How Robust is the Froissart bound?

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Proof of the Froissart theorem is reconsidered in a different way to extract its necessary conditions. Two physical inputs, unitarity and absence of massless intermediate hadrons, are indisputable. Also important are mathematical properties of the Legendre functions. Assumptions on dispersion relations, single or double, appear to be excessive. Instead, one should make assumptions on possible high-energy asymptotics of the amplitude in nonphysical configurations, which have today no firm basis. Asymptotics for the physical amplitude always appear essentially softer than for the nonphysical one. Froissart's paper explicitly assumed the hypothesis of power behavior and obtained asymptotic bound for total cross sections $\sim \log^2(s/s_0)$ with some constant s_0. Our bounds are slightly stronger than original Froissart ones. They show that the scale s_0 should itself slowly grow with s. Under different assumptions about asymptotic behavior of nonphysical amplitudes, the total cross section could grow even faster than $\log^2 s$. The problem of correct asymptotics might be clarified by precise measurements at the LHC and higher energies.

I. INTRODUCTION

One of the cornerstones for the present strong interaction physics is the Froissart theorem [1]. It declares that the total cross sections of any two-hadron scattering cannot grow with energy faster than $(\log s)^2$. The expectation agrees with current experimental data [2]. However, the same data can also be fitted so to have an asymptotical increase as s^δ with $\delta \sim 0.08$ [3]. Of course, some authors suggest that the power behavior is only temporal and will change at some very high energies. If, however, such a fit continued infinitely, it would violate the canonical Froissart bound. Thus, experimental validity of this bound stays an open question. Measurements at LHC may help to clarify it.

Theoretically, the Froissart theorem was initially proved for $2 \to 2$ amplitudes [1] in the framework of the double-dispersion representation (the Mandelstam representation) [4] with a finite number of subtractions. Such representation is true in the nonrelativistic quantum mechanics with Yukawa-type potentials [5]. However, it has never been proven mathematically for any relativistic amplitude.

Froissart's bound for total cross sections was reproduced by Martin [6] without any dispersion representations, and even without considering amplitudes. In his proof, the unitarity condition and absence of angular singularities in the physical region were applied only to the absorptive part of an elastic amplitude. He explicitly assumed also that the absorptive part in the physical region may grow with energy not faster than some power of s.

Later, however, Martin returned to the investigation of the amplitude as a whole [7] (sure, the amplitude contains more information than its absorptive part). Using the basic principles of axiomatic local field theory, he extended the analyticity domain of the scattering amplitude so that it reveals at least part of the Mandelstam cuts. This has appeared to be sufficient for confirming Froissart's results for the amplitude. Moreover, such an approach has allowed many new results to be obtained (*e.g.*, Ref. [8]; see also the recent paper [9] and references therein). It was further suggested that a more accurate account for unitarity could even improve existing bounds [10].

There is, however, an interesting problem in the Martin approach. Axiomatics of the quantum field theory suggest, in particular, that the theory is constructed from local quantized fields, which are related with isolated (quasifree) asymptotic *in* and *out* states.

Meanwhile, Froissart-Martin boundaries are usually applied to hadron processes. It is common belief now that the hadron interactions are underlaid by the quantum chromodynamics (QCD). However, QCD is hard to consider an axiomatic theory. Indeed, it deals with quark and gluon fields, which are local, but (because of confinement) cannot have isolated one-quark and/or one-gluon states.

On the other hand, hadrons, consisting of quarks and gluons, cannot be pointlike. Therefore, "effective QCD", dealing directly with hadrons, should contain some non-locality (imagine description of atoms without explicit use of charged nuclei and electrons).

Thus, application of the ideas and methods of axiomatic local field theory to hadron properties might look dubious, as well as application of dispersion relations. That is why we reconsider here the derivation of Froissart's results, without any hypotheses on double-dispersion, or even single-dispersion, representations, or axioms of quantum field theory. In this way we clarify the origin and necessary inputs for the Froissart bound and can discuss their reliability. The present approach allows us also to use stricter inequalities for the Legendre functions and, thus, slightly

improve the original Froissart bound for total cross sections or other observables.

The presentation here goes as follows. In Sec. II we demonstrate that only a finite number of partial-wave amplitudes are essential at each particular energy. Section III shows construction of bounds for amplitudes in different configurations, either studied before or not, in terms of the number of essential partial waves.

II. MODIFIED FROISSART DERIVATION

To begin with, we go along Froissart's lines as close as possible. Just as in Ref. [1], we consider a reaction of the type

$$a + b \rightarrow c + d \qquad (1)$$

among scalar particles. We shall assume that all masses are equal to m as we deal only with asymptotic properties, where the difference between the masses is expected to be negligible. We introduce the familiar Mandelstam variables $s = (p_a + p_b)^2$, $t = (p_c - p_a)^2$ and $u = (p_d - p_a)^2$. Then $s + t + u = 4m^2$. Evidently, s is the c.m. energy squared for the reaction (1), which is called the s-channel reaction. A similar role is played by t and u for two cross-channel reactions

$$a + \bar{c} \rightarrow \bar{b} + d, \qquad \bar{c} + b \rightarrow \bar{a} + d, \qquad (2)$$

respectively the t-channel and u-channel ones. For the s-channel reaction, t and u are momentum transfers squared, related to the reaction angle θ_s as

$$\cos\theta_s = 1 + (t/2q_s^2) = -1 - (u/2q_s^2), \qquad (3)$$

where $q_s^2 = (s - 4m^2)/4$, q_s being the s-channel c.m. momentum. The s-channel physical region is given by

$$q_s^2 > 0, \quad |\cos\theta_s| \leq 1; \quad \text{or} \quad s > 4m^2, \quad t \leq 0, \quad u \leq 0.$$

Let us relate the reaction amplitude with the partial-wave amplitudes (we use the same normalization and relations for the amplitudes as Froissart [1]):

$$A(s, \cos\theta_s) = \frac{\sqrt{s}}{\pi q_s} \sum_{l=0}^{\infty} a_l(s)(2l + 1)P_l(\cos\theta_s), \qquad (4)$$

$$a_l(s) = \frac{\pi q_s}{2\sqrt{s}} \int_{-1}^{+1} A(s, \cos\theta_s')P_l(\cos\theta_s')d(\cos\theta_s'). \qquad (5)$$

For any physical (integer) l, $|a_l|$ is bounded by one, a_l being an element of the unitary S-matrix.

At the next step, Froissart [1] uses the momentum transfer dispersion relation for the amplitude at fixed s to show that a_l exponentially decreases at large l. We go another way, without any dispersion relations. Note, first of all, that $P_l(z)$ are analytical functions of z in the whole z-plane and

$$|P_l(z)| < |P_l(\pm 1)| = 1 \quad \text{at} \quad -1 < z < 1.$$

If the series (4) for the amplitude and for its angular derivative are convergent in the end points $\cos\theta_s = \pm 1$, then they are convergent also for any physical $\cos\theta_s$, and the amplitude $A(s, \cos\theta_s)$ is an analytical function of $\cos\theta_s$ inside the whole physical region (and nearby).

As is well known, in addition to $P_l(z)$, the Legendre functions of the 1st kind, there exist also the Legendre functions of the 2nd kind, $Q_l(z)$. They are also analytical functions of z, but have branch points. At integer values of l, the 1st Riemann sheet contains only one cut, between -1 and $+1$. The jump over this cut is

$$\frac{1}{2i}[Q_l(x + i\epsilon) - Q_l(x - i\epsilon)] = -\frac{\pi}{2}P_l(x), \quad -1 < x < +1. \qquad (6)$$

Therefore, expression (5) may be rewritten as

$$a_l(s) = \frac{q_s}{2i\sqrt{s}} \oint A(s, z')Q_l(z')dz', \qquad (7)$$

where integration runs along the closed contour going counterclockwise around the cut of $Q_l(z)$ between -1 and $+1$. Such a contour crosses the real z-axis both at $z > +1$, and at $z < -1$. Let us define

$$z' = \cosh(\alpha + i\varphi')$$

with real α and φ'. We can choose $\alpha \geq 0$ and construct the contour in Eq. (7) so as to have a constant value of α on the whole contour, integration running over φ', say, from $-\pi$ to $+\pi$. In the z-plane, such a contour is an ellipse

$$\left(\frac{\text{Re}z}{\cosh\alpha}\right)^2 + \left(\frac{\text{Im}z}{\sinh\alpha}\right)^2 = 1, \qquad (8)$$

having semiaxes $\cosh\alpha$, $\sinh\alpha$ and foci at $z = -1$, $z = +1$; at the contour

$$dz' = i\sinh(\alpha + i\varphi') \cdot d\varphi'.$$

Now we can use integral (7) to investigate behavior of a_l at large values of l. The Legendre function Q_l may be written in the form [11]

$$Q_l(\cosh\beta) = \sqrt{\pi} \cdot \frac{\Gamma(l + 1)}{\Gamma(l + \frac{3}{2})} \cdot \frac{e^{-\beta(l+1)}}{\sqrt{1 - e^{-2\beta}}}$$

$$\times {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; l + \frac{3}{2}; \frac{1}{1 - e^{2\beta}}\right), \qquad \text{Re}\beta > 0, \qquad (9)$$

which provides the asymptotic expression

$$\sinh(\alpha + i\varphi') \cdot Q_l(z')|_{l \rightarrow +\infty}$$

$$= e^{-(\alpha + i\varphi')(l+1/2)} \cdot \sqrt{\frac{\pi}{2l}} \sinh(\alpha + i\varphi') \cdot [1 + \mathcal{O}(l^{-1})]. \qquad (10)$$

