4$ dimensions, containing only one potential $w(r,t)$. This agrees with the result found by taking an $SO(4)\times SO(2)$ truncation of the general Ansatz (2.14) and (2.15).

Then the perturbed variables can be written as

$$\begin{aligned}
m(r,t) &= m(r) + \epsilon m_1(r)e^{i\Omega t} + \ldots,\\
\sigma(r,t) &= \sigma(r)(1 + \epsilon \sigma_1(r)e^{i\Omega t}) + \ldots,\\
w(r,t) &= w(r) + \epsilon w_1(r)e^{i\Omega t} + \ldots,\\
V(r,t) &= V(r) + \epsilon V_1(r)e^{i\Omega t} + \ldots,
\end{aligned} \tag{4.12}$$

with $m(r)$, $\sigma(r)$, $w(r)$, and $V(r)$ static solutions and ϵ a small parameter.

After replacing in the general EYMCS equations (2.7), one finds the following relations valid to first order in ϵ:

$$\begin{aligned}
&w_1'' + \left(\frac{1}{r} + \frac{N'}{N} + \frac{\sigma'}{\sigma}\right)w_1' + \frac{8\kappa(1-w^2)}{rN\sigma}V_1' - \frac{w'}{r^2N}m_1' + w'\sigma_1' + \frac{1}{N}\left(\frac{\Omega^2}{N\sigma^2} + \frac{2}{r^2}(1-3w^2) - \frac{16\kappa w V'}{r\sigma}\right)w_1\\
&\quad + \left(\frac{w'}{r^2N}\left(1 - \frac{r\sigma'}{\sigma}\right) - \frac{w''}{r^2N}\right)m_1 - \frac{8\kappa(1-w^2)V'}{rN\sigma}\sigma_1 = 0,\\
&m_1 = 3rNw'w_1, \qquad \sigma_1' = \frac{3\sigma w'}{r}w_1', \qquad V_1' = \frac{24\kappa\sigma}{r^3}(w^2-1)w_1 + \sigma V'\sigma_1.
\end{aligned} \tag{4.13}$$

Thus the functions m_1, V_1, and σ_1 can be eliminated in favor of $w_1(r)$, leading to a single Schrödinger equation for w_1:

$$-\frac{d^2\chi}{dr_\star^2} + U_\Omega \chi = \Omega^2\chi, \tag{4.14}$$

where $\chi = w_1\sqrt{r}$ and a new radial coordinate $r_\star$ is introduced via $\frac{d}{dr_\star} = N\sigma\frac{d}{dr}$. The expression of the potential in (4.14) is

$$U_\Omega = \frac{N\sigma^2}{r^2}\left[6\left(w^2 - w'^2 - \frac{1}{6}\right) - \frac{5N}{4} + 12(w^2-1)\frac{ww'}{r} + \frac{(1-w^2)^2}{r^2}\left(\frac{9}{2}w'^2 + 192\kappa^2 - \frac{3}{4}\right) + \frac{16\kappa V'}{\sigma}(rw - 3(1-w^2)w') - \frac{r^2V'^2(1-6w'^2)}{4\sigma^2}\right]. \quad (4.15)$$

One can verify that for $\kappa = V = 0$, the above relations reduce to those found in [21] for a $D = 5$ EYM system (note that, however, the unperturbed solutions there have infinite mass).

The potential above is regular in the entire range $-\infty < r_\star < \infty$. Near the event horizon, one finds $U_\Omega \to 0$; for large values of $r_\star$ the potential is positive and bounded. Standard results from quantum mechanics [45] further imply that there are no negative eigenvalues for Ω^2 (and then no unstable modes) if the potential U_Ω is everywhere positive.

Indeed, our numerical results show the existence of black hole solutions with a positive potential $U(r_\star)$ for all values of $\kappa > 1/8$ we have considered. As expected, all stable solutions we could find in this way have no nodes of the magnetic gauge potential $w(r)$ (see Fig. 9 for such configurations). Therefore, at least some of our solutions are linearly stable. The picture is, however, quite complicated and depends on the values of κ, Q and of the event horizon radius. For example, for solutions with $\kappa = 0.3$ and $Q = 0.187Q^{(c)}$, all solutions between the critical extremal black hole configuration (with $r_h \simeq 0.702$) and $r_h \simeq 0.95$ have $U(r_\star) > 0$. However, for the same value of the κ, all configurations with $Q = 1.05Q^{(c)}$ have $U(r_\star) < 0$.

At the same time, we cannot predict anything if $U(r_\star)$ is not positive definite. In this case we have to solve numerically Eq. (4.14) as an eigenvalue problem. Very likely, the full picture is complicated and a systematic study would represent a very complex task. We note only that, by using a trial-function approach (see e.g. [46]), we could prove that a number of EYMCS solutions with one node in $w(r)$ are indeed unstable.

E. The globally regular solutions

All $D = 5$ globally regular solutions emerge as a zero event horizon radius limit of the black hole branch with $Q = Q^{(c)}$. In principle, a disconnected branch of solitons with $\kappa < 1/8$ may exist; however, we could not find such configurations. Heuristically, this can be understood as follows. Since, from (3.11), the electric charge is proportional to the CS coupling constant, the existence of a minimal value of κ means that below that value the electrostatic repulsion is too small compared with other interactions for a bound state to exist.

