

**SU(3) breaking in charmless  $B$  decays**Maxime Imbeault<sup>1,\*</sup> and David London<sup>2,†</sup><sup>1</sup>*Département de Physique, Cégep de Baie-Comeau, 537 Boulevard Blanche, Baie-Comeau, Quebec, Canada G5C 2B2*<sup>2</sup>*Physique des Particules, Université de Montréal, C.P. 6128, Succursale Centre-Ville, Montréal, Quebec, Canada H3C 3J7*

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There are many charmless  $B$  decay pairs whose amplitudes are related by U spin ( $d \leftrightarrow s$ ) or flavor SU(3). The theoretical uncertainty in any analysis involving such pairs must take into account U-spin/SU(3) breaking. In the past, such considerations generally used theoretical input, but we show that this can be experimentally measured. We present lists of two- and three-body decay pairs from which the size of the breaking can be obtained. We detail the values of U-spin/SU(3) breaking given by the present experimental data. One pair— $B_d^0 \rightarrow \pi^+ \pi^-$  and  $B_d^0 \rightarrow \pi^- K^+$ —may exhibit large nonfactorizable breaking, depending on the value taken for the  $B_d^0 \rightarrow \pi^+ \pi^-$  direct  $CP$  asymmetry. We present other signals of SU(3) breaking in two- and three-body decays, and discuss further tests for nonfactorizable effects. Finally, we also point out that the pure-penguin decay  $B_s^0 \rightarrow \bar{K}^0 \bar{K}^0 K^0$  is intriguing because it can be used to cleanly probe the  $B_s^0$ - $\bar{B}_s^0$  mixing phase.

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**I. INTRODUCTION**

In the standard model (SM),  $CP$  violation is due to the presence of a nonzero complex phase in the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix  $V$ . This phase information is elegantly displayed in the unitarity triangle, in which the  $CP$ -violating interior angles are  $\alpha$ ,  $\beta$ , and  $\gamma$  [1]. By measuring these  $CP$  phases in many different ways, one can test the SM.

Much theoretical work has gone into elucidating clean methods for obtaining  $\alpha$ ,  $\beta$ , and  $\gamma$  from  $B$  decays. In 1999, it was pointed out that, apart from CKM matrix elements, the amplitudes for the decays  $B_d^0 \rightarrow \pi^+ \pi^-$  and  $B_s^0 \rightarrow K^+ K^-$  are equal under U-spin symmetry ( $d \leftrightarrow s$ ) [2]. With one additional piece of information, the phase  $\gamma$  can be obtained. Subsequently, all  $B$  decay pairs that are related by U spin were tabulated [3], and another method for extracting weak-phase information using a different U-spin pair ( $B_s^0 \rightarrow \pi^+ K^-$  and  $B_d^0 \rightarrow \pi^- K^+$ ) was proposed [4].

In order to determine the theoretical uncertainty of a particular method, it is necessary to address the issue of U-spin breaking. In general, theoretical input is used. However, one of the purposes of the present paper is to note that, in fact, this can be experimentally measured. The point is that, under U-spin symmetry, four of the experimental observables—the branching ratios and direct  $CP$  asymmetries of the two decays—are related, i.e. they are not independent. Thus, the experimental values of these observables, and the extent to which the relation among them is not satisfied, gives a measure of U-spin breaking. Note: This is not a completely new result.

The relation among the four observables already appears in a number of papers. However, in general, it is used as a theoretical constraint, rather than an experimental result.

In addition, one can go further. If one neglects annihilation- and exchange-type diagrams (which are expected to be small) in the  $B$  decay amplitudes, there are other pairs of amplitudes which are equal, apart from CKM matrix elements [5]. In this case, it is not U spin that is assumed, but rather full flavor SU(3) symmetry.<sup>1</sup> Here there are many more pairs whose amplitudes are related. Because the relation among the four observables holds in the SU(3) limit, it is possible to measure SU(3)-breaking effects using any of these decay pairs.

In fact, there are a number of two-body  $B$  decay pairs for which this information is presently available. Furthermore, in such decays, the factorizable contribution to the breaking is often under good theoretical control. If this is taken into account, the measurement of U-spin/SU(3) breaking then tells us the size of nonfactorizable effects. In most cases, the data shows that such effects are small. However, as we show below, there is one decay pair— $B_d^0 \rightarrow \pi^+ \pi^-$  and  $B_d^0 \rightarrow \pi^- K^+$ —for which the effect is not clear. There may or may not be large nonfactorizable breaking, depending on whether one uses the  $BABAR$  or Belle measurement of the  $B_d^0 \rightarrow \pi^+ \pi^-$  direct  $CP$  asymmetry. Although this result is not definitive, it does raise questions about analyses which neglect nonfactorizable U-spin/SU(3) breaking.

We begin in Sec. II with a discussion of U spin and U-spin breaking as it applies to a pair of charmless  $\bar{b} \rightarrow \bar{d}$  and  $\bar{b} \rightarrow \bar{s}$  decays. We show how the measurement of the

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<sup>1</sup>Note: Because isospin is a good symmetry, in practice there is little difference between U spin and SU(3).

branching ratios and direct  $CP$  asymmetries of these two decays allows one to experimentally measure the breaking. In Sec. III, we turn to an examination of two-body  $B$  decays. We present lists of five U-spin pairs and 11 additional SU(3) pairs whose U-spin/SU(3) breaking can be measured using this method. We show the latest data for five of these pairs. For two of these, the measurements are reasonably accurate, and one pair may show signs of significant nonfactorizable U-spin/SU(3) breaking. Finally, we discuss several pairs of decays whose amplitudes are equal, including CKM factors, within SU(3). A measure of SU(3) breaking is given by comparing the branching ratios of the two decays, as well as the direct  $CP$  asymmetries.

We discuss three-body decays in Sec. IV. There are seven decay pairs whose amplitudes are related by U spin—the amount of breaking can be measured experimentally using the above method. In passing, we note that the pure-penguin decay  $B_s^0 \rightarrow \bar{K}^0 \bar{K}^0 K^0$  is interesting. Given that the final state is a  $CP$  eigenstate, the measurement of the indirect  $CP$  asymmetry in this decay cleanly probes the  $B_s^0$ - $\bar{B}_s^0$  mixing phase, and might be easier experimentally than what is done at the moment. We also present the list of an additional 24 decay pairs whose amplitudes are related by SU(3). In this case, all final-state particles are identical, and so permutations of these particles must be considered. We show that, in (almost) all cases, the amplitudes are equal only for the totally symmetric final state  $|S\rangle$ , so that this state must be isolated experimentally in order to measure SU(3) breaking. We also point out the decay pairs whose amplitudes are equal, including CKM factors, within SU(3) for  $|S\rangle$ . In principle, these can also give information about SU(3) breaking. We conclude in Sec. V.

## II. U SPIN AND U-SPIN BREAKING

Consider charmless  $\bar{b} \rightarrow \bar{d}$  and  $\bar{b} \rightarrow \bar{s}$  decays.<sup>2</sup> Their amplitudes can be written as

$$\begin{aligned} A(\bar{b} \rightarrow \bar{d}) &= A_u \lambda_u^{(d)} + A_c \lambda_c^{(d)} + A_t \lambda_t^{(d)}, \\ A(\bar{b} \rightarrow \bar{s}) &= A'_u \lambda_u^{(s)} + A'_c \lambda_c^{(s)} + A'_t \lambda_t^{(s)}, \end{aligned} \quad (1)$$

where the  $A_i$  and  $A'_i$  ( $i = u, c, t$ ) each represent a linear combination of diagrams, and  $\lambda_p^{(q)} = V_{pb}^* V_{pq}$ . Using the unitarity of the CKM matrix ( $\lambda_u^{(q)} + \lambda_c^{(q)} + \lambda_t^{(q)} = 0$ ), we can reduce the number of terms in the amplitudes from three to two. For instance, if the  $\lambda_c^{(q)}$  piece is eliminated, we have

<sup>2</sup>Much of the material in this section can be found in Refs. [3,6].

$$\begin{aligned} A(\bar{b} \rightarrow \bar{d}) &= (A_u - A_c) \lambda_u^{(d)} + (A_t - A_c) \lambda_t^{(d)} \\ &= (A_u - A_c) \left[ |\lambda_u^{(d)}| e^{i\gamma} + \frac{(A_t - A_c)}{(A_u - A_c)} |\lambda_t^{(d)}| e^{-i\beta} \right] \\ &= C [|\lambda_u^{(d)}| e^{i\gamma} + |\lambda_t^{(d)}| r e^{i\delta_r} e^{-i\beta}], \\ A(\bar{b} \rightarrow \bar{s}) &= (A'_u - A'_c) \lambda_u^{(s)} + (A'_t - A'_c) \lambda_t^{(s)} \\ &= (A'_u - A'_c) \left[ |\lambda_u^{(s)}| e^{i\gamma} - \frac{(A'_t - A'_c)}{(A'_u - A'_c)} |\lambda_t^{(s)}| \right] \\ &= C' [|\lambda_u^{(s)}| e^{i\gamma} - |\lambda_t^{(s)}| r' e^{i\delta'_r}], \end{aligned} \quad (2)$$

where  $C \equiv (A_u - A_c)$ ,  $C' \equiv (A'_u - A'_c)$ ,  $r e^{i\delta_r} \equiv (A_t - A_c)/(A_u - A_c)$ , and  $r' e^{i\delta'_r} \equiv (A'_t - A'_c)/(A'_u - A'_c)$ . Above we have explicitly written the weak-phase dependence, including the minus sign from  $V_{tb}^* V_{ts}$ .

If the two amplitudes are given by a similar combination of diagrams, then under U-spin symmetry, which exchanges  $d$  and  $s$  quarks, we have  $A'_i = A_i$ , so that

$$C' = C, \quad r' = r, \quad \delta'_r = \delta_r, \quad (3)$$

and the two amplitudes are described by four unknown theoretical parameters:  $\gamma$ ,  $|C|$ ,  $r$ ,  $\delta_r$  ( $\beta$  has been measured quite accurately through the indirect  $CP$  asymmetry in  $B_d^0 \rightarrow J/\psi K_S$  [1], and is therefore taken to be known).

In general, there are four observables in the  $\bar{b} \rightarrow \bar{d}$  and  $\bar{b} \rightarrow \bar{s}$  processes:

$$\begin{aligned} B_d &= |A(\bar{b} \rightarrow \bar{d})|^2 + |A(b \rightarrow d)|^2, \\ B_s &= |A(\bar{b} \rightarrow \bar{s})|^2 + |A(b \rightarrow s)|^2, \\ A_d &= \frac{|A(\bar{b} \rightarrow \bar{d})|^2 - |A(b \rightarrow d)|^2}{|A(\bar{b} \rightarrow \bar{d})|^2 + |A(b \rightarrow d)|^2}, \\ A_s &= \frac{|A(\bar{b} \rightarrow \bar{s})|^2 - |A(b \rightarrow s)|^2}{|A(\bar{b} \rightarrow \bar{s})|^2 + |A(b \rightarrow s)|^2}. \end{aligned} \quad (4)$$

$B_d$  and  $B_s$  are related to the  $CP$ -averaged  $\bar{b} \rightarrow \bar{d}$  and  $\bar{b} \rightarrow \bar{s}$  decay rates, while  $A_d$  and  $A_s$  are direct  $CP$  asymmetries. The  $CP$ -conjugate amplitude  $A(\bar{b} \rightarrow \bar{q})$  is obtained from  $A(b \rightarrow q)$  by changing the signs of the weak phases.

