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Applying generalized Padé approximants in analytic QCD models

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A method of resummation of truncated perturbation series, related to diagonal Padé approximants but giving results independent of the renormalization scale, was developed more than ten years ago by us with a view of applying it in perturbative QCD. We now apply this method in analytic QCD models, i.e., models where the running coupling has no unphysical singularities, and we show that the method has attractive features, such as a rapid convergence. The method can be regarded as a generalization of the scale-setting methods of Stevenson, Grunberg, and Brodsky-Lepage-Mackenzie. The method involves the fixing of various scales and weight coefficients via an auxiliary construction of diagonal Padé approximant. In low-energy QCD observables, some of these scales become sometimes low at high order, which prevents the method from being effective in perturbative QCD, where the coupling has unphysical singularities at low spacelike momenta. There are no such problems in analytic QCD.

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I. INTRODUCTION

Extending the applicability of QCD from high energies, where it can be consistently treated by perturbation methods, down to the low-energy regime is one of the main tasks of theoretical hadronic physics. A simple-minded utilization of perturbation series is clearly forbidden, not just by the sheer size of the expansion parameter (the running coupling parameter $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ at low momentum transfer $Q^2 \equiv -q^2$), but even more so by the existence of unphysical (Landau) singularities of the coupling parameter in the complex Q^2 plane. These singularities are inferred from the renormalization group equation when the corresponding beta function is expressed in terms of a truncated perturbation series. These singularities are unphysical because they do not reflect correctly the analytic properties of spacelike observables $\mathcal{D}(Q^2)$, properties based on the general principles of local quantum field theories [1,2]. Consequently, the most straightforward procedure for applying QCD to low-energy quantities consists in removing this unwanted nonanalyticity by some kind of analytization of the coupling parameter $a(Q^2) \mapsto \mathcal{A}_1(Q^2)$. The analytic coupling parameter $\mathcal{A}_1(Q^2)$ can differ significantly from the perturbative one, $a(Q^2)$, only at low momenta $|Q^2| \lesssim 1\ \mathrm{GeV}^2$. Several constructions of such analytic QCD models, i.e., of $\mathcal{A}_1(Q^2)$, have been made during the last 15 yr—starting from the seminal papers of Shirkov, Solovtsov, and Milton [3–5]. For reviews of various types of analytic QCD models see Refs. [6–9]. On the other hand, handling the physics of hadrons at low energies by simply utilizing an appropriately modified, "analytized," coupling parameter (together with its higher order analogs) within perturbative approaches is a very ambitious task, since it implicitly rests on the assumption that the low-Q^2 behavior of $\mathcal{A}_1(Q^2)$ can be defined in a way that all nonperturbative effects are effectively included—at least for inclusive quantities. Of particular interest here is the behavior of $\mathcal{A}_1(Q^2)$ for $Q^2 \to 0$, and this question was the subject of intensive studies during the last several years, based either on analytic methods (Schwinger-Dyson equations [10], Banks-Zaks expansion [11,12]) or on numerical lattice approaches [13]. They have finally led to the strong suspicion of "freezing" of the coupling parameter near $Q^2 = 0$. If one wants to go a step further, however, and specify $\mathcal{A}_1(Q^2)$ for the whole range $|Q^2| \lesssim Q^2_{as}$ (Q^2_{as} denotes the momentum transfer where asymptotic freedom should start to dominate), such that all nonperturbative effects get included, one clearly has to utilize as much as possible external information, both on the side of empirical constraints and on the side of general physical principles, such as causality, unitarity, analyticity, asymptotic freedom, operator product expansion, renormalization scale, scheme independence, etc.

Within the present paper, we focus mainly on the analytical structure and on the renormalization scale (RScl) independence of the resulting physical quantities. We apply, in various analytic QCD models, a global (i.e., nonpolynomial in the coupling) RScl-invariant resummation/evaluation method, which we developed in the context of perturbative QCD more than ten years ago [14,15], and we compare this evaluation method with other methods. In Sec. II, we recapitulate the aforementioned RScl-invariant resummation method for spacelike observables

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(in perturbative QCD). The presentation this time is somewhat less formal and, perhaps, more intuitive. In Sec. III, we describe the minimal adjustments needed for the method to be used in analytic QCD models. In that section, we also argue why we should expect our resummation method to work significantly better in analytic QCD than in perturbative QCD. In Sec. IV, we apply the method to the evaluation of the derivative of the massless (vector) current-current correlation function, i.e., the Adler function, both in perturbative QCD and in various motivated analytic QCD models. First, the evaluations are made for the leading-β_0 part of the Adler function, where we know the exact result within each analytic QCD model, so this case is used as a test case for our resummation method to rather high values of the order index M. Subsequently, we apply our method to the truncated series of the Adler function, where only the first three full coefficients (beyond the leading term) are known. In Sec. V, we summarize the results and present conclusions.

II. RECAPITULATION OF THE METHOD

In this section, we present the resummation method developed in Refs. [14,15] in a somewhat simpler and, perhaps, more intuitive way. We consider a massless spacelike physical observable $\mathcal{D}(Q^2)$, whose perturbation series in powers of the perturbative QCD (pQCD) coupling, $a(Q^2) \equiv \alpha_s(Q^2)/\pi$,

$$\mathcal{D}(Q^2)_{\rm pt} = a(Q^2) + \sum_{j=1}^{\infty} d_j a(Q^2)^{j+1} \tag{1}$$

is known up to $\sim a^{2M}$, such that we are faced with the truncated perturbation series $\mathcal{D}(Q^2)^{[2M]}_{\rm pt}$,

$$\mathcal{D}(Q^2)^{[2M]}_{\rm pt} = a(Q^2) + \sum_{j=1}^{2M-1} d_j a(Q^2)^{j+1}. \tag{2}$$

Here we have chosen the RScl μ^2 to be equal to the physical scale Q^2 of the process ($\mu^2 = Q^2$). For a general RScl μ^2, the full and the truncated perturbation series read

$$\mathcal{D}(Q^2)_{\rm pt} = a(\mu^2) + \sum_{j=1}^{\infty} d_j(\mu^2/Q^2) a(\mu^2)^{j+1} \tag{3}$$

$$\mathcal{D}(Q^2;\mu^2)^{[2M]}_{\rm pt} = a(\mu^2) + \sum_{j=1}^{2M-1} d_j(\mu^2/Q^2) a(\mu^2)^{j+1}. \tag{4}$$

This truncated series has a residual RScl dependence due to truncation. The μ^2-dependence of $d_j(\mu^2/Q^2)$ is dictated by the μ^2-independence of the full series $\mathcal{D}(Q^2)_{\rm pt}$ and the μ^2-dependence of $a(\mu^2)$ given by the well-known renormalization group equation (RGE)

$$\begin{aligned}\frac{da(\mu^2)}{d\ln\mu^2} &= -\sum_{j\ge 2}\beta_{j-2} a(\mu^2)^j \\ &= -\beta_0 a(\mu^2)^2(1 + c_1 a(\mu^2) + c_2 a(\mu^2)^2 + \ldots),\end{aligned} \tag{5}$$

where the right-hand side is the beta function $\beta(a)$, and we denoted $c_j \equiv \beta_j/\beta_0$. In particular, we obtain (we denote throughout: $d_j(1) \equiv d_j$ and $d_0 = d_0(\mu^2/Q^2) = 1$)

$$d_1(\mu^2/Q^2) = d_1 + \beta_0 \ln(\mu^2/Q^2), \tag{6}$$

$$d_2(\mu^2/Q^2) = d_2 + \sum_{k=1}^{2} \frac{2!}{k!(2-k)!}\beta_0^k \ln^k\left(\frac{\mu^2}{Q^2}\right) d_{2-k} + \beta_1 \ln\left(\frac{\mu^2}{Q^2}\right), \tag{7}$$

$$\begin{aligned}d_3(\mu^2/Q^2) = d_3 &+ \sum_{k=1}^{3} \frac{3!}{k!(3-k)!}\beta_0^k \ln^k\left(\frac{\mu^2}{Q^2}\right) d_{3-k} \\ &+ \beta_1\left[2d_1 \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{5}{2}\beta_0 \ln^2\left(\frac{\mu^2}{Q^2}\right)\right] \\ &+ \beta_2 \ln\left(\frac{\mu^2}{Q^2}\right),\end{aligned} \tag{8}$$

etc. Note that $a(\mu^2)$ and $d_j(\mu^2/Q^2)$ are not only RScl dependent, but also renormalization scheme (RSch) dependent (as are also $d_j \equiv d_j(1)$), i.e., they are functions of μ^2, $c_2 = \beta_2/\beta_0$, $c_3 = \beta_3/\beta_0$, etc. The RSch dependence of $d_j(\mu^2/Q^2)$ and d_j involves $c_2, \ldots, c_j$ (when $j \ge 2$). The first two coefficients β_0 and β_1 are universal in the mass-independent schemes: $\beta_0 = (11 - 2n_f/3)/4$, $\beta_1 = (102 - 38n_f/3)/16$.

