PHYSICAL REVIEW D **84,** 063506 (2011)

Standard cosmology in Chern-Simons gravity

F. Gomez,* P. Minning, and P. Salgado†

Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile

(Received 16 April 2011; published 6 September 2011)

We consider a five-dimensional action which is composed of a gravitational sector and a sector of matter, where the gravitational sector is given by a Chern-Simons gravity action instead of the Einstein-Hilbert action. In the event that the matter action is the action for a perfect fluid, it is shown that the standard five-dimensional Friedmann-Robertson-Walker (FRW) equations and some of their solutions can be obtained, in a certain limit, from the so-called Chern-Simons-FRW field equations, which are the cosmological field equations corresponding to a Chern-Simons gravity theory. It is also shown, using a compactification procedure known as dynamic compactification, that the cosmological field equations obtained from the Chern-Simons gravity theory lead, in a certain limit, to the usual four-dimensional FRW equations.

DOI: 10.1103/PhysRevD.84.063506 PACS numbers: 04.50.−h, 04.50.Kd, 98.80.−k, 98.80.Jk

I. INTRODUCTION

Three of the four fundamental forces of nature are consistently described by Yang-Mills (YM) quantum theories. Gravity, the fourth fundamental interaction, resists quantization in spite of general relativity (GR) and YM theories having a similar geometrical foundation. There exists, however, a very important difference between YM theory and GR (for a thorough discussion, see, e.g., Ref. [1]).

YM theories rely heavily on the existence of the "stage"—the fixed, nondynamical, background metric structure with which the space-time manifold M is assumed to be endowed.

In GR the space-time is a dynamical object which has independent degrees of freedom, and is governed by dynamical equations, namely, the Einstein field equations. This means that in GR the geometry is dynamically determined. Therefore, the construction of a gauge theory of gravity requires an action that does not consider a fixed space-time background. An action for gravity fulfilling these conditions, albeit only in odd-dimensional space-time, $d = 2n + 1$, was proposed long ago by Chamseddine [2–4]. In the first-order formalism, where the independent fields are the vielbein e^a and the spin connection ω^{ab}, the Lagrangian can be written as

$$L^{(2n+1)} = \kappa \varepsilon_{a_1 \cdots a_{2n+1}} \sum_{k=0}^{n} \frac{c_k}{\ell^{2(n-k)+1}} \times R^{a_1 a_2} \cdots R^{a_{2k-1} a_{2k}} e^{a_{2k+1}} \cdots e^{a_{2n+1}}, \quad (1)$$

where κ and c_k are dimensionless constants and ℓ is a length parameter. As it stands, the Lagrangian (1) is invariant under the local Lorentz transformations $\delta e^a = \lambda^a{}_b e^b$, $\delta \omega^{ab} = -\mathrm{D}_\omega \lambda^{ab}$, where $\lambda^{ab} = -\lambda^{ba}$ are the real, local, infinitesimal parameters that define the transformation and D_ω stands for the Lorentz covariant derivative. When the c_k constants are chosen as

$$c_k = \frac{1}{2(n-k)+1} \binom{n}{k},$$

the Lagrangian (1) can be regarded as the Chern-Simons form for the anti–de Sitter (AdS) algebra, and its invariance is enlarged accordingly to include AdS "boosts."

If Chern-Simons theories are the appropriate gauge theories to provide a framework for the gravitational interaction, then these theories must satisfy the correspondence principle; namely, they must be related to general relativity. Interesting research in this direction has recently been carried out in Refs. [5–7]. In the first two references it was found that the modification of Chern-Simons theory for AdS gravity according to the expansion method of Ref. [8] is not sufficient to produce a direct link with general relativity due to the presence of higher order curvature terms in the action.

However, in Ref. [7] it was shown that the standard, five-dimensional general relativity (without a cosmological constant) can be obtained from Chern-Simons gravity theory for a certain Lie algebra $\mathcal{B}$. The Chern-Simons Lagrangian is built from a $\mathcal{B}$-valued, one-form gauge connection A that depends on a scale parameter l which can be interpreted as a coupling constant that characterizes different regimes within the theory. The $\mathcal{B}$ algebra, on the other hand, is obtained from the AdS algebra and a particular semigroup S by means of the S-expansion procedure introduced in Refs. [9,10]. The field content induced by $\mathcal{B}$ includes the vielbein e^a, the spin connection ω^{ab}, and two extra bosonic fields h^a and k^{ab}. In Ref. [7] it was then shown that it is possible to recover odd-dimensional Einstein gravity theory from a Chern-Simons gravity theory in the limit where the coupling constant l tends to zero while keeping the effective Newton's constant fixed.

*fernagomez@udec.cl
†pasalgad@udec.cl

It is the purpose of this work to find some solutions for flat cosmological field equations, which were obtained from an action for Chern-Simons gravity theory, studied in Ref. [7]. Using a compactification procedure, known as dynamic compactification, we show that the cosmological field equations obtained from Chern-Simons gravity theory lead, in a certain limit, to the usual four-dimensional Friedmann-Robertson-Walker (FRW) equations.

This paper is organized as follows: In Sec. II we briefly review how to recover the five-dimensional general relativity from Chern-Simons gravity. In Sec. III we obtain the field equations for the Lagrangian $L = L_g + L_M$, where L_g is the Chern-Simons gravity Lagrangian and L_M is the corresponding matter Lagrangian. In the case where L_M is a perfect fluid, we obtain the so-called Chern-Simons Friedmann-Robertson-Walker field equations in Sec. IV, together with some of their solutions, which lead, in a certain limit, to the usual five-dimensional FRW equations. In Sec. V we consider a cosmological model based on the Kaluza-Klein theory: We study a metric in which the scale factor of the compact space evolves as an inverse power of the radius of the observable Universe. A summary and three appendixes conclude this work.

II. GENERAL RELATIVITY FROM CHERN-SIMONS GRAVITY

In this section we briefly review how to recover the five-dimensional general relativity from Chern-Simons gravity. The Lagrangian for five-dimensional Chern-Simons AdS gravity can be written as

$$L^{(5)}_{\rm AdS} = \kappa\Big(\frac{1}{5l^5}\epsilon_{a_1\cdots a_5}e^{a_1}\cdots e^{a_5} + \frac{2}{3l^3}\epsilon_{a_1\cdots a_5}R^{a_1a_2}e^{a_3}\cdots e^{a_5} + \frac{1}{l}\epsilon_{a_1\cdots a_5}R^{a_1a_2}R^{a_3a_4}e^{a_5}\Big), \quad (2)$$

where e^a corresponds to the one-form *vielbein*, and $R^{ab} = d\omega^{ab} + \omega^a{}_c\omega^{cb}$ to the Riemann curvature in the first-order formalism.

The Lagrangian (2) is off-shell invariant under the AdS-Lie algebra SO(4,2), whose generators $\tilde{J}_{ab}$ of Lorentz transformations and $\tilde{P}_a$ of AdS boosts satisfy the commutation relationships

$$[\tilde{J}_{ab}, \tilde{J}_{cd}] = \eta_{cb}\tilde{J}_{ad} - \eta_{ca}\tilde{J}_{bd} + \eta_{db}\tilde{J}_{ca} - \eta_{da}\tilde{J}_{cb}, \quad (3)$$

$$[\tilde{J}_{ab}, \tilde{P}_c] = \eta_{cb}\tilde{P}_a - \eta_{ca}\tilde{P}_b, \quad (4)$$

$$[\tilde{P}_a, \tilde{P}_b] = \tilde{J}_{ab}. \quad (5)$$

In order to interpret the gauge field associated with a translational generator $\tilde{P}_a$ as the vielbein, one is *forced* to introduce a length scale l in the theory (for details see [7]). Therefore, following Refs. [2,3], the one-form gauge field A of the Chern-Simons theory is given, in this case, by

$$A = \frac{1}{l}e^a\tilde{P}_a + \frac{1}{2}\omega^{ab}\tilde{J}_{ab}. \quad (6)$$

It is important to notice that once the length scale l is brought into the Chern-Simons theory, the Lagrangian splits into several sectors, each one of them proportional to a different power of l, as we can see directly in Eq. (1) with

$$c_k = \frac{1}{2(n-k)+1}\binom{n}{k}.$$

From Lagrangian (2) it is apparent that neither the $l\to\infty$ nor the $l\to 0$ limit yields the Einstein-Hilbert term $\epsilon_{a_1\cdots a_5}R^{a_1a_2}e^{a_3}\cdots e^{a_5}$ alone. Rescaling κ properly, those limits will lead either to the Gauss-Bonnet term (Poincaré Chern-Simons gravity) or to the cosmological constant term by itself, respectively.

We arrive at the Lagrangian (2) as the Chern-Simons form for the AdS algebra in five dimensions. This algebra choice is crucial, since it permits the interpretation of the gauge fields e^a and ω^{ab} as the fünfbein and the spin connection, respectively. It is, however, not the only possible choice: As it explicitly shown in [7], there exist other Lie algebras that also allow for a similar identification and lead to a Chern-Simons Lagrangian that touches upon the Einstein-Hilbert term in a certain limit.