The properties of these solutions are uniquely fixed by the CS parameter κ and are somehow different from other nA solitons in the literature. For example, their mass is an almost linear function of κ, while the value at the origin of the metric function g_{tt} is very close to -1; see Fig. 10. The shooting parameter $b = -w''(0)/2$ and the value at infinity of the electric potential also take small values. It would be interesting to find an analytical understanding of these numerical results. A typical $D = 5$ globally regular solution is shown below in Fig. 14 (left panel).

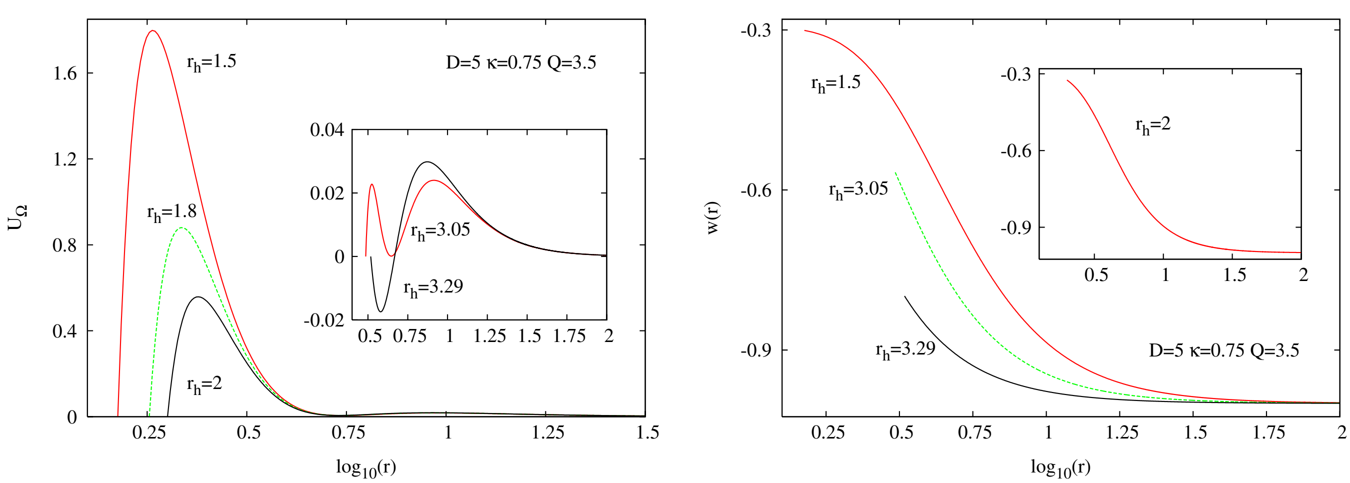

FIG. 9 (color online). The potentials for the perturbation equation (4.14) are shown together with the corresponding magnetic gauge potentials of the unperturbed solutions for several values of r_h and fixed values of κ, Q. For $r_h = 3.29$, there is a negative region of the potential, $V(r) < 0$.

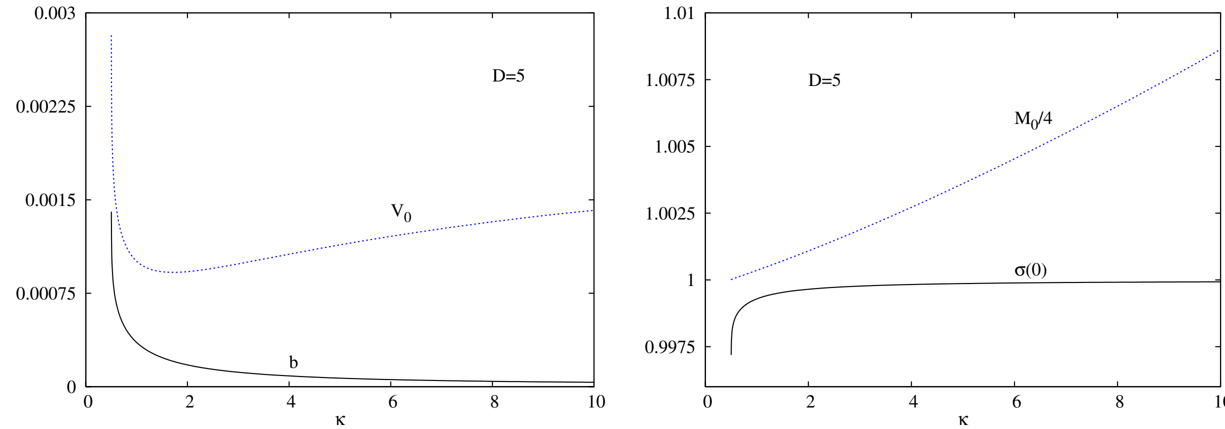

FIG. 10 (color online). A number of parameters of the $D = 5$ globally regular solutions are plotted as a function of the Chern-Simons coupling constant κ.

Concerning the stability of solitons, the formalism proposed above for black holes applies also in the zero event horizon case. However, at this stage we cannot say something precise about their stability, since all globally regular solutions we have investigated have a negative potential. Also, the fact that there are no nodeless solitons [since $w(0) = 1$ and $w(\infty) = -1$] strongly suggests that all particlelike solutions are expected to be unstable.