In the following, we will mainly be interested in the RScl dependence of the different (perturbation) series. Therefore, it will prove advantageous to use logarithmic derivatives of the pQCD coupling a instead of powers a^n. Specifically, we introduce[1]

$$\tilde{a}_{n+1}(Q^2) \equiv \frac{(-1)^n}{\beta_0^n n!}\frac{d^n a(Q^2)}{d(\ln Q^2)^n} \tag{9}$$

and reorganize the (truncated) perturbation series (3) and (4) into the "modified (truncated) perturbation series" ("mpt," in equations)

$$\mathcal{D}(Q^2)_{\rm mpt} = a(\mu^2) + \sum_{j=1}^{\infty} \tilde{d}_j(\mu^2/Q^2)\tilde{a}_{j+1}(\mu^2), \tag{10}$$

[1] Note that the factor in front of the right-hand side is chosen such that $\tilde{a}_1 \equiv a$ and $\tilde{a}_{n+1} = a^{n+1} + \mathcal{O}(a^{n+2})$ for $n \ge 1$. Only at one-loop level approximation we have $\tilde{a}_{n+1} = a^{n+1}$, but in general $\tilde{a}_{n+1} \ne a^{n+1}$.

$$\mathcal{D}(Q^2;\mu^2)^{[2M]}_{\mathrm{mpt}} = a(\mu^2) + \sum_{j=1}^{2M-1} \tilde{d}_j(\mu^2/Q^2)\tilde{a}_{j+1}(\mu^2). \quad (11)$$

Here, the coefficients $\tilde{d}_j(\mu^2/Q^2)$ are chosen so that the expressions (3) and (10) are formally identical. The advantage of using here the logarithmic derivatives (9) and the expansions (10) and (11),[2] as opposed to the expansions (3) and (4), lies principally in the simple recursion relations for $\tilde{a}_n$'s

$$\frac{d}{d\ln\mu^2}\tilde{a}_n(\mu^2) = -\beta_0 n \tilde{a}_{n+1}, \quad (12)$$

whereas for the powers a^n, the relation is more complicated:

$$\frac{d}{d\ln\mu^2}a(\mu^2)^n = -n\beta_0 a(\mu^2)^{n+1}(1 + c_1 a(\mu^2) + c_2 a(\mu^2)^2 + \cdots), \quad (13)$$

the right-hand side here being the consequence of the RGE (5). When we use the fact that the full series $\mathcal{D}(Q^2)_{\mathrm{mpt}}$ in Eq. (10) is RScl independent

$$\frac{d}{d\ln\mu^2}\mathcal{D}(Q^2)_{\mathrm{mpt}} = 0, \quad (14)$$

we obtain a set of differential equations

$$\frac{d}{d\ln\mu^2}\tilde{d}_n(\mu^2/Q^2) = n\beta_0\tilde{d}_{n-1}(\mu^2/Q^2) \qquad (n = 1, 2, \ldots), \quad (15)$$

whose integration gives (we denote throughout $\tilde{d}_j(1) \equiv \tilde{d}_j$ and $\tilde{d}_0 = 1$)

$$\tilde{d}_n(\mu^2/Q^2) = \tilde{d}_n + \sum_{k=1}^{n}\frac{n!}{k!(n-k)!}\beta_0^k \ln^k\left(\frac{\mu^2}{Q^2}\right)\tilde{d}_{n-k}. \quad (16)$$

We note that the relations (16) for $\tilde{d}_n(\mu^2/Q^2)$, in contrast to those for $d_n(\mu^2/Q^2)$ in Eqs. (6)–(8), do not involve any higher-loop beta coefficients β_j ($j \geq 1$). Therefore, it is suggestive to compare the situation with the one-loop limit of QCD (where $\beta_1 = \beta_2 = \ldots = 0$). In that limit, the perturbative coupling, now denoted as $a_{1\ell}(\mu^2)$, has the one-loop RGE running from a given value, $a(Q^2)$, at the scale Q^2 to the scale μ^2:

$$a_{1\ell}(\mu^2) = \frac{a(Q^2)}{1 + \beta_0 \ln(\mu^2/Q^2)a(Q^2)}. \quad (17)$$

Furthermore, in this case we have $\tilde{a}_{n+1,1\ell}(\mu^2) = a_{1\ell}(\mu^2)^{n+1}$, where $\tilde{a}_{n+1,1\ell}(\mu^2)$ are the logarithmic derivatives of $a_{1\ell}(\mu^2)$ analogous to Eq. (9).

Consequently, if we define the (auxiliary) quantity $\tilde{\mathcal{D}}(Q^2)$ via the following power series:

$$\tilde{\mathcal{D}}(Q^2)_{\mathrm{pt}} = a_{1\ell}(\mu^2) + \sum_{j=1}^{\infty}\tilde{d}_j(\mu^2/Q^2)a_{1\ell}(\mu^2)^{j+1}, \quad (18)$$

then Eq. (16) represents the correct μ^2 dependence of the coefficients, so as to ensure μ^2 independence of the auxiliary quantity $\tilde{\mathcal{D}}(Q^2)$. Phrased differently, the auxiliary quantity (18) is exactly invariant under the combined RScl transformations

$$\tilde{d}_j \to \tilde{d}_j(\mu^2/Q^2) \quad \text{via Eq. (16)}, \qquad a(Q^2) \to a_{1\ell}(\mu^2) \quad \text{via Eq. (17)}. \quad (19)$$

Note that Eq. (17) has the form of a homographic transformation. The latter observation leads to an appropriate way for treating truncated series, which are in general μ^2 dependent due to truncation, in particular $\tilde{\mathcal{D}}(Q^2;\mu^2)^{[2M]}_{\mathrm{pt}}$ (we consider truncated series with an even number of terms). Namely, it is well known in mathematics that the diagonal Padé approximants (dPA's), being ratios of two polynomials (P_M, R_M), both of order M,

$$[M/M](x) = P_M(x)/R_M(X), \quad (20)$$

remain dPA's under the homographic transformation

$$x \mapsto \bar{x} = x/(1 + Kx), \quad (21)$$

(where K is an arbitrary constant). This means that

$$[M/M](\bar{x}) = \mathcal{P}_M(x)/\mathcal{R}_M(x), \quad (22)$$

where $\mathcal{P}_M(x)$ and $\mathcal{R}_M(x)$ are again two polynomials both of order M. More explicitly, if $[M/M]_{\bar{f}}(x)$ is the dPA of a function $\bar{f}(x)$, whose Taylor expansion around $x = 0$ exists ($\bar{f}(x) - [M/M]_{\bar{f}}(x) \sim x^{2M+1}$), then there exists a function $F(\neq \bar{f})$, such that $[M/M]_{\bar{f}}(\bar{x}) = [M/M]_F(x)$. As a consequence, it can be shown that for any function f (with Taylor expansion around $x = 0$), the following identity holds:[3]

$$[M/M]_f(x) = [M/M]_{\bar{f}}(\bar{x}), \quad (23)$$

where $\bar{x} = x/(1 + Kx)$ and $\bar{f}(\bar{x}) = f(x)$. In our case of $\tilde{\mathcal{D}}(Q^2)_{\mathrm{pt}}$ and its expansion (18), we identify: $x = a(Q^2)$, $\bar{x} = a_{1\ell}(\mu^2) = x/(1 + Kx)$ $[K = \beta_0 \ln(\mu^2/Q^2);\ \mu^2 = Q^2 \exp(K/\beta_0)]$, and $\tilde{\mathcal{D}}(Q^2)_{\mathrm{pt}} = f(x) = \bar{f}(\bar{x})$. The latter identification holds because $\tilde{\mathcal{D}}(Q^2)_{\mathrm{pt}} = x + \sum_{j=1}^{\infty}\tilde{d}_j x^{j+1} = \bar{x} + \sum_{j=1}^{\infty}\tilde{d}_j(\mu^2/Q^2)\bar{x}^{j+1}$. The identity (23) means that

[2] The logarithmic derivatives of the coupling and the expansions of the type (10) and (11) were used systematically in Refs. [16,17] (in the context of analytic QCD), and in Ref. [18] (in the context of pQCD).