Since $\sinh\alpha < |\sinh(\alpha + i\varphi')| < \cosh\alpha$, for large real l there is the upper boundary

$$|\sinh(\alpha + i\varphi')Q_l(z')|_{l\to+\infty} < e^{-\alpha(l+1/2)}\sqrt{\frac{\pi}{2l}\cosh\alpha}, \quad (11)$$

independent of φ'. Therefore,

$$|a_l| < \frac{q_s}{\sqrt{s}} \cdot B_\alpha(s) \cdot \sqrt{e^{-\alpha}\cosh\alpha} \cdot \frac{e^{-\alpha l}}{\sqrt{l}}, \quad (12)$$

where

$$B_\alpha(s) = \sqrt{\frac{\pi}{8}} \int_{-\pi}^{+\pi} |A(s, z')| d\varphi', \qquad z' = \cosh(\alpha + i\varphi'). \quad (13)$$

Evidently, the upper boundary (12) decreases with increasing l. The larger α is, the faster is the decrease of the boundary, which means the stricter limitation for $|a_l|$ at large l. If the amplitude $A(s, z)$ has a singularity nearest to the physical region at

$$z = z_0 \equiv \cosh(\alpha_0 + i\varphi_0),$$

then the contour in Eq. (7) may be blown up until it touches this nearest singularity. If the singularity is integrable and B_α stays finite in this limit, we can drag α to α_0 (the value of φ_0 may influence only the limiting value of B_α). For the amplitude $A(s, z)$, the ellipse with $\alpha = \alpha_0$ is just the Lehmann ellipse of analyticity in z [12].

Let us consider, in more detail, possible singularities in the z-plane. Unitarity predicts that an amplitude has singularities in both s-, and t- or u-channels. Those may be poles, corresponding to one-particle states, or branch points, corresponding to two-particle or multiparticle thresholds. Any of such singularities has a position described by a definite value of the corresponding Mandelstam invariant, say, t_0, independent of other Mandelstam invariants. In addition, there can be anomalous singularities (Landau singularities) [13], whose positions, say, in the t-plane, depend on the s-value. However, each leading anomalous singularity in reaction (1) is related to some threshold, and at $s \to +\infty$ it tends toward the corresponding threshold point. Since we are interested here just in large positive s, we will neglect possibility of s-dependence for all t- or u-singularities meaningful in our present problem.

According to Eq. (3), all t- and/or u-channel singularities reveal themselves in the s-channel as singularities in the z-plane. One-particle and threshold singularities, related to stable particles, have real non-negative values of t or u. This means that the t-channel (u-channel) generates real z-singularities at $z > +1$ ($z < -1$). Unstable particles generate complex singularities, but they are not leading (nearest) ones, being positioned at secondary Riemann sheets (we assume initial and final particles in reaction (1) to be stable). Anomalous singularities also can be complex, but the nearest ones are real. Thus, in conventional opinion, the nearest z-singularity has either $\varphi_0 = 0$, for t-channel, or

$\varphi_0 = \pm\pi$, for u-channel. As was explained, the φ_0-value may influence the boundary (12) only through the coefficient B. In what follows, we assume, for simplicity, that the nearest singularity is related to the t-channel; it has $\varphi_0 = 0$ and is positioned at fixed point $t = t_0 \geq 0$.

If there exist massless particles, as in QED, then some amplitudes may have a pole at $t_0 = 0$, i.e., at the edge of the physical region, at $z = +1$. Then both the corresponding forward amplitude and the total cross section are infinite at any value of the s-channel energy, their boundaries being meaningless. Multiphoton exchanges are related to thresholds, also at $t_0 = 0$. Such singularities are also at the edge of the physical region, but they are integrable and do not provide infinities of the forward amplitudes and/or total cross sections. Applicability of high-energy boundaries for QED amplitudes without one-photon exchanges needs special investigation.

If there are no massless particles, then $t_0 > 0$, and $z_0 = \cosh\alpha_0 = 1 + (t_0/2q_s^2) > 1$. There is a finite interval of α, from 0 to

$$\alpha_0 = \ln\left(z_0 + \sqrt{z_0^2 - 1}\right),$$

where Eq. (7) is applicable and the boundary (12) is operative. The limit $\alpha \to \alpha_0$ may be reached if the effective value of B_α stays finite. Let us consider this problem in some more detail. If t_0 corresponds to a pole, one can separate the pole contribution and continue to blow up the remaining contour (7) further, till the next singularity. The pole term provides then the inequality (12) with $\alpha = \alpha_0$ and the coefficient B expressed through the pole residue (below we will explicitly consider this case). Of course, contribution of the continued contour decreases with l faster than the pole one. For the nonpole-leading singularity, both threshold- and anomalous-leading singularities are integrable, and the corresponding expression (13) is finite even at $\alpha = \alpha_0$, when the singularity lies just at the integration contour. Therefore, in all practical cases, one can use the boundary (12) with $\alpha = \alpha_0$ and some finite coefficient $B_0(s)$. At high energies $\alpha_0 \approx \sqrt{t_0}/q_s \approx 2\sqrt{t_0/s}$. The factor

$$\sqrt{e^{-\alpha_0}\cosh\alpha_0} = \sqrt{1 - e^{-\alpha_0}\sinh\alpha_0}$$

is always lower than unity, and tends toward unity at high energies. Therefore, for our purpose here (for finding upper boundaries at high energies), we can change this factor by unity.

Thus, after all we have two upper boundaries for the partial amplitudes:

$$|a_l| \leq 1 \quad \text{and} \quad |a_l| < \frac{q_s}{\sqrt{s}}B_0(s)\frac{e^{-\alpha_0 l}}{\sqrt{l}}. \quad (14)$$

The first inequality is true for any l, while the second one is applicable only at sufficiently large values of l. It is interesting to compare these inequalities for partial amplitudes with those of Froissart [1].

Of course, the first inequality is the same in both cases. But the second one is slightly different. Froissart's Eq. (4) may be rewritten as

$$|a_l| < \frac{q_s}{\sqrt{s}} B_{\text{Fr}}(s) \frac{e^{-\alpha_0(l-N)}}{\sqrt{l-N}},$$

with integer positive N equal to the number of subtractions. (Of course, $l > N$; it seems, that N appeared here because Froissart worked with the infinite integration intervals, so his dispersion integrals were subtracted; our integrals (7) and (13) run over the final φ'-interval and need no subtractions.) Our parameter α_0 is simply related with Froissart's parameter x_0:

$$\alpha_0 = \log\left(x_0 + \sqrt{x_0^2 - 1}\right).$$

The quantity $B_{\text{Fr}}(s)$ is constructed differently than our $B_0(s)$, but it is also linearly related with the amplitude in nonphysical configurations.

Note that Froissart's factor $\exp[-\alpha_0(l-N)]/\sqrt{l-N}$ is somewhat larger than our $\exp[-\alpha_0 l]/\sqrt{l}$. Therefore, his Eq. (4) is somewhat weaker than our second boundary (14). Moreover, to simplify further calculations, Froissart additionally changed the factor $1/\sqrt{l-N} < 1$ by just unity (see Ref. [1], lower left column on p. 1055). Thus, for the l-dependence in the second inequality (14) he effectively used the purely exponential factor $\exp(-\alpha_0 l)$, instead of the smaller factor $\exp(-\alpha_0 l) \cdot l^{-1/2}$. Martin [6] also applied softened boundaries for the Legendre functions.

In difference, the present approach allows us to use inequality (11) which is the strictest boundary for the asymptotics (10). In what follows, we retain the resulting boundaries (14) for the partial-wave amplitudes as they are, without any further simplifications.

III. BOUNDARIES FOR AMPLITUDES AND CROSS SECTIONS

Evidently, at very large l, the latter of boundaries (14) is stricter than the former. Let us denote L to be the minimal value of l, for which the former boundary is above the latter (we assume L to be sufficiently large, so that both inequalities (14) are applicable near L). Then

$$\frac{q_s}{\sqrt{s}} B_0(s) \frac{e^{-\alpha_0 L}}{\sqrt{L}} < 1 < \frac{q_s}{\sqrt{s}} B_0(s) \frac{e^{-\alpha_0(L-1)}}{\sqrt{L-1}},$$

or

$$1 < \frac{\sqrt{s}}{q_s} \cdot \frac{e^{\alpha_0 L}\sqrt{L}}{B_0(s)} < e^{\alpha_0}\sqrt{\frac{L}{L-1}}. \quad (15)$$

We see that generally L depends on energy s. If $B_0(s)$ increases with energy, so does the corresponding value of L. In such a case, the interval between the upper and lower boundaries (15) shrinks. Then, at high energies we can write

$$e^{\alpha_0 L}\sqrt{L} = \frac{1}{2} B_0(s), \quad (16)$$

keeping in mind that the correct value of L is the nearest integer number above the solution of equality (16).