Following the definitions of Ref. [9], let us consider the S expansion of the Lie algebra SO(4,2) using as a semigroup $S^{(3)}_{\rm E}$. After extracting a resonant subalgebra and performing its 0_S reduction, one finds a new Lie algebra, call it $\mathcal{B}$, whose generators $\{J_{ab}, P_a, Z_{ab}, Z_a\}$ satisfy the commutation relationships

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{cb}J_{ad} - \eta_{ca}J_{bd} + \eta_{db}J_{ca} - \eta_{da}J_{cb},\\
[J_{ab}, P_c] &= \eta_{cb}P_a - \eta_{ca}P_b,\\
[P_a, P_b] &= Z_{ab},\\
[J_{ab}, Z_{cd}] &= \eta_{cb}Z_{ad} - \eta_{ca}Z_{bd} + \eta_{db}Z_{ca} - \eta_{da}Z_{cb},\\
[J_{ab}, Z_c] &= \eta_{cb}Z_a - \eta_{ca}Z_b,\\
[Z_{ab}, P_c] &= \eta_{cb}Z_a - \eta_{ca}Z_b,\\
[P_a, Z_b] &= 0,\\
[Z_{ab}, Z_c] &= 0,\\
[Z_{ab}, Z_{cd}] &= 0,\\
[Z_a, Z_b] &= 0,
\end{aligned} \quad (7)$$

where these new generators can be written as $J_{ab} = \lambda_0\otimes\tilde{J}_{ab}$, $Z_{ab} = \lambda_2\otimes\tilde{J}_{ab}$, $P_a = \lambda_1\otimes\tilde{P}_a$, $Z_a = \lambda_3\otimes\tilde{P}_a$.

Here $\tilde{J}_{ab}$ and $\tilde{P}_a$ correspond to the original generators of SO(4,2), and the λ_α belong to a discrete, Abelian semigroup. The semigroup elements $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ are *not* real numbers, and they are *dimensionless*. In this particular case, they obey the multiplication law given by $\lambda_\alpha\lambda_\beta = \lambda_{\alpha+\beta}$ when $\alpha+\beta\leq 4$, and $\lambda_\alpha\lambda_\beta = \lambda_4$ when $\alpha+\beta > 4$.

In order to write down a Chern-Simons Lagrangian for the $\mathcal{B}$ algebra, we start from the one-form gauge connection

$$A = \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{l}e^a P_a + \frac{1}{2}k^{ab}Z_{ab} + \frac{1}{l}h^a Z_a \qquad (8)$$

and the two-form curvature (for details see [7])

$$F = \frac{1}{2}R^{ab}J_{ab} + \frac{1}{l}T^a P_a + \frac{1}{2}\left(D_\omega k^{ab} + \frac{1}{l^2}e^a e^b\right)Z_{ab} + \frac{1}{l}(D_\omega h^a + k^a{}_b e^b)Z_a, \qquad (9)$$

where D is the covariant derivative with respect to the Lorentz piece of the connection, and $T^a = De^a$.

Using the extended Cartan homotopy formula as in Ref. [11], and integrating by parts, it is possible to write down the Chern-Simons Lagrangian in five dimensions for the $\mathcal{B}$ algebra as

$$L^{(5)}_{\rm ChS} = \alpha_1 l^2 \varepsilon_{abcde}R^{ab}R^{cd}e^e + \alpha_3\varepsilon_{abcde}\left(\tfrac{2}{3}R^{ab}e^c e^d e^e + 2l^2 k^{ab}R^{cd}T^e + l^2 R^{ab}R^{cd}h^e\right), \qquad (10)$$

where we can see that (i) if one identifies the field e^a with the vielbein, the system consists of the Einstein-Hilbert action plus nonminimally coupled matter fields given by h^a and k^{ab}, and (ii) it is possible to recover the odd-dimensional Einstein gravity theory from a Chern-Simons gravity theory in the limit where the coupling constant l equals zero while keeping the effective Newton's constant fixed.

The variation of the Lagrangian (10), modulo boundary terms, is given by

$$\delta L^{(5)}_{\rm ChS} = \varepsilon_{abcde}(2\alpha_3 R^{ab}e^c e^d + \alpha_1 l^2 R^{ab}R^{cd} + 2\alpha_3 l^2 D_\omega k^{ab}R^{cd})\delta e^e + \alpha_3 l^2\varepsilon_{abcde}R^{ab}R^{cd}\delta h^e + 2\alpha_3 l^2\varepsilon_{abcde}R^{cd}T^e\delta k^{ab} + 2\varepsilon_{abcde}(\alpha_1 l^2 R^{cd}T^e + \alpha_3 l^2 D_\omega k^{ab}T^e + \alpha_3 e^c e^d T^e + \alpha_3 l^2 R^{cd}D_\omega h^e + \alpha_3 l^2 R^{cd}k^e{}_f e^f)\delta\omega^{ab}. \qquad (11)$$

Therefore, in the limit where the coupling constant l equals zero, we obtain

$$\delta L^{(5)}_{\rm ChS} = 2\alpha_3\varepsilon_{abcde}R^{ab}e^c e^d + 2\alpha_3\varepsilon_{abcde}e^c e^d T^e\delta\omega^{ab}; \qquad (12)$$

i.e., the limit where $l \to 0$ leads us to just the Einstein-Hilbert dynamics in the vacuum. It is interesting to observe that the argument given here is not just a five-dimensional accident. In every odd dimension, it is possible to perform the S expansion in the way sketched here, take the vanishing coupling constant limit $l \to 0$, and recover Einstein-Hilbert gravity [7].

III. EINSTEIN-CHERN-SIMONS FIELD EQUATIONS FOR THE FRW METRIC

In this section we consider the field equations for the Lagrangian $L = L_g + L_M$, where L_g is the Chern-Simons gravity Lagrangian $L^{(5)}_{\rm ChS}$ and L_M is the corresponding matter Lagrangian. If $T^a = 0$ and $k^{ab} = 0$, Eq. (11) takes the form

$$\delta L^{(5)}_{\rm ChS} = \varepsilon_{abcde}(2\alpha_3 R^{ab}e^c e^d + \alpha_1 l^2 R^{ab}R^{cd})\delta e^e + \alpha_3 l^2\varepsilon_{abcde}R^{ab}R^{cd}\delta h^e + 2\varepsilon_{abcde}\delta\omega^{ab}\alpha_3 l^2 R^{cd}Dh^e. \qquad (13)$$

In Eq. (13) the fields e^a, ω^{ab} (through R^{ab}), and h^a are present. If we wish to find a FRW-type cosmological solution, then we must demand that the three fields satisfy the cosmological principle.

Five-dimensional FRW metric

We consider first the fields e^a and ω^{ab} (through R^{ab}). In five dimensions, the FRW metric is given by [12–14]

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\{d\chi^2 + r^2(\chi)[d\theta_2^2 + sen^2\theta_2 d\theta_3^2 + sen^2\theta_2 sen^2\theta_3 d\theta_4^2]\} \\ &= \eta_{ab}e^a e^b \eta_{ab} = \mathrm{diag}(-1, +1, +1, +1, +1). \end{aligned} \qquad (14)$$

Introducing an orthonormal basis, we have

$$\begin{aligned} &e^0 = dt, \qquad e^1 = a(t)d\chi, \qquad e^2 = a(t)r(\chi)d\theta_2, \\ &e^3 = a(t)r(\chi)sen\theta_2 d\theta_3, \\ &e^4 = a(t)r(\chi)sen\theta_2 sen\theta_3 d\theta_4. \end{aligned} \qquad (15)$$

Taking the exterior derivatives, we get

$$\begin{aligned} &de^0 = 0, \qquad de^1 = \frac{\dot a}{a}e^0 e^1, \qquad de^2 = \frac{\dot a}{a}e^0 e^2 + \frac{r'}{ar}e^1 e^2, \\ &de^3 = \frac{\dot a}{a}e^0 e^3 + \frac{r'}{ar}e^1 e^3 + \frac{1}{ar}\cot\theta_2 e^2 e^3, \\ &de^4 = \frac{\dot a}{a}e^0 e^4 + \frac{r'}{ar}e^1 e^4 + \frac{1}{ar}\cot\theta_2 e^2 e^4 + \frac{1}{ar}\frac{1}{sen\theta_2}\cot\theta_3 e^3 e^4, \end{aligned} \qquad (16)$$

where a prime denotes a derivative with respect to r. The next step is to use Cartan's first structural equation

$$T^a = de^a + \omega^a{}_b e^b = 0$$

and the antisymmetry of the connection forms, $\omega^{ab} = -\omega^{ba}$, to find the nonzero connection forms. The calculations give

$$\begin{aligned}\omega^0{}_p &= \frac{\dot{a}}{a}e^p = \omega^p{}_0, \quad p = 1, 2, 3, 4;\\ \omega^1{}_p &= \frac{r'}{ar}e^p = -\omega^p_1, \quad p = 2, 3, 4;\\ \omega^2{}_p &= -\frac{1}{ar}\cot\theta_2 e^p = -\omega^p{}_2, \quad p = 3, 4;\\ \omega^3{}_4 &= -\frac{1}{ar}\frac{1}{sen\theta_2}\cot\theta_3 e^4 = -\omega^4{}_3.\end{aligned} \tag{17}$$

