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Randall-Sundrum limit of $f(R)$ brane-world models

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By setting some special boundary conditions in the variational principle we obtain junction conditions for the five-dimensional $f(R)$ gravity that in the Einstein limit $f(R) \to R$ transform into the standard Randall-Sundrum junction conditions. We apply these junction conditions to a particular model of a Friedmann universe on the brane and show explicitly that the limit gives the standard Randall-Sundrum model Friedmann equation.

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I. INTRODUCTION

Brane universes are now one of the most interesting option for unifying gauge interactions with gravity at the TeV scale [1]. They were nicely introduced by Randall and Sundrum [2] and further developed cosmologically [3]. However, it is not trivial to combine brane theories with higher-order gravity theories such as $f(R)$ or $f(R, R_{ab}R^{ab}, R_{abcd}R^{abcd})$ gravities [4–6], which are theories with Lagrangians dependent on the curvature invariants. The obstacles are the ambiguities of the quadratic delta function contributions to the field equations. The only cases that naturally avoid these ambiguities are curvature invariants combinations that form Lovelock densities [7,8]. However, due to some new approaches (improvement of the continuity properties of the metric on the brane or reduction to a second-order theory), the obstacles were challenged successfully in Ref. [9] and the Israel junction conditions [10] were obtained following earlier discussion of Refs. [11–14]. The method of reduction $f(R)$ theory to a second-order theory is possible by the introduction of an extra scalar field–the scalaron. In this approach $f(R)$ theory becomes the scalar-tensor Brans-Dicke gravity [15] with a Brans-Dicke parameter $\omega = 0$, and the induced scalaron potential with the scalaron playing the role of the Brans-Dicke field. The junction conditions obtained in Ref. [9] generalized both the junction conditions obtained in Refs. [16,17] for the Brans-Dicke field without a scalar field potential, and also the conditions derived in Refs. [12,18,19] for $f(R)$ brane gravity. In Refs. [20,21] previously derived junction conditions in [9] were applied to cosmology.

It is important to say that the junction conditions for $f(R)$ gravity given in [9,20] were obtained by using the boundary conditions that assumed unrestricted variation of the metric and the scalaron on the brane. In fact, the freedom of the variation of the scalaron on the brane leads to the continuity of trace of the extrinsic curvature on it (see Eq. (2.7) in [20]). This property makes it impossible to obtain the standard Israel junction conditions in the Einstein limit $f(R) \to R$. However, the boundary conditions used in Refs. [9,20] are not unique and can be replaced by some different ones–those which allow the Einstein limit; and this is the task of the present paper.

In Sec. II we discuss the variational principle for $f(R)$ brane models reduced to second-order Brans-Dicke models, and we derive the junction conditions that allow the Einstein limit. In Sec. III we apply these junction conditions to the Friedmann geometry on the brane. In Sec. IV we solve for the bulk anti–de Sitter geometry, and in Sec. V we explicitly show how the proposed junction conditions work in the Einstein-Randall-Sundrum limit. In Sec. VI we give our conclusions.

II. BOUNDARY CONDITIONS AND JUNCTION CONDITIONS WITH AN RANDALL-SUNDRUM LIMIT

We start with the action of the $f(R)$ theory [4] in five dimensions which reads as

$$S_p = \frac{1}{2\kappa_5^2}\int_{M_p} d^5x\sqrt{-g}f(R) + S_{\text{bulk},p}, \tag{2.1}$$

where R is the Ricci scalar, κ_5^2 is a five-dimensional Einstein constant, $S_{\text{bulk},p}$ is the bulk matter action, and M_p ($p = 1, 2$) is the spacetime volume. Since the action (2.1) gives fourth-order field equations, then it is advisable to use an equivalent action

$$\bar{S}_p = \int_{M_p} d^5x\sqrt{-g}\{f'(Q)(R-Q) + f(Q)\}, \tag{2.2}$$

where Q is an extra field that plays the role of a Lagrange multiplier, and $f'(Q) \equiv df(Q)/dQ$. Varying the action (2.2) with respect to g_{ab} and Q we obtain the equations of motion

$$\tfrac{1}{2}g^{ab}f(Q) - f'(Q)R^{ab} - g^{ab}\Box f'(Q) + f'(Q)^{;ab} = 0, \tag{2.3}$$