Since there are four unknown theoretical parameters in the amplitudes in the U-spin limit, one might imagine that these can be determined from measurements of  $B_{d,s}$  and  $A_{d,s}$ . However, this is not true. It is straightforward to show that, in this limit,  $X = 1$ , where

$$X \equiv -\frac{A_s B_s}{A_d B_d}. \quad (5)$$

Thus, there are only three independent observables. This implies that

$$-\frac{|A(\bar{b} \rightarrow \bar{s})|^2 - |A(b \rightarrow s)|^2}{|A(\bar{b} \rightarrow \bar{d})|^2 - |A(b \rightarrow d)|^2} = 1. \quad (6)$$

Explicitly, we have

$$\frac{|A(\bar{b} \rightarrow \bar{s})|^2 - |A(b \rightarrow s)|^2}{|A(\bar{b} \rightarrow \bar{d})|^2 - |A(b \rightarrow d)|^2} = \frac{|\lambda_u^{(s)}| |\lambda_t^{(s)}| |\sin\gamma| |C'|^2 r' \sin\delta'_r}{|\lambda_u^{(d)}| |\lambda_t^{(d)}| |\sin\alpha| |C|^2 r \sin\delta_r}. \quad (7)$$

Now, the sine law associated with the unitarity triangle gives

$$\frac{\sin\gamma}{|\lambda_t^{(d)}|} = \frac{\sin\alpha}{|\lambda_c^{(d)}|} = \frac{\sin\beta}{|\lambda_u^{(d)}|}. \quad (8)$$

We therefore have

$$\begin{aligned} \frac{|A(\bar{b} \rightarrow \bar{s})|^2 - |A(b \rightarrow s)|^2}{|A(\bar{b} \rightarrow \bar{d})|^2 - |A(b \rightarrow d)|^2} &= \frac{|\lambda_u^{(s)}| |\lambda_t^{(s)}| |C'|^2 r' \sin\delta'_r}{|\lambda_u^{(d)}| |\lambda_c^{(d)}| |C|^2 r \sin\delta_r} \\ &= \frac{|V_{us}| |V_{tb}| |V_{ts}| |C'|^2 r' \sin\delta'_r}{|V_{ud}| |V_{cb}| |V_{cd}| |C|^2 r \sin\delta_r} \\ &= \frac{|C'|^2 r' \sin\delta'_r}{|C|^2 r \sin\delta_r}, \end{aligned} \quad (9)$$

where  $|V_{us}| |V_{tb}| |V_{ts}| / |V_{ud}| |V_{cb}| |V_{cd}| = 1$ . The above ratio equals 1 only in the U-spin limit. Thus,  $(X - 1)$  is a measure of U-spin breaking.

Until now, when this breaking was taken into account, it was only through theoretical estimates (e.g. see Refs. [2,7]). However, in fact it can be obtained from the experimental data. This can be combined with the theoretical calculations to look for large nonfactorizable corrections (we will see this explicitly in Sec. III E). Furthermore, if the theoretical prediction of U-spin breaking is accurate, one can use the measurement of  $(X - 1)$  to search for new physics [6].

### III. TWO-BODY DECAYS

#### A. U-spin pairs

We begin with  $B \rightarrow PP$  decays ( $P =$  pseudoscalar), focusing on those  $\bar{b} \rightarrow \bar{d}$  and  $\bar{b} \rightarrow \bar{s}$  processes that are related by U spin. [It is straightforward to extend our analysis to other two-body decays, such as  $B \rightarrow VP$  ( $V =$  vector).] There are five U-spin pairs:

- (1)  $B_d^0 \rightarrow \pi^+ \pi^-$  and  $B_s^0 \rightarrow K^+ K^-$ ,
- (2)  $B_s^0 \rightarrow \pi^+ K^-$  and  $B_d^0 \rightarrow \pi^- K^+$ ,
- (3)  $B^+ \rightarrow K^+ \bar{K}^0$  and  $B^+ \rightarrow \pi^+ K^0$ ,
- (4)  $B_d^0 \rightarrow K^0 \bar{K}^0$  and  $B_s^0 \rightarrow \bar{K}^0 K^0$ ,
- (5)  $B_d^0 \rightarrow K^+ K^-$  and  $B_s^0 \rightarrow \pi^+ \pi^-$ .

The first (second) decay is  $\bar{b} \rightarrow \bar{d}$  ( $\bar{b} \rightarrow \bar{s}$ ). In all cases, the two decays within a pair are related by U-spin reflection ( $d \leftrightarrow s$ ). This applies not only to the particles in the process (e.g.  $\pi^+ \leftrightarrow K^+$ ,  $B_d^0 \leftrightarrow B_s^0$ , etc.), but also to the individual diagrams involved. For any pair, one can measure the two branching ratios and direct  $CP$  asymmetries in order to obtain  $X$  [Eq. (5)], and measure U-spin breaking.

#### B. SU(3) pairs

U-spin pairs have been discussed at some length in Refs. [3,6]. However, one can go further. First, one pair which is not included in the list in Sec. III A, but appears in Refs. [3,6], is  $B_s^0 \rightarrow \pi^0 \bar{K}^0$  and  $B_d^0 \rightarrow \pi^0 K^0$ . The reason it is not included is that the two decays are not related by U spin. There are a number of ways to see this. First,  $\pi^0 = (d\bar{d} - u\bar{u})/\sqrt{2}$ , so that it does not transform into itself under U spin. Second, one diagram that contributes to  $B_s^0 \rightarrow \pi^0 \bar{K}^0$  is the penguin  $P$ , involving the quark-level transition  $\bar{b} \rightarrow \bar{d}d\bar{d}$ . Under U-spin reflection, this becomes  $\bar{b} \rightarrow \bar{s}s\bar{s}$ , which does not contribute to  $B_d^0 \rightarrow \pi^0 K^0$ . What is going on is the following: it is true that the amplitudes for  $B_s^0 \rightarrow \pi^0 \bar{K}^0$  and  $B_d^0 \rightarrow \pi^0 K^0$  have the same diagrammatic decomposition [5], and so they satisfy Eq. (3). However, the diagrams assume isospin invariance in addition to U spin, so that the symmetry is really flavor SU(3). Thus,  $B_s^0 \rightarrow \pi^0 \bar{K}^0$  and  $B_d^0 \rightarrow \pi^0 K^0$  is not a U-spin pair, but is in fact an SU(3) pair.

Second, it is standard to express the amplitudes for  $B \rightarrow PP$  decays in terms of diagrams [5]. Certain of these diagrams—those of annihilation and exchange type—are expected to be smaller than the others. If these diagrams are neglected, then there are additional pairs of decays which satisfy Eq. (3). These are not related by U spin, but are instead related by SU(3). The complete list of SU(3) pairs (which includes some U-spin pairs) is

- (i)  $(B_d^0 \rightarrow \pi^+ \pi^-, B_s^0 \rightarrow \pi^+ K^-)$  and  $(B_d^0 \rightarrow \pi^- K^+, B_s^0 \rightarrow K^+ K^-)$ ,
- (ii)  $(B_d^0 \rightarrow \pi^0 \pi^0, B_s^0 \rightarrow \pi^0 \bar{K}^0, B_s^0 \rightarrow \eta_8 \bar{K}^0)$  and  $(B_d^0 \rightarrow \pi^0 K^0, B_d^0 \rightarrow \eta_8 K^0)$ ,
- (iii)  $(B_d^0 \rightarrow K^0 \bar{K}^0, B^+ \rightarrow K^+ \bar{K}^0, B_d^0 \rightarrow \pi^0 \eta_8)$  and  $(B^+ \rightarrow \pi^+ K^0, B_s^0 \rightarrow K^0 \bar{K}^0)$ .

[Here,  $\eta_8$  is a member of the octet of SU(3). The physical  $\eta$  and  $\eta'$  are linear combinations of  $\eta_8$  and the SU(3) singlet,  $\eta_0$ .] The decays in the first (second) parentheses are  $\bar{b} \rightarrow \bar{d}$  ( $\bar{b} \rightarrow \bar{s}$ ) transitions. [Note that, depending on the pair, there may be an additional factor (e.g.  $\sqrt{2}$ ) in relating the  $\bar{b} \rightarrow \bar{d}$  and  $\bar{b} \rightarrow \bar{s}$  decays.] From this list, we see that there are, in fact, 16 possible pairs of decays rather than the five of Sec. III A.

If  $(X - 1)$  is obtained using a pair from Sec. III A, then U-spin breaking is measured. However, if an SU(3) pair is used, then what is probed is not U-spin breaking, but rather SU(3) breaking. Interestingly, we have data for a number of the decays in the above list, so that it is possible to get  $X$ , and obtain an experimental measurement of U-spin/SU(3) breaking in these decays. This is done in Sec. III D.

#### C. Estimates of $A_{s,d}$

As described above, one can measure U-spin/SU(3) breaking through  $X$ . This quantity involves the direct  $CP$  asymmetries  $A_{d,s}$ , which arise due to the interference of two amplitudes with different weak and strong phases. The

maximal size of  $A_{d,s}$  is roughly equal to the ratio of the magnitudes of the two interfering amplitudes.

In two-body decays, the  $\bar{b} \rightarrow \bar{s}$  diagrams<sup>3</sup> are expected to obey the approximate hierarchy [5],

$$\begin{aligned} 1: & |P'_{tc}|, \\ \bar{\lambda}: & |T'|, |P'_{EW}|, \\ \bar{\lambda}^2: & |C'|, |P'_{uc}|, |P'_{EW}|, \end{aligned} \quad (10)$$

where  $\bar{\lambda} \approx 0.2$ . Since all  $\bar{b} \rightarrow \bar{s}$  decays in the list in Sec. III B receive contributions from  $P'_{tc}$ ,  $A_s$  is sizable [ $\approx O(\bar{\lambda}) \sim 20\%$ ] only if the decay amplitude also includes  $T'$ . If there is no  $T'$ , but only  $C'$  or  $P'_{uc}$ , then  $A_s$  is small [ $\approx O(\bar{\lambda}^2) \sim 5\%$ ]. In this case, the relative experimental error will necessarily be large, which will then translate into a large error on  $(X - 1)$ .

The expected approximate hierarchy<sup>4</sup> of the  $\bar{b} \rightarrow \bar{d}$  diagrams is [5]

$$\begin{aligned} 1: & |T|, \\ \bar{\lambda}: & |C|, |P_{tc}|, |P_{uc}|, \\ \bar{\lambda}^2: & |P_{EW}|, \\ \bar{\lambda}^3: & |P_{EW}^C|. \end{aligned} \quad (11)$$

Since all  $\bar{b} \rightarrow \bar{d}$  decays in the list in Sec. III B receive penguin contributions,  $A_d$  is always sizable [at least  $\approx O(\bar{\lambda}) \sim 20\%$ ].

Thus, the most promising pairs for measuring U-spin/SU(3) breaking are those whose  $\bar{b} \rightarrow \bar{s}$  decay amplitude includes a  $T'$ . These are given in the first entry in the list in Sec. III B.