F. The case $\kappa = \alpha/8$: An exact solution

In what follows, we find it interesting to restore the (G, e) factors in the general expressions. For $\kappa = \alpha/8$, one finds the following exact solution of the EYMCS equations within the metric Ansatz (2.9) employing isotropic coordinates,[15] and a gauge group $SO(4) \times SO(2)$:

$$w(r) = \frac{J - 2r^2}{J + 2r^2}, \qquad V(r) = \frac{e}{4}\sqrt{\frac{3}{\pi G}} f(r),$$
$$f_0(r) = f^2(r), \qquad f_1(r) = \frac{f_2(r)}{r^2} = \frac{1}{f(r)},$$
$$\text{with} \quad f(r) = \left[1 + \frac{Q^{(c)2}}{2r^2}\left(\frac{Q}{Q^{(c)}} - \frac{J^2}{(2r^2 + J)^2}\right)\right]^{-1}, \quad (4.16)$$

where $Q > Q^{(c)}$ and J are arbitrary parameters, and $Q^{(c)} = 16\kappa/e$ is the critical charge parameter, $Q^{(c)} = \frac{8}{e}\sqrt{\frac{\pi G}{3}}$. This describes an extremal black hole with nA hair, the regular event horizon being at $r = 0$ (in these isotropic coordinates). The mass, electric charge, chemical potential, and entropy of this solution are

$$\mathcal{M} = \frac{\sqrt{3}\pi^{3/2} Q}{e\sqrt{G}} = \frac{8\pi^2}{e^2}\frac{Q}{Q^{(c)}}, \qquad \mathcal{Q} = \frac{4\pi^2 Q}{e},$$
$$\Phi = \frac{1}{4}\sqrt{\frac{3}{\pi G}} = \frac{2}{eQ^{(c)}},$$
$$S = \frac{\pi^2}{2G}\left(4\sqrt{\frac{\pi G}{3}}\frac{Q}{e} - \frac{32\pi G}{3e^2}\right)^{3/2} = \frac{\pi^2}{2G}\left(\frac{1}{2}(Q - Q^{(c)})Q^{(c)}\right)^{3/2}, \quad (4.17)$$

such that $\mathcal{M} = \Phi\mathcal{Q}$. This exact solution has a number of interesting properties. For example, one can show that the horizon is regular, with the following behavior as $r \to 0$,

$$f_1(r) = \frac{f_2}{r^2} = \frac{(Q - Q^{(c)})Q^{(c)}}{2r^2} + 1 + \frac{2Q^{(c)2}}{J} + O(r^2),$$
$$f_0(r) = \frac{4r^4}{(Q - Q^{(c)})^2 Q^{(c)2}} + O(r^6) \quad (4.18)$$

for the metric functions, and

$$R = \frac{4}{Q^{(c)}(Q^{(c)} - Q)} + O(r^2),$$
$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{304}{(Q^{(c)} - Q)^2 Q^{(c)2}} + O(r^2) \quad (4.19)$$

for the curvature and Kretschmann scalars. Also, the magnetic field is vanishing on the horizon (this property is also shared by the extremal black holes with $Q > Q^{(c)}$ and $\kappa > \alpha/8$), with $w(r) = 1 - 4r^2/J + O(r^4)$.

The RN solution is found by taking $J = 0$ in (4.16). The mass, electric charge, and chemical potential of this solution are still given by (4.17), while the entropy of the RN solution is $S_{\mathrm{RN}} = \frac{\pi^2}{2G}(\frac{QQ^{(c)}}{2})^{3/2}$, which is higher than the entropy of the corresponding nA configuration. Thus the extremal Abelian solution is thermodynamically favored, a

[15] In principle, this solution can also be written in the Schwarzschild-type coordinate system (2.10). For example, the relation between the Schwarzschild radial coordinate $\bar{r}$ and the radial coordinate r in (4.16) is $\bar{r} = r/\sqrt{f(r)}$. However, this results in a much more complicated expression of the solution.

feature which seems to be shared by other extremal solutions with $\kappa > \alpha/8$ that we have constructed numerically.

In the limit $Q = Q^{(c)}$, the solution (4.16) describes a particlelike soliton, in which case

$$f(r) = \left[1 + \frac{2Q^{(c)2}(J + r^2)}{(J + 2r^2)^2}\right]^{-1}, \tag{4.20}$$

while, from (4.17), $\mathcal{M} = 8\pi^2/e^2$, $\mathcal{Q} = 32\sqrt{G}\pi^{5/2}/(\sqrt{3}e^2)$, and $\Phi = \sqrt{3/(\pi G)}/4$. One can show that $r = 0$ is a regular origin, with the following behavior in that limit:

$$\begin{aligned} f_1(r) &= \frac{f_2}{r^2} = 1 + \frac{2Q^{(c)2}}{J} + O(r^2), \\ f_0(r) &= \frac{J^2}{(J + 2Q^{(c)2})^2} + O(r^2), \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} R &= \frac{48Q^{(c)2}}{(J + 2Q^{(c)2})^2} + O(r^2), \\ R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= \frac{5760Q^{(c)4}}{(J + 2Q^{(c)2})^4} + O(r^2). \end{aligned} \tag{4.22}$$

One should note that the parameter J which appears via the magnetic gauge potential $w(r)$ does not enter any global quantity (although these nA solutions are supported by a nonzero J). However, J has a physical meaning, since it also enters the metric components. Then, although the mass and charge are the same, a local observer could distinguish between the nA solution (4.16) and the corresponding RN solution via the motion of a test particle.