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$$Q = R, \tag{2.4}$$

with the condition that $f''(Q) \neq 0$, and we interpret $f'(Q)$ as an extra scalar field—the scalaron:

$$\phi = f'(Q) = f'(R). \tag{2.5}$$

Using the scalaron, the action (2.2) can be rewritten in the form of a Brans-Dicke action with a Brans-Dicke parameter $\omega = 0$, i.e.,

$$\bar{S}_p = \int_{M_p} d^5x\sqrt{-g}\{\phi R - V(\phi)\} + S_{\text{bulk},p}, \tag{2.6}$$

where $V(\phi) = -\phi R(\phi) + f(R(\phi))$ [9].

In our previous papers [9,20] we derived junction conditions for brane-world model action (2.1), but none of them possessed a standard Randall-Sundrum limit. Here we suggest an approach that allows one to do so at the expense of choosing some special boundary conditions while varying the brane-world action with an appropriate Hawking-Lutrell boundary term that for the action (2.6) is [22]

$$S_{\text{HL}_p} = -2(-1)^p\epsilon\int_{\partial M_p}\sqrt{-h}\phi K d^4x, \tag{2.7}$$

where K is the trace of the extrinsic curvature tensor K_{ab}, h is the determinant of the induced metric $h_{ab} = g_{ab} - \epsilon n_a n_b$, n^a is a unit normal vector to a boundary ∂M_p, and $\epsilon = 1$ ($\epsilon = -1$) for a timelike (a spacelike) brane, respectively. The total action of the theory is then

$$\bar{S}_{\text{tot}_p} = \bar{S}_p + S_{\text{HL}_p}. \tag{2.8}$$

The variation of the action (2.8) leads to

$$\delta S_{\text{tot}_p} = -\int_{\partial M_p} d^4x\sqrt{-h}(-1)^p\{\epsilon[(g^{ab} + \epsilon n^a n^b)(\phi_{;e}n^e) + 2n^b h^{ea}\phi_{,e} + \phi K h^{ab} - \phi K^{ab} - 2n^{(a}\phi^{,b)}]\delta g_{ab} - 2(-1)^p\epsilon\int_{\partial M_p} d^4x\sqrt{-h}K\delta\phi, \tag{2.9}$$

where the bulk parts have been omitted. Full variation over the bulk space, separated by a brane, requires the variation of both of these parts separately (i.e. first for $p = 1$, and then for $p = 2$). This means that the full action is

$$\bar{S}_{\text{tot}} = S_{\text{tot}_1} + S_{\text{tot}_2} + S_{\text{brane}}, \tag{2.10}$$

where

$$\delta S_{\text{brane}} = \kappa_5^2\int_{\partial M_p} d^4x\sqrt{-h}S^{ab}\delta g_{ab}, \tag{2.11}$$

and S_{ab} is an energy-momentum tensor of the matter on the brane.

Now, there is a crucial point that allows one to obtain the Randall-Sundrum limit of the $f(R)$ brane junction conditions. Namely, we vary the total action (2.10) in such a way that we set boundary conditions as follows:

$$\delta\phi \to 0 \quad \text{for } w \to 0, \tag{2.12}$$

where w is a coordinate normal to the brane. The physical meaning of (2.12) is that we impose the variation of the scalaron to vanish on the brane but not in the bulk. In fact, such a choice allows one to kill the last term in the variation (2.9) leaving the other terms untouched (since we did not assume that $\delta g_{ab} = 0$ on the brane) and also leaving the freedom of a choice for the trace of the extrinsic curvature that not necessarily has to be continuous on the brane. Such a choice of the boundary conditions allows one to obtain the following junction conditions [20]:

$$-(g^{ab} + \epsilon n^a n^b)[\phi_{;c}n^c] - 2n^{(a}h^{eb)}[\phi_{,e}] - [\phi K]h^{ab} + [\phi K^{ab}] + 2n^{(a}[\phi^{,b)}] = \epsilon\kappa_5^2 S^{ab}, \tag{2.13}$$

where $[A] \equiv A^+ - A^-$ for any quantity A with its values A^+ and A_- right and left of the brane, respectively.