There are two types of contributions to U-spin/SU(3) breaking—factorizable and nonfactorizable. The factorizable effects depend essentially on form factors and decay constants, and can be reliably calculated. It has been shown that factorization holds well for  $T/T'$  diagrams [8]. Thus, for those decay pairs which include these diagrams—i.e. the most promising for measuring  $X$ —the ratio  $|C'/C|$  is dominated by factorizable U-spin/SU(3) breaking.

The U-spin relations  $r' = r$  and  $\delta'_r = \delta_r$  are not affected by factorizable breaking effects,<sup>5</sup> as the various form factors and decay constants cancel [2,7]. On the other hand, they could be altered by nonfactorizable effects, and these cannot be calculated theoretically. Still, it is thought that nonfactorizable U-spin/SU(3) breaking is not large, being higher order in  $1/m_b$ . As we show below, this can be checked experimentally through the measurement of  $(X - 1)$ .

<sup>3</sup>The diagrams include the magnitudes of the associated CKM matrix elements.

<sup>4</sup> $C'$  and  $C$  in Eqs. (10) and (11) represent color-suppressed tree diagrams, and are not the parameters in Eq. (2).

<sup>5</sup>Decays such as  $B_d^0 \rightarrow \pi^0 K^0$  constitute an exception to this rule, as they can be factorized in two different ways. However, there are very few such decays.

## D. Numerical analysis

The four quantities required for the measurement of  $X$  are  $B_{d,s}$  and  $A_{d,s}$  [Eq. (4)]. The  $B_{d,s}$ 's are related to the branching ratios by

$$\tau_{(q)} p_{c(q)} B_q = 8\pi m_{B(q)}^2 \mathcal{B}_{(q)}, \quad (12)$$

where, for a  $\bar{b} \rightarrow \bar{q}$  process ( $q = d, s$ ),  $\tau_{(q)}$  is the  $B$ -meson lifetime,  $p_{c(q)}$  is the momentum of the final-state mesons in the  $B$  rest frame,  $m_{B(q)}$  is the rest mass of the  $B$  meson, and  $\mathcal{B}_{(q)}$  is the  $CP$ -averaged branching ratio. The  $A_{d,s}$ 's are equal to  $-C_{CP}$ , where  $C_{CP}$  is the direct  $CP$  asymmetry in a given decay.

At present, there are five different pairs of two-body decays for which we have the data required by the method of Sec. II for measuring U-spin/SU(3) breaking:

- (1)  $B_d^0 \rightarrow \pi^+ \pi^-$  and  $B_d^0 \rightarrow \pi^- K^+$ ,
- (2)  $B_s^0 \rightarrow \pi^+ K^-$  and  $B_d^0 \rightarrow \pi^- K^+$ ,
- (3)  $B^+ \rightarrow K^+ \bar{K}^0$  and  $B^+ \rightarrow \pi^+ K^0$ ,
- (4)  $B_d^0 \rightarrow K^0 \bar{K}^0$  and  $B^+ \rightarrow \pi^+ K^0$ ,
- (5)  $B_d^0 \rightarrow \pi^0 \pi^0$  and  $B_d^0 \rightarrow \pi^0 K^0$ .

The current experimental values are given in Table I. The values of the  $B$  masses and lifetimes can be found in Ref. [1].

With these inputs, one can compute the value of  $(X - 1)$  obtained for each of the five decay pairs using Eq. (5). The results are shown in Table II. Note that, as described in Sec. III C, the direct  $CP$  asymmetries in  $B^+ \rightarrow \pi^+ K^0$  and  $B_d^0 \rightarrow \pi^0 K^0$  are expected to be quite small, leading to a very large error on  $(X - 1)$ . This is indeed what is found [pairs (3), (4), and (5)].

Finally, the decays  $B_d^0 \rightarrow \pi^+ \pi^-$  and  $B_s^0 \rightarrow K^+ K^-$  form a U-spin pair. From the updated QCD light-cone sum-rule calculation of Ref. [12], we have

$$\left| \frac{C'}{C} \right|_{\text{fact}} = \frac{f_K f_{B_s K}^+(M_K^2)}{f_\pi f_{B_d \pi}^+(M_\pi^2)} \left( \frac{M_{B_s}^2 - M_K^2}{M_{B_d}^2 - M_\pi^2} \right) = 1.41_{-0.11}^{+0.20}. \quad (13)$$

Here and below, we take  $f^+(M_K^2) \simeq f^+(M_\pi^2) \simeq f^+(0)$  since the variation of the form factors over this range of  $q^2$  falls well within the errors of their calculation [13]. Thus, using the data from Table I and Eq. (20) below, we expect

$$\begin{aligned} A_{CP}(B_s^0 \rightarrow K^+ K^-) &= - \left| \frac{C'}{C} \right|_{\text{fact}}^2 A_{CP}(B_d^0 \rightarrow \pi^+ \pi^-) \\ &\quad \times \frac{\mathcal{B}(B_d^0 \rightarrow \pi^+ \pi^-)}{\mathcal{B}(B_s^0 \rightarrow K^+ K^-)} \\ &= \begin{cases} -0.11 \pm 0.04 & (\text{BABAR}), \\ -0.23 \pm 0.07 & (\text{Belle}). \end{cases} \end{aligned} \quad (14)$$

Similarly, the decays  $B_s^0 \rightarrow \pi^+ K^-$  and  $B_s^0 \rightarrow K^+ K^-$  form an SU(3) pair, so that

TABLE I. Input values for the experimental quantities [1,9] (unless otherwise indicated, averaged over all measurements). Data for the  $B_d^0 \rightarrow \pi^+ \pi^-$  direct  $CP$  asymmetry taken from Refs. [10] (*BABAR*) and [11] (*Belle*). For asymmetric error bars, we take the average of both errors and assume a Gaussian distribution.

Decay	$\mathcal{B}[\times 10^6]$	$-C_{CP}$	$p_c$ [MeV/c]
$B_d^0 \rightarrow \pi^+ \pi^-$	$5.16 \pm 0.22$	$0.25 \pm 0.08$ ( <i>BABAR</i> ) $0.55 \pm 0.09$ ( <i>Belle</i> )	2636
$B_d^0 \rightarrow \pi^- K^+$	$19.4 \pm 0.6$	$-0.098_{-0.011}^{+0.012}$	2615
$B_s^0 \rightarrow \pi^+ K^-$	$5.0 \pm 1.1$	$0.39 \pm 0.17$	2659
$B^+ \rightarrow K^+ \bar{K}^0$	$1.36_{-0.27}^{+0.29}$	$0.12_{-0.18}^{+0.17}$	2593
$B^+ \rightarrow \pi^+ K^0$	$23.1 \pm 1.0$	$0.009 \pm 0.025$	2614
$B_d^0 \rightarrow K^0 \bar{K}^0$	$0.96_{-0.19}^{+0.21}$	$0.06 \pm 0.26$	2592
$B_d^0 \rightarrow \pi^0 \pi^0$	$1.55 \pm 0.19$	$0.43_{-0.24}^{+0.25}$	2636
$B_d^0 \rightarrow \pi^0 K^0$ ( <i>BABAR</i> )	$10.1 \pm 0.6 \pm 0.4$	$-0.13 \pm 0.13 \pm 0.03$	2615
$B_d^0 \rightarrow \pi^0 K^0$ ( <i>Belle</i> )	$8.7 \pm 0.5 \pm 0.6$	$0.14 \pm 0.13 \pm 0.06$	

TABLE II. Output values for the quantity  $(X - 1)$  for the five pairs of decays.

Decay pair	$(X - 1)$
(1)	$0.49 \pm 0.47$ ( <i>BABAR</i> ) $-0.32 \pm 0.13$ ( <i>Belle</i> )
(2)	$-0.08 \pm 0.42$
(3)	$-2.3 \pm 3.6$
(4)	$-4 \pm 16$
(5)	$1.0 \pm 2.1$ ( <i>BABAR</i> ) $-2.8 \pm 2.0$ ( <i>Belle</i> )

$$\begin{aligned}
 A_{CP}(B_s^0 \rightarrow K^+ K^-) &= - \left| \frac{C'}{C} \right|_{\text{fact}}^2 A_{CP}(B_s^0 \rightarrow \pi^+ K^-) \\
 &\times \frac{\mathcal{B}(B_s^0 \rightarrow \pi^+ K^-)}{\mathcal{B}(B_s^0 \rightarrow K^+ K^-)} \\
 &= -0.12 \pm 0.06, \quad (15)
 \end{aligned}$$

where  $|C'/C|_{\text{fact}} = f_K/f_\pi$ . These predictions are in agreement with one another and will be tested when  $A_{CP}(B_s^0 \rightarrow K^+ K^-)$  is measured.

### E. Measurement of nonfactorizable SU(3) breaking

The theoretical expression for  $X$  is given in Eq. (9). As discussed above, within factorization, only the ratio  $|C'/C|$  contributes to  $X$ . Therefore, given an experimental measurement of  $X$  and a theoretical calculation of  $|C'/C|_{\text{fact}}$ , one can obtain

$$\frac{r' \sin \delta'_r}{r \sin \delta_r} = \left| \frac{C}{C'} \right|_{\text{fact}}^2 X \quad (16)$$

and see whether it is consistent with 1 (small nonfactorizable U-spin/SU(3) breaking).

For the first two pairs of the previous subsection, which yield reasonably precise measurements of  $X$ , we have

$$\begin{aligned}
 \text{pair (1): } \left| \frac{C'}{C} \right|_{\text{fact}} &= \frac{f_K}{f_\pi} \frac{f_{B_d \pi}^+(M_K^2)}{f_{B_d \pi}^+(M_\pi^2)} \left( \frac{M_{B_d}^2 - M_\pi^2}{M_{B_d}^2 - M_K^2} \right) \\
 &\approx \frac{f_K}{f_\pi} = 1.20,
 \end{aligned}$$

$$\begin{aligned}
 \text{pair (2): } \left| \frac{C'}{C} \right|_{\text{fact}} &= \frac{f_K}{f_\pi} \frac{f_{B_d \pi}^+(M_K^2)}{f_{B_s K}^+(M_\pi^2)} \left( \frac{M_{B_d}^2 - M_\pi^2}{M_{B_s}^2 - M_K^2} \right) \\
 &= 1.01_{-0.15}^{+0.07}. \quad (17)
 \end{aligned}$$

The ratio  $f_K/f_\pi$  and the value in the second line are taken from Ref. [12]. (We have neglected small errors in  $f_K/f_\pi$ .) These give

$$\begin{aligned}
 \text{pair (1): } \frac{r' \sin \delta'_r}{r \sin \delta_r} &= \begin{cases} 1.0 \pm 0.3 & (\text{BABAR}), \\ 0.47 \pm 0.09 & (\text{Belle}), \end{cases} \quad (18) \\
 \text{pair (2): } \frac{r' \sin \delta'_r}{r \sin \delta_r} &= 0.90 \pm 0.43.
 \end{aligned}$$

For pair (2), the theoretical prediction for factorizable U-spin breaking is consistent with the experimental measurement of Table II. However, for pair (1), things are more complicated. With the *BABAR* direct  $CP$  asymmetry measurement of Table I, there is no evidence of a nonfactorizable SU(3)-breaking correction. On the other hand, with the *Belle* measurement, there is a  $6\sigma$  deviation of the value of  $|C'/C|_{\text{fact}}^2 X$  from 1, which implies large nonfactorizable breaking. There is thus a significant discrepancy between the two measurements, and so it will be important to pay attention to this pair in the future. This does illustrate the importance of measuring  $X$  experimentally, and this in as many different decay pairs as possible.