Note that Froissart also introduced the interfacial number L [1]. His value L_{Fr} also increases with s, but is different from ours. Because of the stronger second boundary (14), our inequalities (15) provide a lower value of L than Froissart's L_{Fr}. Martin's value of L (denoted as $\bar{L}$) [6] is also larger than ours.

Now, to construct various bounds for the scattering amplitude (4), first of all we separate its series into two parts, below and above L (again, similar to Froissart [1]), and then estimate

$$|A(s, \cos\theta_s)| \leq \frac{\sqrt{s}}{\pi q_s} \left[\sum_{l=0}^{L-1} (2l+1)|a_l(s)| \cdot |P_l(\cos\theta_s)| \right.$$
$$\left. + \sum_{l=L}^{\infty} (2l+1)|a_l(s)| \cdot |P_l(\cos\theta_s)| \right]. \quad (17)$$

To the partial amplitudes in each part, we apply the corresponding boundary (14).

A. Forward (backward) amplitude

For the forward (or backward) amplitude, with $|P_l(\pm 1)| = 1$, we obtain

$$|A(s, \pm 1)| < \frac{\sqrt{s}}{\pi q_s} \cdot \sum_{l=0}^{L-1} (2l+1)$$
$$+ \frac{B_0(s)}{\pi} \cdot \sum_{l=L}^{\infty} e^{-\alpha_0 l}(2l+1)l^{-(1/2)}. \quad (18)$$

The first term sums to $L^2\sqrt{s}/(\pi q_s)$. The second term, with $l = L + l'$, can be rewritten as

$$\frac{1}{\pi} B_0(s) \frac{e^{-\alpha_0 L}}{\sqrt{L}} \cdot \sum_{l'=0}^{\infty} e^{-\alpha_0 l'} \left[2L\left(1 + \frac{l'}{L}\right)^{1/2} + \left(1 + \frac{l'}{L}\right)^{-(1/2)} \right],$$

which is, due to the left inequality (15), smaller than

$$\frac{\sqrt{s}}{\pi q_s} \cdot \sum_{l'=0}^{\infty} e^{-\alpha_0 l'} \left[2L\left(1 + \frac{l'}{L}\right)^{1/2} + \left(1 + \frac{l'}{L}\right)^{-(1/2)} \right].$$

This sum converges only due to the decreasing exponential factor. If we define $y = \alpha_0 l'$, $Y = \alpha_0 L$ then at small α_0 (i.e., at high energy) the sum tends toward the integral (see Appendix)

$$I(s) = \frac{1}{\alpha_0} \int_0^{\infty} dy\, e^{-y} \left[2L\left(1 + \frac{y}{Y}\right)^{1/2} + \left(1 + \frac{y}{Y}\right)^{-(1/2)} \right]. \quad (19)$$

Thus, our boundary for the forward (backward) amplitude takes the form

$$|A(s, \pm1)| < \frac{\sqrt{s}}{\pi q_s}[L^2 + I(s)]. \qquad (20)$$

Its high-energy behavior directly depends on properties of L (and $Y = \alpha_0 L$).

Let us consider various possibilities. A finite limit of L at $s \to \infty$ would mean that $B_0(s)$ has also a finite limiting value. Such a case would lead to decreasing total cross section and is not interesting here.

If L increases with s, but Y stays finite (or even decreases), then $B_0(s)$ also grows, but not faster than $(1/\alpha_0)^{1/2} \sim s^{1/4}$. The corresponding total cross section cannot infinitely grow with energy. It tends to be constant (or may even slowly decrease in asymptotics).

In connection with the Froissart theorem, the most interesting is the case when the total cross section does increase with energy, without any finite limit. Then both L and Y should grow. Integral (19) can then be approximately calculated as

$$I(s) = \frac{1}{\alpha_0^2}[2Y + 1 + \mathcal{O}(1/Y)],$$

and boundary (20) at high energies takes the simple form

$$|A(s, \pm1)| < \frac{2}{\pi}\left(\frac{Y+1}{\alpha_0}\right)^2 \approx \frac{2q_s^2}{\pi t_0}(Y+1)^2 \approx \frac{s}{2\pi t_0}(Y+1)^2. \qquad (21)$$

Therefore, at forward or backward angles, the modulus of the amplitude behaves the most like sY^2, when $s \to +\infty$. If the considered amplitude corresponds to elastic scattering, then one can use the optical theorem to derive that the total cross section behaves the most like Y^2, as s goes to infinity.

B. Fixed-angle amplitude

For the fixed-angle configuration we will consider only the case of an infinitely increasing total cross section, corresponding to growing values of L and Y. At nonforward (nonbackward) angles we also begin with the inequality (17). Since L is growing with s, we can fix some finite number l_0 and subdivide the first term, again into two parts, with $0 \leq l < l_0$ and with $l_0 \leq l < L$. We choose the value l_0 so that $P_l(\cos\theta)$ for $l \geq l_0$ can be, with good accuracy, presented in its large-l asymptotic form [11]. This form corresponds to combining Eqs. (6) and (10); for $\epsilon < \theta < \pi - \epsilon$ it provides the estimate

$$|P_l(\cos\theta)|_{l\to+\infty} = \sqrt{\frac{2}{\pi l \sin\theta}} \cdot \left|\left\{\cos\left[\left(l + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right]\right.\right.$$
$$\left.\left. + \mathcal{O}(l^{-1})\right\}\right| < \sqrt{\frac{2}{\pi l \sin\theta}}. \qquad (22)$$

In Eq. (17), contributions with $l < l_0$ do not grow at $s \to +\infty$. Thus, the amplitude increasing at high energy should be related to (and bounded by) two sums:

$$|A(s, \cos\theta_s)| < \frac{\sqrt{s}}{\pi q_s} \cdot \sqrt{\frac{2}{\pi \sin\theta_s}} \sum_{l=l_0}^{L-1}(2l^{1/2} + l^{-(1/2)})$$
$$+ \frac{B_0(s)}{\pi} \cdot \sqrt{\frac{2}{\pi \sin\theta_s}} \sum_{l=L}^{\infty} e^{-\alpha_0 l}(2 + l^{-1}). \qquad (23)$$

Just as for the forward amplitude, one can use the left inequality (15) to rewrite

$$|A(s, \cos\theta_s)| < \frac{\sqrt{s}}{\pi q_s} \cdot \sqrt{\frac{2}{\pi \sin\theta_s}}\left[\sum_{l=l_0}^{L-1}(2l^{1/2} + l^{-(1/2)})\right.$$
$$\left. + \sum_{l'=0}^{\infty} e^{-\alpha_0 l'}\left(2L^{1/2} + \frac{L^{-(1/2)}}{1 + l'/L}\right)\right].$$

The most singular high-energy behavior of these sums is determined by their first terms. At $s \to +\infty$ they can be approximated as (again, see Appendix)

$$|A(s, \cos\theta_s)| < \frac{\sqrt{s}}{\pi q_s} \cdot \sqrt{\frac{2}{\pi \sin\theta_s}}\left(\sum_{l=l_0}^{L-1} 2l^{1/2} + 2L^{1/2}\sum_{l'=0}^{\infty} e^{-\alpha_0 l'}\right)$$
$$\approx \frac{\sqrt{s}}{\pi q_s} \cdot \sqrt{\frac{2}{\pi \sin\theta_s}}\left(\frac{2}{\alpha_0^{3/2}}\int_0^Y dy\sqrt{y} + \frac{2\sqrt{L}}{1 - e^{-\alpha_0}}\right),$$

which takes the final form

$$|A(s, \cos\theta_s)| < \frac{2\sqrt{s}}{\pi q_s} \cdot \sqrt{\frac{2}{\pi \sin\theta_s}}\left(\frac{Y}{\alpha_0}\right)^{3/2}\left(\frac{2}{3} + \frac{1}{Y}\right). \qquad (24)$$

Evidently, the main term in the fixed-angle bound comes from the sum over $l < L$, just as for the forward (backward) amplitude. Presence of cosine in the asymptotic expression (22) gives evidence for possibility of oscillating angular distributions in elastic scattering with increasing total cross section. If so, boundary (24) limits the upper edges of those oscillations.

Singularities of this boundary at the ends of the physical angular interval, at $\theta = 0$ or π, are related, of course, to the change in the possible energy behavior: $\sim(q_s Y)^{3/2}$ (or $\sim L^{3/2}$) inside the interval and $\sim(q_s Y)^2$ (or $\sim L^2$) at its ends.

To understand how this works, let us consider in more detail our boundary at very small angles. Note that near $\theta = 0$ (i.e., $\cos\theta$ near unity) both all $P_l(\cos\theta)$ and all their derivatives are positive, and we may eliminate signs of modulus for the Legendre polynomials in the right-hand side of inequality (17). Moreover, one can differentiate it over $\cos\theta_s$. The adequate expression for $P_l(z)$ near $z = 1$ is [11]

$$P_l(z) = {}_2F_1\left(l + 1, -l; 1; \frac{1 - z}{2}\right).$$

It shows that every additional differentiation of $P_l(z)$ over z at $z = 1$ provides an additional factor which is quadratic in l.