From Cartan's second structural equation

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c\omega^c{}_b$$

we can calculate the curvature matrix. The nonzero components are

$$\begin{aligned}R^0{}_p &= \frac{\ddot{a}}{a}e^0e^p = R^p{}_0, \quad p = 1, 2, 3, 4;\\ R^1{}_p &= \left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{r''}{a^2 r}\right]e^1e^p = -R^p{}_1, \quad p = 2, 3, 4;\\ R^q{}_p &= \left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{(ar)^2} - \left(\frac{r'}{ar}\right)^2\right]e^qe^p = -R^p{}_q, \quad q = 2, 3;\\ & p = q + 1, 4.\end{aligned} \tag{18}$$

From the assumption of isotropy, the spatial directions should be equal for the orthonormal frame. Hence, the components of the curvature matrix should be equal for the four directions. This means that we must have

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{r''}{a^2 r} = \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{(ar)^2} - \left(\frac{r'}{ar}\right)^2, \quad r'' - \frac{1}{r}r'^2 + \frac{1}{r} = 0. \tag{19}$$

Integrating, one finds

$$r' \equiv \frac{dr}{d\chi} = (1 - Kr^2)^{1/2}. \tag{20}$$

Introducing (20) in (19) we have

$$r'' = -kr \tag{21}$$

with $k = +1, 0, -1$. Therefore

$$R^{0p} = \left(\frac{\ddot{a}}{a}\right)e^0e^p, \quad p = 1, 2, 3, 4; \tag{22}$$

$$R^{qp} = \left(\frac{\dot{a}^2 + k}{a^2}\right)e^qe^p, \quad q = 1, 2, 3, \quad p = q + 1, 4. \tag{23}$$

The field h^a

We consider now the field h^a. Writing the field h^a in the vielbein basis, we have

$$h^a = h^a{}_\mu e^\mu = \eta^{a\mu}h_{\mu\nu}e^\nu. \tag{24}$$

Using Eq. 13.4.6 of Ref. [15], we have

$$h^0 = f(t)e^0, \qquad h^p = g(t)e^p, \qquad p = 1, \ldots, 4. \tag{25}$$

From (17) and (25) we can see that

$$Dh^0 = 0, \qquad Dh^p = \left[(g - f)\frac{\dot{a}}{a} + \dot{g}\right]e^0e^p, \tag{26}$$

where a, f, and g are functions dependent on time t. The next step is to introduce (22), (23), and (26) into

$$\begin{aligned}\delta L &= \delta L_{\mathrm{ChS}} + \delta L_M,\\ \delta L &= \delta L_{\mathrm{ChS}} + \frac{\delta L_M}{\delta e^a}\delta e^a + \frac{\delta L_M}{\delta h^a}\delta h^a + \frac{\delta L_M}{\delta\omega^{ab}}\delta\omega^{ab},\end{aligned} \tag{27}$$

where $\delta L^{(5)}_{\mathrm{ChS}}$ is given by (13), and where $\frac{\delta L_M}{\delta e^a}$ is associated with the energy-momentum tensor $T_{\mu\nu}$, $\frac{\delta L_M}{\delta h^a}$ is associated with the energy-momentum tensor for the field h^a, which we denote as $T^{(h)}_{\mu\nu}$, and $\frac{\delta L_M}{\delta\omega^{ab}}$ is associated with the spin tensor $S_{\mu\nu\lambda}$.

The calculations give

$$48\alpha_3\left(\frac{\dot{a}^2 + k}{a^2}\right) + 24\alpha_1 l^2\left(\frac{\dot{a}^2 + k}{a^2}\right)^2 = \beta_1 T_{00}, \tag{28}$$

$$-24\alpha_3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}^2 + k}{a^2}\right)\right] - 24\alpha_1 l^2\frac{\ddot{a}}{a}\left(\frac{\dot{a}^2 + k}{a^2}\right) = \beta_1 T_{11}, \tag{29}$$

$$24\alpha_3 l^2\left(\frac{\dot{a}^2 + k}{a^2}\right)^2 = \beta_2 T^{(h)}_{00}, \tag{30}$$

$$-24\alpha_3 l^2\frac{\ddot{a}}{a}\left(\frac{\dot{a}^2 + k}{a^2}\right) = \beta_2 T^{(h)}_{11}, \tag{31}$$

$$24\alpha_3 l^2\left(\frac{\dot{a}^2 + k}{a^2}\right)\left[(g - f)\frac{\dot{a}}{a} + \dot{g}\right] = 0, \tag{32}$$

where in the last equation we took into account the fact that $T^a = 0$, which means that the spin tensor is zero.

IV. DYNAMICS OF HOMOGENEOUS AND ISOTROPIC COSMOLOGIES

The cosmological principles state that our Universe is homogeneous and isotropic. To solve the Einstein-Chern-Simons field equations under this assumption, we need an energy-momentum tensor which is also homogeneous and isotropic. The most general forms of the energy-momentum tensors, $T_{\mu\nu}$ and $T^{(h)}_{\mu\nu}$, compatible with homogeneity and isotropy are given by [12]

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P\eta_{\mu\nu}, \tag{33}$$

$$T^{(h)}{}_{\mu\nu} = (\rho^{(h)} + P^{(h)})u_\mu u_\nu + P^{(h)}\eta_{\mu\nu}, \tag{34}$$

where ρ is the proper energy (or mass) density of the usual fluid and P its pressure. $\rho^{(h)}$ and $P^{(h)}$ are the energy density and pressure for the perfect fluid associated with the field

h^a. Homogeneity implies that the pressure and density should be position independent on the spatial hypersurfaces. Hence, they can only be time dependent. If the vector u_μ, which is the five-velocity of the fluid, has a spatial component, then the fluid has a spatial direction compared to the hypersurfaces. This would violate our assumption of spatial isotropy. Thus the vector u_μ has only a time component; the fluid flow is orthogonal to the hypersurfaces: $u_\mu = (1, 0, 0, 0, 0)$. The energy-momentum tensor is therefore diagonal, i.e.,

$$\begin{aligned} &T_{00} = \rho, \qquad T_{11} = T_{22} = T_{33} = T_{44} = P, \\ &T^{(h)}{}_{00} = \rho^{(h)}, \qquad T^{(h)}_{11} = T^{(h)}_{22} = T^{(h)}_{33} = T^{(h)}_{44} = P^{(h)}. \end{aligned} \tag{35}$$

Introducing (35) in (28)–(32), we have

$$6\left(\frac{\dot{a}^2 + k}{a^2}\right) + \alpha l^2 \left(\frac{\dot{a}^2 + k}{a^2}\right)^2 = \kappa_1 \rho, \tag{36}$$

$$3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}^2 + k}{a^2}\right)\right] + \alpha l^2 \frac{\ddot{a}}{a}\left(\frac{\dot{a}^2 + k}{a^2}\right) = -\kappa_1 P, \tag{37}$$

$$l^2 \left(\frac{\dot{a}^2 + k}{a^2}\right)^2 = \kappa_2 \rho^{(h)}, \tag{38}$$

$$l^2 \frac{\ddot{a}}{a}\left(\frac{\dot{a}^2 + k}{a^2}\right) = -\kappa_2 P^{(h)}, \tag{39}$$

$$(g - f)\frac{\dot{a}}{a} + \dot{g} = 0, \tag{40}$$

where

$$\alpha = \frac{3\alpha_1}{\alpha_3}, \qquad \kappa_1 = \frac{\beta_1}{8\alpha_3}, \qquad \kappa_2 = \frac{\beta_2}{24\alpha_3}. \tag{41}$$

We note that κ_1 is the gravitational constant which is positive; α and κ_2 are constants whose values should be determined. We should note that Eqs. (36) and (37) have been found in the study of the FRW equations for an Einstein plus Gauss-Bonnet action in $4 + d$ dimensions (see [16]). From Eqs. (36)–(40) we notice the following:

(a) If $l = 0$ and $h^a = 0$ we have that $f = g = \rho^{(h)} = P^{(h)} = 0$. Thus Eqs. (36)–(40) take the form

$$6\left(\frac{\dot{a}^2 + k}{a^2}\right) = \kappa_1 \rho, \tag{42}$$

$$3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}^2 + k}{a^2}\right)\right] = -\kappa_1 P; \tag{43}$$

i.e., in the limit $l = 0$ we recover the usual five-dimensional FRW equations.

