Minimal modification to tribimaximal mixing

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We explore some ways of minimally modifying the neutrino mixing matrix from tribimaximal, characterized by introducing at most one mixing angle and a CP violating phase thus extending our earlier work. One minimal modification, motivated to some extent by group theoretic considerations, is a simple case with the elements $V_{\alpha 2}$ of the second column in the mixing matrix equal to $1/\sqrt{3}$. Modifications by keeping one of the columns or one of the rows unchanged from tribimaximal mixing all belong to the class of minimal modification. Some of the cases have interesting experimentally testable consequences. In particular, the T2K and MINOS collaborations have recently reported indications of a nonzero θ_{13}. For the cases we consider, the new data sharply constrain the CP violating phase angle δ, with δ close to 0 (in some cases) and π disfavored.

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I. INTRODUCTION

Mixing of different neutrino species has been established by various experiments [1]. Recently the T2K [2] and MINOS [3] collaborations have reported indications of a nonzero θ_{13} providing new evidence for neutrino mixing and new information about the mixing pattern. The mixing can be represented by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) [4] mixing matrix V in the charged current interaction of the W boson with left-handed charged leptons l_L and neutrinos ν_L, $\mathcal{L} = -(g/\sqrt{2})\bar{l}_L\gamma^\mu V\nu_L W^+_\mu +$ H.c.. The elements of the unitary matrix V are usually indicated by $V_{\alpha j}$ with $\alpha = e, \mu, \tau, \ldots$ and $j = 1, 2, 3, \ldots$. With three neutrinos, there are three mixing angles, one Dirac phase, and possibly two Majorana phases if neutrinos are Majorana particles.

It is possible to fit data from various experiments [5], except for possible anomalies in the LSND, MiniBoone [6], and MINOS [7] data. The pre-T2K data, with modified Gallium cross section used by the SAGE Collaboration [8], on mixing and mass parameters, can be summarized at the $1\sigma(3\sigma)$ level as [5,9]

$$\begin{aligned}
\Delta m^2_{21} &= 7.59 \pm 0.20\binom{+0.61}{-0.69} \times 10^{-5}\ \mathrm{eV}^2,\\
\Delta m^2_{31} &= -2.36 \pm 0.11(\pm 0.37)\\
&\quad \times [+2.46 \pm 0.12(\pm 0.37)] \times 10^{-3}\ \mathrm{eV}^2,\\
\theta_{12} &= 34.5 \pm 1.0\binom{+3.2}{-2.8}^\circ,\\
\theta_{23} &= 42.8^{+4.7}_{-2.9}\binom{+10.7}{-7.3}^\circ,\\
\theta_{13} &= 5.1^{+3.0}_{-3.3}(\leq 12.0)^\circ. \qquad (1)
\end{aligned}$$

Here the mixing parameters are those of the Particle Data Group (PDG) parametrization [1] (with $\theta_{12} \equiv \theta_{e2}$, $\theta_{23} \equiv \theta_{\mu 3}$, $\theta_{13} \equiv \theta_{e3}$). There is no direct experimental information on the phases δ_{PDG}.

The mixing pattern is well approximated by the so-called tribimaximal mixing pattern of the form [10]

$$V_{\mathrm{TB}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \qquad (2)$$

Note that with this mixing pattern, θ_{13} is equal to zero.

The recent T2K data on θ_{13} (and therefore $|V_{e3}|$) show that [2] at 90% C.L. it is not zero with $\sin^2 2\theta_{13}$ in the range of 0.03 (0.04)–0.28 (0.34) for normal (inverted) neutrino mass hierarchy. Data from MINOS [3] also disfavor $\theta_{13} = 0$ at 89% C.L. This information provides an important constraint on theoretical model building for neutrino mixing [11]. Combining the recent T2K and MINOS data with previous neutrino oscillation data and using the new reactor flux in Ref. [12], more stringent constraints on the mixing parameters than those given in Eq. (1) have been obtained [13]. These new constraints are shown in Table I. We will use them for our later discussions.