Derivation of boundaries, considered above, shows that most efficient in the case of the growing cross section are values $l \sim L$. Therefore, the n-th derivative of the boundary over $\cos\theta_s$ at $\cos\theta_s = 1$ grows with energy as $(L^2)^{n+1}$. This means that the angular dependence of the boundary reveals a narrow forward peak which rapidly shrinks with energy growing. Formally, the same is true for the backward scattering. We will return to this situation when considering the fixed-t configuration.

C. Fixed-t (or -u) amplitude

Up to now, following Froissart [1], we considered high-energy boundaries for amplitudes in two regimes: either forward (backward) amplitudes, related in elastic cases to the total cross sections; or amplitudes of two-particle processes at a fixed angle. Here we consider another interesting regime, not discussed by Froissart or any other author. It is the case of fixed-momentum transfer, t or u. We begin, again, with two-term Eq. (17).

For definiteness, let us take at first a fixed value of t and denote the corresponding amplitude as $A(s, t)$. At high energy, according to Eq. (3), the corresponding $\cos\theta_s \to 1$, *i.e.*, $\theta_s \to 0$. Then, one can approximately express the Legendre polynomials with sufficiently large $l \geq l_0$ through the Bessel functions [11]

$$P_l(\cos\theta_s) = J_0(\xi) + \mathcal{O}(\theta_s^2), \qquad (25)$$

with $\xi = (l + 1/2)\theta_s$. At high energy, $\xi \approx y\sqrt{(-t)/t_0}$, where again $y = \alpha_0 l$.

Now, in analogy with previous subsections, we obtain the estimate

$$|A(s, t)| < \frac{\sqrt{s}}{\pi q_s} \cdot \sum_{l=l_0}^{L-1} (2l + 1)|J_0(\xi)|$$
$$+ \frac{B_0(s)}{\pi} \cdot \sum_{l=L}^{\infty} e^{-\alpha_0 l}(2l + 1)l^{-(1/2)}|J_0(\xi)|,$$

which can, after using the left inequality (15) and the fact that essential values of $l - L$ at high energy are much less than L, be rewritten as

$$|A(s, t)| < \frac{\sqrt{s}}{\pi q_s} \Bigg[\sum_{l=l_0}^{L-1} (2l + 1) \cdot |J_0(l\theta_s)|$$
$$+ \sum_{l'=0}^{\infty} e^{-\alpha_0 l'}(2L + l' + 1) \cdot |J_0(L\theta_s + l'\theta_s)| \Bigg]$$

(compare to inequality (18) and its transforms). High-energy asymptotics of the right-hand side can be, again, expressed through integrals

$$|A(s, t)| < \frac{\sqrt{s}}{\pi q_s} \cdot \frac{2}{\alpha_0^2} \cdot \Bigg[\int_0^Y dy \cdot y \left| J_0\left(y\sqrt{\frac{-t}{t_0}}\right) \right|$$
$$+ \int_0^{\infty} dy' \cdot e^{-y'}\left(Y + \frac{y'}{2Y}\right) \left| J_0\left(\frac{Y + y'}{\sqrt{t_0}}\sqrt{-t}\right) \right| \Bigg].$$

As in other cases considered until now, the second integral here is less singular at high energies and can be neglected. Then, our final high-energy estimate for the fixed-t amplitude is

$$|A(s, t)| < \frac{2}{\pi} \cdot \frac{Y^2}{\alpha_0^2} \cdot \int_0^1 dx \left| J_0\left(Y\sqrt{\frac{(-t)x}{t_0}}\right) \right|, \qquad (26)$$

with $x = (y/Y)^2$ (the limit $\sqrt{s}/q_s \to 2$ at $s \to \infty$ is also used here). At $t = 0$ this inequality coincides with the estimate (21) for the forward amplitude. At nonzero finite values of t the argument of the Bessel function infinitely increases with energy, and we can use the asymptotic expression [14]

$$|J_0(\xi)|_{\xi \to +\infty} = \sqrt{\frac{2}{\pi\xi}} \cdot \left| \left[\cos\left(\xi - \frac{\pi}{4}\right) + \mathcal{O}(\xi^{-1}) \right] \right| < \sqrt{\frac{2}{\pi\xi}} \qquad (27)$$

(compare to Eqs. (22) and (25)). Then, finally, we obtain

$$|A(s, t)| < \frac{4}{3\alpha_0^2}\left(\frac{2Y}{\pi}\right)^{3/2} \cdot \left(\frac{t_0}{-t}\right)^{1/4}. \qquad (28)$$

Energy behavior of this boundary ($\sim Y^{3/2}/\alpha_0^2$) is intermediate between the forward boundary ($\sim Y^2/\alpha_0^2$) and the fixed-angle one ($\sim Y^{3/2}/\alpha_0^{3/2}$). It shows also a rather slow decrease with increasing momentum transfer, $\sim (-t)^{-1/4}$.

Again, similar to the fixed-angle case, presence of cosine in Eq. (27) may provide evidence for a possible oscillating t-distribution. Then the boundary (28) limits the upper edges of those oscillations.

Similar to the fixed-angle case at $\theta_s \to 0$, the fixed-t bound is singular at $(-t) \to 0$. This singularity, again, is spurious, related to the change of the energy behavior. The boundary (26) for the amplitude in the small $|t|$-region increases with energy very differently at $t = 0$ ($\sim Y^2/\alpha_0^2$) and at small finite value of $|t|$ ($\sim Y^{3/2}/\alpha_0^2$). This means that the amplitude boundary (26) at small $|t|$ reveals a narrow peak which shrinks when energy grows.

To understand the structure of this peak, let us consider in more detail relation

$$|A(s.t)| < A^{(\max)}(s, t).$$

Near $t = 0$ the $A^{(\max)}(s, t)$ corresponds to the right-hand side (26). At sufficiently small $|t|$ and fixed s (and Y as well) the Bessel function in Eq. (26) is positive, and the signs of modulus may be omitted. For small arguments [14]

$$J_0(z) \approx 1 - \frac{z^2}{4},$$

and we obtain

$$\frac{d}{dt} A^{(\text{max})}(s, t)\Big|_{t=0} = \frac{2}{\pi} \cdot \frac{Y^2}{\alpha_0^2} \cdot \frac{Y^2}{8t_0} = A^{(\text{max})}(s, 0) \cdot \frac{Y^2}{8t_0}. \tag{29}$$

When considering a differential cross section, it is familiar to parameterize its near-forward t-dependence as $\exp(bt)$ (recall that $t < 0$ in the physical region and that we do not account for spins). Since $d\sigma(s, t)/dt \propto |A(s, t)|^2$, we can express the slope of the forward peak as

$$b = \frac{d}{dt} \log\left[\frac{d\sigma(s, t)}{dt}\right]_{t=0} = 2\frac{d}{dt} \log|A(s, t)|_{t=0}. \tag{30}$$

In analogy, we can define the slope related to the boundary $A^{(\text{max})}(s, t)$ as

$$b^{(\text{max})} = 2\frac{d}{dt} \log[A^{(\text{max})}(s, t)]_{t=0}. \tag{31}$$

Of course, $b^{(\text{max})}$ does not necessarily provide a bound for b, though $A^{(\text{max})}$ is the bound for $|A|$. The reason is evident: differentiation may violate inequalities. Now, Eq. (29) shows that at high energies $b^{(\text{max})} = Y^2/(4t_0)$. It is interesting to note that the slope $b^{(\text{max})}$ has the high-energy behavior $\sim Y^2$, exactly the same as $\sigma_{\text{tot}}^{(\text{max})}$, the boundary for σ_{tot}. Therefore,

$$\left[\frac{b^{(\text{max})}}{\sigma_{\text{tot}}^{(\text{max})}}\right]_{s \to +\infty} = \text{const.}$$

There is, however, an essential difference between the two quantities: the physical total cross section σ_{tot} is always bounded by $\sigma_{\text{tot}}^{(\text{max})}$, while the physical slope b may be either larger or smaller than $b^{(\text{max})}$. But if σ_{tot} is saturated and indeed increases $\sim Y^2$, then the diffraction peak slope b should be also saturated and increase with energy not slower than $b^{(\text{max})}$. Thus, in the saturated regime $b \geq b^{(\text{max})}$. Otherwise, the amplitude boundary (26) at fixed $t < 0$ might become violated when energy grows.

The high-energy scattering at fixed t corresponds to angles near $\vartheta_s = 0$, i.e., to forward scattering. The case of backward scattering, for angles near $\vartheta_s = \pi$, may be considered in a similar way, with change $t \to u$.

D. Amplitude inside the Lehmann ellipse

Our approach allows us to discuss one more case: asymptotics of the amplitude outside the physical region, but inside the Lehmann ellipse. Though this case is not of direct physical interest, it may have theoretical interest. Here we again apply Eq. (17), but instead of $P_l(\cos\vartheta_s)$ we use $P_l(z)$ with $z = \cosh(\alpha + i\varphi)$; by our convention, $\alpha > 0$.