In Ref. [7], the discrepancy between the *BABAR* and *Belle* measurements was noted. It was observed that, within SU(3) (including factorizable corrections),  $A_{CP}(B_d^0 \rightarrow \pi^+ \pi^-)$  is equal to

$$\begin{aligned}
 & - \left( \frac{f_\pi}{f_K} \right)^2 \frac{\mathcal{B}(B_d^0 \rightarrow \pi^- K^+)}{\mathcal{B}(B_d^0 \rightarrow \pi^+ \pi^-)} A_{CP}(B_d^0 \rightarrow \pi^- K^+) \\
 & = 0.26 \pm 0.03.
 \end{aligned} \tag{19}$$

In addition, LHCb recently measured the direct  $CP$  asymmetry in  $B_s^0 \rightarrow \pi K$ , which differs from  $B_d^0 \rightarrow \pi^+ \pi^-$  only in its spectator quark, to be  $0.27 \pm 0.08$  [14]. Both of these results are consistent with the  $BABAR$  measurement, and suggest that this result is favored (which implies no non-factorizable correction).

Although there is no definitive evidence of nonfactorizable  $SU(3)$ -breaking corrections, this does call into question any analysis which does not include such effects. The point is that it is straightforward to take this into account. Within U-spin/ $SU(3)$  symmetry, the four observables  $B_{d,s}$  and  $A_{d,s}$  are not independent. However, if one allows U-spin/ $SU(3)$  breaking, this no longer holds. If one assumes that  $\delta'_r = \delta_r$ , i.e. the phase is unaffected by the breaking, and takes  $|C'/C|$  from factorization, then non-factorizable U-spin/ $SU(3)$  breaking contributes only to  $r'/r$ . That is, there is one additional theoretical parameter ( $r'/r$ ), but there is one additional measurement, so that the nonfactorizable breaking can be obtained. This is essentially just the measurement of  $X$ .

Now, pair (1) is useful for another reason. As detailed previously, it is not possible to obtain the theoretical parameters in the amplitudes solely from the measurements of  $B_{d,s}$  and  $A_{d,s}$ —additional input is needed. This has been discussed for two of the U-spin pairs. For  $B_d^0 \rightarrow \pi^+ \pi^-$  and  $B_s^0 \rightarrow K^+ K^-$ , it has been noted that  $\gamma$  can be extracted through the additional measurement of the indirect  $CP$  asymmetry in  $B_d^0 \rightarrow \pi^+ \pi^-$  [2,7]. Similarly,  $\gamma$  can be obtained from  $B_s^0 \rightarrow \pi^+ K^-$  and  $B_d^0 \rightarrow \pi^- K^+$  with the added information coming from the measurement of the branching ratio of  $B^+ \rightarrow \pi^+ K^0$  [4].

Both of these pairs appear in the list in Sec. III A. However, if one expands the symmetry from U spin to  $SU(3)$ , they can be combined, producing the pair  $B_d^0 \rightarrow \pi^+ \pi^-$  and  $B_d^0 \rightarrow \pi^- K^+$  [pair (1), in the list in Sec. III B].  $\gamma$  can then be extracted using the method of Ref. [2], taking  $B_d, A_d$ , and  $A_{\text{ind}}^{CP}$  from  $B_d^0 \rightarrow \pi^+ \pi^-$ , and  $B_s$  from  $B_d^0 \rightarrow \pi^- K^+$  instead of  $B_s^0 \rightarrow K^+ K^-$ . Since [9,15]

$$\begin{aligned}
 \mathcal{B}(B_d^0 \rightarrow \pi^- K^+) &= (19.4 \pm 0.6) \times 10^{-6}, \\
 \mathcal{B}(B_s^0 \rightarrow K^+ K^-) &= (23.9 \pm 3.9) \times 10^{-6},
 \end{aligned} \tag{20}$$

one sees that the first (experimental) error is smaller than the second one. Thus, the error on  $\gamma$  is also smaller. Alternatively, suppose that the technique of Ref. [4] is used, taking  $B_s$  and  $A_s$  from  $B_d^0 \rightarrow \pi^- K^+$ , and  $B_d$  from  $B_d^0 \rightarrow \pi^+ \pi^-$  instead of  $B_s^0 \rightarrow \pi^+ K^-$  [information from  $\mathcal{B}(B^+ \rightarrow \pi^+ K^0)$  is added]. The error on  $\gamma$  will still be smaller since [9]

$$\begin{aligned}
 \mathcal{B}(B_d^0 \rightarrow \pi^+ \pi^-) &= (5.16 \pm 0.22) \times 10^{-6}, \\
 \mathcal{B}(B_s^0 \rightarrow \pi^+ K^-) &= (5.0 \pm 1.1) \times 10^{-6}.
 \end{aligned} \tag{21}$$

The point is that the branching ratios of  $B_d^0$  decays are measured much more accurately than those of  $B_s^0$  decays, so that the extracted value of  $\gamma$  is more precise if pair (1) is used, rather than either U-spin pair.

In fact, this method was proposed many years ago, in 1995 [16]. In this paper, information from both  $A_{\text{ind}}^{CP}(B_d^0 \rightarrow \pi^+ \pi^-)$  and  $\mathcal{B}(B^+ \rightarrow \pi^+ K^0)$  is added simultaneously. In addition, perfect  $SU(3)$  symmetry is not imposed, so there are a total of six independent measurements. It is assumed that  $|C'/C| = f_K/f_\pi$  [Eq. (17)] and that  $\delta'_r = \delta_r$ , but  $r'$  and  $r$  are left as independent. This means that the amplitudes are written in terms of four hadronic theoretical parameters and two weak phases. In Ref. [16], it is argued that both weak phases can be extracted. However, this method can be modified: if we assume that  $\beta$  is known from  $A_{\text{ind}}^{CP}(B_d^0 \rightarrow J/\psi K_S)$ , then we have the freedom to take  $\delta'_r$  and  $\delta_r$  as independent. Now there are six equations in six unknowns ( $C, r', r, \delta'_r, \delta_r, \gamma$ ), so that one can solve for the theoretical parameters (numerically, if necessary). This analysis was partially performed in Ref. [17]. We must stress here that no assumption about the size of nonfactorizable effects in  $r'/r$  and  $\sin\delta'_r/\sin\delta_r$  is made here—this information is taken from the experimental data.

## F. Other signals of $SU(3)$ breaking

There are pairs of decays whose amplitudes are equal at the quark level, including CKM factors, under  $SU(3)$ . At the meson level, the processes are those within parentheses in the list in Sec. III B. The amplitudes for the two decays can be written

$$A_i = C_i^{(q)} [\lambda_u^{(q)} + \lambda_t^{(q)} r_i^{(q)} e^{i\delta_{r,i}^{(q)}}], \tag{22}$$

where  $i = 1, 2$  and  $q = d, s$  (the hadronic parameters have primes for  $q = s$ ). Assuming only factorizable  $SU(3)$  breaking,  $r_1^{(q)} = r_2^{(q)}$  and  $\delta_{r,1}^{(q)} = \delta_{r,2}^{(q)}$ . We therefore expect the branching ratios and direct  $CP$  asymmetries for the two decays to satisfy

$$\mathcal{B}_2 = \left| \frac{C_2^{(q)}}{C_1^{(q)}} \right|_{\text{fact}}^2 \mathcal{B}_1, \quad A_{CP,2} = A_{CP,1}. \tag{23}$$

(We neglect any mass and lifetime differences between the two decaying  $B$  mesons.) Any deviation from these relations is a sign of nonfactorizable  $SU(3)$  breaking.

The pairs or amplitude relations are (all experimental data is taken from Ref. [9]):

- (1)  $B_d^0 \rightarrow \pi^- K^+$  and  $B_s^0 \rightarrow K^+ K^-$ :

$$\left| \frac{C_1'}{C_2'} \right|_{\text{fact}} = \frac{f_{B_d\pi}^+(M_K^2)}{f_{B_s K}^+(M_K^2)} \left( \frac{M_{B_d}^2 - M_\pi^2}{M_{B_s}^2 - M_K^2} \right) = 0.85_{-0.12}^{+0.07}. \tag{24}$$

(This is based on the results of Ref. [12].) The data for the branching ratios for these decays are given in Eq. (20). We expect

$$\left| \frac{C'_2}{C'_1} \right|_{\text{fact}}^2 \frac{\mathcal{B}(B_d^0 \rightarrow \pi^- K^+)}{\mathcal{B}(B_s^0 \rightarrow K^+ K^-)} \quad (25)$$

to be consistent with 1. It equals  $1.12 \pm 0.26$ , so there is no evidence of nonfactorizable SU(3) breaking.

We also expect that

$$A_{CP}(B_s^0 \rightarrow K^+ K^-) = A_{CP}(B_d^0 \rightarrow \pi^- K^+) = -0.098_{-0.011}^{+0.012}. \quad (26)$$

- (2)  $B_d^0 \rightarrow \pi^+ \pi^-$  and  $B_s^0 \rightarrow \pi^+ K^-$ : here,  $|C_1/C_2|_{\text{fact}} = 0.85_{-0.12}^{+0.07}$ , as in Eq. (24). The experimental data is:  $\mathcal{B}(B_d^0 \rightarrow \pi^+ \pi^-) = (5.16 \pm 0.22) \times 10^{-6}$ ,  $\mathcal{B}(B_s^0 \rightarrow \pi^+ K^-) = (5.0 \pm 1.1) \times 10^{-6}$ . We expect

$$\left| \frac{C_2}{C_1} \right|_{\text{fact}}^2 \frac{\mathcal{B}(B_d^0 \rightarrow \pi^+ \pi^-)}{\mathcal{B}(B_s^0 \rightarrow \pi^+ K^-)} \quad (27)$$

to be consistent with 1. It equals  $1.43 \pm 0.40$ . We also expect the direct  $CP$  asymmetries to be equal. As noted above, LHCb finds  $A_{CP}(B_s^0 \rightarrow \pi^+ K^-) = 0.27 \pm 0.08$  [14]. This is in good agreement with the *BABAR* measurement of  $A_{CP}(B_d^0 \rightarrow \pi^+ \pi^-)$  in Table I, but not that of Belle. Thus, at this point the question of nonfactorizable SU(3) breaking in this decay pair is not clear.