[3] We have: $f(x) - [M/M]_f(x) \sim x^{2M+1}$, and $\bar{f}(\bar{x}) - [M/M]_{\bar{f}}(\bar{x}) \sim \bar{x}^{2M+1} \sim x^{2M+1}$. Therefore, since $f(x) = \bar{f}(\bar{x})$ and $[M/M]_{\bar{f}}(\bar{x}) = [M/M]_F(x)$, we obtain: $[M/M]_F(x) - [M/M]_f(x) \sim x^{2M+1}$. This implies $[M/M]_f(x) = [M/M]_F(x)$ (i.e., $[M/M]_f(x) = [M/M]_{\bar{f}}(\bar{x})$, Eq. (23)), because the $[M/M](x)$ Padé's are uniquely determined by the coefficients of their expansion in powers x^n for $n \leq 2M$.

dPA's of $\tilde{\mathcal{D}}(Q^2)_{\rm pt}$ have exact independence of the RScl μ^2. Stated differently, when constructing dPA of expansion (18), it does not matter which value of the RScl μ^2 we use in (18).

This fact was noticed by Gardi [19], who, as a result, argued that the truncated perturbation series of the form (3) for physical observables $\mathcal{D}(Q^2)$ can be well approximated by dPA's because the result is approximately RScl independent (i.e., it is exactly RScl independent when the RGE running is approximated to be one-loop). Here, we see that these considerations are valid without approximation for the (RScl-independent) auxiliary quantity $\tilde{\mathcal{D}}(Q^2)$, which is defined via the power series (18). This is related with the fact that the RScl dependence of the coefficients $\tilde{d}_j(\mu^2/Q^2)$ as given by Eq. (16), although involving only β_0 and no higher β_j coefficients, is exact. On the other hand, the RScl dependence of the original coefficients $d_j(\mu^2/Q^2)$ appearing in the power series (3) is more complicated and involves (for $j \geq 2$) higher-loop beta coefficients β_k ($k \leq j-1$), as seen in Eqs. (6)–(8).

The dPA $[M/M]$ of $\tilde{\mathcal{D}}(Q^2)$ has the general form

$$[M/M]_{\tilde{\mathcal{D}}}(a_{1\ell}(\mu^2)) = x\frac{1 + A_1 x + \cdots A_{M-1}x^{M-1}}{1 + B_1 x + \cdots + B_M x^M}\Bigg|_{x=a_{1\ell}(\mu^2)}. \tag{24}$$

We rewrite it by applying a partial fraction decomposition of the fraction on the right-hand side.[4] If we denote the M zeros of the denominator polynomial $(1 + B_1 x + \cdots + B_M x^M)$ by $-1/\tilde{u}_j$ ($j = 1, \ldots, M$), we obtain

$$[M/M]_{\tilde{\mathcal{D}}}(a_{1\ell}(\mu^2)) = \sum_{j=1}^{M} \tilde{\alpha}_j \frac{x}{1 + \tilde{u}_j x}\Bigg|_{x=a_{1\ell}(\mu^2)}, \tag{25}$$

with appropriate "weights" $\tilde{\alpha}_j$ ($j = 1, \ldots, M$). Using Eq. (17) gives us finally

$$[M/M]_{\tilde{\mathcal{D}}}(a_{1\ell}(\mu^2)) = \sum_{j=1}^{M} \tilde{\alpha}_j a_{1\ell}(\tilde{Q}_j^2), \quad \text{where } \tilde{Q}_j^2 = \mu^2 \exp(\tilde{u}_j/\beta_0), \tag{26}$$

i.e., we expressed $[M/M]_{\tilde{\mathcal{D}}}$ as a weighted average of one-loop running couplings defined at specific reference momentum values (gluon virtualities) $\tilde{Q}_j^2$ ($j = 1, \ldots, M$).[5] Since, as argued, the expressions (24)–(26) are exactly independent of the RScl chosen in the original series (18), both the weights $\tilde{\alpha}_j$ and the scales $\tilde{Q}_j^2$ are exactly independent of this RScl.

This observation helps us find an analogous approximant for the true observable $\mathcal{D}$ (or its truncated version $\mathcal{D}^{[2M]}$). By comparing Eq. (10) with (18), we are motivated to define the following approximant:

$$\mathcal{G}_{\mathcal{D}}^{[M/M]}(Q^2) = \sum_{j=1}^{M} \tilde{\alpha}_j a(\tilde{Q}_j^2), \tag{27}$$

i.e., we simply replace in the expression (26) the one-loop running coupling $a_{1\ell}(\tilde{Q}_j^2)$ by the exact (n-loop running, n arbitrary) coupling parameter $a(\tilde{Q}_j^2)$.

The resulting approximant has two important properties:

(1) It is, by sheer construction, exactly RScl invariant (since $\tilde{\alpha}_j$ and $\tilde{Q}_j^2$ are independent of μ^2);

(2) It fulfills the approximation requirement

$$\mathcal{D}(Q^2) - \mathcal{G}_{\mathcal{D}}^{[M/M]}(Q^2) = \mathcal{O}(\tilde{a}_{2M+1}) = \mathcal{O}(a^{2M+1}), \tag{28}$$

i.e., it reproduces the first $2M$ terms of the series (10) and of the series (3). It is relatively straightforward to show the latter fact, by expanding the expression (27) in terms of logarithmic derivatives (see the Appendix).

An approximant of the type (27) was originally introduced in Ref. [14], based on more mathematical considerations. It was called "modified Baker-Gammel approximant" and interpreted as a particularly clever resummation procedure for the physical observable $\mathcal{D}(Q^2)$. In Ref. [14], also a more formal proof of the properties 1 (RScl invariance) and 2 (approximation property) was given. The proof rested on choosing the kernel of the Baker-Gammel approximant to be $k(z, \tilde{u}) = f(\tilde{u})/z$ where $z = a(\mu^2)$, $\tilde{u} = \beta_0 \ln(\tilde{Q}^2/\mu^2)$ and $f(\tilde{u}) = a(\tilde{Q}^2)$.[6] Within the present paper, we constructed the same approximant (27) in a more heuristic and physically motivated manner.

In Ref. [15] we extended the construction of this approximant, so as to be applicable also to the case when an even number of coefficients d_j ($j = 1, \ldots, 2M$) are known in the expansion (3), and in Ref. [22], the method was applied in pQCD.

We can interpret the form (27) as a kind of extension of the previously known scale-setting techniques (principle of minimal sensitivity [23], effective charge method [24] and related approaches [25], and the scale-setting of Brodsky-Lepage-Mackenzie [26] and its extensions [27–29]) to several scales. However, in the presented case, these scales are not fixed by a specific motivated prescription of scale-setting, but are rather based primarily on the successes of diagonal Padé approximants in physics and on the additional requirement of refining the approximate (one-loop)

[4] In MATHEMATICA [20], the command "Apart" achieves this.

[5] In principle, $-1/\tilde{u}_j$'s (and thus $\tilde{Q}_j^2$'s) and $\tilde{\alpha}_j$'s can be sorted into complex conjugate pairs and into real values. In Sec. IV, we apply this approach to the massless Adler function, for which it turns out that all $\tilde{Q}_j^2$ and $\tilde{\alpha}_j$ are real.

[6] For the conventional Baker-Gammel approximants, see, for example, part II of Ref. [21]. Exact RScl invariance of such constructions in the special case of the aforementioned kernel was apparently first shown in Ref. [14].

RScl invariance of the approximant to the exact RScl invariance. These approximants are global, i.e., they go beyond the polynomial form in a, and this is one of the reasons why we expect them to include nonperturbative effects.