To find the large-l asymptotics of $P_l(z)$ with z outside the physical region, we can use the relation between Legendre functions of the 1st and 2nd kinds [11]:

$$Q_l(\cosh\beta) - Q_{-l-1}(\cosh\beta) = \pi \frac{\cos(\pi l)}{\sin(\pi l)} \cdot P_l(\cosh\beta). \tag{32}$$

Substituting the corresponding expressions (9) and tending the value of l to a positive integer number, we obtain relation

$$P_l(\cosh\beta) = \frac{\Gamma(l + \frac{1}{2})}{\sqrt{\pi}\Gamma(l + 1)} \cdot \frac{e^{\beta l}}{\sqrt{1 - e^{-2\beta}}}$$
$$\times {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; -l + \frac{1}{2}; \frac{1}{1 - e^{2\beta}}\right), \quad \text{Re}\,\beta > 0. \tag{33}$$

Of course, this relation is correct only at positive integer values of l, but only such values appear in the sums (17). For large l, it provides the asymptotic form

$$P_l(\cosh\beta)\big|_{l \to +\infty} = \frac{1}{\sqrt{2\pi l}} \cdot \frac{e^{\beta(l + \frac{1}{2})}}{\sqrt{\sinh\beta}} \cdot [1 + \mathcal{O}(l^{-1})],$$
$$\text{Re}\,\beta > 0,$$

which, for $z = \cosh(\alpha + i\varphi)$, gives

$$|P_l(z)|\big|_{l \to +\infty} = \frac{1}{\sqrt{2\pi l}} \cdot \frac{e^{\alpha(l + \frac{1}{2})}}{(\sinh^2\alpha + \sin^2\varphi)^{1/4}} \cdot [1 + \mathcal{O}(l^{-1})],$$
$$\alpha > 0. \tag{34}$$

If we use this expression in the sums

$$|A(s, z)| \leq \frac{\sqrt{s}}{\pi q_s}\left[\sum_{l=0}^{L-1} (2l + 1)|a_l(s)| \cdot |P_l(z)|\right.$$
$$\left. + \sum_{l=L}^{\infty} (2l + 1)|a_l(s)| \cdot |P_l(z)|\right] \tag{35}$$

and apply, as earlier, bounds (14), we see that the second sum converges only at $\alpha < \alpha_0$ (recall that $\alpha = \alpha_0$ is the singular point where the series should diverge). Since $\alpha_0 \to 0$ at $s \to +\infty$, our approach does not allow to investigate high-s behavior of $A(s, z)$ at any fixed z outside the physical region. However, we are able to consider, e.g., the case of a fixed nonphysical value for the momentum transfer t (or u).

The nonphysical interior of the z-plane Lehmann ellipse corresponds to $0 \leq \alpha \leq \alpha_0$ and is described by inequality

$$\left(\frac{\text{Re}\,z}{\cosh\alpha_0}\right)^2 + \left(\frac{\text{Im}\,z}{\sinh\alpha_0}\right)^2 \leq 1 \tag{36}$$

(compare to Eq. (8)). Since $z = 1 + t/(2q_s^2)$ and $\alpha_0 \approx \sqrt{t_0}/q_s$ at high energies, we can rewrite condition (36) for $s \to +\infty$ as

$$\frac{\mathrm{Re}\,t}{t_0} + \left(\frac{\mathrm{Im}\,t}{2t_0}\right)^2 \le 1. \tag{37}$$

The case of equality here corresponds to the limiting form of the Lehmann ellipse. In the complex t-plane, it is the parabola, symmetrical with respect to the real t-axis and directed to the left of $t = t_0$.

For any point t inside this parabola one can define

$$t_r = (|t| + \mathrm{Re}\,t)/2. \tag{38}$$

Then,

$$\frac{\mathrm{Re}\,t}{t_r} + \left(\frac{\mathrm{Im}\,t}{2t_r}\right)^2 = 1$$

(compare to Eq. (37)). Inside the parabola, $0 \le t_r \le t_0$. Further, we can use the familiar relation $z = 1 + t/(2q_s^2)$. At high (but not infinite) s-values and fixed t, the parametrization $z = \cosh(\alpha + i\varphi)$ provides $\alpha \approx \sqrt{t_r}/q_s \approx \alpha_0 \cdot \sqrt{t_r/t_0}$, $|\varphi| \approx \sqrt{(t_r - \mathrm{Re}\,t)}/q_s \approx \alpha_0 \cdot \sqrt{(t_r - \mathrm{Re}\,t)/t_0}$, and $\sinh(\alpha + i\varphi) \approx \sqrt{t}/q_s$. Expression (34) takes the form

$$|P_l(z)|_{l, s\to+\infty} \approx \frac{1}{\sqrt{2\pi l \alpha_0}} \cdot e^{\alpha_0 l \sqrt{t_r/t_0}} \cdot \left(\frac{t_0}{|t|}\right)^{1/4}. \tag{39}$$

Note that in the physical region, at real $t \le 0$, $\alpha = 0$ and $|\varphi| = \theta_s$, as should be.

Now we can return to the inequality (35) and continue construction of the boundary for the amplitude $A(s, t)$. As before, contributions of finite-l terms are inessential for the case of increasing σ_{tot} (i.e., for increasing L), and we will run the summation from some $l = l_0$, which is fixed, but sufficiently large to admit application of the asymptotics (34) and (39). Then

$$|A(s, t)| \le \frac{2}{\pi}\left(\frac{t_0}{|t|}\right)^{1/4}\left[\sum_{l=l_0}^{L-1} \frac{2l+1}{\sqrt{2\pi l \alpha_0}} e^{\alpha_0 l \sqrt{t_r/t_0}} \right.$$
$$\left. + \sum_{l=L}^{\infty} \frac{2l+1}{\sqrt{2\pi l \alpha_0}} B_0(s) \frac{e^{-\alpha_0 l}}{2\sqrt{l}} e^{\alpha_0 l \sqrt{t_r/t_0}}\right].$$

Using for $B_0(s)$ the left inequality (15) (or, equivalently, equality (16)), changing the sums by integrals (as described in Appendix and used in the previous subsections), and discarding inessential contributions, we obtain

$$|A(s, t)| < \left(\frac{2}{\pi}\right)^{3/2}\left(\frac{t_0}{|t|}\right)^{1/4}\left[\frac{1}{\alpha_0^2} \int_0^Y dy \sqrt{y} e^{y\sqrt{t_r/t_0}}\right.$$
$$\left. + \frac{\sqrt{Y} e^{Y\sqrt{t_r/t_0}}}{\alpha_0^2} \int_0^{\infty} dy e^{-y(1-\sqrt{t_r/t_0})}\right].$$

The second integral here diverges at $t_r \to t_0$ and, thus, restricts the region of applicability for our bound as $t_r < t_0$. Now, integration of the second term and transformation of the integral in the first term provides the boundary

$$|A(s, t)| < \frac{1}{\alpha_0^2}\left(\frac{2Y}{\pi}\right)^{3/2}\left(\frac{t_0}{|t|}\right)^{1/4}\left[\frac{2}{3}\int_0^1 dx e^{x^{2/3} Y \sqrt{t_r/t_0}}\right.$$
$$\left. + \frac{e^{Y\sqrt{t_r/t_0}}}{Y(1 - \sqrt{t_r/t_0})}\right], \tag{40}$$

where $x = (y/Y)^{3/2}$.

It is interesting to compare this boundary with the similar fixed-t boundary (28) for real negative t-values of the physical region. As was explained, the physical region may be reached by the limit $t_r \to 0$ at $|t| \ne 0$. When we formally apply this limit to the boundary (40), the second term becomes a parametrically small ($\sim 1/Y$) correction to the first one, and we can neglect it. Then the boundary (40) takes the same functional structure as the boundary (28), but being twice as small. The difference can be traced to the different large-l asymptotics of the Legendre polynomials $P_l(z)$ inside the physical interval ($z = x$, $-1 < x < +1$) and outside it. The cosine, present in the asymptotic expression (22) (and in the related expression (27)), is the combination of two exponentials, both of which should be taken into account. Only one of those exponentials is essential in the similar asymptotic expression (34), for z outside the physical region. Therefore, transition between asymptotics inside and outside the physical region may be noncontinuous (the Stokes phenomenon). Note, however, that the modulus of the cosine reveals oscillations, which frequency increases with energy. Their averaging provides the factor $1/2$ and could make the asymptotics be continuous between physical and nonphysical regions.