The reason for the existence of the exact solution (4.16) can be understood by noticing that the solutions reported in this work admit an interesting connection with a model considered in [29]. This connection follows from the observation that, for static configurations, the action (2.1) with an $SO(4) \times SO(2)$ gauge group reduces essentially to an Einstein–Yang-Mills-Maxwell model, with a Chern-Simons–type coupling term between the $U(1)$ and nA fields. Basically, with this group contraction down from $SO(6)$, the CS density (2.5) reduces to the hybrid CS density in [29]. Moreover, as noticed in that paper, this model with $\kappa = \alpha/8$ corresponds to the coupling of the super-YM theory to the $D = 5$ supergravity [47].

It is shown in [29] that for this value of the CS coupling constant, any flat space self-dual solution of the $D = 4$ YM equation with a gauge group[16] $SO(4)$ can be uplifted to $D = 5$ and promoted to soliton solutions of the full model.

A slight generalization of this construction can be summarized as follows. Let us consider a $D = 4$ seed configuration consisting in a geometry described by a Ricci-flat line element $d\sigma^2$ (parametrized by the $\vec{x}$ coordinates) with Euclidean signature, and an $SO(4)$ nA field satisfying the self-duality equations[17]

$$F_{ij} = \tfrac{1}{2}\sqrt{\det\sigma}\,\gamma_5\varepsilon_{ijkl}F^{kl}, \tag{4.23}$$

in a metric background given by $d\sigma^2$.

Then this configuration can be uplifted to solutions of the $D = 5$ EYMCS model in this work, for an $SO(4) \times O(2)$ gauge group. The five-dimensional line element is

$$ds^2 = \frac{1}{f(\vec{x})}d\sigma^2 - f^2(\vec{x})dt^2, \tag{4.24}$$

while the purely electric $SO(2)$ potential reads

$$V(\vec{x}) = \frac{e}{4}\sqrt{\frac{3}{\pi G}}f(\vec{x}). \tag{4.25}$$

In the above relations, $f(\vec{x})$ is a solution of the Poisson equation

$$\nabla^2\left(\frac{1}{f}\right) = -\frac{4\pi G}{3}\frac{1}{2}\,\mathrm{Tr}\{F_{ij}F^{ij}\}, \tag{4.26}$$

where the operator ∇^2 is taken with respect to the four-dimensional metric $d\sigma^2$. The $D = 5$ YM gauge field is $F = F_{ij}dx^i \wedge dx^j + (dV \wedge dt)\Sigma^{(\pm)}_{5,6}$.

Then one can easily prove that the full set of equations (2.7) are satisfied, provided κ takes the special value

$$\kappa = \frac{e}{2}\sqrt{\frac{\pi G}{3}} = \frac{\alpha}{8}. \tag{4.27}$$

In principle, this approach can be used to uplift to $D = 5$ all four-dimensional self-dual solutions of the YM equations, including configurations displaying no symmetries and multicenter solutions. Note that only soliton solutions of Eq. (4.26) were considered in Ref. [29]. Here, this has been extended to the construction of black hole solutions by adding to $1/f$ an extra part which is a solution of the homogeneous equation $\nabla^2(\frac{1}{f}) = 0$ with suitable boundary conditions.

The simplest case is found by taking $d\sigma^2$ to be the four-dimensional Euclidean space and F_{ij} the BPST instanton [48]. This leads to the spherically symmetric configuration (4.16) discussed above. However, a variety of other physically interesting solutions can be obtained in a similar way (this includes configurations with a $D = 4$ nonflat base space, e.g. the Euclideanized Schwarzschild metric and the $D = 4$ YM instantons in [49,50]). Moreover, one can

[16] In [29], in fact, the YM field takes its values in one or the other chiral representation of $SO(4)$, namely, for self-dual and antiself-dual $SU(2)$ fields. Here, by contrast, our magnetic YM connection is fully $SO(4)$ valued, leading to the appearance of $\Sigma_{5,6} = \gamma_5$ in (4.23).

[17] Thus, all known $D = 4$, $SU(2)$ YM instantons can provide a solution of (4.23).

generalize this framework by including rotation in the $D = 5$ metric Ansatz, in which case it is possible, e.g., to find a hairy generalization of the BMPV black hole [51]. Such solutions are found beyond the simple Ansatz in this work and will be reported elsewhere.

V. ON $D > 5$ SOLUTIONS

Our results confirm the existence of both black holes and soliton solutions of Eqs. (3.1) for $D = 7, 9$ as well. Then we expect that the EYMCS model considered possesses asymptotically flat, spherically symmetric solutions with finite mass for any $D = 2n + 1 \geq 5$. The numerical methods employed in this case are similar to those discussed for $D = 5$, though as expected the numerical difficulties increase with D.

Determining the pattern of the $D > 5$ solutions in the parameter space represents a very complex task which is outside the scope of this paper. Instead, we analyzed in detail a few particular classes of $D = 7, 9$ solutions which, hopefully, reflect at least some of the relevant properties of the general pattern.

Not entirely surprisingly, it turns out that the case $D = 5$ has some special properties. For example, we have seen already that for $D > 5$, unlike in $D = 5$, the branches of nA solutions do not end in RN black holes. Also, according to our description of the heuristic argument concerning Derrick scaling at the end of Sec. II B, finite mass/energy solutions can be constructed in spacetime dimensions $D \geq 7$ even in the absence of the gravitational term in the Lagrangian, i.e. for a fixed Minkowski background.