- (3)  $A(B_d^0 \rightarrow \pi^0 K^0) = \sqrt{3}A(B_d^0 \rightarrow \eta_8 K^0)$ : we expect  $\mathcal{B}(B_d^0 \rightarrow \pi^0 K^0) = |C'_1/C'_2|_{\text{fact}}^2 3\mathcal{B}(B_d^0 \rightarrow \eta_8 K^0)$  and  $A_{CP}(B_d^0 \rightarrow \pi^0 K^0) = A_{CP}(B_d^0 \rightarrow \eta_8 K^0)$ .
- (4)  $A(B_d^0 \rightarrow \pi^0 \pi^0) = A(B_s^0 \rightarrow \pi^0 \bar{K}^0) = \sqrt{3}A(B_s^0 \rightarrow \eta_8 \bar{K}^0)$ : this leads to the prediction that  $A_{CP}(B_s^0 \rightarrow \pi^0 \bar{K}^0) = A_{CP}(B_s^0 \rightarrow \eta_8 \bar{K}^0) = 0.43_{-0.24}^{+0.25}$ . Also, we expect that  $\mathcal{B}(B_s^0 \rightarrow \pi^0 \bar{K}^0) = (1.55 \pm 0.16) \times 10^{-6}$ ,  $\mathcal{B}(B_s^0 \rightarrow \eta_8 \bar{K}^0) = (0.52 \pm 0.05) \times 10^{-6}$ , modulo factorizable SU(3) corrections.
- (5)  $A(B^+ \rightarrow \pi^+ K^0) = A(B_s^0 \rightarrow K^0 \bar{K}^0)$ , so that the direct  $CP$  asymmetries are expected to be equal for these decays, and we expect  $\mathcal{B}(B^+ \rightarrow \pi^+ K^0) = |C'_1/C'_2|_{\text{fact}}^2 \mathcal{B}(B_s^0 \rightarrow K^0 \bar{K}^0)$ .
- (6)  $A(B_d^0 \rightarrow K^0 \bar{K}^0) = A(B^+ \rightarrow K^+ \bar{K}^0) = \sqrt{3}A(B_d^0 \rightarrow \pi^0 \eta_8)$ : we expect the direct  $CP$  asymmetries for these three decays to be equal. Also, we expect that  $\mathcal{B}(B_d^0 \rightarrow K^0 \bar{K}^0) = \mathcal{B}(B^+ \rightarrow K^+ \bar{K}^0) = 3\mathcal{B}(B_d^0 \rightarrow \pi^0 \eta_8)$ , modulo factorizable SU(3) corrections.

Note: It would not be a surprise to see evidence of significant nonfactorizable effects in the decays in items 4–6, as these are dominated by diagrams for which factorization is not expected to hold.

#### IV. THREE-BODY DECAYS

We now turn to  $B \rightarrow PPP$  decays. In the past, such decays were little studied—most of the theoretical work looking at clean methods for obtaining the weak phases focused on two-body  $B$  decays. This is essentially for

two reasons: (i) final states such as  $\psi K_S$ ,  $\pi^+ \pi^-$ , etc. are  $CP$  eigenstates, and (ii) when there is a second decay amplitude, with a different weak phase, it has been possible to find methods to remove this ‘‘pollution,’’ and cleanly extract weak-phase information.

Things are not the same in the case of three-body  $B$  decays. First, because there are three particles, final states such as  $K_S \pi^+ \pi^-$  are not  $CP$  eigenstates—the value of its  $CP$  depends on whether the relative  $\pi^+ \pi^-$  angular momentum is even or odd. Second, even if it were possible to distinguish the states of  $CP +$  and  $-$ , one still has the problem of removing the pollution due to additional decay amplitudes. For these reasons, the conventional wisdom has been that it is not possible to obtain clean weak-phase information from three-body decays.

Recently, it was shown that, by doing a diagrammatic analysis, one can address these two problems [18]. First, a Dalitz-plot analysis can be used to experimentally separate the  $CP +$  and  $-$  final states. Second, one can often remove the pollution of additional diagrams and cleanly measure the  $CP$  phases. In Ref. [19], the procedure for extracting  $\gamma$  from  $B \rightarrow K \pi \pi$  decays was described in detail. Thus, in fact, it is possible to use three-body decays to obtain weak-phase information and search for new physics.

In this paper, the goal is to find pairs of  $\bar{b} \rightarrow \bar{d}$  and  $\bar{b} \rightarrow \bar{s}$  decays which satisfy Eq. (3) and permit the measurement of  $X$ . As we will see, in order to do this with three-body decays, the diagrammatic decomposition of Ref. [18] is necessary.

##### A. U-spin pairs

As with  $B \rightarrow PP$  decays (Sec. III A), we look for pairs of  $\bar{b} \rightarrow \bar{s}$  and  $\bar{b} \rightarrow \bar{d}$  decays that are related by U-spin reflection. We find that there are seven such pairs of three-body decays:

- (1)  $B_s^0 \rightarrow K^+ K^- \bar{K}^0$  and  $B_d^0 \rightarrow K^0 \pi^+ \pi^-$ ,
- (2)  $B_s^0 \rightarrow \bar{K}^0 \pi^+ \pi^-$  and  $B_d^0 \rightarrow K^+ K^0 K^-$ ,
- (3)  $B_d^0 \rightarrow K^0 K^- \pi^+$  and  $B_s^0 \rightarrow K^+ \bar{K}^0 \pi^-$ ,
- (4)  $B_d^0 \rightarrow K^+ \bar{K}^0 \pi^-$  and  $B_s^0 \rightarrow K^0 K^- \pi^+$ ,
- (5)  $B^+ \rightarrow \pi^+ \pi^+ \pi^-$  and  $B^+ \rightarrow K^+ K^+ K^-$ ,
- (6)  $B^+ \rightarrow K^+ K^- \pi^+$  and  $B^+ \rightarrow K^+ \pi^+ \pi^-$ ,
- (7)  $B_s^0 \rightarrow \bar{K}^0 \bar{K}^0 K^0$  and  $B_d^0 \rightarrow K^0 K^0 \bar{K}^0$ .

In order to show that these pairs do indeed satisfy Eq. (3), one has to compare the amplitudes of the decays within a pair.

Under U spin, the  $d$  and  $s$  quarks are in a doublet, as are  $\bar{s}$  and  $-\bar{d}$ . Thus,  $K^+$  and  $\pi^+$ , and  $K^-$  and  $\pi^-$ , are considered to be identical particles. We therefore see that the final states of pairs 1–4 contain no identical particles. One can straightforwardly compare the amplitudes of the decays within these pairs. We refer to Ref. [18] for a description of the diagrams; here we label each diagram  $D$  by an index  $q$  ( $q = u, d, s$ ) denoting the flavor of the quark pair ‘‘popped’’ from the vacuum. Under isospin symmetry,  $D_u = D_d$ , under U spin,  $D_d = D_s$ , and under full SU(3),  $D_u = D_d = D_s$ . We have:

pair 1:

$$\begin{aligned} A(B_s^0 \rightarrow K^+ K^- \bar{K}^0) &= -T_{1,s} e^{i\gamma} - C_{1,s} e^{i\gamma} - \hat{P}_{a;uc} e^{i\gamma} - \hat{P}_{a;tc} e^{-i\beta} - \frac{2}{3} P_{EW1,s} e^{-i\beta} \\ &\quad + \frac{1}{3} P_{EW1,u} e^{-i\beta} - \frac{2}{3} P_{EW1,s}^C e^{-i\beta} + \frac{1}{3} P_{EW2,u}^C e^{-i\beta}, \\ A(B_d^0 \rightarrow K^0 \pi^+ \pi^-) &= -T'_{1,d} e^{i\gamma} - C'_{1,d} e^{i\gamma} - \tilde{P}'_{a;uc} e^{i\gamma} + \tilde{P}'_{a;tc} + \frac{2}{3} P'_{EW1,d} - \frac{1}{3} P'_{EW1,u} + \frac{2}{3} P'^C_{EW1,d} - \frac{1}{3} P'^C_{EW2,u}, \end{aligned} \quad (28)$$

pair 2:

$$\begin{aligned} A(B_s^0 \rightarrow \bar{K}^0 \pi^+ \pi^-) &= -T_{2,d} e^{i\gamma} - C_{1,d} e^{i\gamma} - \tilde{P}_{b;uc} e^{i\gamma} - \tilde{P}_{b;tc} e^{-i\beta} - \frac{2}{3} P_{EW1,d} e^{-i\beta} + \frac{1}{3} P_{EW1,u} e^{-i\beta} \\ &\quad + \frac{1}{3} P_{EW1,u}^C e^{-i\beta} - \frac{2}{3} P_{EW2,d}^C e^{-i\beta}, \\ A(B_d^0 \rightarrow K^+ K^0 K^-) &= -T'_{2,s} e^{i\gamma} - C'_{1,s} e^{i\gamma} - \hat{P}'_{b;uc} e^{i\gamma} + \hat{P}'_{b;tc} + \frac{2}{3} P'_{EW1,s} - \frac{1}{3} P'_{EW1,u} - \frac{1}{3} P'^C_{EW1,u} + \frac{2}{3} P'^C_{EW2,s}, \end{aligned} \quad (29)$$

pair 3:

$$\begin{aligned} A(B_d^0 \rightarrow K^0 K^- \pi^+) &= -T_{2,s} e^{i\gamma} - \hat{P}_{b;uc} e^{i\gamma} - \hat{P}_{b;tc} e^{-i\beta} + \frac{1}{3} P_{EW1,u}^C e^{-i\beta} - \frac{2}{3} P_{EW2,s}^C e^{-i\beta}, \\ A(B_s^0 \rightarrow K^+ \bar{K}^0 \pi^-) &= -T'_{2,d} e^{i\gamma} - \tilde{P}'_{b;uc} e^{i\gamma} + \tilde{P}'_{b;tc} - \frac{1}{3} P'^C_{EW1,u} + \frac{2}{3} P'^C_{EW2,d}, \end{aligned} \quad (30)$$

pair 4:

$$\begin{aligned} A(B_d^0 \rightarrow K^+ \bar{K}^0 \pi^-) &= -T_{1,s} e^{i\gamma} - \hat{P}_{a;uc} e^{i\gamma} - \hat{P}_{a;tc} e^{-i\beta} - \frac{2}{3} P_{EW1,s}^C e^{-i\beta} + \frac{1}{3} P_{EW2,u}^C e^{-i\beta}, \\ A(B_s^0 \rightarrow K^0 K^- \pi^+) &= -T'_{1,d} e^{i\gamma} - \tilde{P}'_{a;uc} e^{i\gamma} + \tilde{P}'_{a;tc} + \frac{2}{3} P'^C_{EW1,d} - \frac{1}{3} P'^C_{EW2,u}, \end{aligned} \quad (31)$$

where

$$\tilde{P}_a \equiv P_{1,d} + P_{2,u}, \quad \tilde{P}_b \equiv P_{1,u} + P_{2,d}, \quad \hat{P}_a \equiv P_{1,s} + P_{2,u}, \quad \hat{P}_b \equiv P_{1,u} + P_{2,s}. \quad (32)$$

For  $\bar{b} \rightarrow \bar{d}$  transitions, the diagrams are written without primes; for  $\bar{b} \rightarrow \bar{s}$  transitions, they are written with primes. (The overall signs of the amplitudes assume  $\bar{u}$  is negative, as with isospin. If one takes  $\bar{d}$  to be negative, as with U spin, one may obtain a different overall sign. But the physics does not change.)