To find explicit high-energy behavior of the boundary (40), we need to calculate its integral. Using decomposition for the exponential, we obtain [11]

$$\int_0^1 dx e^{x^{2/3} Z} = \sum_{n=0}^{\infty} \frac{Z^n}{n!} \cdot \frac{1}{\frac{2}{3}n + 1} = \Phi\left(\frac{3}{2}, \frac{5}{2}; Z\right),$$
$$\Phi\left(\frac{3}{2}, \frac{5}{2}; Z\right)_{Z\to+\infty} = \frac{3}{2} \cdot \frac{e^Z}{Z}[1 + \mathcal{O}(Z^{-1})],$$

where $Z = Y\sqrt{t_r/t_0}$. Finally, at fixed t with $0 < t_r < t_0$, we obtain

$$|A(s, t)|_{s\to+\infty} < \left(\frac{2}{\pi}\right)^{3/2}\left(\frac{t_0}{|t|}\right)^{1/4} \cdot \frac{\sqrt{Y}}{\alpha_0^2}$$
$$\times e^{Y\sqrt{t_r/t_0}} \cdot \left(\frac{1}{\sqrt{t_r/t_0}} + \frac{1}{1 - \sqrt{t_r/t_0}}\right). \tag{41}$$

Thus, after all, the high-energy asymptotics is not continuous between physical and nonphysical regions. Presence of singularities in the boundary (41) at $t_r = 0$ and $t_r = t_0$, similar to previous cases, is related to the change of the asymptotic behavior.

It is interesting to note the difference between bounds for the nonphysical interior of the Lehmann ellipse and all the cases considered before. The second sum

(contributions of partial waves with $l \geq L$; in the brackets of Eq. (41) they provide the second term) has asymptotically the same functional behavior with respect to energy-dependent parameters Y and α_0, as the first sum (coming from waves with $l < L$; in the brackets of Eq. (41), see the first term). Numerically, the second term becomes even larger than the first one, if t_r is close to the singularity point t_0. In the physical region, contribution of waves with $l \geq L$ was parametrically smaller than contribution of waves with $l < L$.

Evidently, the amplitude in the nonphysical region can grow much faster than in the physical region, either inside it or at the edge. This is directly related to difference in the high-l asymptotics of the Legendre polynomials $P_l(z)$. In the whole physical region, they all do not exceed unity. Moreover, as a function of l inside the physical region, they oscillate at physical arguments and slowly decrease with growing l. On the other hand, outside the physical region, $P_l(z)$ exponentially increases with growing l. This is just the origin of the exponential factor in the bound (41).

In this subsection, we have studied the case of fixed t inside the Lehmann ellipse. In terms of the z-plane, this corresponds to $z \to +1$. The case of fixed u-value, which corresponds to $z \to -1$, may be considered in a similar way. However, we can discuss also other points inside the Lehmann ellipse. They may be characterized by

$$t - u = 4q_s^2 z$$

(recall that $t + u = -s + 4m^2$). Then the condition (36) for the ellipse interior takes the form

$$\left[\frac{\mathrm{Re}(t - u)}{4q_s^2 \cosh\alpha_0}\right]^2 + \left[\frac{\mathrm{Im}(t - u)}{4q_s^2 \sinh\alpha_0}\right]^2 < 1, \qquad (42)$$

which shows that $\mathrm{Re}(t - u)$ and $\mathrm{Im}(t - u)$ inside the ellipse can grow with energy not faster than $\sim q_s^2$ and $\sim q_s$, correspondingly. The fixed-t (or -u) case is just a particular case of such extreme possibility. Now, if we parametrize, as before, $z = \cosh(\alpha + i\varphi) = \cosh\alpha \cdot \cos\varphi + i\sinh\alpha \cdot \sin\varphi$, then

$$\mathrm{Re}(t - u) = 4q_s^2 \cosh\alpha \cdot \cos\varphi,$$
$$\mathrm{Im}(t - u) = 4q_s^2 \sinh\alpha \cdot \sin\varphi.$$

Since $0 < \alpha < \alpha_0$ inside the Lehmann ellipse, α should decrease at $s \to +\infty$ as $\sim q_s^{-1}$ or faster, while φ may stay fixed. In this limit $z \to \cos\varphi$, and $|\varphi|$ may be confronted with θ_s. If we construct a high-energy boundary for the fixed-φ amplitude, starting from $\alpha > 0$ and using the asymptotic expression (34), the exponential factor will contain the parameter $\alpha L = (\alpha/\alpha_0)Y$. For the case of α decreasing faster than q_s^{-1}, the boundary appears to have the same functional structure as the fixed-angle one (24), but being twice as small (compare with the relation between the two fixed-t boundaries, physical boundary (28) and nonphysical one (40) at $t_r \to 0$).

For α decreasing as q_s^{-1}, it is reasonable again to define t_r by the relation

$$1 + \frac{t_r}{2q_s^2} = \cosh\alpha = \frac{1}{4q_s^2}(|t| + |u|). \qquad (43)$$

The latter equality results from Eq. (8) and relations between t, u, and z. The value t_r defined in such a way is the t-value corresponding to the real positive z-point being on the same z-plane ellipse as the given point t (or u). The earlier expression (38) for t_r arises as the high-energy limit of Eq. (43) at fixed t. As in the nonphysical fixed-t case, we have $0 < t_r < t_0$ and, in the high-energy limit,

$$\alpha \approx \frac{\sqrt{t_r}}{q_s}, \qquad \frac{\alpha}{\alpha_0} \approx \sqrt{\frac{t_r}{t_0}}.$$

Now, to the case of fixed φ and t_r, we can apply the same procedure as described above for the fixed-t case, starting from Eqs. (34) and (35) to construct the high-energy boundary. In this manner we obtain

$$|A(s,z)| < \left(\frac{2}{\pi\alpha_0}\right)^{3/2}\left(\sin^2\varphi + \frac{t_r}{t_0}\alpha_0^2\right)^{-(1/4)}\left[\int_0^Y dy\sqrt{y}\,e^{y\sqrt{t_r/t_0}}\right.$$
$$\left. + \sqrt{Y}e^{Y\sqrt{t_r/t_0}}\int_0^\infty dy\,e^{-y(1-\sqrt{t_r/t_0})}\right].$$

High-Y asymptotics at fixed values of t_r and φ gives the final bound

$$|A(s,z)|_{s\to+\infty} < \left(\frac{2}{\pi\alpha_0}\right)^{3/2}\left(\sin^2\varphi + \frac{t_r}{t_0}\alpha_0^2\right)^{-(1/4)}$$
$$\times \frac{\sqrt{Y}e^{Y\sqrt{t_r/t_0}}}{\sqrt{t_r/t_0}(1 - \sqrt{t_r/t_0})}. \qquad (44)$$

If $\sin\varphi \neq 0$, the relation between the nonphysical boundaries (41) and (44) for the fixed-t and fixed-(φ, t_r) cases is essentially the same as between the physical boundaries (28) and (24) for the fixed-t and fixed-θ_s cases (recall that in the physical region t_r equals zero and, thus, is always fixed). If $\varphi \to 0$ faster than q_s^{-1}, then t tends toward t_r; inequality (44) takes the same form as the previous inequality (41) for nonphysical positive t-values. If $\varphi \to \pm\pi$, also tending toward the limit faster than q_s^{-1}, then $u \to t_r$, and expression (44) comes to correspond with the nonphysical fixed-u case, for real positive u-values.

IV. HIGH-ENERGY BEHAVIOR OF AMPLITUDES

In the previous section we constructed high-energy upper boundaries for a $2 \to 2$ amplitude in different physical or nonphysical configurations: forward or backward, fixed-angle scattering, fixed momentum transfer, and the nonphysical interior of the Lehmann ellipse. In all the cases, the boundaries have been expressed through two energy-dependent parameters, α_0 and Y.

One of them, α_0, has the clear and very simple energy dependence. But it depends also on the parameter t_0,

related to the position of a crossed-channel singularity, which we assume to be energy-independent. We have not discussed, however, how the kind of the singularity could influence the amplitude asymptotics.

The other parameter, $Y = \alpha_0 L$, depends on the number L of the partial-wave amplitudes that could be essential. Dependence of L (and Y) on energy is much less clear than for α_0. It is definitely not fixed by kinematics or any physical principles.

Now we are going to discuss both problems in some detail.

A. Role of different singularities

Up to now we have assumed t_0 to be related with the position of any z-singularity, nearest to the physical region. At very high values of l, contribution of this singularity is definitely the largest one. However, at l-values near L, it may appear less essential than that of some more remote, but more intensive singularity. Consider this point more specifically.

Let the nearest singularity be a pole corresponding to a one-particle exchange, say, in the t-channel. If so, we can present the amplitude as

$$A(s, t) = \frac{r(s)}{t - t_0} + \tilde{A}(s, t) = \frac{1}{2q_s^2} \cdot \frac{r(s)}{z - z_0} + \tilde{A}(s, z),$$

where $\tilde{A}(s, z)$ has no z-singularities inside the Lehmann ellipse related to z_0. Substituting this form into Eq. (7), we separate the simple pole contribution

$$a_l^{(\text{pole})} = -\frac{\pi r(s)}{2q_s\sqrt{s}} \cdot Q_l(z_0), \qquad (45)$$

while contribution of other singularities retains the form similar to the contour integral (7), with $\alpha = \tilde{\alpha} > \alpha_0$. The value of $\tilde{\alpha}$ may be increased up to α_1, corresponding to the next nearest singularity. Now, applying again the asymptotic relation (10), we can rewrite boundary (12) as

$$|a_l| < \frac{q_s}{\sqrt{s}} \cdot \frac{|r(s)|}{q_s\sqrt{s}} \left(\frac{\pi}{2}\right)^{3/2} \cdot \frac{1}{\sqrt{e^{\alpha_0}\sinh\alpha_0}} \cdot \frac{e^{-\alpha_0 l}}{\sqrt{l}}$$
$$+ \frac{q_s}{\sqrt{s}} \cdot \tilde{B}_1(s) \cdot \sqrt{e^{-\alpha_1}\cosh\alpha_1} \cdot \frac{e^{-\alpha_1 l}}{\sqrt{l}}, \qquad (46)$$

where $\tilde{B}_1(s)$ is related with $\tilde{A}(s, z)$. At very large l, the first term in the right-hand side is always the leading one, since $\alpha_1 > \alpha_0$. But this right-hand side itself is then small and decreasing. The situation may be different, however, at $l \approx L$, where the right-hand side is near unity.