The reason is that only in $D = 5$ is it necessary to have the Einstein-Hilbert term in the Lagrangian to satisfy the (heuristic) Derrick scaling requirement. In that case, the Yang-Mills term scales as L^{-4} so that in four spacelike dimensions the scaling L^{-5} of the CS term is not balanced in the absence of gravity, the latter scaling as L^{-2}. Generally, the CS terms scales as $L^{-(2n+1)}$, in $2n$ spacelike dimensions. Then the Derrick scaling requirement can be satisfied if the Yang-Mills term scales as L^{-4p}, provided that

$$2n + 1 > 2n > 4p, \tag{5.1}$$

i.e., when $D > 4p + 1$. The Yang-Mills terms that scale as L^{-4p} are the pth members of the YM hierarchy [43]

$$\mathcal{L}^{(p)}_{\rm YM} = \frac{1}{2 \cdot (2p)!} \, {\rm Tr}\{F(2p)^2\}. \tag{5.2}$$

Then the Lagrangian that satisfies the requirement (5.1) is

$$\mathcal{L}_{\rm matter} = \mathcal{L}^{(p)}_{\rm YM} + \kappa \mathcal{L}^{(2n+1)}_{\rm CS}, \qquad n > 2p. \tag{5.3}$$

This is possible to satisfy for $n \geq 3$, i.e. $D \geq 7$. Thus for $D = 7$ and $D = 9$, the only choice of the YM term is the usual $p = 1$ member of the YM hierarchy [i.e. the one in (2.1)], while for $D > 9$ either one or both of $p = 1$ and $p = 2$ YM terms (5.2) are possible choices, etc. All properties of these solitons are fixed by the value of the CS coupling constant, the relation (3.31) being valid also in that case. A detailed discussion of the nongravitating YMCS solutions, including an existence proof, will be presented elsewhere.

Returning to gravitating configurations, one notes that, however, to some extent, the families of $D > 5$ black holes resemble their five-dimensional counterparts. In particular, the profiles of the solutions look similar to those exhibited in Sec. IV and hence will not be plotted here again. As in $D = 5$, no multinode solutions of the magnetic gauge potential w were observed for $D > 5$. There are, however, marked qualitative differences in some properties of the gravitational solutions in $D > 5$, vs those in $D = 5$. These are discussed below.

Starting with the dependence of black hole properties on the electric charge parameter Q for a fixed event horizon radius r_h, we plot in Fig. 11 the results of the numerical integration for a given value of the CS coupling constant κ. It seems that, for any r_h, the solutions exist for a finite range[18] of Q. (Thus, as expected, one cannot find black holes with an arbitrarily large electric charge.)

The limiting behavior of these solutions at the limit of the Q interval is qualitatively different from the case $D = 5$. In terms of the value w_h of the magnetic gauge potential on the horizon, the solutions exist for $w_h \in (w_{h,\min}, 1)$, where the minimal value $w_{h,\min}$ depends on r_h (although always with $w_{h,\min} \neq -1$; i.e. the branch of nA solutions does not join the RN configuration). The numerical results suggest that in this case the limiting configurations consist of extremal black holes with a regular horizon; see Fig. 11 (left panel). Another feature of the solutions which is worth mentioning is that for intermediate values of the horizon radius, there exists a large region of the parameter w_h for which both $V'(r_h)$ and $w'(r_h)$ are very close to zero. Moreover, for some intermediate values of Q, we notice the existence of two different solutions with the same charge parameter.

The behavior of solutions as a function of the event horizon radius is shown in Fig. 12 for several values of the charge parameter Q and a given κ. In terms of the scaled event horizon radius a_H and scaled temperature t_H, the sets of nA black holes with $Q \neq Q^{(c)}$ interpolate between two extremal configurations, and one can notice the existence of three branches of solutions. The first branch of nA solutions starts from an extremal black hole with $w(r_h)$ taking a minimal value. This solution extends up to a maximal value of r_h, where a second branch of nonextremal solutions emerges, extending backwards in r_h. For the

[18]Although the solutions in Fig. 11 have $Q > Q^{(c)}$, for other values of r_h we could find solutions with $Q < Q^{(c)}$ as well.