The combined data show that θ_{13} is nonzero at 3σ level. In fact the pre-T2K data already provide a hint that θ_{13} is nonzero at a more-than-1σ level [9]. A nonzero V_{e3} would rule out tribimaximal mixing. Now the tribimaximal mixing prediction for the angle θ_{12} is outside the 1σ level. Also, because one of the elements in the mixing matrix is zero, the tribimaximal mixing does not allow CP violation to be manifest in neutrino oscillation, i.e. the Jarlskog parameter [14] J is identically equal to zero. The combined

TABLE I. Ranges for mixing parameters obtained in Ref. [13].

Parameter	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\sin^2\theta_{13}$
Best fit	0.312	0.42	0.025
1σ range	0.296–0.329	0.39–0.50	0.018–0.032
2σ range	0.280–0.347	0.36–0.60	0.012–0.041
3σ range	0.265–0.364	0.34–0.64	0.005–0.050

experimental efforts [15–19] will be able to measure *CP* violation in neutrino oscillation [20]. Were *CP* violation to be established in the future, it would definitively rule out tribimaximal mixing. We also recall that *CP* violation in the lepton sector has a profound implication regarding why our Universe is dominated by matter. There is thus an additional motivation to study deviations from tribimaximal mixing.

Phenomenologically, small deviations from the tribimaximal pattern can be easily parametrized in terms of three small parameters and studied [21,22]. In [23], we studied a minimal modification with one complex parameter. Here we extend this discussion.

Theoretically, many attempts have been made to derive the tribimaximal mixing, but in our opinion a simple and compelling construction is still sorely lacking. Many of the attempts were based on the tetrahedral group A_4 first studied by Ma and Rajasekaran for neutrino models [24], and subsequently by others [25]. A group theoretic discussion was given in [26] attributing the difficulty to a clash between the residual Z_2 in one sector and Z_3 in another. Residual discrete symmetries in the context of neutrino mixing have also been extensively discussed in Ref. [27]. Authors in Ref. [28] have named this the "sequestering problem." Within the context of A_4, it was shown [29] that two assumptions were needed to obtain a one-parameter family of mixing matrices which contains tribimaximal mixing. In other words, to obtain tribimaximal mixing, it is necessary to find one reason or another to set this particular parameter to zero. This suggests, or at least motivates, studying various one-parameter modifications to tribimaximal mixing.

II. THE FORM OF MINIMAL MODIFICATIONS FOR V_{TB}

In our 2006 work [23], which we will review briefly in the appendixes, we were naturally led in the context of A_4, assuming that the sequestering problem of [28] could be solved, to a modification of tribimaximal mixing in which the middle (1, 1, 1) column was left unchanged.

It has not escaped the notice of many authors that the three columns, $(-2, 1, 1)$, $(0, 1, -1)$, $(1,\ 1,\ 1)$, of V_{TB} furnish the two-dimensional and the one-dimensional representations of the permutation group S_3. Historically, this observation led Wolfenstein [30] long ago to guess, based on a sense that somehow (1, 1, 1) is special, a mixing matrix that consists of V_{TB} with its last two columns interchanged. Another early guess was put forward by Yamanaka, Sugawara, and Pakvasa [31]. The mutual orthogonality of these three columns of course also means that they correspond to the three Gell-Mann diagonal generators of SU(3). Similarly, these three columns also appear [29,32] in a table of Clebsch-Gordon coefficients for SU(2). These may all indicate some deeper reasons for tribimaximal mixing, such as the possibility [29] that neutrinos are composite.

With so little known about the underlying theory of neutrino mixing, we take here a purely phenomenological approach. If we take the column vectors as reflecting some basic feature of the neutrino, a minimal modification may be to keep the column vectors unchanged as much as possible. Some phenomenological implications have been studied in Ref. [33]. In this paper we analyze these minimal modifications to the tribimaximal mixing in light of the recent T2K data. Because of unitarity, we can leave at most one column unchanged. Minimal modification to the tribimaximal mixing can therefore be characterized by which column we leave unchanged. Some special cases have been considered in the literature [23,29,32,34,35]. We refer to them as case V^a, case V^b, and case V^c, corresponding to keeping the third, second, and first columns unchanged, respectively. This class of modifications can be obtained by multiplying a two-generation mixing matrix from the right of V_{TB}, 2 and 3, 1 and 3, and 1 and 2 mixing patterns. These modifications can be viewed as perturbation to the tribimaximal mixing by modifying the neutrino mass matrix [33]. One can also motivate minimal modification by perturbing the charged lepton mass matrix in a similar fashion, which would result in one of the rows of V_{TB} being unchanged. One such example has been studied by Friedberg and Lee in [35]. Therefore there are another three types of minimal modification, keeping the first row or the second row or the third row indicated by case W^a, case W^b, and case W^c, respectively.