(b) We need an equation relating the energy, the pressure, and the scale factor. The energy-momentum tensor has to be divergence-free, which signals the conservation of energy. This follows automatically from Eqs. (36) and (37) from which we find the following relation:

$$\dot{\rho} + 4\frac{\dot{a}}{a}(\rho + P) = 0. \tag{44}$$

Analogously, from Eqs. (38) and (39) we find the following conservation equation:

$$\dot{\rho}^{(h)} + 4\frac{\dot{a}^3}{a(\dot{a}^2 + k)}\rho^{(h)} + 4\frac{\dot{a}}{a}P^{(h)} = 0. \tag{45}$$

A. Solutions of FRW-ChS field equations: Case $k = 0$

The observational evidence seems to indicate that the Universe could be flat (see [17,18]). Now try to solve Eqs. (36)–(40) for the case $k = 0$. If $k = 0$ Eqs. (36)–(40) take the form

$$6\frac{\dot{a}^2}{a^2} + \alpha l^2 \frac{\dot{a}^4}{a^4} = \kappa_1 \rho, \tag{46}$$

$$\dot{\rho} + 4\frac{\dot{a}}{a}(\rho + P) = 0, \tag{47}$$

$$l^2 \frac{\dot{a}^4}{a^4} = \kappa_2 \rho^{(h)}, \tag{48}$$

$$\dot{\rho}^{(h)} + 4\frac{\dot{a}}{a}(\rho^{(h)} + P^{(h)}) = 0, \tag{49}$$

$$(g - f)\frac{\dot{a}}{a} + \dot{g} = 0. \tag{50}$$

It seems that the most convenient way to approach the solution of Eqs. (36)–(40) is as follows: First solve the system of Eqs. (36) and (37), in the same way as is done in general relativity, and then replace $a(t)$ in (38)–(40). Following the same procedure used in general relativity, we further assume that the perfect fluid obeys the barotropic equation of state,

$$P = \omega \rho, \tag{51}$$

where ω can be a time-dependent function or a constant. If ω is a constant then, introducing (51) in (47), we obtain

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{4(\omega + 1)}, \tag{52}$$

where the subscript zero means evaluation at the present time t_0.

Using Eqs. (46), (47), and (51) we obtain

$$(1 + \omega)\left[6 + \alpha l^2 \frac{\dot{a}^2}{a^2}\right]\frac{\dot{a}^2}{a^2} = -\left[3 + \alpha l^2 \frac{\dot{a}^2}{a^2}\right]\frac{d}{dt}\left(\frac{\dot{a}}{a}\right). \tag{53}$$

This equation gives the behavior of the scale factor $a = a(t)$ and is to be solved for the cases when the parameter is uniquely $\omega = -1$ and the general case when $\omega \neq -1$.

1. Solutions for $\omega = -1$

If $\omega = -1$, Eq. (53) reduces to

$$\left[3 + \alpha l^2 \frac{\dot{a}^2}{a^2}\right] \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = 0. \tag{54}$$

To find a solution to (54) there are two possibilities:

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = 0, \tag{55}$$

$$\left[3 + \alpha l^2 \frac{\dot{a}^2}{a^2}\right] = 0. \tag{56}$$

It is straightforward to see that both equations lead to similar results. The solution of (56) is a particular case of the solution (55). The solution of (55) is given by

$$a(t) = Ce^{H_0 t} \tag{57}$$

which is a de Sitter-type solution, where C is a constant and $H_0 = H = \frac{\dot{a}}{a}$ is the Hubble constant. Rewriting Eq. (46) as

$$\alpha l^2 \left(\frac{\dot{a}^2}{a^2}\right)^2 + 6\frac{\dot{a}^2}{a^2} - \kappa_1 \rho = 0 \tag{58}$$

we see that it is a quadratic equation in $\dot{a}^2/a^2$ whose solution yields a value for the Hubble parameter of the form

$$\frac{\dot{a}^2}{a^2} = -\frac{3}{\alpha l^2}\left\{1 \pm \left[1 \pm l^2 \frac{\alpha}{9}\kappa_1 \rho\right]^{1/2}\right\}. \tag{59}$$

If we consider the case of a small l^2 limit, we can expand the root to first order in l^2. In the expansion we can see that it is necessary to take the negative sign to recover the FRW equations when $l^2 = 0$. Thus in the first-order approximation, Eq. (59) takes the form

$$H^2 = \frac{\dot{a}^2}{a^2} \approx \frac{\rho\kappa_1}{6}\left\{1 - l^2 \frac{\alpha}{36}\kappa_1 \rho\right\} \tag{60}$$

or

$$H_0^2 \approx \frac{\rho_0 \kappa_1}{6}\left\{1 - l^2 \frac{\alpha}{36}\kappa_1 \rho_0\right\}, \tag{61}$$

where in the $l^2 = 0$ limit, we get the same result that we obtain using the Einstein equations in five dimensions.

2. Solutions for $\omega \neq -1$

We consider now the behavior of the scale factor for the general case when ω is left as a free parameter. By integrating Eq. (53) with $\omega \neq -1$ we obtain

$$\frac{\dot{a}}{a} = \frac{1}{\gamma}\tan\left\{\frac{1}{\gamma}\left[\frac{a}{\dot{a}} - 2(1+\omega)t\right]\right\}, \tag{62}$$

where we have defined $\gamma = \sqrt{\frac{\alpha l^2}{6}}$. In the small l^2 limit, we can expand Eq. (62). In fact, taking the arctan of each side of (62) we have

$$\arctan\left(\gamma\frac{\dot{a}}{a}\right) = \frac{1}{\gamma}\left[\frac{a}{\dot{a}} - 2(1+\omega)t\right] \tag{63}$$

and, carrying out the expansion to first order in γ, we obtain

$$\gamma^2 \frac{\dot{a}^2}{a^2} + 2(1+\omega)\frac{\dot{a}}{a}t - 1 \approx 0. \tag{64}$$

Solving for the Hubble parameter, we obtain

$$\frac{\dot{a}}{a} \approx -\frac{(1+\omega)}{\gamma^2}t\left[1 \pm \sqrt{1 + \left(\frac{\gamma}{(1+\omega)t}\right)^2}\right]. \tag{65}$$

Again, expanding the square root in (65) to first order in l^2, and considering the negative sign to recover the five-dimensional FRW equations, we have

$$\frac{\dot{a}}{a} \approx -\frac{(1+\omega)}{\gamma^2}t\left[1 \pm \sqrt{1 + \left(\frac{\gamma}{(1+\omega)t}\right)^2}\right]. \tag{66}$$

Integrating (66), we obtain a value for the scale factor of the form

$$a(t) \approx Ct^{1/2(1+\omega)}\left[1 + \frac{\alpha l^2}{12}\left(\frac{1}{2(1+\omega)}\right)^3 \frac{1}{t^2}\right], \tag{67}$$

where

$$C = a_0\left\{[2(1+\omega)]^2 \frac{\kappa_1 \rho_0}{6}\right\}^{1/4(1+\omega)}.$$

From Eq. (67) we notice the following:

(i) The cases of greatest physical interest are those with $\omega = 0$ and $\omega = 1/4$, which are in the category $\omega \neq -1$. These cases are usually called the eras of matter and radiation, respectively.

(ii) For small values of l^2 and for values of t^2 that are not small, we have that the term on the right in (67) is negligible compared to the first term, and we recover the usual solutions to the five-dimensional FRW equations.

(iii) In the case that t^2 is of the order of l^2, we have that the term on the right in (67) is not negligible compared to the first term, and therefore, it becomes important in the description of evolution: This is a notable difference from the results obtained from general relativity. If the term on the right in (67) takes a value greater than zero, then it is possible that this term is important for the description of an inflationary period in the early stages of the Universe.

(iv) We should note that this solution corresponds to a valid theory in five dimensions which describes the evolution of five-dimensional space-time.

3. Solution without matter, $\rho = p = 0$

We consider the case $\rho = p = 0$. This case is described by the equation of state $p = \omega\rho$. If $\rho = 0$ and assuming that $\dot{a}/a \neq 0$, we have that Eq. (46) takes the form

$$6\frac{\dot{a}^2}{a^2} + \alpha l^2 \frac{\dot{a}^4}{a^4} = 0 \Rightarrow \frac{\dot{a}^2}{a^2} = -\frac{6}{\alpha l^2}. \tag{68}$$

Since $\frac{\dot{a}^2}{a^2} > 0$, we have that α must be less than zero. Writing $\alpha = -|\alpha|$, Eq. (68) takes the form

$$H_0^2 = \frac{\dot{a}^2}{a^2} = \frac{6}{|\alpha| l^2} \tag{69}$$

whose solution is given by

$$a(t) = Ce^{H_0 t} \tag{70}$$

which corresponds to a model of de Sitter type. In the context of general relativity there is no similar solution for the case $\rho = p = 0$.