Let us consider the high-energy behavior of the boundary (46). The asymptotics of the t-channel pole residue $r(s)$ is $\sim s^J$ being directly related to the spin J of the corresponding hadron. There exist only two kinds of hadrons which are stable under strong interactions and can, thus, generate poles on the physical Riemann sheet. They are basic pseudoscalar mesons with $J = 0$ (pions, kaons, eta,

and so on) or basic baryons with $J = 1/2$. Neither of the corresponding exchanges can produce increasing total cross sections. Moreover, their contribution to the boundary (46) (see the first term) is vanishing when s grows at fixed l. Increasing total cross sections could be induced by exchanges of the higher-spin hadrons, but such hadrons reveal themselves only as resonances. Therefore, the corresponding poles are positioned at nonphysical Riemann sheets and contribute to $\tilde{B}_1(s)$. Thus, to provide an increasing cross section, the role of the second term should grow at high s, due to growing $\tilde{B}_1(s)$.

If $\tilde{B}_1(s)$ grows indeed with s, the value of L at sufficiently high energy becomes determined by the second term in the right-hand side (46), while the first term becomes inessential. If the singularity at $\alpha = \alpha_1$ is also a pole, we can separate it as well. After all, possible increase of the total cross sections appears to be related to the nearest threshold, even if there is a nearer pole. Therefore, we can use formulas of the preceding sections, assuming that α_0 always corresponds to the nearest threshold and not to a pole. Note that the Lukaszuk-Martin boundary [8] for total cross sections, widely discussed in the literature, uses just the nearest threshold, which is the two-pion threshold.

B. Energy dependence of boundaries

Now we return to the explicit high-energy behavior of parameters α_0 and Y, which determine upper boundaries for amplitudes in different scattering configurations. At high energies,

$$\alpha_0 \approx \frac{\sqrt{t_0}}{q_s} \approx 2\sqrt{\frac{t_0}{s}}, \qquad (47)$$

where t_0, as explained, is related to the nearest threshold in the crossed channel.

Less-evident energy dependence of the other parameter, Y, is seen in Eq. (16), which can be rewritten as

$$e^Y \cdot \sqrt{Y} = \frac{B_0(s)\sqrt{\alpha_0}}{2}. \qquad (48)$$

The quantity $B_0(s)$ is linearly related to the amplitude integrated over nonphysical configurations, as determined by Eq. (13) with $\alpha = \alpha_0$. Evidently, the energy behavior of Y (and, therefore, of boundaries for the amplitude in physical configurations) directly depends on the (unknown) energy behavior of $B_0(s)$.

The original Froissart boundaries have similar structure. Instead of our parameter Y, those boundaries contain $Y_{\text{Fr}} = \alpha_0 L_{\text{Fr}}$. Recall that in Froissart's notations $\alpha_0 = \log(x_0 + \sqrt{x_0^2 - 1})$. The value of L_{Fr} at high energies was chosen so that

$$e^{Y_{\text{Fr}}} = B_{\text{Fr}}(s)e^{N\alpha_0},$$

where N is the number of necessary subtractions (see Ref. [1], upper right column on p. 1055). Though both

$B_{\mathrm{Fr}}(s)$ and our $B_0(s)$, by construction, are linearly related with the amplitude in nonphysical configurations, those configurations are, generally, different (they correspond to different values of z or, equivalently, t). Therefore, $B_{\mathrm{Fr}}(s)$ looks not identical to $B_0(s)$. But if we assume an increasing total cross section, both $B_{\mathrm{Fr}}(s)$ and $B_0(s)$ should increase as well. It is reasonable to assume that they have similar high-energy asymptotics. Then $Y_{\mathrm{Fr}} > Y$, and Froissart's boundaries are higher than ours. In particular,

$$\sigma_{\mathrm{tot}} < CY^2 < CY_{\mathrm{Fr}}^2.$$

Discussion (and/or derivation) of any dispersion relation always contains two ingredients. The main one is, of course, knowledge of positions for singularities. According to standard assumptions, the character and position of a singularity is determined by the unitarity condition. Then, the amplitude singularities are only poles and branch points, corresponding to one-particle states or to several-particle thresholds, respectively. Their positions are determined, therefore, by the particle masses. Any singularity of other kinds (*e.g.*, essential singularity) is not suggested by the unitarity and, hence, is not expected to appear at some final distance.

Unitarity, by itself, says nothing about analytical properties of the infinite energy point. Nevertheless, a familiar assumption is that this point also has no essential singularity. It means, in particular, that when $s \rightarrow \infty$ along any direction at the physical Riemann sheet, the amplitude $A(s, t)$ can increase not faster than some limited power of s (at any fixed value of t or u). This is the second important ingredient, after which the dispersion relation in s (with some limited number of subtractions) arises just as a simple manifestation of the Cauchy theorem.

If $A(s, t)$ is restricted by s^n, then this is true also for both $B_{\mathrm{Fr}}(s)$ and $B_0(s)$. This implies that Y_{Fr} grows no faster than $n \cdot \log s$, and σ_{tot} grows no faster than $\log^2 s$. It is just the canonical formulation of the Froissart theorem.

Our Eq. (48) gives a more complicated, but somewhat stronger, restriction for Y:

$$Y < \left(n - \frac{1}{4}\right)\log s - \frac{1}{2}\log Y \approx \left(n - \frac{1}{4}\right)\log s - \frac{1}{2}\log\log s. \tag{49}$$

Of course, it is smaller than the Froissart boundary $n \log s$. If we describe the high-energy boundary for Y by the standard parametrization $\sim \log(s/s_0)$, then Eq. (49) means that the scale s_0 itself should be energy-dependent: it should slowly (logarithmically) increase with energy.

Recall now that, according to Eq. (21), the high-energy boundary for σ_{tot} is determined by Y^2. Then we see that σ_{tot}, indeed, cannot grow faster than $\sim\log^2(s/s_0)$ with a fixed scale s_0. But such growing log-squared behavior can be saturated only if the scale s_0 used here also increases with energy.

V. DISCUSSION OF THE RESULTS

Let us summarize the above results for quantum amplitudes of $2 \rightarrow 2$ processes.

(i) Dispersion relations, single or double, are not necessary for the Froissart theorem. Moreover, in the above considerations we have not assumed any specific nature of the underlying interaction. And even more, all the above relations are consequences of the rather general quantum picture. They could be equally applied either to the nonrelativistic Schrödinger equation (using the energy E instead of the invariant s) or to relativistic interaction(s) of (non-)elementary particles.

(ii) Unitarity is known since the original paper [1] to be a necessary input for the Froissart theorem. In all cases, it restricts the partial-wave amplitudes, which contribute to the total amplitude.

(iii) One more necessary input for the Froissart theorem is absence of singularities in the physical region of $z = \cos\theta$ (inside or on the edges of the interval $[-1, +1]$). In the nonrelativistic case, this may be ensured by properties of the potential (as, *e.g.*, for the Yukawa potential). In the relativistic case, this may be provided by the unitarity condition in the crossed channel(s), if no massless exchanges are possible (note the double-sided role of the unitarity in the relativistic case). It is just the reason why the Froissart theorem may be applied to strong interactions (having finite-mass pions as the lightest particles), but not to electroweak interactions (having the massless photon). Absence of any physical-region singularities guarantees the exponential smallness of high-l partial-wave amplitudes. As a result, only a finite number of partial waves may be "essential" at each given energy.

(iv) A very important ingredient of the Froissart theorem comes from the mathematical properties of the Legendre functions. Our calculations clearly demonstrate that high-energy asymptotics for amplitudes in different configurations is directly coupled with high-l behavior of the Legendre functions, which have $l = \infty$ as the essential singularity. It is well-known that the Legendre polynomials $P_l(z)$ at large l behave very differently inside the physical region (the real interval $-1 \leq z \leq +1$), at its edge ($z = \pm 1$), and outside it. At the edge $|P_l(\pm 1)| = 1$, while inside the region $|P_l(z)| < 1$ and decreases $\sim l^{-1/2}$ with growing l. Outside the physical region, it exponentially grows. Just these well-known facts imply that the quantum amplitude has very different behavior inside the physical region, outside it, or at the edge. They also explain why high-energy asymptotics of the amplitude is

much more moderate in the physical region than outside it.

(v) The most disputable input is given by assumptions on the high-energy behavior of the amplitude in the nonphysical region. The familiar assumption is the power boundary for the increase in s, with some restricted power which is universal, in the sense that it is applicable for any value of t (or $\cos\theta_s$), physical or nonphysical, and even complex. The canonical log-squared bound for the total cross sections arises as a consequence of such a restriction for the amplitude at nonphysical values of t.