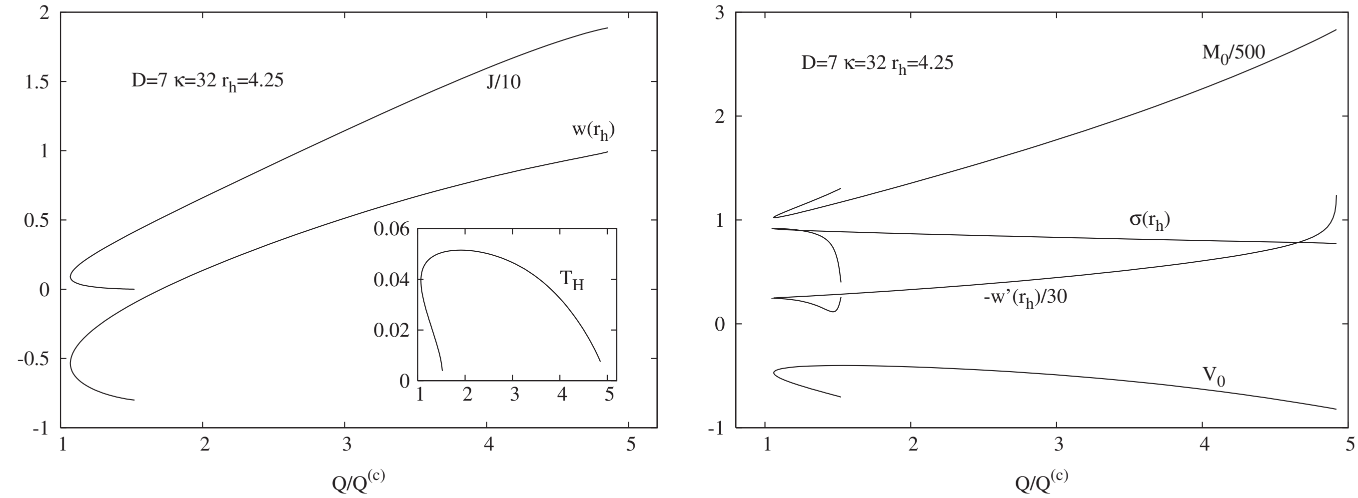

FIG. 11. A number of parameters of $D = 7$ black hole solutions are plotted as a function of the electric charge parameter Q for fixed values of (κ, r_h). Here and in Fig. 12 $Q^{(c)} = -64\kappa/5$ is the critical value of the electric charge parameter.

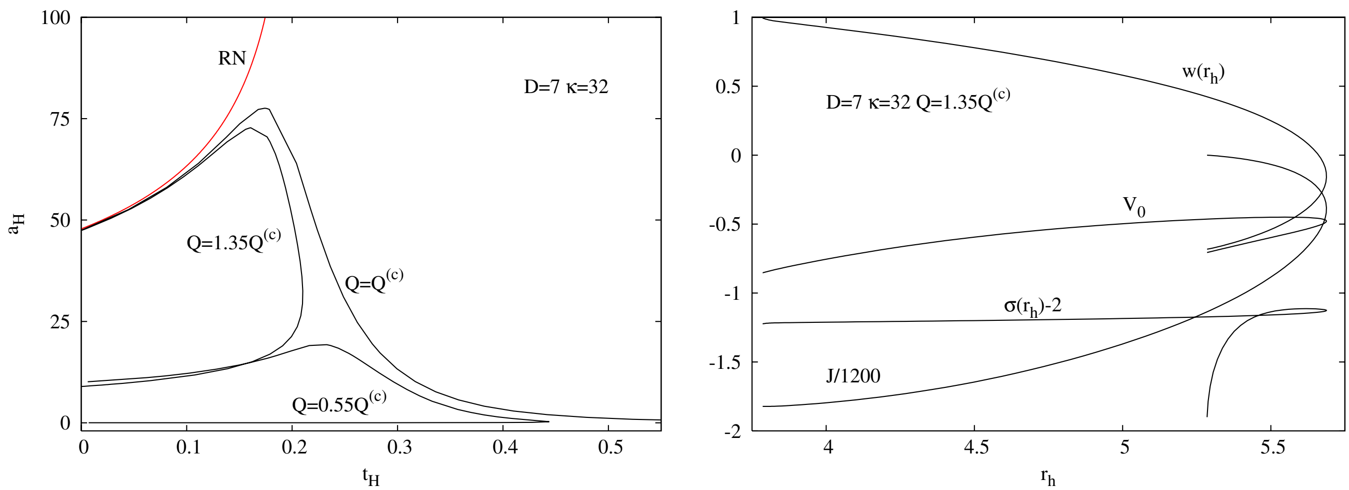

FIG. 12 (color online). A number of parameters of $D = 7$ black hole solutions are shown for fixed values of κ, Q and a varying r_h.

same r_h, the mass of one of these solutions is larger than the corresponding mass on the first branch (when they both exist). Also, this secondary branch has a negative specific heat, while the entropy increases with the temperature for the first branch. Finally, a third branch of solutions emerges for a critical r_h, which has again a positive specific heat. This branch ends in an extremal configuration with a minimal r_h.

The picture is different for $Q = Q^{(c)}$, in which case the third branch is absent and the second branch of solutions extends to $r_h \to 0$; see Fig. 12 (left panel). The limit of a vanishing event horizon radius corresponds to a globally regular, particlelike configuration. (This feature is similar to the $D = 5$ result.)

The $D = 9$ solutions we have studied possess a similar pattern. A number of results in this case are shown in Fig. 13 as a function of the parameter $w(r_h)$. One can see that the limits of the domain of existence correspond to extremal configurations with $T_H = 0$.

We did not consider the issue of stability of $D > 5$ solutions. In principle, this is a straightforward extension of the work in $D = 5$, the problem reducing again to a single Schrödinger equation. The fact that for any D we have found nodeless solutions suggests the existence in all dimensions of configurations which are stable against spherically symmetric perturbations.

As mentioned already, similar to $D = 5$, one also finds a different class of solutions describing globally regular solitons with $Q = Q^{(c)}$. In Fig. 14 (right panel) we show the profile of such a configuration in $D = 7$ spacetime dimensions. One can see some differences between this solution and the one in $D = 5$, the distortion of the space-time geometry being more pronounced in the higher dimensional case.