V. FOUR-DIMENSIONAL FRW EQUATIONS OF STANDARD COSMOLOGY FROM FIVE-DIMENSIONAL FRW-CHS EQUATIONS: CASE $k = 0$

So far we have found some solutions for flat cosmological field equations, which were obtained from a Lagrangian for a Chern-Simons gravity theory, studied in Ref. [7]. One problem with these solutions is that they are valid only in a five-dimensional space. In this section we consider a space-time metric which contains as a subspace the usual FRW metric in four dimensions. As in the previous sections we study the case $k = 0$. The metric will be written in a convenient way so that it can achieve the compactness of the fifth dimension. Following Refs. [16,19] we consider the following five-dimensional metric:

$$ds^2 = -dt^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + b^2(t)dx^2. \tag{71}$$

Introducing an orthonormal basis, we get

$$e^0 = dt, \qquad e^p = a dx^p, \quad p = 1, 2, 3, \qquad e^4 = b dx^4. \tag{72}$$

Taking the exterior derivatives, we get

$$de^0 = 0, \qquad de^p = \frac{\dot{a}}{a} e^0 e^p, \quad p = 1, 2, 3, \qquad de^4 = \frac{\dot{b}}{b} e^0 e^4. \tag{73}$$

We introduce (72) and (73) into Cartan's first structural equation

$$T^a = de^a + \omega^a{}_b e^b = 0 \tag{74}$$

and the antisymmetry of the connection forms, $\omega^{ab} = -\omega^{ba}$, to find the nonzero connection forms. The calculations give

$$\omega^0{}_p = \omega^p{}_0 = \frac{\dot{a}}{a} e^p, \quad p = 1, 2, 3, \qquad \omega^0_4 = \omega^4_0 = \frac{\dot{b}}{b} e^4. \tag{75}$$

From Cartan's second structural

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \omega^{cb} \tag{76}$$

we can calculate the curvature matrix. The nonzero components are

$$\begin{aligned} R^{0p} &= -R^{p0} = \frac{\ddot{a}}{a} e^0 e^p, \quad p = 1, 2, 3, \\ R^{04} &= -R^{40} = \frac{\ddot{b}}{b} e^0 e^4, \\ R^{pq} &= -R^{qp} = \frac{\dot{a}^2}{a^2} e^p e^q, \quad p, q = 1, 2, 3, \\ R^{p4} &= -R^{4p} = \frac{\dot{a}}{a}\frac{\dot{b}}{b} e^0 e^p, \quad p = 1, 2, 3. \end{aligned} \tag{77}$$

We consider now the field h^a. Writing the h^a field in the vielbein basis we have $h^a = h^a{}_\mu e^\mu = \eta^{a\mu} h_{\mu\nu} e^\nu$. Using the procedure developed in chapter 13 of Ref. [15], we have

$$\begin{aligned} h^0 &= f e^0 + g e^4, \\ h^p &= r e^p, \quad p = 1, 2, 3, \\ h^4 &= m e^0 + q e^4, \end{aligned} \tag{78}$$

where $f = f(t)$, $g = g(t)$, $r = r(t)$, $m = m(t)$, $q = q(t)$.

From (75) and (78) we can see that

$$\begin{aligned} Dh^0 &= \frac{\dot{b}}{b}(g - m) e^0 e^4, \\ Dh^p &= \left[\dot{r} + (r - f)\frac{\dot{a}}{a}\right] e^0 e^p + g \frac{\dot{a}}{a} e^p e^4, \quad p = 1, 2, 3, \\ Dh^4 &= \left[\dot{q} + \frac{\dot{b}}{b}(q - f)\right] e^0 e^4. \end{aligned} \tag{79}$$

The next step is to introduce (77) and (79) into

$$\delta L = \delta L_{\text{ChS}} + \frac{\delta L_M}{\delta e^a} \delta e^a + \frac{\delta L_M}{\delta h^a} \delta h^a + \frac{\delta L_M}{\delta \omega^{ab}} \delta \omega^{ab}, \tag{80}$$

where $\delta L^{(5)}_{\text{ChS}}$ is given by (13), and where $\frac{\delta L_M}{\delta e^a}$ is associated with the energy-momentum tensor $T_{\mu\nu}$, $\frac{\delta L_M}{\delta h^a}$ is associated with the energy-momentum tensor for the field h^a which we denote as $T^{(h)}_{\mu\nu}$, and $\frac{\delta L_M}{\delta \omega^{ab}}$ is associated with the spin tensor $S_{\mu\nu\lambda}$.

The calculations give

$$\left[\frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a}\frac{\dot{b}}{b}\right] + l^2 \frac{\alpha_1}{\alpha_3} \frac{\dot{a}^3}{a^3} \frac{\dot{b}}{b} = \frac{\beta_1}{24\alpha_3} \rho, \tag{81}$$

$$\left[2\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + 2\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{a}^2}{a^2}\right] + l^2 \frac{\alpha_1}{\alpha_3}\left[2\frac{\ddot{a}}{a}\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{a}^2}{a^2}\frac{\ddot{b}}{b}\right] = -\frac{\beta_1}{8\alpha_3} P, \tag{82}$$

$$\left[\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right]+l^2\frac{\alpha_1}{\alpha_3}\frac{\ddot{a}}{a}\frac{\dot{a}^2}{a^2}=-\frac{\beta_1}{24\alpha_3}P_d, \tag{83}$$

$$l^2\frac{\dot{a}^3}{a^3}\frac{\dot{b}}{b}=\frac{\beta_2}{24\alpha_3}\rho^{(h)}, \tag{84}$$

$$l^2\left[2\frac{\ddot{a}}{a}\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}^2}{a^2}\frac{\ddot{b}}{b}\right]=-\frac{\beta_2}{24\alpha_3}P^{(h)}, \tag{85}$$

$$l^2\frac{\ddot{a}}{a}\frac{\dot{a}^2}{a^2}=-\frac{\beta_1}{24\alpha_3}P_d^{(h)}, \tag{86}$$

$$\frac{\dot{a}}{a}\left[\dot{q}+(q-f)\frac{\dot{b}}{b}\right]+2\frac{\dot{a}}{a}\frac{\dot{b}}{b}\left[\dot{r}+(r-f)\frac{\dot{a}}{a}\right]=0, \tag{87}$$

$$\frac{\dot{a}}{a}\left[\dot{r}+\frac{\dot{a}}{a}(r-f)\right]=0, \tag{88}$$

$$g\frac{\dot{a}^2}{a^2}=0, \tag{89}$$

$$\left[2\frac{\ddot{a}}{a}\frac{\dot{a}}{a^2}g+\frac{\dot{a}^2}{a^2}\frac{\dot{b}}{b}(g-m)\right]=0, \tag{90}$$

where we have considered $T_{\mu\nu}=(\rho,P,P,P,P_d)$ and $T^{(h)}_{\mu\nu}=(\rho^{(h)},P^{(h)},P^{(h)},P^{(h)},P^{(h)}_d)$ and where P_d and $P^{(h)}_d$ are related to the pressure in the fifth dimension.

Assuming that $\frac{\dot{a}}{a}\neq 0$, we have that Eqs. (87)–(90) are reduced to

$$\dot{q}+\frac{\dot{b}}{b}(q-f)=0, \tag{91}$$

$$\dot{r}+\frac{\dot{a}}{a}(r-f)=0, \tag{92}$$

$$g=m=0 \tag{93}$$

so that the system of Eqs. (81)–(90) takes the following form:

$$3\left[\frac{\dot{a}^2}{a^2}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}\right]-12\varepsilon\frac{\dot{a}^3}{a^3}\frac{\dot{b}}{b}=\frac{\rho}{2\kappa_1}, \tag{94}$$

$$\left[2\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+2\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}^2}{a^2}\right]-4\varepsilon\left[2\frac{\ddot{a}}{a}\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}^2}{a^2}\frac{\ddot{b}}{b}\right]=-\frac{P}{2\kappa_1}, \tag{95}$$

$$-3\left[\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right]+12\varepsilon\frac{\ddot{a}}{a}\frac{\dot{a}^2}{a^2}=-\frac{P_d}{2\kappa_1}, \tag{96}$$

$$\varepsilon\frac{\dot{a}^3}{a^3}\frac{\dot{b}}{b}=-\kappa_2\rho^{(h)}, \tag{97}$$

$$\frac{\varepsilon}{3}\left[2\frac{\ddot{a}}{a}\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}^2}{a^2}\frac{\ddot{b}}{b}\right]=-\kappa_2P^{(h)}, \tag{98}$$

$$\varepsilon\frac{\ddot{a}}{a}\frac{\dot{a}^2}{a^2}=-\kappa_2P_d^{(h)}, \tag{99}$$

$$\dot{q}+\frac{\dot{b}}{b}(q-f)=0, \tag{100}$$

$$\dot{g}+\frac{\dot{a}}{a}(g-f)=0, \tag{101}$$

where we have defined

$$g=r,\qquad \varepsilon=-\frac{1}{4}\frac{\alpha_1}{\alpha_3}l^2,\qquad \kappa_1=4\frac{\alpha_3}{\beta_1},\qquad \kappa_2=\frac{\beta_2}{96}\frac{\alpha_1}{\alpha_3^2}. \tag{102}$$

From Eqs. (94)–(101) we can see that it is necessary to have an equation relating the energy, the pressure, and the scale factor. The energy-momentum tensor has to be divergence-free which signals the conservation of energy. This follows automatically from Eqs. (94)–(100) from which we find the following conservation equation:

$$\dot{\rho}+3\frac{\dot{a}}{a}(\rho+P)+\frac{\dot{b}}{b}(\rho+P_d)=0. \tag{103}$$

Analogously from Eqs. (97)–(99), we find the following conservation equation:

$$\dot{\rho}^{(h)}+3\frac{\dot{a}}{a}(\rho^{(h)}+P^{(h)})+\frac{\dot{b}}{b}(\rho^{(h)}+P_d^{(h)})=0. \tag{104}$$

A. Dynamical compactification

Following the compactification procedure developed in Ref. [19], we consider the case when the scale factor $b(t)$ is given by

$$b(t)=\frac{1}{a^n},\qquad n>0. \tag{105}$$

The parameter n must be positive for dynamical compactification to take place. Therefore, b gets smaller as the radius of our Universe a become bigger.