Projecting Eq. (2.13) onto the directions tangent to the brane by multiplying it by $h_{ac}h_{bd}$, we obtain:

$$-h_{ab}\left[\frac{\partial\phi}{\partial w}\right] - [\phi K]h_{ab} + [\phi K_{ab}] = \epsilon\kappa_5^2 S_{ab}. \tag{2.14}$$

Next, assuming that a brane is timelike ($\epsilon = 1$), and that the scalaron is continuous on it, i.e. that

$$[\phi] = 0, \tag{2.15}$$

and combining (2.14) with its contraction, we obtain the final junction conditions in a more convenient form

$$[K_{ab}] = \frac{\kappa_5^2(S_{ab} - \frac{1}{3}h_{ab}S) - \frac{h_{ab}}{3}[\frac{\partial\phi}{\partial w}]}{\phi}. \tag{2.16}$$

Finally, we can see that these $f(R)$ theory junction conditions (2.16) transform in the Einstein (or Randall-Sundrum) limit $f(R) \to R$, i.e., for [cf. (2.5)]

$$\phi \to 1, \qquad \partial\phi/\partial w \to 0 \tag{2.17}$$

into the standard Israel junction conditions for a five-dimensional spacetime [3]:

$$[K_{ab}] = \kappa_5\left(S_{ab} - \frac{1}{3}h_{ab}S\right). \tag{2.18}$$

In Ref. [20] we derived different types of junction conditions. Since the condition (2.12) was not imposed, then the scalar field part of the brane boundary term (2.9) vanished provided that $[K] = 0$, i.e., the trace of the extrinsic curvature was assumed to be continuous on the brane (Eqs. (2.7)–(2.10) of Ref. [20]). After imposing a mirror symmetry the trace of the extrinsic curvature had additionally to vanish $K = 0$ (Eqs. (2.11)–(2.13) of

Ref. [20]), which further under the requirement of vanishing the curvature scalar $R = 0$ gave the special junction conditions obtained in Ref. [18], for example.

However, the junction conditions (2.16) are different and they allow the Randall-Sundrum limit that is quite beneficial for developing $f(R)$ cosmology on the brane.

III. $f(R)$ FRIEDMANN COSMOLOGY ON THE BRANE

In order to discuss Friedmann cosmology on the brane we start with a five-dimensional spherically symmetric bulk metric given in the form [23]

$$ds^2 = -h(r)dT^2 + \frac{dr^2}{h(r)} + r^2 d\Omega_3^2, \tag{3.1}$$

where

$$d\Omega_3^2 = d\chi^2 + sin^2\chi(d\theta^2 + sin^2\theta d\phi^2)$$

is the metric of a three-dimensional unit sphere (which means that we assume the Friedmann curvature index $k = +1$ here). After making a coordinate transformation $T = T(w, \tau)$, $r = r(w, \tau)$, the metric reads as

$$ds^2 = \left\{-h(r)T'^2 + \frac{r'^2}{h(r)}\right\}dw^2 + \left\{-h(r)\dot{T}^2 + \frac{\dot{r}^2}{h(r)}\right\}d\tau^2 + \left\{-2h(r)T'\dot{T} + \frac{2}{h(r)}r'\dot{r}\right\}dwd\tau + r^2 d\Omega_3^2. \tag{3.2}$$