Let us consider in some more detail the problem of discontinuities between asymptotics for different configurations. Froissart's calculations [1] show different high-energy behavior for $\theta_s = 0$ and $\theta_s \neq 0$. Our calculations confirm this result and present it also as discontinuities of asymptotics between $t < 0$, $t = 0$, and $t > 0$. At first sight, this looks strange since $t = 0$ is a nonsingular point, where the amplitude is analytic (and continuous). Therefore, it would be natural to expect the s-asymptotics to be continuous as well. However, the Legendre functions clearly demonstrate just the opposite behavior. The point $z = 1$ is always an analyticity point for $P_l(z)$. Nevertheless, the high-l asymptotics is discontinuous near $z = 1$. Indeed, at the real axis below this point

$$P_l(\cos\theta) \approx \sqrt{\frac{2}{\pi l \sin\theta}} \cdot \cos\left[\left(l + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right];$$

at the point itself,

$$P_l(1) = 1;$$

at last, above this point,

$$P_l(\cosh\beta) \approx \frac{1}{\sqrt{2\pi l}} \cdot \frac{e^{\beta(l+1/2)}}{\sqrt{\sinh\beta}}, \qquad \mathrm{Re}\,\beta > 0.$$

Discontinuities of the asymptotics are evident here. However, they appear only in the limit $l \to \infty$ at fixed z. If one takes l to be large but fixed, $P_l(z)$ is, of course, continuous in z, as seen from one more well-known approximate relation (25). It shows that the different asymptotics join in a narrow intervals $(z - 1) \sim l^{-2}$.

A similar conclusion is true for the amplitude as well. Equation (26) is continuous near $t = 0$, if Y is finite. But essential changes of the right-hand side take place in a very narrow interval $\Delta t \sim t_0/Y^2$. In the limit $Y \to \infty$ we obtain discontinuous boundaries for different values of t near zero. Analogous is the transition from physical real values of $t < 0$ to nearby complex values of t.

Now we briefly discuss the problem of power high-energy behavior for the amplitude. In quantum field theory, such behavior could not be deduced from any general principles (in particular, unitarity can say nothing

on this problem). The only motivation for the power behavior near infinity is that it allows us to write dispersion relations.

In difference, for quantum mechanics, the high-energy asymptotics of the amplitude can be found somehow, if the potential is given. Dispersion relation in energy is true for the forward scattering amplitude with many quantum-mechanical potentials [15]. Potentials which admit dispersion relations for nonforward scattering seem to be much rarer. But at least for Yukawa-like potentials, the amplitude satisfies even the Mandelstam representation, which is the double-dispersion relation in energy and momentum transfer [5].

In the relativistic case, the problem of high-energy asymptotics for two-particle amplitudes is still open. Even the single-dispersion relation in energy has been mathematically proven only for the pion-nucleon elastic scattering in the forward direction or in some finite interval of real negative (*i.e.*, physical) values of t [16]. Note, however, that this does not prove the Froissart log-squared behavior of the pion-nucleon σ_{tot}. For the forward dispersion relation to be true, σ_{tot} may grow faster than the canonical Froissart bound, but not faster than some finite power of s.

For better understanding of the situation, it is interesting to look for hints from the perturbation theory. Summation of the Feynman diagrams was most intensively investigated for QED and perturbative QCD (pQCD). In both cases, sums of essential logarithms for "one-tower" diagrams provide the power behavior of the high-energy asymptotics for σ_{tot}. The corresponding exponents are small: $\sim\alpha^2$ in QED [17–19] and $\sim\alpha_s$ in pQCD [20]. The difference is due to different forms of interactions between the corresponding gauge bosons: through electron loop(s) for the photons in QED, and through gluon exchange(s) for the gluons in pQCD.

The authors of Ref. [18] believe that "unitarity is violated" because of such power behavior. Therefore they consider it to be transient. As they hope, it will be changed by the log-squared behavior after summing up all "multitower" diagrams, though the authors agree that "this method has no mathematical justification".

We have seen, however, that violation of the log-squared asymptotics is not necessarily related to violation of unitarity. It may mean violation of power bounds for nonphysical amplitudes. On the Lehmann ellipse, the values of momentum transfers may reach large complex values $|t| \approx |u| \approx s/2$, $|\mathrm{Im}\,t| \approx |\mathrm{Im}\,u| \approx \sqrt{st_r}$, with t_r being fixed at the given Lehmann ellipse. High-energy behavior for amplitudes in such configurations has never been investigated.

There is one more reason why cross section estimates in QED and/or pQCD might be doubtful. Both theories provide massless (photon/gluon) exchanges, which generate z-singularity at the edge of the physical region. For scattering of charged (colored) objects this singularity is

nonintegrable (it is the pole due to one-photon/one-gluon exchange) and makes the total cross section infinite at any energy. Thus, discussion of any bounds for σ_{tot} becomes meaningless.

However, for neutral (colorless) objects, the corresponding singularities, though being also on the edge of the physical region, are integrable and do not provide a permanently infinite cross section. In such situations, the nearest singularity(ies), just at the edge of the physical region, may appear less essential for the asymptotics than more distant singularities (we have seen above how the nearest pole could be inessential at high energies, as compared with more distant contributions). Then, the Froissart approach might be applicable for neutral (colorless) objects in QED (pQCD), even despite the possibility of massless exchanges. This needs, however, special investigation. If confirmed, results of diagram summation for QED and pQCD, briefly described above, could give indeed serious theoretical hints for power (though rather slow) increase of hadron cross sections.

Strong interaction phenomenology presents also some other evidences, though indirect, for power increase of the total cross sections. For example, an essential input to prove the power asymptotics (and the Mandelstam representation) of amplitudes for nonrelativistic scattering in Yukawa-type potentials was a restricted value of real parts for Regge trajectories in such potentials [5]. The phenomenological evidence for linearity of hadron Regge trajectories, if true, means that the relativistic amplitudes, at least in some configurations, may grow faster than any power of energy. Correspondingly, the total cross sections may grow faster than $(\log s)^2$.

In summary, the real content of the Froissart theorem is the much softer high-energy behavior of physical amplitudes (and total cross sections) as compared to behavior of nonphysical amplitudes. This is implied by the physical requirements of unitarity and absence of massless exchanges, together with the mathematical properties of the Legendre functions. Dispersion relations, either in energy or in momentum transfers, are not necessary. Moreover, the nature of interaction is, by itself, inessential; however, strong interactions are marked out by the absence of massless particles.

The specific form of the high-energy bound for amplitudes in physical configurations (and, thus, for the total cross section as well) is directly correlated with the high-energy behavior for amplitudes in nonphysical configurations (in particular, at large and complex t). The canonical log-squared bound corresponds to power asymptotics of the nonphysical amplitudes (which can be "hidden" in dispersion relations with a finite number of subtractions). Its violation, contrary to folklore in the literature, would not mean violation of unitarity. It may mean only that the amplitude in nonphysical configurations can grow with energy faster than any power of s. Such possibility does not seem to contradict any basic principles. Precise measurements of cross sections at very high energies (at LHC, in particular) can possibly help to discriminate between logarithmic and/or power asymptotics. Other high-energy observables may also be helpful.

APPENDIX A: SUMS AND INTEGRALS FOR BOUNDARIES

When constructing boundaries for amplitudes in different configurations, we need to calculate the high-energy behavior for sums of the form

$$S = \sum_{l_1}^{l_2} l^k \cdot f(\alpha l),$$

where summation runs on integer values of l, and $\alpha \to 0$ at $s \to +\infty$. Let us rewrite the sum as

$$S = \sum_{l_1}^{l_2} l^k \cdot f(\alpha l) \cdot \Delta l,$$

where $\Delta l = 1$. Now, define the new variable $y = \alpha l$. Then the sum takes the form

$$S = \alpha^{-(k+1)} \sum_{y_1}^{y_2} y^k \cdot f(y) \cdot \Delta y,$$

with summation running on the y-points, corresponding to the integer values of l, with intervals $\Delta y = \alpha$. When $s \to +\infty$ (*i.e.*, $\alpha \to 0$), the latter sum tends toward the integral

$$S \approx \alpha^{-(k+1)} \int_{y_1}^{y_2} y^k \cdot f(y) \cdot dy.$$

For all our boundaries we have used just such integrals. As an illustration, let us consider the simple sum

$$\sum_{0}^{L-1} (2l + 1) = L^2.$$

The above procedure, with $Y = \alpha L$, transforms it into the sum of two integrals

$$\alpha^{-2} \int_0^{(Y-\alpha)} 2y dy + \alpha^{-1} \int_0^{(Y-\alpha)} dy$$

$$= \left(\frac{Y-\alpha}{\alpha}\right)^2 + \left(\frac{Y-\alpha}{\alpha}\right) = \frac{Y(Y-\alpha)}{\alpha^2} = L^2\left(1 - \frac{1}{L}\right).$$

If L is growing at $s \to +\infty$, the main term is correctly reproduced.

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