A. One of the columns in V_{TB} unchanged

Leaving one of the columns in V_{TB} unchanged, we have the three possibilities:

$$V^a = V_{\mathrm{TB}}\begin{pmatrix} \cos\tau & \sin\tau & 0 \\ -\sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$V^b = V_{\mathrm{TB}}\begin{pmatrix} \cos\tau & 0 & \sin\tau e^{i\delta} \\ 0 & 1 & 0 \\ -\sin\tau e^{-i\delta} & 0 & \cos\tau \end{pmatrix}, \tag{3}$$
$$V^c = V_{\mathrm{TB}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & \sin\tau e^{i\delta} \\ 0 & -\sin\tau e^{-i\delta} & \cos\tau \end{pmatrix}.$$

We will use the abbreviation $c = \cos\tau$ and $s = \sin\tau$.

Note that for V^a, with the third column in $V_{\rm TB}$ unchanged, $V_{e3} = 0$ and there is no nonremovable phase leading to a vanishing Jarlskog parameter, $J = 0$. No CP violation phenomena can show up in oscillation related processes.

B. One of the rows in $V_{\rm TB}$ unchanged

Keeping the neutrino mass matrix unchanged and perturbing the charged lepton mass matrix for the tribimaximal mass matrices, one obtains a V in the form of a unitary U multiplied from left to the tribimaximal mixing $V = UV_{\rm TB}$. The three cases with one of the rows unchanged from tribimaximal mixing can be written in the following forms:

$$W^a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} V_{\rm TB},$$
$$W^b = \begin{pmatrix} c & 0 & se^{i\delta} \\ 0 & 1 & 0 \\ -se^{-i\delta} & 0 & c \end{pmatrix} V_{\rm TB}, \tag{4}$$
$$W^c = \begin{pmatrix} c & se^{i\delta} & 0 \\ -se^{-i\delta} & c & 0 \\ 0 & 0 & 1 \end{pmatrix} V_{\rm TB}.$$

Since there is no nonremovable phase in W^a, no CP violation phenomena can show up in oscillation related processes for this case.

As mentioned earlier, the case W^c has been motivated theoretically and studied by Friedberg and Lee [35].

III. PHENOMENOLOGICAL IMPLICATIONS

As was mentioned in the introduction, of these six cases, we are theoretically prejudiced in favor of V^b, which we studied in Ref. [23]. As far as we know, some of the other cases are not well motivated and we analyzed them merely as straw men to be knocked down by future precision experiments.

There are no phases for cases V^a and W^a. They are given by

$$V^a = \begin{pmatrix} \frac{2c}{\sqrt{6}} - \frac{s}{\sqrt{3}} & \frac{c}{\sqrt{3}} + \frac{2s}{\sqrt{6}} & 0 \\ -\frac{c}{\sqrt{6}} - \frac{s}{\sqrt{3}} & \frac{c}{\sqrt{3}} - \frac{s}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{c}{\sqrt{6}} - \frac{s}{\sqrt{3}} & \frac{c}{\sqrt{3}} - \frac{s}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix},$$
$$W^a = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{c}{\sqrt{6}} - \frac{s}{\sqrt{6}} & \frac{c}{\sqrt{3}} + \frac{s}{\sqrt{3}} & \frac{c}{\sqrt{2}} - \frac{s}{\sqrt{2}} \\ -\frac{c}{\sqrt{6}} + \frac{s}{\sqrt{6}} & \frac{c}{\sqrt{3}} - \frac{s}{\sqrt{3}} & -\frac{c}{\sqrt{2}} + \frac{s}{\sqrt{2}} \end{pmatrix}. \tag{5}$$