Now, we transform the metric (3.2) into the form that defines the Gaussian normal coordinates [23]

$$ds^2 = dw^2 - d\tau^2 + r^2 d\Omega_3^2,$$

by taking

$$-h(r)T'^2 + \frac{r'^2}{h(r)} = 1, \tag{3.3}$$

$$h^2(r)T'\dot{T} = r'\dot{r} \tag{3.4}$$

(the prime is the derivative with respect to w and the dot with respect to τ) and further assuming that τ is a proper time on the brane i.e.

$$-h(r)\dot{T}^2 + \frac{\dot{r}^2}{h(r)} = -1. \tag{3.5}$$

In order to calculate junction conditions (2.16) we need to calculate the components of the extrinsic curvature for the metric (3.2), i.e.,

$$K_{ab} = -\Gamma^w{}_{ab} = \frac{1}{2}\frac{\partial h_{ab}}{\partial w}, \tag{3.6}$$

which after specifying the components of the induced metric h_{ab}

$$h_{ab} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\chi & 0 \\ 0 & 0 & 0 & r^2\sin^2\chi\sin^2\theta \end{vmatrix} \tag{3.7}$$

allows the only nonvanishing term

$$K_{22} = r\frac{\partial r}{\partial w} = \pm rh(r)\dot{T} = \pm r\sqrt{\dot{r}^2 + h(r)}. \tag{3.8}$$

After assuming that there is a mirror symmetry ($A^+ = -A^- = A$, i.e. $[A] = 2A$ for any quantity A) and that the matter on the brane is in the form of a perfect fluid, the junction conditions (2.16) read as

$$\pm 2r\sqrt{\dot{r}^2 + h(r)} = \frac{\kappa_5^2 r^2\rho - [\frac{\partial\phi}{\partial w}]r^2}{3\phi}. \tag{3.9}$$

In order to proceed further and specify the function $h(r)$ we need to solve field equations in the bulk.

IV. THE SOLUTION OF A VACUUM FIVE-DIMENSIONAL FIELD EQUATION FOR $f(R)$ THEORY

The action (2.2) with $S_{\text{bulk},p} = 0$ for the spherically symmetric metric (3.1) can be expressed as

$$\int r^3\sin^2\chi\cos\theta\{f(Q) - f'(Q)(Q + h''(r) + 6\frac{h'(r)}{r} + 6\frac{h(r)}{r^2} - \frac{6}{r^2})\}d^5x, \tag{4.1}$$

where for this metric [24]

$$R = -h''(r) - 6\frac{h'(r)}{r} - 6\frac{h(r)}{r^2} + \frac{6}{r^2}. \tag{4.2}$$

Varying the action (4.1) with respect to $h(r)$, one obtains [cf. Ref. [24]-Eq. (15)] for the scalaron that

$$f'(Q) = ar + b, \tag{4.3}$$

where a and b are constants. Now, substituting (4.3) to the field Eqs. (2.3) we obtain the differential equation for $h(r)$ as follows:

$$(ar + b)(r^2h''(r) + 4) + r(b + 2ar)h'(r) - 2(2b + 3ar)h(r) = 0. \tag{4.4}$$

The most general solution of (4.4) is

$$h(r) = -\frac{a^2B}{2b^3} - \frac{B}{4br^2} + \frac{aB}{3b^2r} + \frac{a^3Br}{b^4} + Ar^2 + \frac{a^4Br^2\ln(r)}{b^5} - \frac{a^4Br^2\ln(ar+b)}{b^5} - \left(2br^2 + \frac{4ar^3}{3}\right)\left(-\frac{a^2}{2b^3} - \frac{1}{4br^2} + \frac{a}{3b^2r} + \frac{a^3r}{b^4} + \frac{a^4r^2\ln(r)}{b^5} - \frac{a^4r^2\ln(ar+b)}{b^5}\right) + \frac{b(3b^4 - 2ab^3r + 8a^3br^3 + 8a^4r^4) + 4a^2r^2(ar+b)^2(2ar-b)\ln(r) - 4a^2r^2(ar+b)^2(2ar-b)\ln(ar+b)}{6b^5}, \quad (4.5)$$

which in the case of a constant Ricci curvature $R = \text{const}$ [i.e., $a = 0$ and $f'(Q) = b$] gives the solution for a five-dimensional anti–de Sitter space in the form [23]

$$h(r) = 1 - \frac{C}{4r^2} + Ar^2, \quad (4.6)$$

with A, B, and $C = B/b$ being constants.