Substituting (105) into the metric (71) we have

$$ds^2=-dt^2+a^2(t)[(dx^1)^2+(dx^2)^2+(dx^3)^2]+\frac{dx^2}{a^{2n}(t)}. \tag{106}$$

From (105) we can see that

$$\frac{\dot{b}}{b}=-n\frac{\dot{a}}{a};\qquad \frac{\ddot{b}}{b}=n(n+1)\frac{\dot{a}^2}{a^2}-n\frac{\ddot{a}}{a} \tag{107}$$

so that by introducing (107) into Eqs. (94)–(104) we obtain

$$3(1-n)\frac{\dot{a}^2}{a^2}+12\varepsilon n\frac{\dot{a}^4}{a^4}=\frac{\rho}{2\kappa_1}, \tag{108}$$

$$(1-n)\left[2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right]+4n\varepsilon\left[4\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right]\frac{\dot{a}^2}{a^2}=-\frac{\tilde{P}}{2\kappa_1}, \tag{109}$$

$$-3\left[\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}\right]+3\frac{\dot{a}^2}{a^2}+12\varepsilon\frac{\ddot{a}}{a}\frac{\dot{a}^2}{a^2}=\frac{P_d}{2\kappa_1}, \tag{110}$$

$$\varepsilon n\frac{\dot{a}^4}{a^4}=-\kappa_2\rho^{(h)}, \tag{111}$$

$$\varepsilon\left[\frac{n(n+1)}{3}\frac{\dot{a}^2}{a^2}-n\frac{\ddot{a}}{a}\frac{\dot{a}^2}{a^2}\right]=-\kappa_2 P^{(h)}, \tag{112}$$

$$\varepsilon\frac{\ddot{a}}{a}\frac{\dot{a}^2}{a^2}=\kappa_2 P_d^{(h)}, \tag{113}$$

$$\dot{q}-n\frac{\dot{a}}{a}(q-f)=0, \tag{114}$$

$$\dot{g}+\frac{\dot{a}}{a}(g-f)=0, \tag{115}$$

and the conservation equations take the form

$$\dot{\rho}+3\frac{\dot{a}}{a}(\rho+\tilde{P})=0, \tag{116}$$

$$\dot{\rho}^{(h)}+3\frac{\dot{a}}{a}(\rho^{(h)}+\tilde{P}^{(h)})=0, \tag{117}$$

where, following Refs. [16,19], we have defined the effective pressures $\tilde{P}$ and $\tilde{P}^{(h)}$ as

$$\begin{aligned}\tilde{P}&=P-\frac{n}{3}\frac{\dot{a}}{a}(\rho+P_d),\\ \tilde{P}^{(h)}&=P^{(h)}-\frac{n}{3}\frac{\dot{a}}{a}(\rho^{(h)}+P_d^{(h)}).\end{aligned} \tag{118}$$

From (108)–(117) we notice the following:

(a) If $l=0$ (i.e. $\varepsilon=0$), the equations that do not vanish identically are given by

$$3(1-n)\frac{\dot{a}^2}{a^2}=\frac{\rho}{2\kappa_1}, \tag{119}$$

$$(1-n)\left[2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right]=-\frac{\tilde{P}}{2\kappa_1}, \tag{120}$$

$$-3\left[\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}\right]+3\frac{\dot{a}^2}{a^2}=\frac{P_d}{2\kappa_1}, \tag{121}$$

$$\dot{\rho}+3\frac{\dot{a}}{a}(\rho+\tilde{P})=0. \tag{122}$$

Equations (119), (120), and (122) have the same form as the equations obtained using general relativity in four dimensions. However, there is a difference. The pressure in these equations is the effective pressure $\tilde{P}$ defined in (118), which may take negative values when P and P_d are positive so that, in principle, it could be used to describe an accelerated expansion of the Universe (see Ref. [19]).

(b) If $l\to 0$ (i.e., $\varepsilon\to 0$) the Friedmann equations obtained using four-dimensional general relativity are recovered (with $\tilde{P}$ instead of P). In this limit, in addition to the Friedmann equations, a system of equations appears describing the behavior of the field h^a.

Now, we consider the solution of the Eqs. (108), (109), and (116). Following the same procedure used in general relativity, we further assume that the perfect fluid obeys the barotropic equation of state

$$\tilde{P}=\omega\rho, \tag{123}$$

where ω can be a time-dependent function or a constant. If ω is a constant then, introducing (123) in (116), we obtain

$$\rho=\rho_0\left(\frac{a_0}{a}\right)^{3(\omega+1)}, \tag{124}$$

where the subscript zero means evaluation at the present time t_0.

Using (108), (116), and (123) we obtain

$$\begin{aligned}(1+\omega)&\left[3(1-n)+12\varepsilon n\frac{\dot{a}^2}{a^2}\right]\frac{\dot{a}^2}{a^2}\\ &=-\frac{2}{3}\left[3(1-n)+24\varepsilon n\frac{\dot{a}^2}{a^2}\right]\frac{d}{dt}\left(\frac{\dot{a}}{a}\right),\end{aligned} \tag{125}$$

giving the behavior of the scale factor $a=a(t)$ and that to be solved for the cases when the parameter is uniquely $\omega=-1$ and the general case when $\omega\neq-1$.

1. Solution $\omega=-1$

If $\omega=-1$, Eq. (125) reduces to

$$\left[3(1-n)+24\varepsilon n\frac{\dot{a}^2}{a^2}\right]\frac{d}{dt}\left(\frac{\dot{a}}{a}\right)=0. \tag{126}$$

To find a solution to (126) there are two possibilities:

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right)=0, \tag{127}$$

$$3(1-n)+24\varepsilon n\frac{\dot{a}^2}{a^2}=0. \tag{128}$$

It is straightforward to see that both equations lead to similar results. The solution of (128) is a particular case of the solution (127). The solution of (127) is given by

$$a(t)=Ce^{H_0 t} \tag{129}$$

which is a de Sitter-type solution, where C is a constant and $H_0=H=\frac{\dot{a}}{a}$ is the Hubble constant. Rewriting Eq. (108) as

$$12\varepsilon n\left(\frac{\dot{a}^2}{a^2}\right)^2 + 3(1-n)\frac{\dot{a}^2}{a^2} - \frac{\rho}{2\kappa_1} = 0 \tag{130}$$

we see that it is a quadratic equation in $\dot{a}^2/a^2$ whose solution yields a value for the Hubble parameter of the form

$$\frac{\dot{a}^2}{a^2} = -\frac{3(1-n)}{24\varepsilon n}\left\{1 \pm \left[1 \pm \varepsilon\frac{8n}{3(1-n)^2}\frac{\rho}{\kappa_1}\right]^{1/2}\right\}. \tag{131}$$

If we consider the case of a small l^2 limit, we can expand the root to first order in l^2. In the expansion we can see that it is necessary to take the negative sign to recover the FRW equations when $l^2 = 0$. Thus in the first-order approximation, Eq. (131) takes the form

$$H = \frac{\dot{a}^2}{a^2} \approx \frac{\rho}{3(1-n)\kappa_1}\left\{1 - \varepsilon\frac{2n}{3(1-n)^2}\frac{\rho}{\kappa_1}\right\} \tag{132}$$

or

$$H_0 \approx \frac{\rho_0}{3(1-n)\kappa_1}\left\{1 - \varepsilon\frac{2n}{3(1-n)^2}\frac{\rho}{\kappa_1}\right\}, \tag{133}$$

an expression in the $l^2 = 0$ limit which is identical to that obtained when using the Einstein equations in four dimensions.

2. Solutions for $\omega \neq -1$

We consider now the behavior of the scale factor for the general case when ω is left as a free parameter. By integrating Eq. (125) with $\omega \neq -1$ we obtain

$$\frac{\dot{a}}{a} = \frac{1}{\gamma}\tan\left\{\frac{1}{\gamma}\left[\frac{a}{\dot{a}} - \frac{3}{2}(1+\omega)t\right]\right\}, \tag{134}$$

where we have defined $\gamma = \sqrt{\frac{4n\varepsilon}{(1-n)}}$. In the small l^2 limit, we can expand Eq. (134). In fact, taking the arctan of each side of (134) we have

$$\arctan\left(\gamma\frac{\dot{a}}{a}\right) = \frac{1}{\gamma}\left[\frac{a}{\dot{a}} - \frac{3}{2}(1+\omega)t\right], \tag{135}$$

and carrying out the expansion to first order in γ we obtain

$$\gamma^2\frac{\dot{a}^2}{a^2} + \frac{3}{2}(1+\omega)\frac{\dot{a}}{a}t - 1 \approx 0. \tag{136}$$

Solving for the Hubble parameter, we obtain

$$\frac{\dot{a}}{a} \approx -\frac{3(1+\omega)}{4\gamma^2}t\left[1 \pm \sqrt{1 + \left(\frac{4\gamma}{3(1+\omega)t}\right)^2}\right]. \tag{137}$$

Again, expanding the square root in (137) to first order in l^2, and considering the negative sign to recover the four-dimensional FRW equations, we have

$$\frac{\dot{a}}{a} \approx -\frac{2}{3(1+\omega)}\frac{1}{t}\left[1 - \frac{4n\varepsilon}{(1-n)}\left(\frac{2}{3(1+\omega)}\frac{1}{t}\right)^2\right]. \tag{138}$$

Integrating (138), we obtain a value for the scale factor of the form

$$a(t) \approx Ct^{1/3(1+\omega)}\left[1 + \frac{2n\varepsilon}{(1-n)}\left(\frac{2}{3(1+\omega)}\right)^3\frac{1}{t^2}\right], \tag{139}$$

where

$$C = a_0\left\{\left[\frac{3}{2}(1+\omega)\right]^2\frac{\rho_0}{6(1-n)\kappa_1}\right\}^{1/4(1+\omega)}.$$

From Eq. (139) we notice the following:

(i) The cases of greatest physical interest are those with $\omega = 0$ and $\omega = 1/3$, which are in the category $\omega \neq -1$. These cases are usually called the eras of matter and radiation, respectively.