The modification to tribimaximal mixing is represented by a nonzero $s = \sin\tau$ ($c = \cos\tau$). For case V^a, the main predictions of this mixing pattern are that $|V_{\mu 3}| = 1/\sqrt{2}$ which is at the boundary of the 1σ allowed range. Present data constrain the modification mixing parameter s (with $c > 0$), which can modify the value for V_{e2}, to be in the range -0.04–0.002 (-0.075–0.0) at 1σ (3σ) with the best fit value of -0.02. For case W^a, a definitive prediction for this case is that $V_{e2} = 1/\sqrt{3}$. This is outside the 1σ range, but within the 2σ range. A nonzero s will modify $V_{\mu 3}$ away from $1/\sqrt{2}$. The current data allow s to be in the range 0–0.08 (-0.14–0.105) at the 1σ (3σ) with the best fit value to be 0.08. For both cases V^a and W^a, V_{e3} are predicted to be zero. They are in conflict with data at the 3σ level. Also there are no phases for V^a and W^a. There is no CP violation in neutrino oscillation. This provides another test for cases V^a and W^a. Should future experiments find CP violation in neutrino oscillations, these two cases would be ruled out.

The other four cases have two parameters; one can use available data on the magnitude of the elements in V to constrain them and to predict other observables, in particular, the Jarlskog CP violating parameter $J = {\rm Im}(V_{e1}V_{e2}^*V_{\mu 1}^*V_{\mu 2})$. Complete determination of parameters related to neutrino physics includes the mixing angles and the CP violating Dirac phase, and also the absolute neutrino masses and possible Majorana phases. Since not much information can be used to constrain the Majorana phases, in the following we will use the combined pre-T2K and the recent T2K and MINOS data to study some phenomenological implications for the mixing parameters of the other four minimal modifications described in the previous section.

Case V^b: In this case the mixing matrix V^b is given by

$$V^b = \begin{pmatrix} \frac{2c}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{2se^{i\delta}}{\sqrt{6}} \\ -\frac{c}{\sqrt{6}} - \frac{se^{-i\delta}}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{c}{\sqrt{2}} - \frac{se^{i\delta}}{\sqrt{6}} \\ -\frac{c}{\sqrt{6}} - \frac{se^{-i\delta}}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{c}{\sqrt{2}} - \frac{se^{i\delta}}{\sqrt{6}} \end{pmatrix}. \tag{6}$$

A definitive prediction of this mixing pattern is that $V_{e2} = 1/\sqrt{3}$. This is outside the 1σ range, but consistent with data within the 2σ level. Since V_{e2} is fixed to be $1/\sqrt{3}$, $|V_{e3}|$ can be expressed as a function of V_{e1} with

$$V_{e3} = \sqrt{2/3 - |V_{e1}|^2}. \tag{7}$$

This can be used as an additional check.

In this case there is room for a nonzero V_{e3} and also a nonzero Jarlskog parameter given by $J = |V_{e1}||V_{e3}|\sin\delta/2\sqrt{3}$ for CP violation. Using the allowed range for θ_{13}, we can easily obtain information for s with the best fit value given by 0.177 and the 1σ (3σ) allowed range 0.16–0.22 (0.09–0.27).

Expressing V_{e3} and $V_{\mu 3}$ in terms of the mixing angle τ (through c and s) and CP violating phase, we have

$$V_{e3} = \sqrt{\frac{2}{3}} se^{i\delta}, \qquad V_{\mu 3} = \frac{1}{\sqrt{2}} c - \frac{1}{\sqrt{6}} se^{i\delta}. \tag{8}$$