There are two truly identical particles in the final states in pair 5 ( $\pi^+$  in  $B^+ \rightarrow \pi^+ \pi^+ \pi^-$  and  $K^+$  in  $B^+ \rightarrow K^+ K^+ K^-$ ), so the overall wave function must be symmetric with respect to the exchange of these two particles:

$$\begin{aligned} A(B^+ \rightarrow \pi^+ \pi^+ \pi^-)_{\text{sym}} &= -T_{2,d} e^{i\gamma} - C_{1,d} e^{i\gamma} - \tilde{P}_{b;uc} e^{i\gamma} - \tilde{P}_{b;tc} e^{-i\beta} - \frac{2}{3} P_{EW1,d} e^{-i\beta} + \frac{1}{3} P_{EW1,u} e^{-i\beta} \\ &\quad + \frac{1}{3} P_{EW1,u}^C e^{-i\beta} - \frac{2}{3} P_{EW2,d}^C e^{-i\beta}, \\ A(B^+ \rightarrow K^+ K^+ K^-)_{\text{sym}} &= -T'_{2,s} e^{i\gamma} - C'_{1,s} e^{i\gamma} - \hat{P}'_{b;uc} e^{i\gamma} + \hat{P}'_{b;tc} + \frac{2}{3} P'_{EW1,s} - \frac{1}{3} P'_{EW1,u} - \frac{1}{3} P'^C_{EW1,u} + \frac{2}{3} P'^C_{EW2,s}. \end{aligned} \quad (33)$$

The penguin diagrams are defined in Eq. (32).

The final states of pair 6 contain the identical particles (under U spin)  $K^+$  and  $\pi^+$ . The overall wave function of the final  $K^+ \pi^+$  pair must be symmetrized with respect to the exchange of these two particles. If the relative angular momentum is even (odd), the U-spin state must be symmetric (antisymmetric):

$$\begin{aligned} A(B^+ \rightarrow K^+ K^- \pi^+)_{\text{sym}} &= -T_{2,s} e^{i\gamma} - C_{1,s} e^{i\gamma} - \hat{P}_{b;uc} e^{i\gamma} - \hat{P}_{b;tc} e^{-i\beta} + \frac{1}{3} P_{EW1,u} e^{-i\beta} - \frac{2}{3} P_{EW1,s} e^{-i\beta} \\ &\quad + \frac{1}{3} P_{EW1,u}^C e^{-i\beta} - \frac{2}{3} P_{EW2,s}^C e^{-i\beta}, \\ A(B^+ \rightarrow K^+ K^- \pi^+)_{\text{anti}} &= T_{2,s} e^{i\gamma} + C_{1,s} e^{i\gamma} + \hat{P}_{b;uc} e^{i\gamma} + \hat{P}_{b;tc} e^{-i\beta} + \frac{1}{3} P_{EW1,u} e^{-i\beta} - \frac{2}{3} P_{EW1,s} e^{-i\beta} \\ &\quad + \frac{1}{3} P_{EW1,u}^C e^{-i\beta} - \frac{2}{3} P_{EW2,s}^C e^{-i\beta}, \\ A(B^+ \rightarrow K^+ \pi^+ \pi^-)_{\text{sym}} &= -T'_{2,d} e^{i\gamma} - C'_{1,d} e^{i\gamma} - \tilde{P}'_{b;uc} e^{i\gamma} + \tilde{P}'_{b;tc} - \frac{1}{3} P'_{EW1,u} + \frac{2}{3} P'_{EW1,d} - \frac{1}{3} P'^C_{EW1,u} + \frac{2}{3} P'^C_{EW2,d}, \\ A(B^+ \rightarrow K^+ \pi^+ \pi^-)_{\text{anti}} &= -T'_{2,d} e^{i\gamma} - C'_{1,d} e^{i\gamma} - \tilde{P}'_{b;uc} e^{i\gamma} + \tilde{P}'_{b;tc} + \frac{1}{3} P'_{EW1,u} - \frac{2}{3} P'_{EW1,d} + \frac{1}{3} P'^C_{EW1,u} - \frac{2}{3} P'^C_{EW2,d}, \end{aligned} \quad (34)$$

where, for the antisymmetric amplitudes, diagrams with the  $K^+$  above (below) the  $\pi^+$  are multiplied by +1 (−1). The penguin diagrams are defined in Eq. (32).

Both the  $\bar{K}^0$  and  $K^0$  are contained in a U-spin triplet, and so these are considered as identical particles. Thus, the final states of the decays in pair 7 contain three identical particles and the group  $S_3$  must be used to describe their permutations. Fortunately, for these decays, the situation is less complicated. For  $K^0 K^0 \bar{K}^0$ , in any diagram, the position of the  $\bar{K}^0$  cannot change, so that only exchanges of the two  $K^0$ 's need be considered. Things are similar for  $\bar{K}^0 \bar{K}^0 K^0$ . Thus, in order to show



that these decays do indeed form a pair which respects Eq. (1), it is sufficient to examine the amplitudes which are symmetric in the exchange of the two truly identical particles. We have

$$\begin{aligned} A(B_s^0 \rightarrow \bar{K}^0 \bar{K}^0 K^0)_{\text{sym}} &= \mathcal{P}_{a;uc} e^{i\gamma} + \mathcal{P}_{a;tc} e^{-i\beta} - \frac{1}{3} P_{EW1,s} e^{-i\beta} - \frac{1}{3} P_{EW1,d} e^{-i\beta} - \frac{1}{3} P_{EW1,s}^C e^{-i\beta} - \frac{1}{3} P_{EW2,d}^C e^{-i\beta}, \\ A(B_d^0 \rightarrow K^0 K^0 \bar{K}^0)_{\text{sym}} &= \mathcal{P}'_{b;uc} e^{i\gamma} - \mathcal{P}'_{b;tc} + \frac{1}{3} P'_{EW1,s} + \frac{1}{3} P'_{EW1,d} + \frac{1}{3} P'_{EW1,d}^C + \frac{1}{3} P'_{EW2,s}^C, \end{aligned} \quad (35)$$

where

$$\mathcal{P}_a \equiv P_{1,s} + P_{2,d}, \quad \mathcal{P}_b \equiv P_{1,d} + P_{2,s}. \quad (36)$$

Now, under U spin, primed diagrams are equal to unprimed diagrams with the exchange  $d \leftrightarrow s$ , i.e. they differ only by  $\lambda_p^{(d)} \leftrightarrow \lambda_p^{(s)}$ . Thus,  $D'_s \sim D_d$ ,  $D'_d \sim D_s$ ,  $D'_u \sim D_u$ ,  $\tilde{P}'_a \sim \tilde{P}_a$ ,  $\tilde{P}'_s \sim \tilde{P}_s$ ,  $\tilde{P}'_d \sim \tilde{P}_d$ ,  $\tilde{P}'_b \sim \tilde{P}_b$ , and  $\mathcal{P}'_a \sim \mathcal{P}_b$ . We therefore see that (almost all) the amplitudes for the  $\bar{b} \rightarrow \bar{d}$  and  $\bar{b} \rightarrow \bar{s}$  decays in pairs 1–7 have the same form, modulo CKM factors (recall that the  $\bar{b} \rightarrow \bar{s}$  amplitudes include the minus sign from  $V_{tb}^* V_{ts}$  [ $P'_{tc}$  and electroweak-penguin diagrams (EWP's)]). The single exception is  $A(B^+ \rightarrow K^+ K^- \pi^+)_{\text{anti}}$  and  $A(B^+ \rightarrow K^+ \pi^+ \pi^-)_{\text{anti}}$  in pair 6. Here, recall that the contribution of a diagram is positive (negative) if the  $K^+$  is above (below) the  $\pi^+$  in that diagram. However, since the U-spin transformation switches  $K^+ \leftrightarrow \pi^+$ , we expect the antisymmetric amplitudes to have a relative  $-$  sign, and this is indeed what is found. We therefore see that, in all pairs, the amplitudes of the  $\bar{b} \rightarrow \bar{d}$  and  $\bar{b} \rightarrow \bar{s}$  decays respect Eq. (3), so that  $(X - 1)$  (U-spin breaking) can be measured using these processes.

Previously, in discussing two-body decays, we noted that the U-spin/SU(3) corrections could be separated into two types—factorizable and nonfactorizable—and that the factorizable corrections could be reliably calculated. As such, the measurement of  $X$  can be translated into a determination of the nonfactorizable corrections. In principle, this can be applied to three-body decays. In practice, however, things are more complicated. In particular, while the  $T^{(i)}$  diagram in two-body decays is, within factorization, proportional to the product of a decay constant and a form factor, in three-body decays, new structures appear. The  $T_1^{(i)}$  diagram is proportional to the product of a  $\langle 2 \text{ particles } |(V - A)|0 \rangle$  matrix element and a form factor, and the  $T_2^{(i)}$  diagram is proportional to the product of a decay constant and a  $\langle 2 \text{ particles } |(V - A)|B \rangle$  matrix element. To date, there have been no definitive calculations of these matrix elements. They have been studied in Ref. [20], but more work is clearly needed.

To this end, the measurement of  $X$  can help. Given that nonfactorizable U-spin/SU(3) breaking is expected to be subdominant compared to factorizable breaking,  $X$  as measured in the above decay pairs can be considered to be a factorizable correction (especially pairs 1–6, which have  $T_1^{(i)}/T_2^{(i)}$  contributions). The knowledge of the precise values of such factorizable effects will guide the calculation of the new matrix elements.

Finally, a natural question is whether clean weak-phase information can be extracted from these decays. For example, the pair  $B_s^0 \rightarrow K^+ K^- \bar{K}^0$  and  $B_d^0 \rightarrow K^0 \pi^+ \pi^-$  is the three-body equivalent of  $B_d^0 \rightarrow \pi^+ \pi^-$  and  $B_s^0 \rightarrow K^+ K^-$ . Can one adapt the method of Ref. [2] to obtain  $\gamma$ ? Unfortunately, the answer is no. In two-body decays, additional information is provided by the measurement of the indirect  $CP$  asymmetry in  $B_d^0 \rightarrow \pi^+ \pi^-$ . Here, however, because  $B_d^0 \rightarrow K^0 \pi^+ \pi^-$  is a three-body decay, the relative  $\pi^+ \pi^-$  angular momentum is not fixed, and so the final state is not a  $CP$  eigenstate. Thus, the measurement of the indirect  $CP$  asymmetry in this decay does not give clean information. The situation is the same for the second pair,  $B_s^0 \rightarrow \bar{K}^0 \pi^+ \pi^-$  and  $B_d^0 \rightarrow K^+ K^0 K^-$ .

In a similar vein,  $B_d^0 \rightarrow K^0 K^- \pi^+$  and  $B_s^0 \rightarrow K^+ \bar{K}^0 \pi^-$  is the three-body equivalent of  $B_s^0 \rightarrow \pi^+ K^-$  and  $B_d^0 \rightarrow \pi^- K^+$ . Can the method of Ref. [4], in which additional information comes from  $\mathcal{B}(B^+ \rightarrow \pi^+ K^0)$ , be adapted to this situation? Unfortunately, here too the answer is no. Unlike the two-body situation, here there is no other three-body decay which provides the appropriate additional information. This holds as well for pairs 4–6.