V. RANDALL-SUNDRUM LIMIT OF THE VACUUM $f(R)$ THEORY ON THE BRANE

In order to discuss a practical transition from the $f(R)$ junction conditions (2.16) to the Randall-Sundrum junction conditions (2.18) we first calculate a jump of the scalaron (4.3) as follows:

$$\left[\frac{\partial\phi}{\partial w}\right] = [(ar(|w|,\tau) + b)_{,w}] = a\frac{\partial r}{\partial|w|}\left[\frac{\partial|w|}{\partial w}\right] = 2ar' = 2ah(r)\dot{T} = \pm 2a\sqrt{\dot{r}^2 + h(r)}. \quad (5.1)$$

Substituting (5.1) into (3.9), we obtain the cosmological equation on the brane as

$$\left(\frac{\dot{r}^2 + h(r)}{r^2}\right)(4ar + 3b)^2 = \left(\frac{\kappa_5^2}{2}\right)^2\rho^2. \quad (5.2)$$

Finally, by taking the limit $a \to 0$ and $b \to 1$ [which is equivalent to the limit $\phi \to 1$, $\partial\phi/\partial w \to 0$ for (2.16)], the Eq. (5.2) takes the form

$$\frac{\dot{r}^2}{r^2} = \frac{\kappa_5^2}{36}\rho^2 - A + \frac{B}{4r^4} - \frac{1}{r^2}, \quad (5.3)$$

which further by assuming $A = -\Lambda_5/6$ and $B = 4U$ becomes the cosmological equation of the well-known Randall-Sundrum-Friedmann brane-world model [3]

$$\frac{\dot{r}^2}{r^2} = \frac{\kappa_5^2}{36}\rho^2 + \frac{\Lambda_5}{6} - \frac{1}{r^2} + \frac{U}{r^4}, \quad (5.4)$$

where Λ_5 is the five-dimensional bulk cosmological constant, and U is the integration constant that refers to the dark radiation (note that the spatial curvature index is $k = +1$ here). In fact, the bulk cosmological constant is induced by geometry of $f(R)$ models despite that $S_{\text{bulk},p} = 0$ in the action (2.1).

VI. CONCLUSIONS

We have proposed new types of brane models that are based on some special boundary conditions in the variational principle. Namely, while varying the Gibbons-Lutrell boundary term, we imposed the condition that the variation of the scalaron should vanish on the brane (but not in the bulk). This allowed us to have less restrictive requirements related to the continuity of the trace of extrinsic curvature on the brane. Because of that we obtained junction conditions for the five-dimensional $f(R)$ gravity that in the Einstein limit $f(R) \to R$ transform into the standard Randall-Sundrum junction conditions. Further, we applied these junction conditions for a particular model of a Friedmann universe on the brane and show explicitly that the limit gives the standard Randall-Sundrum-Friedmann equation. The result is quite beneficial for developing $f(R)$ cosmology on the brane.

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[1] M. Visser, Phys. Lett. **159B**, 22 (1985); N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **436**, 257 (1998); N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Rev. D **59**, 086004 (1999).

[2] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).

[3] P. Binétruy, C. Deffayet, and D. Langlois, Nucl. Phys. **B565**, 269 (2000); P. Binétruy, C. Deffayet, and D. Langlois, Phys. Lett. B **477**, 285 (2000); M. Sasaki, T. Shiromizu, and K. I. Maeda, Phys. Rev. D **62**, 024008 (2000); T. Shiromizu, K. I. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000); S. Mukohyama, T. Shiromizu, and K. I. Maeda, Phys. Rev. D **62**, 024028 (2000); A. N. Aliev and A. E. Gümrükçüoğlu, Classical Quantum Gravity **21**, 5081 (2004).