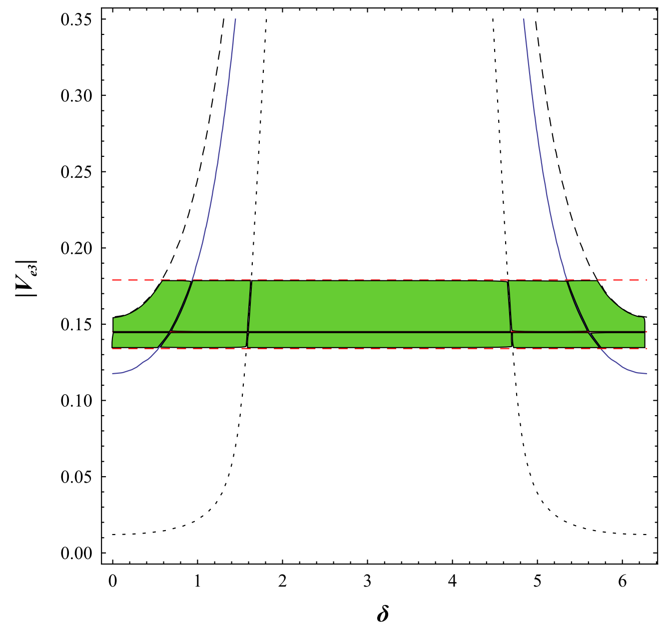

FIG. 1 (color online). Case V^b. $|V_{e3}|$ as a function of δ for $|V_{\mu 3}|$ equals to 0.617 (1σ lower bound, dashed line), 0.641 (best fit value, solid line) and 0.701 (1σ upper bound, dotted line). The solid and dashed horizontal lines are for the best value and the 1σ bounds of $|V_{e3}|$, respectively.

One could use Eq. (8) to eliminate τ and relate $|V_{e3}|$, $|V_{\mu 3}|$ and δ,

$$\begin{aligned}|V_{\mu 3}| &= \left(\frac{1}{2}c^2 + \frac{1}{6}s^2 - \frac{1}{\sqrt{3}}cs\cos\delta\right)^{1/2}\\ &= \left[\left(\frac{1}{2}|V_{e3}|\cos\delta - \frac{1}{\sqrt{2}}\sqrt{1-3|V_{e3}|^2/2}\right)^2\right.\\ &\quad \left. + \frac{1}{4}|V_{e3}|^2\sin^2\delta\right]^{1/2}. \end{aligned} \tag{9}$$

Using the known ranges of $\sin\theta_{23}$ and $\sin\theta_{13}$, the 1σ range for $V_{\mu 3}$ is determined to be 0.617–0.701. We could plot $|V_{e3}|$ in terms of the unknown δ for some typical values of $V_{\mu 3}$ in the allowed range. In solving for $|V_{e3}|$ we have to pick the branch consistent with Eq. (8) of course. We will work in the convention where s and c are all positive. The result is shown in Fig. 1. $|V_{e3}|$ is symmetric in the region of 0 to π and π to 2π for δ. The allowed ranges in Table I rule out some portion of allowed δ. Regions of δ close to π are not allowed at the 1σ level, but there are ranges of δ which can be consistent with data. Improved data can further narrow down the allowed range.

Since in this case there are only two unknown parameters, s and δ, in the model, the four parameters θ_{12}, θ_{23}, θ_{13}, and the CP violating parameter $\delta_{\rm PDG}$ of the general parametrization are related. The boundaries in Fig. 1 represent the 1σ allowed ranges for the parameters of the model. The situations are similar for the other three cases in our later discussions. Note also that the analysis we are carrying out is insensitive to whether the neutrino mass hierarchy is normal or inverted.

One can extract useful information on the CP violating parameter J by using the relation

$$J = \frac{1}{3\sqrt{2}}\sin\delta|V_{e3}|\sqrt{1-3|V_{e3}|^2/2}. \tag{10}$$

Note that J is simply proportional to $\sin\delta$. The results are shown in Fig. 2. On the left panel of Fig. 2, J is plotted as a function of δ for three values of $|V_{e3}|$. The absolute value of J can be as large as 0.04 for $|V_{e3}|$ and takes its 1σ upper value of 0.179.

For $|V_{\mu 3}|$ close to its lower bound, δ close to 0 and 2π are favored. For δ close to π, $|V_{\mu 3}|$ is outside of its 1σ allowed range. There are overlaps for the regions allowed in the right panel of Fig. 2 and those in Fig. 1. Combining