Also interesting to note is the connection of our approximant (27) with Neubert's resummation method [30] which is defined by integration over the momentum flow within the running coupling parameter and the connected momentum distribution function $w_{\mathcal{D}}$:

$$\mathcal{D}^{(\mathrm{LB})}(Q^2)_{\mathrm{pt}} = \int_0^\infty dt w_{\mathcal{D}}(t) a(tQ^2 e^{\bar{\mathcal{C}}}). \tag{29}$$

Here, $\bar{\mathcal{C}} = -5/3$ if the "$\overline{\mathrm{MS}}$" convention for the scale Λ_{QCD} is used. When expanding the parameter $a(tQ^2e^{\bar{\mathcal{C}}})$ around $a(\mu^2)$, it turns out that this expression represents exactly the leading-β_0 part (LB) of the "modified perturbation expansion" (10) (cf. Ref. [17], and Eq. (41) later in the present paper). We see that our approximant (27) is equivalent to an approximation of the distribution function $w_{\mathcal{D}}(t)$ in the integrand in (29) in terms of the weighted sum of delta functions

$$w_{\mathcal{D}}(t) \approx \sum_{j=1}^{M} \tilde{\alpha}_j \delta(t - t_j), \tag{30}$$

where the delta peaks are located at t_j's such that $t_j Q^2 e^{\bar{\mathcal{C}}} = \tilde{Q}_j^2$ $(j = 1, \ldots, M)$.

III. APPLICATION TO ANALYTIC QCD MODELS

In general, the perturbative QCD coupling $a(Q^2)$ has a cut in the complex Q^2 plane along the negative semiaxis up to the positive Landau branching point Λ_{L}^2. On the other hand, by the general principles of the local and causal quantum field theory [1,2], the spacelike observables $\mathcal{D}(Q^2)$ (such as the Adler function, sum rules, etc.) must be analytic functions in the Q^2 complex plane with the exception of the cut on the negative semiaxis $Q^2 \in \mathbb{C}\backslash(-\infty, 0]$. This analyticity property, however, is not reflected by the $a(Q^2)$, which has a cut on a part of the positive axis $[0, \Lambda_{\mathrm{L}}^2]$. Therefore, various analytic QCD models have been constructed where the nonanalytic $a(Q^2)$ is replaced by an analytic $\mathcal{A}_1(Q^2)$, which has no singularities for $Q^2 \in \mathbb{C}\backslash(-\infty, 0]$ and, at high $|Q^2| \gg \Lambda^2$, (approximately) agrees with $a(Q^2)$. For details on some of such models, we refer to various references: minimal analytic (MA) model [3–5,31]; modified minimal analytic model [32]; analytic perturbative models [33]; and a specific ("close to perturbative") analytic model [34]. Reviews of analytic QCD models are given in Refs. [6–9]. Calculational techniques applicable to any analytic QCD model (the latter being defined via a specification of $\mathcal{A}_1(Q^2)$ only) are described in Refs. [16,17,35].

It is natural to ask: how do our approximants $\mathcal{G}^{[M/M]}$ fare in such analytic QCD models? As mentioned above, these approximants (27) choose specific scales which, for low-energy observables, are often close to or inside the (unphysical) Landau singularity regime of $a(Q^2)$. Therefore, the hope is that our approximants fare much better or even develop all their potential in analytic QCD models where they look simply as

$$\mathcal{G}_{\mathcal{D}}^{[M/M]}(Q^2; \mathrm{an}) = \sum_{j=1}^{M} \tilde{\alpha}_j \mathcal{A}_1(\tilde{Q}_j^2). \tag{31}$$

The other intriguing aspect is that, in any analytic QCD model,[7] the analytization of the higher powers a^n goes in fact via the analytization of the logarithmic derivatives (9), cf. Refs. [16,17]

$$\tilde{a}_{n+1} \mapsto \tilde{\mathcal{A}}_{n+1} \qquad (n = 0, 1, 2, \ldots), \tag{32}$$

where $\tilde{\mathcal{A}}_{n+1}$ are the logarithmic derivatives of the analytic coupling $\mathcal{A}_1$:

$$\tilde{\mathcal{A}}_{n+1}(Q^2) \equiv \frac{(-1)^n}{\beta_0^n n!} \frac{\partial^n \mathcal{A}_1(Q^2)}{\partial(\ln Q^2)^n}, \qquad (n = 0, 1, 2, \ldots), \tag{33}$$

and not via the naive replacement $a^n \mapsto \mathcal{A}_1^n$.[8] This means that the evaluated observables in analytic QCD have the (truncated) "modified analytic" ("man," in equations) series form analogous to the modified (truncated) perturbation series form in pQCD (10) and (11)

$$\mathcal{D}(Q^2)_{\mathrm{man}} = \mathcal{A}_1(\mu^2) + \sum_{j=1}^{\infty} \tilde{d}_j(\mu^2/Q^2) \tilde{\mathcal{A}}_{j+1}(\mu^2), \tag{34}$$

$$\mathcal{D}(Q^2; \mu^2)_{\mathrm{man}}^{[2M]} = \mathcal{A}_1(\mu^2) + \sum_{j=1}^{2M-1} \tilde{d}_j(\mu^2/Q^2) \tilde{\mathcal{A}}_{j+1}(\mu^2). \tag{35}$$

In view of the presented resummation method (27), this is intriguing, because it shows that the series in logarithmic

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[7] We regard the specification of the coupling function $\mathcal{A}_1(Q^2)$ in the complex Q^2 plane as the full specification of an analytic QCD model.

[8] The analytic analogs $\mathcal{A}_n(Q^2)$ of powers $a(Q^2)^n$ are obtained from the relations $\mathcal{A}_n = \tilde{\mathcal{A}}_n + \sum_{m\geq 1} \tilde{k}_m(n) \tilde{\mathcal{A}}_{n+m}$, where the coefficients $\tilde{k}_m(n)$ are obtained from the corresponding pQCD RGE equations (with $\mathcal{A}_n \mapsto a^n$, $\tilde{\mathcal{A}}_{n+m} \mapsto \tilde{a}_{n+m}$). These relations were presented for any analytic QCD model in Refs. [16,17] in the case of integer n, and in Ref. [35] for noninteger $n = \nu$. The recurrence relations leading to the above relations, for integer n and within the context of the MA model of Refs. [3–5,31], were presented in Refs. [7,36]. Such construction of higher power analogs $\mathcal{A}_n$, not as powers of $\mathcal{A}_1$ but rather as linear (in $\mathcal{A}_1$) operations on $\mathcal{A}_1$, reflects a very desirable functional feature: their compatibility with linear integral transformations (such as Fourier or Laplace) [37]. On the other hand, in linear tranformations, the image of a power is in general not the power of the image.

derivatives of the coupling play a central role both in the mentioned resummation method [cf. Eqs (10) and (18)] and in the evaluation procedure in analytic QCD models [Eqs. (32)–(34)].

The reason for the necessity, in the analytic QCD models, of the evaluation of the observables via Eq. (35) originates from the fact that the unphysical renormalization scheme ("RS," in equations) dependence of the truncated series (35) is[9]

$$\frac{\partial \mathcal{D}(Q^2;\mathrm{RS})^{[N]}_{\mathrm{man}}}{\partial(\mathrm{RS})} = \tilde{k}_N(\mu^2/Q^2)\tilde{\mathcal{A}}_{N+1}(\mu^2) + \mathcal{O}(\tilde{\mathcal{A}}_{N+2})(\sim \mathcal{A}_{N+1}), \quad (\mathrm{RS} = \ln\mu^2; c_2; c_3; \ldots), \tag{36}$$

and from the fact that in analytic QCD models we have the hierarchy $\mathcal{A}_1(\mu^2) > |\tilde{\mathcal{A}}_2(\mu^2)| > |\tilde{\mathcal{A}}_3(\mu^2)| \cdots$ at all complex μ^2. We stress that the expression on the right-hand side of Eq. (36) contains only terms $\tilde{\mathcal{A}}_j(\mu^2)$ ($j \geq N+1$) and no other type of terms. For example, if $\mathrm{RS} = \ln\mu^2$, the right-hand side of Eq. (36) is exactly $-\beta_0 N \tilde{d}_{N-1}(\mu^2/Q^2)\tilde{\mathcal{A}}_{N+1}(\mu^2)$. If we performed the evaluation by the replacement $a^n \mapsto \mathcal{A}_1^n$ ($n \geq 2$), the resulting truncated analytic power series,

$$\mathcal{D}(Q^2;\mathrm{RS})^{[2M]}_{\mathrm{anTPS}} = \mathcal{A}_1(\mu^2) + \sum_{j=1}^{2M-1} d_j(\mu^2/Q^2)\mathcal{A}_1(\mu^2)^{j+1}, \tag{37}$$

would possess, in general, an increasingly strong RSch dependence when the order of the truncation N increases

$$\frac{\partial \mathcal{D}(Q^2;\mathrm{RS})^{[N]}_{\mathrm{anTPS}}}{\partial(\mathrm{RS})} = k_N(\mu^2/Q^2)\mathcal{A}_1^{N+1}(\mu^2) + \mathcal{O}(\mathcal{A}_1^{N+2}) + \mathrm{NP}_N, \tag{38}$$

where the terms NP_N denote nonperturbative terms ($\sim (\Lambda^2/\mu^2)^k$), which in general become more complicated and increase in their value when N increases. The origin of such terms is the difference $\mathcal{A}_1(\mu^2) - a(\mu^2) \sim (\Lambda^2/\mu^2)^m$ at $\mu^2 > \Lambda^2$.