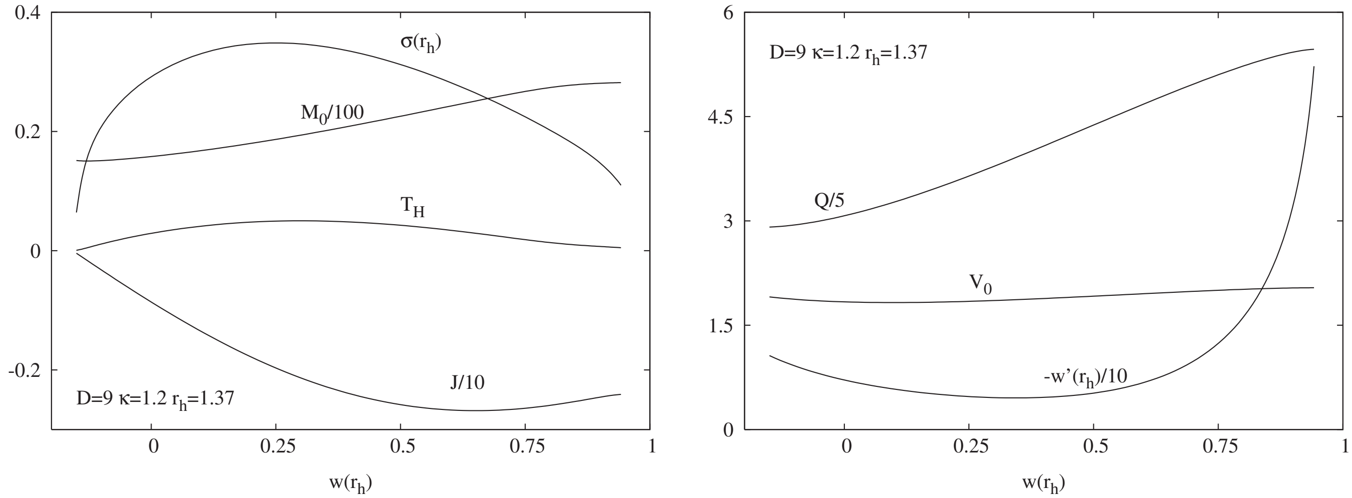

FIG. 13. A number of parameters of $D = 9$ black hole solutions are plotted as a function of the value $w(r_h)$ of the magnetic potential on the horizon for fixed values of κ, r_h.

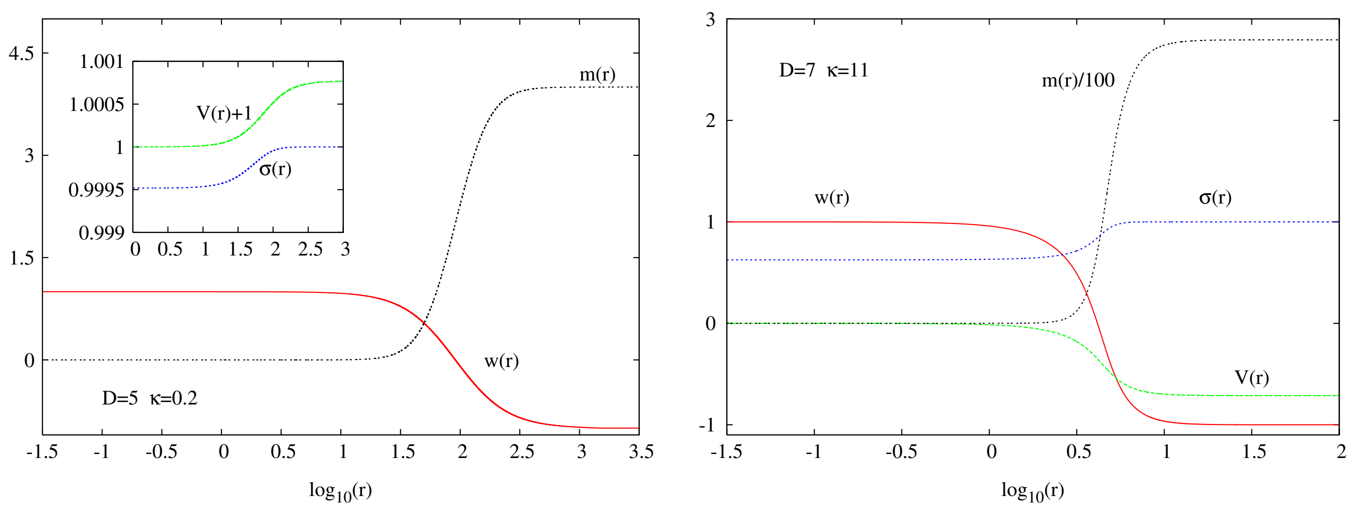

FIG. 14 (color online). The profiles of typical, globally regular, particlelike EYMCS solutions are presented as a function of the radial coordinate r for $D = 5$, 7 spacetime dimensions.

VI. CONCLUSIONS AND FURTHER REMARKS

The main purpose of this work was to present a general study of static, spherically symmetric, asymptotically Minkowskian solutions in a simple EYMCS model in $D = 2n + 1$ dimensions. Our choice of gauge group is $SO(D - 1) \times SO(2)$, which is just sufficient to support a nonvanishing Chern-Simons density. While the smallest simplest gauge group with this property is $SO(D + 1)$, the asymptotic analysis suggested that there were no well-behaved solutions in that case. This is in contrast with the asymptotically AdS solutions encountered in [36], which were found for the full $SO(D + 1)$ gauge group (with $D = 5$).