(ii) For small values of ε and for values of t^2 that are not small, we have that the term on the right in (139) is negligible compared to the first term, and we recover the usual solutions to the four-dimensional FRW equations.

(iii) In the case that t^2 is the order of ε, we have that the term on the right in (139) is not negligible compared to the first and therefore becomes important in the description of evolution. This is a notable difference from the results obtained from general relativity. If the term on the right in (139) takes a value greater than zero, then it is possible that this term is important for the description of an inflationary period in the early stages of the Universe.

3. Solution without matter $\rho = \tilde{P} = 0$

We consider the case $\rho = \tilde{P} = 0$. This case is described by the equation of state $\tilde{P} = \omega\rho$. If $\rho = 0$ and assuming that $\dot{a}/a \neq 0$, we have that Eq. (125) takes the form

$$\frac{\dot{a}^2}{a^2} = -\frac{(n-1)}{4n\varepsilon} = H_0^2 \tag{140}$$

whose solution is given by

$$a(t) = Ce^{H_0 t} \tag{141}$$

which corresponds to a model of de Sitter type. In the context of general relativity there is no similar solution for the case $\rho = P = 0$.

VI. SUMMARY AND OUTLOOK

We have considered a five-dimensional action $S = S_g + S_M$ which is composed of a gravitational sector and a sector of matter, where the gravitational sector is given by a Chern-Simons gravity action instead of the Einstein-Hilbert action and where the matter sector is given by the so-called perfect fluid. We have studied the implications of replacing the Einstein-Hilbert action by the Chern-Simons action on the cosmological evolution for a Friedmann-Robertson-Walker metric.

We have found some solutions for cosmological field equations, which were obtained from the action $S = S_g + S_M$, where S_g is the action for the Chern-Simons gravity theory, studied in Ref. [7]. In the event that the matter action is the action for a perfect fluid, we have shown that the standard five-dimensional FRW equations can be obtained from the above-mentioned cosmological field equations. From the solution (67) we can conclude the following. (a) For small values of l^2 and for values of t^2 that are not small, we have that the term on the right in (67) is negligible compared to the first term, and we recover the usual solutions to the five-dimensional FRW equations. (b) In the case that t^2 is the order of l^2, we have that the term on the right in (67) is not negligible compared to the first term and therefore becomes important in the description of evolution: This is a notable difference from the results obtained from general relativity. If the term on the right in (67) takes a value greater than zero, then it is possible that this term is important for the description of an inflationary period in the early stages of a five-dimensional universe [20].

We have also shown, using a compactification procedure known as dynamic compactification, that the cosmological field equations obtained from the Chern-Simons gravity theory lead, in a certain limit, to the usual four-dimensional FRW equations. From the solution (139) we notice the following. (a) For small values of ε and for values of t^2 that are not small, we have that the term on the right in (139) is negligible compared to the first term, and we recover the usual solutions of the four-dimensional FRW equations. (b) In the case that t^2 is the order of ε, we have that the term on the right in (139) is not negligible compared to the first term and therefore becomes important in the description of evolution. This is a notable difference from the results obtained from general relativity. If the term on the right in (139) takes a value greater than zero, then it is possible that this term is important for the description of an inflationary period in the early stages of the Universe [20]. For the case $\rho = P = 0$ the solution to the field equations corresponds to a model of de Sitter type. It should be noted that such a solution does not exist in the context of general relativity.

This article leads to interesting questions such as the following: What implications will the replacement of the Einstein-Hilbert action by the Chern-Simons action have on the evolution of wormholes described by metrics given in Refs. [21–23]?

ACKNOWLEDGMENTS

This work was supported in part by Dirección de Investigación, Universidad de Concepción through Grant No. 210.011.053-1.0 and in part by FONDECYT through Grant No. 1080530.

APPENDIX A: INTERPRETATION OF THE FIELD h^a, CASE 1

So far we have interpreted the field h^a as a field of matter whose nature has not been specified.

It is known that for space-time of five dimensions in the presence of a gravitational field, a scalar field φ satisfies the field equations obtained from the action

$$S = \int d^5x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right] \tag{A1}$$

because φ is a scalar field and therefore $\partial_\mu\varphi = \nabla_\mu\varphi$. Now we consider the action (A1) for the case under study. The FRW metric with $k = 0$, written in Cartesian coordinates, is given by

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2 + dw^2], \tag{A2}$$

where we see that $\det g_{\mu\nu} = -a^8$, so that $\sqrt{-g} = a^4$. Since in a homogeneous space φ depends only on the time coordinate, we have

$$\begin{aligned} S &= \int d^5x a^4\left[-\frac{1}{2}g^{00}\partial_0\varphi\partial_0\varphi - V(\varphi)\right] \\ &= \int d^5x a^4\left[\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right]. \end{aligned} \tag{A3}$$

Varying this action we obtain the following equation of motion:

$$\ddot{\varphi} + 4\frac{\dot{a}}{a}\dot{\varphi} + \frac{dV}{d\varphi} = 0. \tag{A4}$$

The energy-momentum tensor associated with the φ field is given by

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}[\tfrac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + V(\varphi)] \tag{A5}$$

so that

$$T_{00} = \tfrac{1}{2}\dot{\varphi}^2 + V(\varphi), \tag{A6}$$

$$T_{11} = T_{22} = T_{33} = T_{44} = \tfrac{1}{2}\dot{\varphi}^2 - V(\varphi). \tag{A7}$$

Since for a perfect fluid T_{00} and T_{ii} are related to the energy density ρ and pressure P, we have

$$\begin{aligned} &T_{00} = \rho_\varphi = \tfrac{1}{2}\dot{\varphi}^2 + V(\varphi), \\ &T_{11} = T_{22} = T_{33} = T_{44} = P_\varphi = \tfrac{1}{2}\dot{\varphi}^2 - V(\varphi). \end{aligned} \tag{A8}$$

On the other hand, we know that the h^a field is associated with the conservation equation (45) which, for a flat model i.e. with $k = 0$, takes the form (49),

$$\dot{\rho}^{(h)} + 4\frac{\dot{a}}{a}(\rho^{(h)} + P^{(h)}) = 0. \tag{A9}$$

If we identify $\rho^{(h)}$ with the energy density and $P^{(h)}$ with the pressure for a scalar field φ, that is, if we postulate

$$\rho^{(h)} = \tfrac{1}{2}\dot{\varphi}^2 + V(\varphi), \qquad P^{(h)} = \tfrac{1}{2}\dot{\varphi}^2 - V(\varphi), \tag{A10}$$

then if we introduce (A10) in (49) we obtain

$$\dot{\varphi}\,\ddot{\varphi} + 4\frac{\dot{a}}{a}\dot{\varphi}^2 + \dot{\varphi}\frac{dV}{d\varphi} = 0, \tag{A11}$$

that is,

$$\ddot{\varphi} + 4\frac{\dot{a}}{a}\dot{\varphi} + \frac{dV}{d\varphi} = 0, \tag{A12}$$

which is identical to Eq. (A4) found for the scalar field. This allows us to associate a scalar field with the field h^a. For example, one could make the identification by choosing the components of h^a in the form $f \equiv 0$ and $g \equiv \varphi$.