It is evident that our approximant in analytic QCD, Eq. (31), is RScl invariant (since $\tilde{\alpha}_j$ and $\tilde{Q}_j^2$ are). Furthermore, in complete analogy with the pQCD case, we can show that it fulfills the approximation requirement analogous to Eq. (28):

$$\mathcal{D}(Q^2)_{\mathrm{man}} - \mathcal{G}^{[M/M]}_{\mathcal{D}}(Q^2;\mathrm{an}) = \mathcal{O}(\tilde{\mathcal{A}}_{2M+1}), \tag{39}$$

where the right-hand side has only terms of the form $\tilde{\mathcal{A}}_j(Q^2)$ ($j \geq 2M+1$). The relation (39), together with the aforementioned hierarchy of $\tilde{\mathcal{A}}_j$'s in analytic QCD, gives us additional hope that our approximants (31) will give us values increasingly close to the full value $\mathcal{D}(Q^2)_{\mathrm{man}}$, Eq. (34), in any chosen analytic QCD model. We will see in the next section, on the example of the Adler function at low momenta ($Q^2 = 2\ \mathrm{GeV}^2$), that this hope is well grounded.

[9]The relation (36) can be obtained in complete analogy with the perturbative QCD, under the correspondence (32).

IV. NUMERICAL CHECKS OF THE QUALITY OF THE APPROXIMANTS

In this Section we will investigate how our approximants (31) [and (27)] work when applied to a spacelike QCD observable whose perturbation series is known to a sufficiently high order. Specifically, we will consider the massless Adler function $\mathcal{D}(Q^2)$ at low Q^2 ($Q^2 = 2\ \mathrm{GeV}^2$) and perform numerical evaluations of our approximants both in pQCD and in three different analytic QCD (anQCD) models, namely:

(i) MA model of Refs. [3–5];
(ii) the approximately perturbative anQCD model of Ref. [34] ("CCEM" model);
(iii) the perturbative anQCD model type "EE" (whose beta function involves exponential functions) in two variants, of Ref. [33].

The characteristics of these different models will be specified in more detail later in this section. Beforehand, we sketch the general procedure: we will consider first the LB resummation part of $\mathcal{D}$ whose expression in pQCD is

$$\mathcal{D}^{(\mathrm{LB})}(Q^2)_{\mathrm{pt}} = \int_0^\infty \frac{dt}{t} F_{\mathcal{D}}(t) a(tQ^2 e^{\bar{\mathcal{C}}}) \tag{40}$$

$$= a(Q^2) + \tilde{d}_{1,1}\beta_0 \tilde{a}_2(Q^2) + \cdots + \tilde{d}_{n,n}\beta_0^n \tilde{a}_{n+1}(Q^2) + \cdots. \tag{41}$$

Here, $F_{\mathcal{D}}(t) \equiv w_{\mathcal{D}}(t)t$ is the characteristic function of the Adler function, whose explicit expression was obtained in Ref. [30] on the basis of the leading-β_0 expansion coefficients $d_n^{(\mathrm{LB})} = \tilde{d}_n^{(\mathrm{LB})} = \tilde{d}_{n,n}\beta_0^n$ obtained from the leading-β_0 Borel transform of Refs. [38,39] at RScl $\mu^2 = Q^2$ in the "V" scale convention.[10] The coefficient $\tilde{d}_n^{(\mathrm{LB})}$ represents simultaneously the leading-β_0 part of $\tilde{d}_n$ and of d_n once these two coefficients are organized in series in powers of n_f and thus of β_0; $\tilde{d}_n^{(\mathrm{LB})}$ is RSch independent but RScl dependent (see also Eq. (16); for details, see Ref. [17]).

The evaluations will be performed in the simplest renormalization scheme $c_2 = c_3 = \cdots = 0$ in various QCD models (pQCD and anQCD's, except the anQCD

[10]Ref. [30] uses the notation $\hat{D}(t) = 4F_{\mathcal{D}}(t)/t$. Note that we use throughout the $\overline{\mathrm{MS}}$ convention for the scale Λ, i.e., $\mathcal{C} = \bar{\mathcal{C}} = -5/3$. Large-$\beta_0$ calculations are usually performed with V scale convention, i.e., $\mathcal{C} = 0$. The relations between the two, at a given RScl μ^2 (e.g., $\mu^2 = Q^2$), are: $\tilde{d}_{n,n}(\bar{\Lambda}) = \tilde{d}_{n,n}(\Lambda_{\mathrm{V}}) + \sum_{k=1}^{n-1}(n!/(k!(n-k)!)(-\bar{\mathcal{C}})^k \tilde{d}_{n-k,n-k}(\Lambda_{\mathrm{V}}) + (-\bar{\mathcal{C}})^n$.

model EE). This is convenient because the expressions are then simple and explicitly related with the Lambert function [12,40]. As the point of reference we take the value $a(M_Z^2, \overline{\mathrm{MS}}) = 0.119/\pi$. This then corresponds to the value $a(\mu_{\mathrm{in}}^2; n_f = 3; c_2 = c_3 = \cdots = 0) \approx 0.2215/\pi$ at the "initial" chosen scale $\mu_{\mathrm{in}} = 3m_c = 3.81$ GeV.

We will assume that $n_f = 3$ in our calculations. At $Q^2 = 2\ \mathrm{GeV}^2$ we obtain $a(2\ \mathrm{GeV}^2) = 0.3479/\pi$.

The practical evaluations can be performed by choosing any value of RScl μ^2, e.g. $\mu^2 = Q^2$. In the leading-β_0 case the choice $\mu^2 = Q^2$ means using the coefficients $\tilde{d}_{n,n} \equiv \tilde{d}_{n,n}(\mu^2/Q^2 = 1)$ in the expansion (41). Nonetheless, as shown, the use of different RScl $\mu^2 \neq Q^2$ gives us identical results, as can be checked numerically as well. We note that by choosing $\mu^2 = Q^2$, the coefficients $d_j \equiv d_j(1)$, $\tilde{d}_j \equiv \tilde{d}_j(1)$ are Q^2 independent. Therefore, the weight coefficients $\tilde{\alpha}_j$ and parameters $\tilde{u}_j$ in Eqs. (27) and (31) are Q^2 independent (when $\mu^2 = Q^2$), and thus the ratio of scales $\tilde{Q}_j^2/Q^2 = \exp(\tilde{u}_j/\beta_0)$ [see Eq. (26), with $\mu^2 = Q^2$] will be Q^2 independent (and, of course, μ^2 independent). In Table I, we give the values of weights $\tilde{\alpha}_j$ and scale ratios $\tilde{Q}_j^2/Q^2$ for various indices M of our approximants. We can see from the table that the scale ratios $\tilde{Q}_j^2/Q^2$ get increasingly spread out when the order index M increases. However, for those ratios which are much smaller or much larger than unity, the corresponding weight factors are small.

The authors of Ref. [41] applied the diagonal Padé approximants to the (auxiliary) power series quantity $\tilde{\mathcal{D}}^{(\mathrm{LB})}(Q^2)_{\mathrm{pt}}$ (at $Q^2 = 2\ \mathrm{GeV}^2$) obtained from the series (41) by the replacement $\tilde{a}_{n+1} \mapsto a^{n+1}$ (the approximation of one-loop RGE running), and compared it with the result of the integration (40) obtained by assuming one-loop RGE running of $a(tQ^2 e^{\tilde{C}})$; the integral is ambiguous in the integration at low t (IR regime) due to the Landau singularity, so they chose the principal value for the integration.