The CS term allows one to avoid the usual Derrick-type scaling argument against the existence of $D > 4$ static nA configurations with finite mass in the usual EYM system. This provides an *alternative* to the higher order curvature terms of the YM hierarchy employed previously to reach the same result [25], presenting, at the same time, a richer pattern.

While we could display a number of analytic results valid for any (odd) values of $D > 3$, our main numerical analysis was restricted to the case of $D = 5$ and, to a lesser extent, to $D = 7$. In addition, we could confirm the existence of solutions for $D = 9$ as well. One should emphasize that the properties of the EYMCS solutions in this work are strikingly different from solutions to other nA models without a CS term, considered in the literature.

The main interesting results are for $D = 5$, in which case they can be summarized as follows:

(i) The black hole solutions emerge as perturbations of the RN solution, which becomes unstable when embedded in a larger gauge group.

(ii) The nA solutions are generically thermodynamically favored over the Abelian configurations.

(iii) Some of the nA configurations were shown to be stable against small perturbations.

(iv) A solution in closed form is found for the minimal value of the CS coupling constant, describing an extremal black hole with nA hair. In the absence of a horizon, this becomes the soliton found in [29].

Our results in $D = 7$ and 9 differ somehow from those in $D = 5$. Most importantly, the RN solution in these dimensions does not become unstable when embedded in the corresponding (larger) nA gauge group (in fact, this property holds for any $D > 5$). As a result, the nA black holes do not emerge as perturbations of the RN solutions. Instead, for a fixed value of the electric charge $Q \neq Q^{(c)}$, they appear to interpolate between two nA extremal black holes. These qualitative differences are a consequence of the choice of EYMCS model made here, which in any case does not seem to be a consistent truncation of a known supergravity theory. However, it is possible to search for different EYMCS models in higher (than five) dimensions, whose solutions are likely to fulfill the properties of the $D = 5$ solutions itemized above. For this we note that in $D = 5$ the CS term scales as L^{-5}, vs the YM term, which scales as L^{-4}. In this respect, it would be useful if in the higher $2n + 1$ dimensions where the CS term scales as $L^{-(2n+1)}$, the corresponding YM term would scale as L^{-2n}. This can be achieved only in $D = 4p + 1$ dimensions, where the CS term scales as $L^{-(4p+1)}$, by replacing the usual YM term $F(2)^2$ (the $p = 1$ member of the YM hierarchy) with the pth member of the YM hierarchy which scales as L^{-4p} [27]. It is obvious that this can only be done when $n = 2p$; thus the properties itemized above cannot be duplicated in $D = 4p + 3$ dimensions.

This remark applies equally to the last of the properties itemized above, namely, that closed form solutions can be constructed in all $4p + 1$ dimensions, for a special value of the CS coupling constant. While in the $p = 1$ case discussed here the (usual) $p = 1$ BPST instanton is employed, in the case of a generic p, the instanton [43] of the pth member of the YM hierarchy is employed. This results in a general class of $p \geq 1$ exact solutions, whose properties are similar to those of the $D = 5$ configurations discussed in Sec. IV F.

It is obvious that the black hole solutions in this work violate the no-hair conjecture; that is, two distinct solutions can exist for a given set of global charges. Moreover, some of the nA solutions are really classically stable, because they have maximum entropy among the black holes with the same mass and charge. This behavior is somehow similar to that found in [52] for a family of monopole black holes in the $D = 4$ Einstein–Yang-Mills–Higgs system. There, too, a branch of monopole nA black holes merges with the *magnetic* RN solutions, which is unstable for some range of the parameters. Thus, similar to the case in [52], the hairy black holes in this work may be relevant for the issue of the final stage of an evaporating $D = 2n + 1$ RN black hole.

In principle, the study in this work can be extended in various directions. For example, it will be interesting to consider more general asymptotics and solutions describing black strings and p-branes. Another possible direction would be to include rotation. However, even for the case of static solutions approaching the Minkowski background at infinity, with a gauge group $SO(D-1) \times SO(2)$, we expect to find a variety of interesting solutions. For example, we expect both EYMCS solitons and black holes to exist, which are static but not spherically symmetric. Indeed, such solutions were found in the $D = 4$ EYM system [14].

We close our discussion with some comments on another intriguing feature of the $D = 5$ solutions. Despite the different asymptotic structure of spacetime and the different horizon topology, these solutions have some similarities with the colorful black holes with charge in AdS space [16,53,54], which provide a model of holographic superconductors. In both cases, an Abelian gauge symmetry is spontaneously broken near a black hole horizon with the appearance of a condensate of nA gauge fields there, leading to a phase transition. Also, one can notice a striking similarity of the $J(T_H)$ curves shown in Fig. 6 with some of those exhibited in the literature on AdS holographic superconductors.

It remains an interesting open problem to clarify if the asymptotically flat EYMCS black holes may also provide useful analogies to phenomena observed in condensed matter physics. The first step in this direction would be to compute the conductivity as a function of frequency. This is obtained by perturbing the YM fields around the horizon. However, given the presence of several branches, the general picture is more complicated for asymptotically flat solutions, already for the fundamental RN set of solutions. Also, in the absence of a cosmological constant, the gauge/gravity duality (which seems to provide the deep reason behind the connection between general relativity solutions and condensed matter physics) is not yet understood. At the same time, some of the features of the AdS holographic duals of superconductors may occur for other asymptotics as well, being generic properties of certain classes of hairy black holes. We hope to return to a study of these aspects in a separate paper.

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