APPENDIX B: INTERPRETATION OF THE FIELD h^a, CASE 2

From the metric (71),

$$ds^2 = -dt^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + b^2(t)dx^2, \tag{B1}$$

we can see that $\det g_{\mu\nu} = -a^6b^2$, that is, $\sqrt{-g} = a^3b$. Since in a homogeneous space φ depends only on the time coordinate, we have that (A1) takes the form

$$\begin{aligned} S &= \int d^5xa^3b\left[-\frac{1}{2}g^{00}\partial_0\varphi\partial_0\varphi - V(\varphi)\right] \\ &= \int d^5xa^3b\left[\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right]. \end{aligned} \tag{B2}$$

Varying this action we obtain the following equation of motion:

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + \frac{\dot{b}}{b}\dot{\varphi} + \frac{dV}{d\varphi} = 0. \tag{B3}$$

The corresponding energy-momentum tensor associated with the field φ is given by

$$\begin{aligned} T_{00} &= \rho_\varphi = \tfrac{1}{2}\dot{\varphi}^2 + V(\varphi), \\ T_{11} &= T_{22} = T_{33} = T_{44} = P_\varphi = \tfrac{1}{2}\dot{\varphi}^2 - V(\varphi). \end{aligned} \tag{B4}$$

On the other hand, we know that the field h^a is associated with the conservation equation (104),

$$\dot{\rho}^{(h)} + 3\frac{\dot{a}}{a}(\rho^{(h)} + P^{(h)}) + \frac{\dot{b}}{b}(\rho^{(h)} + P_d^{(h)}) = 0. \tag{B5}$$

If we identify $\rho^{(h)}$ with the energy density and $P^{(h)}$ with the pressure for a scalar field φ, that is, if we postulate

$$\rho^{(h)} = \tfrac{1}{2}\dot{\varphi}^2 + V(\varphi), \qquad P^{(h)} = P_d^{(h)} = \tfrac{1}{2}\dot{\varphi}^2 - V(\varphi), \tag{B6}$$

then if we introduce (B6) in (B5) we obtain

$$\dot{\varphi}\,\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi}^2 + \frac{\dot{b}}{b}\dot{\varphi}^2 + \dot{\varphi}\frac{dV}{d\varphi} = 0, \tag{B7}$$

that is,

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + \frac{\dot{b}}{b}\dot{\varphi} + \frac{dV}{d\varphi} = 0, \tag{B8}$$

which is identical to Eq. (B3) found for the scalar field. This allows us to associate a scalar field with the field h^a in a manner similar to the case studied in Appendix A.

APPENDIX C: THE COSMOLOGICAL PRINCIPLE

We have considered a space-time in five dimensions whose spatial part is homogeneous and isotropic. This space is of type $R \times R \times S^3$. We have said that the cosmological principle implies that all fields must be homogeneous, i.e., are invariant under translations, as well as isotropic, i.e., are invariant under spatial rotations. This means that these conditions must be satisfied by the fields present in (10).

The cosmological principle implies that the metric should be of the form

$$ds^2 = -dt^2 + f(t)g_{ij}(x^k)dx^idx^j, \quad i, j = 1, 2, \ldots, N = 4, \tag{C1}$$

with $(x^0, x^i) = (t, x^i)$. x^i correspond to the coordinates in the homogeneous and isotropic subspace.

Since there are $N(N+1)/2$ Killing vectors corresponding to spatial translations and rotations, we have that (C1) must be invariant in form under the following coordinate transformations:

$$x^0 \to x'^0 = x^0, \qquad x^i \to x'^i = x^i + \varepsilon\xi^i(x^0, x^j), \tag{C2}$$

where ξ^i are the Killing vectors associated with spatial translations and rotations.

On the other hand, any tensor field invariant in the form under transformations (C2) has a vanishing Lie derivative along the vectors ξ^i, i.e.

$$\mathcal{L}_\xi T_{\mu\nu\cdots} = 0. \tag{C3}$$

This means that condition (C3) is the condition for a field satisfying the cosmological principle. This implies that, if a field $T_{\mu\nu\cdots}$ satisfies the cosmological principle, then it must satisfy the equation

$$\begin{aligned} &\delta_\mu{}^iT^j_{\nu\lambda\cdots} + \delta_\nu{}^iT_\mu{}^j{}_{\lambda\cdots} + \delta_\lambda{}^iT_{\mu\nu\cdots}{}^j\ldots \\ &\quad = \delta_\mu{}^jT_{\nu\lambda\cdots}{}^i + \delta_\nu{}^jT_\mu{}^i{}_{\lambda\cdots} + \delta_\lambda{}^jT_{\mu\nu\cdots}{}^i\cdots. \end{aligned} \tag{C4}$$

Examples

(1) Scalar.—We consider a scalar invariant under (C2) in a five-dimensional space-time, which contains a four-dimensional subspace that is maximally symmetric ($\mu, \nu, \ldots = 0, 1, 2, 3, 4$; $i, j, \ldots = 1, 2, 3, 4$):

$$\xi^0 = 0, \qquad \xi^i = \xi^i(x^0, x^j), \tag{C5}$$

with $N(N+1)/2$ Killing vectors. If we want a scalar S to be maximally invariant in form (in its spatial part), then it must satisfy

$$\mathcal{L}_\xi S = 0 \Rightarrow \xi^\mu\partial_\mu S = 0 \Rightarrow \xi^i\partial_i S = 0. \tag{C6}$$

Since ξ^i is arbitrary, we have $\partial_i S = 0$ for which

$$S = S(x^0) = S(t). \tag{C7}$$

(2) Vectors.—We consider a vector invariant under (C2) in a five-dimensional space-time, which contains a maximally symmetric four-dimensional subspace. From (C4) we can see that the vectors must satisfy

$$\delta_\mu{}^i A^j = \delta_\mu{}^j A^i. \tag{C8}$$

The contraction of μ with i leads us to

$$\delta_i{}^i A^j = \delta_i{}^j A^i \Rightarrow N A^j = A^j \Rightarrow (N-1) A^j = 0 \Rightarrow A^i = 0. \tag{C9}$$

So $\mathcal{L}_\xi A^\mu = 0$ takes the form

$$\mathcal{L}_\xi A^0 = 0 \Rightarrow \xi^\mu \partial_\mu A^0 = 0 \Rightarrow \xi^i \partial_i A^0 = 0, \tag{C10}$$

where

$$\partial_i A^0 = 0 \Rightarrow A^0 = A^0(x^0) = A^0(t). \tag{C11}$$

Thus,

$$A^0 = A^0(t), \qquad A^i = 0. \tag{C12}$$

(3) Tensors of rank 2.—We consider a tensor of rank 2. From (C2) we can see that the tensors of rank 2 must satisfy

$$\delta_\mu{}^i A_\nu^j + \delta_\nu{}^i A_\mu{}^j = \delta_\mu{}^j A^i{}_\nu + \delta_\nu{}^j A_\mu{}^i.$$

The same procedure above tells us that

$$A_{00} = A_{00}(t), \qquad A_{i0} = 0 = A_{0i}, \\ A_{ij} = f(t)\eta_{ij}; \quad i, j = 1, 2, 3, 4. \tag{C13}$$

A specific case of a tensor of rank 2 is the energy-momentum tensor for a perfect fluid,

$$T^{\mu\nu} = (\rho + P) u^\mu u^\nu + P\eta^{\mu\nu}, \quad u^\mu = (1, 0, 0, \ldots, 0) \tag{C14}$$

so that

$$T^{00} = \rho(t) + P(t) = T^{00}(t), \\ T^{0i} = P\eta^{0i} = 0, \\ T^{ij} = P\eta^{ij}. \tag{C15}$$

[1] C. Rovelli, *Quantum Gravity*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2004).

[2] A. H. Chamseddine, Phys. Lett. B **233**, 291 (1989).

[3] A. H. Chamseddine, Nucl. Phys. **B346**, 213 (1990).

[4] J. Zanelli, arXiv:hep-th/0502193.

[5] J. Edelstein, M. Hassaine, R. Troncoso, and J. Zanelli, Phys. Lett. B **640**, 278 (2006).

[6] M. Hassaine and M. Romo, J. High Energy Phys. 06 (2008) 018.

[7] F. Izaurieta, P. Minning, A. Perez, E. Rodriguez, and P. Salgado, Phys. Lett. B **678**, 213 (2009).

[8] J. A. de Azcarraga, J. M. Izquierdo, M. Picon, and O. Varela, Nucl. Phys. **B662**, 185 (2003).

[9] F. Izaurieta, E. Rodriguez, and P. Salgado, J. Math. Phys. (N.Y.) **47**, 123512 (2006).

[10] F. Izaurieta, A. Perez, E. Rodriguez, and P. Salgado, J. Math. Phys. (N.Y.) **50**, 073511 (2009).

[11] F. Izaurieta, E. Rodriguez, and P. Salgado, Lett. Math. Phys. **80**, 127 (2007).

[12] J. N. Islam, *An Introduction to Mathematical Cosmology* (Cambridge University Press, Cambridge, England, 2001), 2nd ed.

[13] P. Coles and F. Lucchin, *Cosmology: The Origin and Evolution of Cosmic Structure* (John Wiley & Sons, New York, 1996).

[14] J. V. Narlikar, *An Introduction to Cosmology* (Cambridge University Press, Cambridge, England, 2002).

[15] S. Weinberg, *Gravitation and Cosmology* (John Wiley & Sons, New York, 1972).

[16] K. Andrew, B. Bolen, and C. Middleton, Gen. Relativ. Gravit. **39**, 2061 (2007).

[17] A. Riess, Astron. J. **116**, 1009 (1998).

[18] D. N. Spergel, Astrophys. J. **170**, 377 (2007).

[19] N. Mohammedi, Phys. Rev. D **65**, 104018 (2002).

[20] M. Cataldo, S. del Campo, F. Gomez, and P. Salgado (work in progress).

[21] M. Cataldo, S. del Campo, P. Labraña, J. Crisostomo, and P. Salgado, Phys. Rev. D **78**, 104006 (2008).

[22] M. Cataldo, S. del Campo, P. Minning, and P. Salgado, Phys. Rev. D **79**, 024005 (2009).

[23] M. Cataldo, S. del Campo, F. Gomez, and P. Salgado (work in progress).