The results of this type of (one-loop) evaluation are given in Table II, for the case $Q^2 = 2\ \mathrm{GeV}^2$. We fix the one-loop running coupling $a_{1\ell}(Q'^2)$ so that it agrees with the aforementioned full a at $Q^2 = 2\ \mathrm{GeV}^2$: $a_{1\ell}(Q^2) = a(Q^2) = 0.3479/\pi$. In addition, we include in the table the corresponding results with the full pQCD evaluation in the $c_2 = c_3 = \cdots 0$ renormalization scheme ("two-loop"), which uses in the integral (40) the full pQCD $a(tQ^2 e^{\tilde{C}})$ and our approximants (27). We can see that the dPA's (in the one-loop case) and our approximants (27) oscillate rather erratically around the corresponding principal value. This has to do with the fact that, at higher order index M ($M \geq 3$) the scales $\tilde{Q}_j^2$ come rather close to the Landau singularity of the running perturbative coupling. In fact, the approximants become even complex in the full case once at least one of the scales $\tilde{Q}_j^2$ hits the unphysical cut $(0, \Lambda_{\mathrm{L}}^2)$ (where: $\Lambda_{\mathrm{L}}^2 \approx 0.150\ \mathrm{GeV}^2$, i.e., $\Lambda_{\mathrm{L}} \approx .388$ GeV), since $a(\tilde{Q}_j^2)$ becomes complex. In the one-loop case, we have a simple Landau pole instead of the cut (with $\Lambda_{\mathrm{L}}^2 \approx 0.036\ \mathrm{GeV}^2$, i.e., $\Lambda_{\mathrm{L}} \approx 0.190$ GeV), so the approximants would remain real even when one of the scales were below the Landau pole. In the parentheses, the results of the corresponding truncated series are given—for the one-loop case the truncated version $\tilde{\mathcal{D}}(Q^2; \mu^2)_{\mathrm{pt}}^{[2M]}$ of the expansion (18), and in the full (loop) case the truncated version $\mathcal{D}(Q^2; \mu^2)_{\mathrm{mpt}}^{[2M]}$ Eq. (11), both with RScl $\mu^2 = Q^2$. We see that these truncated series behave, in general, worse than the resummed versions, and show for larger M asymptotic divergence (in the one-loop case for $M \geq 5$, and in the full loop case for $M \geq 4$).

TABLE I. The weight coefficients $\tilde{\alpha}_j$ and the scale ratios $\tilde{Q}_j^2/Q^2$ for our RScl-invariant approximants, Eqs. (27) and (31), for various order indices ($M = 1, 2, 3, 4$), in the case of leading-β_0 massless Adler function.

M	$(\tilde{\alpha}_1; \tilde{Q}_1^2/Q^2)$	$(\tilde{\alpha}_2, \tilde{Q}_2^2/Q^2)$	$(\tilde{\alpha}_3, \tilde{Q}_3^2/Q^2)$	$(\tilde{\alpha}_4, \tilde{Q}_4^2/Q^2)$
$M = 1$	(1; 0.5001)	$\cdots$	$\cdots$	$\cdots$
$M = 2$	(0.6948; 0.1711)	(0.3052; 5.771)	$\cdots$	$\cdots$
$M = 3$	(0.3579; 0.07969)	(0.6011; 1.0534)	(0.041; 85.77)	$\cdots$
$M = 4$	(0.1376; 0.03803)	(0.6821; 0.3862)	(0.1767; 17.16)	(0.0037; 1518)

TABLE II. The results of the one-loop approach: dPA ([M/M]) with increasing index M, at $Q^2 = 2\ \mathrm{GeV}^2$, for the leading-β_0 massless Adler function $\mathcal{D}(Q^2)$. For comparison, the result of the principal value (PV) of integration (with the estimated IR renormalon ambiguity) is included. In addition, the approximants (27) in the case of full pQCD running $a(tQ^2 e^{\tilde{C}})$ are included, and so is the corresponding principal value. In the parentheses, the corresponding results of the truncated series (41) are given (with RScl $\mu^2 = Q^2$). See the text for details.

Case	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$	$M = 6$	PV
1-loop	0.134(0.13)	0.161(0.155)	0.175(0.164)	0.194(0.16)	$-0.497(0.08)$	$0.156(-0.714)$	0.178 ± 0.02
full	0.14(0.134)	0.2(0.174)	0.532(0.198)	$0.095 - 0.051i(0.107)$	$0.162 - 0.009i\ (-1.79)$	$0.25 - 0.001i\ (-39.8)$	0.174 ± 0.02

There are several analytic QCD models (for $\mathcal{A}_1(Q^2)$) in the literature. The most used one is the model of Shirkov, Solovtsov and Milton [3–5], which keeps for the cut of $\mathcal{A}_1(Q^2)$ on the negative Q^2 axis the discontinuity function of the pQCD coupling $a(Q^2)$, and the unphysical pQCD cut on the positive axis is eliminated:

$$\mathcal{A}_1^{(\mathrm{MA})}(Q^2) = \frac{1}{\pi}\int_0^\infty d\sigma \frac{\rho_1^{(\mathrm{pt})}(\sigma)}{\sigma + Q^2}, \tag{42}$$

where $\rho_1^{(\mathrm{pt})}(\sigma) = \mathrm{Im}a(Q'^2 = -\sigma - i\epsilon)$. This represents, in a sense, the minimal changes (in the cut) with respect to pQCD. Therefore, we call this model the minimal analytic.[11] The only adjustable parameter there is the scale $\bar{\Lambda}$ (in the $\overline{\mathrm{MS}}$ scale convention). In order to reproduce QCD phenomenology at high energies, the value of this scale at $n_f = 5$ in MA is about 260 MeV, which corresponds at $n_f = 3$ to the value of $\bar{\Lambda} \approx 415$ MeV [9]. We will use this value in MA and will use there also the RSch $c_2 = c_3 = \cdots = 0$.

Another analytic QCD model is described in Ref. [34] (CCEM). It differs from MA in the sense that the discontinuity function $\rho_1(\sigma) = \mathrm{Im}\mathcal{A}_1(-\sigma - i\epsilon)$ differs from the pQCD discontinuity function at low $\sigma \lesssim 1\ \mathrm{GeV}^2$ where it is replaced by a delta function. The spacelike coupling $\mathcal{A}_1$ is then

$$\mathcal{A}_1(Q^2) = \frac{f_1^2}{u + s_1} + \frac{1}{\pi}\int_{s_0}^\infty ds \frac{r_1^{(\mathrm{pt})}(s)}{s + u}, \tag{43}$$

where $u \equiv Q^2/\Lambda_{\mathrm{W}}^2$, $s \equiv \sigma/\Lambda_{\mathrm{W}}^2$, $r_1^{(\mathrm{pt})}(s) \equiv \rho_1^{(\mathrm{pt})}(\sigma)$ (in the RSch $c_2 = c_3 = \cdots = 0$), and $\Lambda_{\mathrm{W}} \approx 0.487$ GeV is the scale appearing in the Lambert function $W_{\mp 1}(z_\pm)$. The scale Λ_{W} was fixed basically by the requirement that the high-energy QCD phenomenology be reproduced. The (dimensionless) free parameters (f_1^2, $s_1 \equiv M_1^2/\Lambda_{\mathrm{L}}^2$, $s_0 \equiv M_0^2/\Lambda_{\mathrm{L}}^2$) are fixed in the model in such a way that at high Q^2 the model merges with the pQCD coupling to a high degree of accuracy [$\mathcal{A}_1(Q^2) - a(Q^2) \sim (\Lambda^2/Q^2)^3$] and that, simultaneously, it reproduces the measured value of the semihadronic (massless and strangeless) tau decay ratio[12] $r_\tau(\triangle S = 0, m_q = 0)_{\mathrm{exp}} = 0.203 \pm 0.004$. We note that in MA, we have $\mathcal{A}_1^{(\mathrm{MA})}(Q^2) - a(Q^2) \sim (\Lambda^2/Q^2)$, i.e., at high energies this difference is not quite negligible, and the predicted value of $r_\tau(\triangle S = 0, m_q = 0)$ is about 0.14.

[11] In the literature, it is usually called Analytic Perturbation Theory, and it then involves a specific construction of the analytic analogs of higher powers a^n. The construction can be applied only in MA, and it is in such a case equivalent to the construction presented in Refs. [16,17] (the latter construction being applicable to any anQCD).

[12] We use the variant of the model with the value of $s_0 = 3.858$, which reproduces the measured value of r_τ when the leading-β_0 resummation and the inclusion of the known beyond-the-leading-β_0 terms are performed in the evaluation of r_τ.