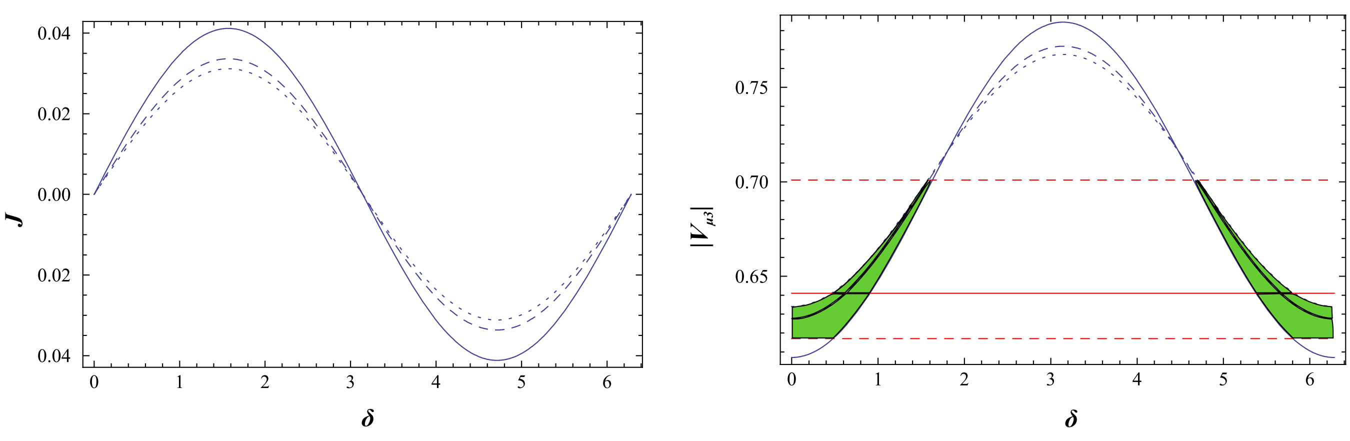

FIG. 2 (color online). Case V^b. Left: J as a function of δ for $|V_{e3}|$ equals to 0.179 (solid line), 0.145 (dashed line), and 0.134 (dotted) line. Right: $|V_{\mu 3}|$ as a function of δ for $|V_{e3}|$ equals to 0.179 (solid line), 0.145 (dashed line), and 0.134 (dotted line). The solid and two dashed horizontal lines are for the best value and the 1σ bounds of $|V_{\mu 3}|$, respectively.

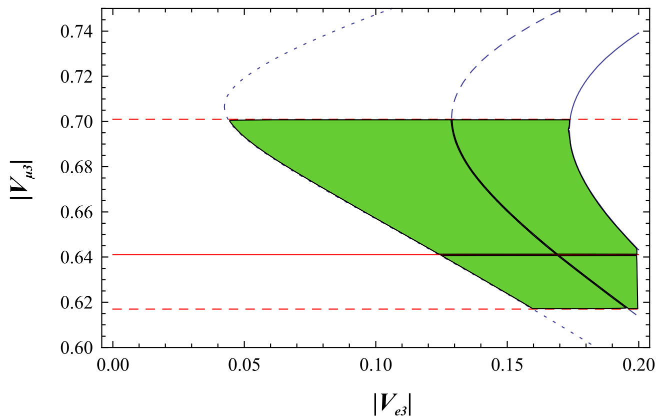

FIG. 3 (color online). Case V^b. Contours of $|V_{e3}|$ and $|V_{\mu 3}|$ for different values of J. The curves are for J equal to ± 0.04 (solid line), ± 0.03 (dashed line), and ± 0.01 (dotted line). The solid and two dashed horizontal lines are for the best value and the 1σ bounds of $|V_{\mu 3}|$, respectively.

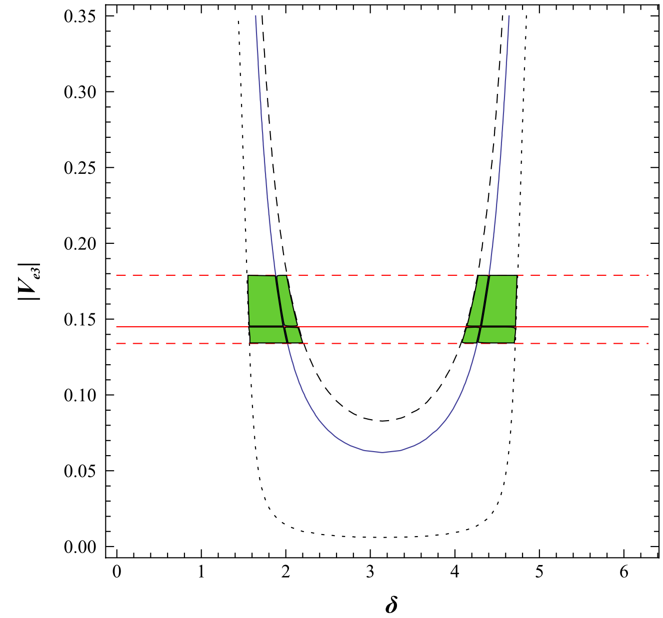

FIG. 4 (color online). Case V^c. $|V_{e3}|$ as a function of δ for $|V_{\mu 3}|$ equal to 0.617 (dashed line), 0.641 (solid line), and 0.701 (dotted line). The solid and dashed horizontal lines are for the best value and the 1σ bounds of $|V_{e3}|$, respectively.

information from V_{e2} discussed earlier, we can conclude that this case is ruled out at the 1σ level, but is consistent with data at the 2σ level.