On the other hand, pair 7,  $B_s^0 \rightarrow \bar{K}^0 \bar{K}^0 K^0$  and  $B_d^0 \rightarrow K^0 K^0 \bar{K}^0$  is intriguing. The key point here is that, because there are truly identical particles in the final state, their relative angular momentum is even, and so the final state is a  $CP$  eigenstate. Now, the diagram contributing to the  $e^{i\gamma}$  piece of the  $\bar{b} \rightarrow \bar{s}$  amplitude is  $\mathcal{P}'_{b;uc}$ . In Sec. III C, we noted that  $|\mathcal{P}'_{uc}|$  is expected to be small in two-body decays, and so a direct  $CP$  asymmetry which is proportional to this diagram will also be small. If the same property holds in three-body decays, the measurement of the indirect  $CP$  asymmetry in the pure-penguin decay  $B_s^0 \rightarrow \bar{K}^0 \bar{K}^0 K^0$  cleanly probes the  $B_s^0$ - $\bar{B}_s^0$  mixing phase (experimentally, this might be easier than performing the angular analysis in  $B_s^0 \rightarrow J/\psi \phi$ , which is presently done). However, if  $\mathcal{P}'_{b;uc}$  is not small, as could happen if there are significant rescattering effects, then  $A_s$  is not negligible, and the method of Ref. [2] can be applied to this pair to obtain  $\gamma$ . Here, U-spin symmetry is assumed, but, as noted above, it is possible to measure  $X$ , which gives the size of U-spin breaking.

## B. SU(3) pairs

Unlike two-body decays, with three-body decays one cannot obtain additional pairs satisfying Eq. (3) by simply neglecting annihilation- and exchange-type diagrams.

However, there is another possibility. If, as in the two-body case, one takes isospin into account in addition to U-spin symmetry, one effectively assumes full flavor SU(3) symmetry. Under this symmetry,  $\pi$ 's and  $K$ 's are identical particles, so that the final state in all decays contains three identical particles. In this case, the six permutations of these particles (the group  $S_3$ ) must be considered. This was analyzed in Ref. [18]. For a given decay, there are six possibilities for the  $S_3$  state of the three particles: a totally symmetric state  $|S\rangle$ , a totally antisymmetric state  $|A\rangle$ , or one of four mixed states  $|M_i\rangle$  ( $i = 1-4$ ). The states are defined as follows. The final-state particles are numbered 1, 2, 3, so that the six possible orders are 123, 132, 312, 321, 231, 213. Under  $S_3$ ,

$$\begin{aligned}
 |S\rangle &\equiv \frac{1}{\sqrt{6}}(|123\rangle + |132\rangle + |312\rangle + |321\rangle + |231\rangle + |213\rangle), \\
 |M_1\rangle &\equiv \frac{1}{\sqrt{12}}(2|123\rangle + 2|132\rangle - |312\rangle - |321\rangle \\
 &\quad - |231\rangle - |213\rangle), \\
 |M_2\rangle &\equiv \frac{1}{\sqrt{4}}(|312\rangle - |321\rangle - |231\rangle + |213\rangle), \\
 |M_3\rangle &\equiv \frac{1}{\sqrt{4}}(-|312\rangle - |321\rangle + |231\rangle + |213\rangle), \\
 |M_4\rangle &\equiv \frac{1}{\sqrt{12}}(2|123\rangle - 2|132\rangle - |312\rangle + |321\rangle \\
 &\quad - |231\rangle + |213\rangle), \\
 |A\rangle &\equiv \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |312\rangle - |321\rangle + |231\rangle - |213\rangle).
 \end{aligned} \tag{37}$$

One can show that certain pairs of decays, related by SU(3) and not by U spin, satisfy Eq. (1), but only for the state  $|S\rangle$  (in most cases). This applies to the following SU(3) pairs<sup>6</sup> (as is standard, we neglect annihilation- and exchange-type diagrams):

- (i)  $(B^+ \rightarrow \pi^+ K^- K^+, B^+ \rightarrow \pi^+ \pi^0 \pi^0, B^+ \rightarrow \pi^+ \pi^+ \pi^-, B_s^0 \rightarrow \bar{K}^0 \pi^+ \pi^-)$  and  $(B_d^0 \rightarrow K^+ K^- K^0, B^+ \rightarrow K^+ K^+ K^-, B^+ \rightarrow K^+ \pi^+ \pi^-)$ ,
- (ii)  $(B^+ \rightarrow K^+ \bar{K}^0 \pi^0, B^+ \rightarrow \bar{K}^0 K^+ \eta_8)$  and  $(B^+ \rightarrow K^0 \pi^+ \pi^0, B_d^0 \rightarrow K^+ \pi^- \pi^0, B^+ \rightarrow K^0 \pi^+ \eta_8)$ ,
- (iii)  $B_s^0 \rightarrow \bar{K}^0 \bar{K}^0 K^0$  and  $(B^+ \rightarrow K^+ K^0 \bar{K}^0, B_d^0 \rightarrow K^0 K^0 \bar{K}^0)$ ,
- (iv)  $B_d^0 \rightarrow \pi^0 \pi^0 \pi^0$  and  $B_d^0 \rightarrow K^0 \pi^0 \pi^0$ ,
- (v)  $(B_d^0 \rightarrow K^- K^+ \pi^0, B_d^0 \rightarrow K^- K^+ \eta_8)$  and  $(B_s^0 \rightarrow \pi^0 \pi^0 \eta_8, B_s^0 \rightarrow \pi^+ \pi^- \eta_8)$ ,
- (vi)  $B_s^0 \rightarrow \bar{K}^0 \pi^0 \pi^0$  and  $B_d^0 \rightarrow K^0 \pi^0 \pi^0$ ,
- (vii)  $B_s^0 \rightarrow \bar{K}^0 \pi^0 \eta_8$  and  $B_d^0 \rightarrow K^0 \pi^0 \eta_8$ ,
- (viii)  $B_s^0 \rightarrow \bar{K}^0 \eta_8 \eta_8$  and  $B_d^0 \rightarrow K^0 \eta_8 \eta_8$ .

<sup>6</sup>Note: This list includes some U-spin pairs. These pairs are related for all  $S_3$  states.

The decays in the first (second) parentheses are  $\bar{b} \rightarrow \bar{d}$  ( $\bar{b} \rightarrow \bar{s}$ ) transitions.

In order to establish which states are the same (modulo CKM factors) for the decays within a pair, one writes the amplitudes for each decay in terms of diagrams, noting the order of the final-state particles for each diagram. It is this order which determines which  $S_3$  states are common to both decays. The state  $|S\rangle$  is symmetric in all possible orders. Thus, as long as the two amplitudes are comprised of the same diagrams, the final-state order is unimportant, and the two decays are related by SU(3) for  $|S\rangle$ . For  $|A\rangle$ , if the first decay amplitude contains the diagram  $D$  with the order  $ijk$ , the second decay amplitude must contain  $D$  with a cyclic permutation of  $ijk$ , or  $-D$  with an anticyclic permutation of  $ijk$ .

The mixed states are more complicated. The six elements of  $S_3$  are:  $I$  (identity),  $P_{12}$  (exchanges particles 1 and 2),  $P_{13}$  (exchanges particles 1 and 3),  $P_{23}$  (exchanges particles 2 and 3),  $P_{\text{cyclic}}$  (cyclic permutation of particle numbers, i.e.  $1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1$ ),  $P_{\text{anticyclic}}$  (anticyclic permutation of particle numbers, i.e.  $1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2$ ). The point is that, under the group transformations,  $|M_1\rangle$  and  $|M_3\rangle$  transform among themselves. Writing

$$|M_1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |M_3\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{38}$$

we can represent each group element by a  $2 \times 2$  matrix:

$$\begin{aligned}
 I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_{12} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \\
 P_{13} &= \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad P_{23} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
 P_{\text{cyclic}} &= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad P_{\text{anticyclic}} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.
 \end{aligned} \tag{39}$$

Similarly, if we write

$$|M_2\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |M_4\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{40}$$

the  $S_3$  matrices take the same form, showing that  $|M_2\rangle$  and  $|M_4\rangle$  also transform among themselves.

From the above matrices, we see that the first rows of the matrices are the same for  $(I, P_{23})$ ,  $(P_{12}, P_{\text{cyclic}})$  and  $(P_{13}, P_{\text{anticyclic}})$ . This indicates that the symmetric mixed states ( $|M_1\rangle$  and  $|M_2\rangle$ ) are the same for the two decays if the particle orders for a given diagram are [(123) or (132)], [(213) or (231)], or [(321) or (312)]. For the antisymmetric mixed states ( $|M_3\rangle$  and  $|M_4\rangle$ ), things are the same, except that there is an additional minus sign if the particle order is anticyclic (this can be seen from the second rows of the matrices).

To demonstrate how this works, we present several examples. First, consider the decays  $B^+ \rightarrow \pi^+ K^- K^+$  and  $B_d^0 \rightarrow K^+ K^- K^0$ . For  $B^+ \rightarrow \pi^+ K^- K^+$  we take particle 1 as  $\pi^+$ , particle 2 as  $K^-$ , and particle 3 as  $K^+$ . The amplitude is

$$A(B^+ \rightarrow \pi^+ K^- K^+) = -T_{2,s} e^{i\gamma}(123) - C_{1,s} e^{i\gamma}(132) - \hat{P}_{b;uc} e^{i\gamma}(123) - \hat{P}_{b;tc} e^{-i\beta}(123) + \frac{1}{3} P_{EW1,u} e^{-i\beta}(231) - \frac{2}{3} P_{EW1,s} e^{-i\beta}(321) + \frac{1}{3} P_{EW1,u}^C e^{-i\beta}(321) - \frac{2}{3} P_{EW2,s}^C e^{-i\beta}(321), \quad (41)$$

where the particle order for each diagram (top to bottom) is given in parentheses. We have continued to label each diagram by an index denoting the flavor of the popped quark pair, but under SU(3), these are all equal. For  $B_d^0 \rightarrow K^+ K^- K^0$ , we take particle 1 as  $K^+$ , particle 2 as  $K^-$ , and particle 3 as  $K^0$ . The amplitude is

$$A(B_d^0 \rightarrow K^+ K^- K^0) = -T'_{2,s} e^{i\gamma}(123) - C'_{1,s} e^{i\gamma}(312) - \hat{P}'_{b;uc} e^{i\gamma}(123) + \hat{P}'_{b;tc}(123) - \frac{1}{3} P'_{EW1,u}(213) + \frac{2}{3} P'_{EW1,s}(123) - \frac{1}{3} P_{EW1,u}^C(321) + \frac{2}{3} P_{EW2,s}^C(321). \quad (42)$$

The penguin diagrams for the two decays are defined in Eq. (32). Comparing the two amplitudes, we see that, due to  $C_{1,s}$  and  $P_{EW1,s}$ ,  $|A\rangle$  and the mixed states are not common. Therefore, the two decays are related only for  $|S\rangle$ .