Yet another analytic QCD model which we will use is the so called EE model of Ref. [33], which is in fact a fully perturbative analytic QCD model [the $\beta(a)$ function is analytic function of $\mathcal{A}_1(Q^2) \equiv a(Q^2)$ at $a = 0$].[13] The beta function has the Ansatz

$$\beta(a) = -\beta_0 a^2 (1 - Y) f(Y)|_{Y \equiv a/a_0}, \tag{44}$$

where $a_0 = a(Q^2 = 0)$ is a finite value (infrared fixed point), $f(Y)$ is analytic at $Y = 0$, and we require analyticity of $a(Q^2)$ at $Q^2 = 0$, which turns out to give the condition $a_0\beta_0 f(1) = 1$. The expansion of $\beta(a)$ in powers of a also has to reproduce the first two universal coefficients β_0 and β_1, cf. Eq. (5). There are at least two variants of the mentioned EE model. In the first variant (''EEv1'') the function $f(Y)$ in the beta function is a combination of (rescaled and translated) functions $(e^Y - 1)/Y$ and $Y/(e^Y - 1)(e^Y - 1)/Y$ and $Y/(e^Y - 1)$:

$$\mathrm{EEv\,1}: f(Y) = \frac{(\exp[-k_1(Y - Y_1)] - 1)}{[k_1(Y - Y_1)]} \times \frac{[k_2(Y - Y_2)]}{(\exp[-k_2(Y - Y_2)] - 1)} \mathcal{K}(k_1, Y_1, k_2, Y_2), \tag{45}$$

where the constant $\mathcal{K}$ ensures the required normalization $f(Y = 0) = 1$. In this variant we have, at first, five real parameters: $a_0 \equiv a(Q^2 = 0)$ and Y_j, k_j ($j = 1, 2$). Two parameters (Y_2 and a_0) are eliminated by the aforementioned conditions: $a_0\beta_0 f(1) = 1$ and the reproduction of the universal β_1 coefficient. The other three parameters are approximately fixed by the condition of analyticity of $a(Q^2)$ and the requirement of obtaining as high a value of $r_\tau(\triangle S = 0, m_q = 0)$ as possible (it is always too low in comparison to the experimental value 0.203 ± 0.004). The obtained values are: $Y_1 = 0.1$, $k_1 = 10.0$, and$k_2 = 11.0$. This results in $a_0 = 0.236$ and the highest possible value $r_\tau(\triangle S = 0, m_q = 0) \approx 0.15$. This latter value is still clearly too low.

The second version (''EEv2'') has the function $f(Y)$ in the beta function modified, in comparison to EEv1, by a factor f_{fact}

$$\mathrm{EEv2}: f_{\mathrm{EEv2}}(Y) = f_{\mathrm{EEv1}}(Y) f_{\mathrm{fact}}(Y), \tag{46}$$

$$\text{with} \quad f_{\mathrm{fact}}(Y) = \frac{(1 + BY^2)}{(1 + (B + K)Y^2)} \qquad (1 \ll K \ll B). \tag{47}$$

This factor has the values of K and B adjusted so that the expansion of the evaluation of $r_\tau(\triangle S = 0, m_q = 0)$, by the inclusion of the LB contribution and of the first three beyond-the-leading-β_0 (bLB) contributions, gives the

[13] Our general construction of $\mathcal{A}_n(Q^2)$ gives in such models: $\mathcal{A}_n = \mathcal{A}_1^n$, as it should be.

TABLE IV. Evaluations of the full massless Adler function in various analytic QCD models, using our RScl-invariant approximants (31) for $M = 1, 2$, at $Q^2 = 2\ \mathrm{GeV}^2$. For comparison, two other evaluations (LB + bLB and TMAn) are included. See the text for details.

model	$M = 1$	$M = 2$	LB + bLB	TMAn
MA	0.1175	0.1196	0.1191	0.1199
CCEM	0.1389	0.1535	0.1528	0.1541
EEv1	0.107	0.1164	0.1183	0.1195
EEv2	0.0972	0.139	0.1584	0.1587

correct r_τ value: $r_\tau(\triangle S = 0, m_q = 0) = 0.203$ ($\Rightarrow B = 1000$ and $K = 5.4$). The factor $f_{\mathrm{fact}}(Y)$ does not destroy the analyticity of $a(Q^2)$, and it does not change substantially the values of $a(Q^2)$, since it is close to the value one for most Y's. However, the price that we pay is high nonetheless: the coefficients $c_j \equiv \beta_j/\beta_0$ of the expansion of the modified beta function are extremely high for $j \geq 4$ ($c_j \gtrsim 10^6$ for $j \geq 4$), implying strong divergence of any evaluation series of observables (including r_τ) when bLB terms of $\sim a^n$ with $n \geq 5$ are included. The factor $f_{\mathrm{fact}}(Y)$ introduces singularities of $\beta(a)$ at rather small values of $|a|$.

For more details on the models CCEM (with $s_0 = 3.858$) and EEv1 and EEv2, we refer to Refs. [33,34], respectively.

The results of our approximants (31) in these analytic QCD models, for the leading-β_0 part of the Adler function $\mathcal{D}(Q^2)$ at $Q^2 = 2\ \mathrm{GeV}^2$, are presented in Table III. For comparison, the exact integrated values

$$\mathcal{D}^{(\mathrm{LB})}_{\mathrm{an}}(Q^2) = \int_0^\infty \frac{dt}{t} F_{\mathcal{D}}(t) \mathcal{A}_1(tQ^2 e^{\bar{C}}) \qquad (48)$$

are also given in the table. Note that the leading-β_0 integration, Eq. (48), has now no ambiguities since no Landau singularities exist, in contrast to the pQCD case (40). Incidentally, the expansion of Eq. (48) is completely analogous to the pQCD expansion (41)

$$\mathcal{D}^{(\mathrm{LB})}(Q^2)_{\mathrm{man}} = \mathcal{A}_1(Q^2) + \tilde{d}_{1,1}\beta_0 \tilde{\mathcal{A}}_2(Q^2) + \cdots + \tilde{d}_{n,n}\beta_0^n \tilde{\mathcal{A}}_{n+1}(Q^2) + \cdots. \qquad (49)$$

In parentheses, we give the results of the corresponding truncated version of the series (49), i.e., $\mathcal{D}^{(\mathrm{LB})}(Q^2)^{[2M]}_{\mathrm{man}}$, with $\mu^2 = Q^2$, for each M. We see in the table that our approximants converge systematically and fast to the exact values when the order index M increases. The truncated series, on the other hand, have divergent behavior, which starts manifesting itself at large M's ($M \geq 7$ in the MA case; $M \geq 5$ in the CCEM and EE cases), since these are analytic QCD models. Despite this divergence, the aforementioned hierarchy of the couplings $|\tilde{\mathcal{A}}_k(Q^2)| > |\tilde{\mathcal{A}}_{k+1}(Q^2)|$, in general, turns out to be true for all relevant indices k in the table ($k = 1, \ldots, 13$), at $Q^2 = 2\ \mathrm{GeV}^2$.

The first three coefficients d_j ($j = 1, 2, 3$) are now exactly known for the Adler function [42–44]. Therefore, we can construct our approximants (31) for the order indices $M = 1$ and $M = 2$ on the basis of these exact four coefficients. The results of this calculation, for the three analytic QCD models, are presented in Table IV. For comparison, we also include the results of the truncated modified analytic (TMAn) series (35), with $\mu^2 = Q^2$ and the more refined "LB+bLB" evaluation which takes into account the leading-β_0 resummation contribution (48) and the three additional known terms (bLB)

$$\mathcal{D}^{(\mathrm{bLB})}(Q^2)_{\mathrm{man}} = \sum_{n=1}^{3} (\tilde{d}_n - \tilde{d}_{n,n}\beta_0^n)\tilde{\mathcal{A}}_{n+1}(Q^2). \qquad (50)$$

We can see that our approximants (31), with index $M = 2$, represent a competitive evaluation of the observable, especially when comparing with the (partially) resummed results LB + bLB and the TMAn).

The results of our method with $M = 2$, in the MA and CCEM cases, deviate from the LB + bLB results less than the TMAn results deviate. Since the analytic models MA and CCEM are in "tame" RSch's [i.e., the ones where the RSch parameters c_j ($j \geq 2$) are very small, in fact, zero], we can expect that both the LB + bLB and TMAn approaches give good estimates of the true value in the model, and that LB + bLB is probably a better approach since it uses significantly more input information than TMAn. However, we recall that our $M = 2$ approximants use as little input information as the TMAn approach, i.e., the first three d_j's, and yet Table IV indicates that our approximants with $M = 2$ are competitive with the LB + bLB approach in the MA and CCEM models.