[4] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980); G. Magnano and L. M. Sokołowski, Phys. Rev. D **50**, 5039 (1994); S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003); G. J. Olmo, Phys. Rev. Lett. **98**, 061101 (2007); S. Capozziello, V. F. Cardone, and A. Troisi, Phys. Rev. D **71**, 043503 (2005); S. Capozziello, S. Nojiri, S. D. Odintsov, and A. Troisi, Phys. Lett. B **639**, 135 (2006); L. Amendola, D. Polarski, and S. Tsujikawa, Phys. Rev. Lett. **98**, 131302 (2007); S. Nojiri and S. D. Odintsov, Phys. Rev. D **74**, 086005 (2006); S. Nojiri and S. D. Odintsov, *Problems of Modern Theoretical Physics, A Volume in Honour of Prof. I. L. Buchbinder in the Occasion of His 60th Birthday* (Tomsk State Pedagogical University Publishing, Tomsk, Russia, 2008), p. 266.

[5] T. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010).

[6] S. Nojiri and S. D. Odintsov, Int. J. Geom. Methods Mod. Phys. **4**, 115 (2007).

[7] D. Lovelock, J. Math. Phys. (N.Y.) **12**, 498 (1971).

[8] N. Deruelle and T. Doležel, Phys. Rev. D **62**, 103502 (2000); K. A. Meissner and M. Olechowski, Phys. Rev. Lett. **86**, 3708 (2001); C. Charmousis and J. F. Dufaux, Classical Quantum Gravity **19**, 4671 (2002); S. C. Davis, Phys. Rev. D **67**, 024030 (2003); A. N. Aliev, H. Cebeci, and T. Dereli, Classical Quantum Gravity **23**, 591 (2006); H. Maeda, V. Sahni, and Yu. Shtanov, Phys. Rev. D **76**, 104028 (2007); P. S. Apostopoulos *et al.*, Phys. Rev. D **76**, 084029 (2007).

[9] A. Balcerzak and M. P. Dąbrowski, Phys. Rev. D **77**, 023524 (2008); A. Balcerzak and M. P. Dąbrowski, J. Cosmol. Astropart. Phys. 01 (2009) 018.

[10] W. Israel, Nuovo Cimento B **44**, 1 (1966).

[11] H. H. V. Borzeszkowski and V. P. Frolov, Ann. Phys. (Leipzig) **492**, 285 (1980).

[12] M. Parry, S. Pichler, and D. Deeg, J. Cosmol. Astropart. Phys. 04 (2005) 014.

[13] E. Dyer and K. Hinterbichler, Phys. Rev. D **79**, 024028 (2009).

[14] S. Nojiri and S. D. Odintsov, J. High Energy Phys. 07 (2000) 049; S. Nojiri, S. D. Odintsov, and S. Ogushi, Phys. Rev. D **65**, 023521 (2001); S. Nojiri and S. D. Odintsov, Gen. Relativ. Gravit. **37**, 1419 (2005).

[15] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

[16] M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, Eur. Phys. J. C **55**, 337 (2008).

[17] M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, arXiv:0811.4609.

[18] N. Deruelle, M. Sasaki, and Y. Sendouda, Prog. Theor. Phys. **119**, 237 (2008).

[19] V. I. Afonso, D. Bazeia, R. Menezes, and A. Yu. Petrov, Phys. Lett. B **658**, 71 (2007).

[20] A. Balcerzak and M. P. Dąbrowski, Phys. Rev. D **81**, 123527 (2010).

[21] A. Balcerzak, Ann. Phys. (Berlin) **19**, 271 (2010).

[22] S. W. Hawking and J. C. Lutrell, Nucl. Phys. **B247**, 250 (1984).

[23] P. D. Mannheim, *Brane Localized Gravity* (World Scientific, Singapore, 2005).

[24] L. Sebastiani and S. Zerbini, Eur. Phys. J. C **71**, 1591 (2011).