Consider  $B_s^0 \rightarrow K^0 \bar{K}^0 K^0$  and  $B^+ \rightarrow K^+ K^0 \bar{K}^0$ . For  $B_s^0 \rightarrow K^0 \bar{K}^0 K^0$ , particle 1 is  $K^0$ , and particles 2 and 3 are  $\bar{K}^0$  (consistent with the choice of mixed states above), so that  $|M_3\rangle = |M_4\rangle = |A\rangle = 0$ . The amplitude is

$$A(B_s^0 \rightarrow K^0 \bar{K}^0 K^0) = \mathcal{P}_{a;uc} e^{i\gamma}(213) + \mathcal{P}_{a;tc} e^{-i\beta}(213) - \frac{1}{3} P_{EW1,s} e^{-i\beta}(123) - \frac{1}{3} P_{EW1,d} e^{-i\beta}(213) - \frac{1}{3} P_{EW1,s}^C e^{-i\beta}(213) - \frac{1}{3} P_{EW2,d}^C e^{-i\beta}(213). \quad (43)$$

For  $B^+ \rightarrow K^+ K^0 \bar{K}^0$ , we take particle 1 as  $K^+$ , particle 2 as  $K^0$ , and particle 3 as  $\bar{K}^0$ . The amplitude is

$$\sqrt{2}A(B^+ \rightarrow K^+ K^0 \bar{K}^0) = \mathcal{P}'_{b;uc} e^{i\gamma}(231) - \mathcal{P}'_{b;tc}(231) + \frac{1}{3} P'_{EW1,s}(231) + \frac{1}{3} P'_{EW1,d}(321) + \frac{1}{3} P_{EW2,s}^C(132) + \frac{1}{3} P_{EW1,d}^C(132). \quad (44)$$

The penguin diagrams for the two decays are defined in Eq. (36). Because of the EWP's, we see that the two decays are related only for  $|S\rangle$ .

Consider  $B^+ \rightarrow \pi^0 \bar{K}^0 K^+$  and  $B^+ \rightarrow \pi^0 K^0 \pi^+$ . For  $B^+ \rightarrow \pi^0 \bar{K}^0 K^+$ , we take particle 1 as  $\pi^0$ , particle 2 as  $\bar{K}^0$ , and particle 3 is  $K^+$ . The amplitude is

$$\sqrt{2}A(B^+ \rightarrow \pi^0 \bar{K}^0 K^+) = -T_{1,s} e^{i\gamma}(321) - C_{2,s} e^{i\gamma}(321) + \mathcal{P}_{b;uc} e^{i\gamma}(123) - \hat{P}_{a;uc} e^{i\gamma}(231) + \mathcal{P}_{b;tc} e^{-i\beta}(123) - \hat{P}_{a;tc} e^{-i\beta}(231) - P_{EW2,s} e^{-i\beta}(123) - \frac{1}{3} P_{EW1,d}^C e^{-i\beta}(321) - \frac{2}{3} P_{EW1,s}^C e^{-i\beta}(132) + \frac{1}{3} P_{EW2,u}^C e^{-i\beta}(132) - \frac{1}{3} P_{EW2,s}^C e^{-i\beta}(321). \quad (45)$$

The penguin diagrams are defined in Eqs. (32) and (36). For  $B^+ \rightarrow \pi^0 K^0 \pi^+$ , we take particle 1 as  $\pi^0$ , particle 2 as  $K^0$ , and particle 3 is  $\pi^+$ . The amplitude is

$$\sqrt{2}A(B^+ \rightarrow \pi^0 K^0 \pi^+) = -T'_{1,d} e^{i\gamma}(321) - C'_{2,d} e^{i\gamma}(321) + P'_{EW2,d}(123) + \frac{1}{3} P_{EW1,u}^C(312) + \frac{2}{3} P_{EW1,d}^C(132). \quad (46)$$

Under SU(3),  $\mathcal{P}_b = \hat{P}_a$ . Thus, in order for the gluonic-penguin contribution to cancel in Eq. (45) above, we need a state which is symmetric in  $(123) \leftrightarrow (231)$ . This is  $|S\rangle$  or  $|A\rangle$ —mixed states are excluded. However,  $|A\rangle$  is itself excluded by the  $P_{EW1}^C$  contribution—apart from CKM factors, it has the same sign in the two amplitudes, despite the particle order being cyclic in one case and anticyclic in the other. Thus, the two decay amplitudes are related only for  $|S\rangle$ .

Finally, consider  $B^+ \rightarrow \pi^- \pi^+ \pi^+$  and  $B^+ \rightarrow \pi^- K^+ \pi^+$ . For  $B^+ \rightarrow \pi^- \pi^+ \pi^+$ , particle 1 is  $\pi^-$ , and particles 2 and 3 are  $\pi^+$ . This implies that  $|M_3\rangle = |M_4\rangle = |A\rangle = 0$ . The amplitude is

$$A(B^+ \rightarrow \pi^- \pi^+ \pi^+) = -T_{2,d} e^{i\gamma}(213) - C_{1,d} e^{i\gamma}(231) - \tilde{P}_{b;uc} e^{i\gamma}(213) - \tilde{P}_{b;tc}(213) + \frac{1}{3} P_{EW1,u}(123) - \frac{2}{3} P_{EW1,d}(213) + \frac{1}{3} P_{EW1,u}^C(213) - \frac{2}{3} P_{EW2,d}^C(213). \quad (47)$$

For  $B^+ \rightarrow \pi^- K^+ \pi^+$ , take particle 1 as  $\pi^-$ , particle 2 as  $K^+$ , and particle 3 is  $\pi^+$ . All six  $S_3$  states allowed. The amplitude is

$$\begin{aligned}
A(B^+ \rightarrow \pi^- K^+ \pi^+) &= -T'_{2,d} e^{i\gamma}(213) - C'_{1,d} e^{i\gamma}(231) - \tilde{P}'_{b;uc} e^{i\gamma}(213) + \tilde{P}'_{b;tc}(213) - \frac{1}{3} P'_{EW1,u}(132) + \frac{2}{3} P'_{EW1,d}(312) \\
&\quad - \frac{1}{3} P'_{EW1,u}(312) + \frac{2}{3} P'_{EW2,d}(312).
\end{aligned} \tag{48}$$

The penguin diagrams for the two decays are defined in Eq. (32). All states with  $2 \leftrightarrow 3$  symmetry are allowed. Thus, unlike the above cases, the two decay amplitudes are related for  $|S\rangle$ ,  $|M_1\rangle$ , and  $|M_2\rangle$ . This is a special case. Here, the processes are identical, save for the flavor of the decay quark ( $d$  or  $s$ ). As a result, the amplitudes are equal for all nonzero states. There is one other pair like this— $B^+ \rightarrow K^- \pi^+ K^+$  and  $B^+ \rightarrow K^- K^+ K^+$ . For all other pairs, the two decay amplitudes are related only for  $|S\rangle$  (or for all  $S_3$  states in the case of U-spin pairs).

Now, in Refs. [18,19] it was shown how the  $S_3$  states can be determined experimentally. Below we review the method, focusing on the state  $|S\rangle$ . Consider the decay  $B^+ \rightarrow \pi^+ K^- K^+$ . The Dalitz-plot events can be described by  $s_+ = (p_{\pi^+} + p_{K^+})^2$  and  $s_- = (p_{\pi^+} + p_{K^-})^2$ , so that the decay amplitude,  $\mathcal{M}(s_+, s_-)$ , can be extracted. We introduce the third Mandelstam variable,  $s_0 = (p_{K^+} + p_{K^-})^2$ . It is related to  $s_+$  and  $s_-$  as follows:

$$s_+ + s_- + s_0 = m_B^2 + m_\pi^2 + 2m_K^2. \tag{49}$$

The totally symmetric SU(3) decay amplitude is then given by

$$\begin{aligned}
|S\rangle &= \frac{1}{\sqrt{6}} [\mathcal{M}(s_+, s_-) + \mathcal{M}(s_-, s_+) + \mathcal{M}(s_+, s_0) \\
&\quad + \mathcal{M}(s_0, s_+) + \mathcal{M}(s_0, s_-) + \mathcal{M}(s_-, s_0)].
\end{aligned} \tag{50}$$

The state  $|S\rangle$  can be determined for the other decays similarly. With this, the size of U-spin/SU(3) breaking can be found through the measurement of  $X$  using any of the SU(3) pairs.

### C. Other signals of SU(3) breaking

Finally, we note that there are certain decays which have identical amplitudes for the totally symmetric state  $|S\rangle$ . They are given by the processes within parentheses in the list in Sec. IV B. For these, the branching ratios and direct  $CP$  asymmetries should be equal in the SU(3) limit. Thus, by obtaining the state  $|S\rangle$  for these decays, the measurement of these quantities constitutes a further test of SU(3) breaking.

## V. CONCLUSIONS

Within U-spin symmetry ( $d \leftrightarrow s$ ), the amplitudes for certain charmless  $\bar{b} \rightarrow \bar{d}$  and  $\bar{b} \rightarrow \bar{s}$  decays are equal, apart from CKM matrix elements. Using this, two methods were proposed for extracting weak-phase information from measurements of particular U-spin decay pairs. The theoretical uncertainty of these methods must include the issue of U-spin breaking. In general, theoretical input is used to address this. However, one of the points of the present

paper is that this breaking can be measured experimentally. Under U spin, the branching ratios and direct  $CP$  asymmetries of the two decays are not independent—there is a relation among them. Thus, one can determine U-spin breaking by measuring the four observables, and seeing the extent to which this relation is not satisfied.

Furthermore, if one neglects annihilation- and exchange-type diagrams, there are additional pairs of  $B$  decays whose amplitudes are equal, apart from CKM matrix elements. In this case, the symmetry is flavor SU(3). Here, too, the relation among the four observables holds in the SU(3) limit, so that SU(3)-breaking effects can be determined from the measurements of these quantities.

In this paper, we present the list of two-body  $B$  decay pairs from which the size of the breaking can be obtained. In fact, there are five such pairs for which these measurements have been done. We present this data, along with the determination of U-spin/SU(3) breaking. In many such decays, the calculation of the factorizable contribution to the breaking is reliable. Taking this into account, one can measure the size of nonfactorizable effects. It is expected that these are small. However, there is one decay pair— $B_d^0 \rightarrow \pi^+ \pi^-$  and  $B_d^0 \rightarrow \pi^- K^+$ —which may exhibit large nonfactorizable breaking, depending on whether one uses the *BABAR* or *Belle* measurement of the  $B_d^0 \rightarrow \pi^+ \pi^-$  direct  $CP$  asymmetry. Although this result is not conclusive, it does raise questions about analyses which neglect nonfactorizable U-spin/SU(3) breaking.

We also present the list of three-body  $B$  decay pairs whose amplitudes are the same, apart from CKM factors. However, here the situation is more complicated. Under SU(3), the final-state particles are all identical, and the equality of amplitudes holds (almost always) only for the totally symmetric final state  $|S\rangle$ . Thus, this state must be isolated experimentally in order to measure SU(3) breaking, and we describe how to do this.

We discuss the decay pairs whose amplitudes are equal, including CKM factors, within SU(3). For two-body decays, the size of SU(3) breaking is indicated by comparing the branching ratios and direct  $CP$  asymmetries of the two decays. For three-body decays, once again the equality of amplitudes holds only for  $|S\rangle$ , so that this state must be distinguished in order to probe SU(3) breaking.

Finally, we note in passing that the pure-penguin decay  $B_s^0 \rightarrow \bar{K}^0 \bar{K}^0 K^0$  is particularly interesting. Here the final state is a  $CP$  eigenstate. Thus, given that the direct  $CP$  asymmetry is expected to be small, the measurement of the indirect  $CP$  asymmetry in this decay cleanly probes the  $B_s^0$ - $\bar{B}_s^0$  mixing phase. This might be easier experimentally than performing the angular analysis in  $B_s^0 \rightarrow J/\psi \phi$ , which is what is done at present.

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