TABLE III. Evaluations of the leading-β_0 massless Adler function $\mathcal{D}^{(\mathrm{LB})}(Q^2)$ in various analytic QCD models, using our RScl-invariant approximants (31), with increasing index M, at $Q^2 = 2\ \mathrm{GeV}^2$. For comparison, the exact result of the integration (48) is included. In parentheses in the table, the values of the corresponding truncated series $\mathcal{D}^{(\mathrm{LB})}(Q^2)^{[2M]}_{\mathrm{man}}$ are given (with RScl $\mu^2 = Q^2$). See the text for details.

model	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$	$M = 6$	$M = 7$	exact
MA	0.1167(0.1147)	0.1222(0.1214)	0.1217(0.1208)	0.1217(0.1205)	0.1217(0.1211)	0.1217(0.1209)	0.1217(0.1174)	0.1217
CCEM	0.1371(0.1321)	0.1649(0.164)	0.165(0.1733)	0.1617(0.1788)	0.1624(−0.0048)	0.1632(−0.0407)	0.1626(11.7)	0.1627
EEv1	0.1062(0.1047)	0.1141(0.1144)	0.1136(0.1146)	0.1131(0.1138)	0.1132(0.1063)	0.1133(0.0842)	0.1133(12.42)	0.1133
EEv2	0.0965(0.0952)	0.1036(0.1035)	0.1035(0.1041)	0.1032(0.1041)	0.1032(0.1018)	0.1032(0.084)	0.1032(−0.4615)	0.1032

On the other hand, the RSch coefficients c_j are increasing fast in the models EEv1 and dramatically fast in EEv2. In that case, the coefficients $\tilde{d}_j$ and $(\tilde{d}_j - \tilde{d}_{j,j}\beta_0^j)$, which depend on c_j via an additive term $-c_j/(j-1)$ (if $j \geq 2$), increase very fast when j increases, so that TMAn and LB + bLB approaches become uncertain.[14] We notice that in the case of EEv2, our approximant (for $M = 2$) is essentially different from the LB + bLB and from the TMAn result. The TMAn series (35) and the truncated bLB series (50) become, in that case, very divergent once we include the terms $\tilde{\mathcal{A}}_{n+1}$ with $n \geq 4$ (cf. Ref. [33] for further details on the divergence of the coefficients $\tilde{d}_n$ in this case). In that case, our approximants, for $M = 2$, are probably comparatively the most reliable estimate of the true result in the EEv2 model.

V. CONCLUSIONS

We tested in various analytic QCD models an earlier-developed [14,15] RScl-invariant resummation method, by applying it to the evaluation of the massless Adler function $\mathcal{D}(Q^2)$ at low energy ($Q^2 = 2\ \mathrm{GeV}^2$). The method is global, i.e., nonpolynomial in the (analytic) coupling parameter. It is related with the method of dPA's, representing an extension of the dPA method by achieving exact RScl independence. The method, applied to spacelike observables, results in a linear combination of coupling parameters at several spacelike momentum scales (each of them RScl-invariant), and thus represents an extension of the well-known scale-setting techniques of Stevenson [23], Grunberg [24], and Brodsky-Lepage-Mackenzie [26]. For observables with low scale Q^2 of the process, the method, when applied within the perturbative QCD, is not very efficient in practice. The reason for this is that the perturbative QCD coupling $a(Q^2)$ has unphysical (Landau) singularities at low positive Q^2, and some of the scales of our approximant turn out to be close or even within this singularity sector. On the other hand, the method turns out to be very efficient in analytic QCD models, because the analytic coupling $\mathcal{A}_1(Q^2)$ has no unphysical singularities. In the case of the leading-β_0 part of the Adler function, the results of the method converge very fast to the exact result within each analytic QCD model. Furthermore, when the method is applied to the truncated (analytic) series of the entire Adler function, whose first three coefficients beyond the leading order are known exactly, the result of the method becomes competitive with the result of the sum of the (exact) LB contribution and the truncated bLB analytic series, although the latter method (LB + bLB) uses significanly more input information than our method. We conclude that our method is at the moment probably the best method, in the analytic QCD frameworks, for the evaluation of spacelike observables when the evaluation is based on the known part of the truncated integer power perturbation series of the observable. The method can be used also for the evaluation of timelike observables (such as the cross section of e^+e^- scattering into hadrons, and semihadronic τ decay ratio r_τ) when the latter are expressed as contour integrals involving spacelike observables.

[14]The $c_j \equiv \beta_j/\beta_0$ coefficients in EEv2 are: $-106.8 (j = 2)$; $326.7 (j = 3)$; $1.72 \cdot 10^6 (j = 4)$; $3.08 \cdot 10^6 (j = 5)$, etc.

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APPENDIX: THE APPROXIMATION REQUIREMENT

Here we show that the approximation requirement, Eq. (28), is fulfilled by our approximant (27). Taylor-expanding $a(\tilde{Q}_j^2)$'s in the approximant around $\ln(\mu^2)$, by using the definitions (9), we obtain

$$\begin{aligned}\mathcal{G}_{\mathcal{D}}^{[M/M]}(Q^2) &= \sum_{j=1}^{M} \tilde{\alpha}_j a(\tilde{Q}_j^2) \\ &= \sum_{j=1}^{M} \tilde{\alpha}_j \sum_{k=0}^{\infty} \tilde{a}_{k+1}(\mu^2)(-\beta_0 \ln(\tilde{Q}_j^2/\mu^2))^k \quad \text{(A1)} \\ &= \sum_{k=0}^{\infty} \tilde{a}_{k+1}(\mu^2) \sum_{j=1}^{M} \tilde{\alpha}_j(-\beta_0 \ln(\tilde{Q}_j^2/\mu^2))^k \\ &= \sum_{k=0}^{\infty} \tilde{a}_{k+1}(\mu^2) \sum_{j=1}^{M} \tilde{\alpha}_j(-\tilde{u}_j)^k. \quad \text{(A2)}\end{aligned}$$

In the last equation we used the fact that $\tilde{u}_j = \beta_0 \ln(\tilde{Q}_j^2/\mu^2)$, see Eqs. (25) and (26). However, Eqs. (24)–(26) and (18) tell us that

$$\tilde{\mathcal{D}}(Q^2)_{\mathrm{pt}} - [M/M]_{\tilde{\mathcal{D}}}(a_{1\ell}(\mu^2)) = \mathcal{O}(a_{1\ell}(\mu^2)^{2M+1}). \quad \text{(A3)}$$

This implies that the expansion of $[M/M]_{\tilde{\mathcal{D}}}(x)$ in powers of $x = a_{1\ell}(\mu^2)$ reproduces[15] the coefficients at powers of x^n for $n = 1, \ldots, 2M$ in the expansion of $\tilde{\mathcal{D}}(Q^2)$, Eq. (18):

$$\sum_{j=1}^{M} \tilde{\alpha}_j(-\tilde{u}_j)^k = \tilde{d}_k(\mu^2/Q^2) \quad \text{for } k = 0, 1, \ldots, 2M-1. \quad \text{(A4)}$$

Note that $\tilde{d}_0(\mu^2/Q^2) \equiv 1$. Inserting the indentities (A4) into Eq. (A2), we obtain

[15]This is the expansion of the expression (25) in powers of x.

$$\mathcal{G}_{\mathcal{D}}^{[M/M]}(Q^2) = \sum_{k=0}^{2M-1} \tilde{a}_{k+1}(\mu^2)\tilde{d}_k(\mu^2/Q^2) + \mathcal{O}(\tilde{a}_{2M+1}). \tag{A5}$$

This, in combination with the expansion (10) of the observable $\mathcal{D}(Q^2)$ in $\tilde{a}_{k+1}(\mu^2)$, gives us immediately

$$\mathcal{D}(Q^2)_{\rm mpt} - \mathcal{G}_{\mathcal{D}}^{[M/M]}(Q^2) = \mathcal{O}(\tilde{a}_{2M+1}) = \mathcal{O}(a^{2M+1}), \tag{A6}$$

i.e., the approximation identity (28). The same proof can be repeated in analytic QCD models (except for the notational change $\tilde{a}_{k+1} \mapsto \tilde{\mathcal{A}}_{k+1}$), i.e., the approximation identity (39) is also valid.

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