In Fig. 3, the contours of $|V_{e3}|$ and $|V_{\mu 3}|$ for different values of J are shown. The contours are degenerate in $\pm|J|$.

Case V^c: For this case, the mixing matrix is

$$V^c = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{c}{\sqrt{3}} & \frac{se^{i\delta}}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{c}{\sqrt{3}} - \frac{se^{-i\delta}}{\sqrt{2}} & \frac{c}{\sqrt{2}} - \frac{se^{i\delta}}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{c}{\sqrt{3}} + \frac{se^{-i\delta}}{\sqrt{2}} & -\frac{c}{\sqrt{2}} + \frac{se^{i\delta}}{\sqrt{3}} \end{pmatrix}. \tag{11}$$

A prediction for the mixing is that $V_{e1} = 2/\sqrt{6}$. V_{e3} is related to V_{e2} by

$$|V_{e3}| = \sqrt{1/3 - |V_{e2}|^2}. \tag{12}$$

Imposing unitarity of the V_{PMNS}, this prediction is in the 1σ region.

Within the 1σ range of V_{e3}, s is allowed to vary from 0.2 to 0.31 with the best fit value of 0.25, and V_{e2} can be within its 1σ range. One can again combine information from $|V_{\mu 3}|$ and $|V_{e3}|$ to constrain the CP violating phase δ and the Jarlskog parameter J. We have

$$|V_{\mu 3}| = \left[\left(|V_{e3}|\cos\delta + \frac{1}{\sqrt{2}}\sqrt{1-3|V_{e3}|^2}\right)^2 + |V_{e3}|^2\sin\delta^2\right]^{1/2},$$

$$J = -\frac{1}{3\sqrt{2}}\sin\delta|V_{e3}|\sqrt{1-3|V_{e3}|^2}. \tag{13}$$

The results are shown in Figs. 4–6. For this case even a larger portion of the δ range is ruled out by data as can seen from Fig. 4. From the right panel of Fig. 5, we also see that a large portion of δ is ruled out. However, there are overlaps for the two regions. This case can be consistent with data at the 1σ level. CP violating information is shown in the left panel of Fig. 5. The largest value of J is 0.04 when $|V_{e3}|$ takes its 1σ upper bound value of 0.179. The correlations of J, $V_{\mu 3}$, and V_{e3} are shown in Fig. 6.

Case W^b: The mixing matrix W^b is given by

$$W^b = \begin{pmatrix} \frac{2c}{\sqrt{6}} - \frac{se^{i\delta}}{\sqrt{6}} & \frac{c}{\sqrt{3}} + \frac{se^{i\delta}}{\sqrt{3}} & -\frac{se^{i\delta}}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{c}{\sqrt{6}} - \frac{2se^{-i\delta}}{\sqrt{6}} & \frac{c}{\sqrt{3}} - \frac{se^{-i\delta}}{\sqrt{3}} & -\frac{c}{\sqrt{2}} \end{pmatrix}. \tag{14}$$

This case predicts $V_{\mu 3} = 1/\sqrt{2}$. This prediction is at the boundary of the 1σ allowed range. But it is different than case V^a since V_{e3} is not zero and J can be nonzero. s is in the range of 0.19–0.25 at the 1σ level with the best fit value given by 0.20. In this case correlations of $|V_{e2}|$, $|V_{e3}|$, and δ or J are given by

$$|V_{e2}| = \frac{1}{\sqrt{3}}\left[\left(\sqrt{2}|V_{e3}|\cos\delta + \sqrt{1-2|V_{e3}|^2}\right)^2 + 2|V_{e3}|^2\sin^2\delta\right]^{1/2},$$

$$J = \frac{1}{3\sqrt{2}}\sin\delta|V_{e3}|\sqrt{1-2|V_{e3}|^2}. \tag{15}$$

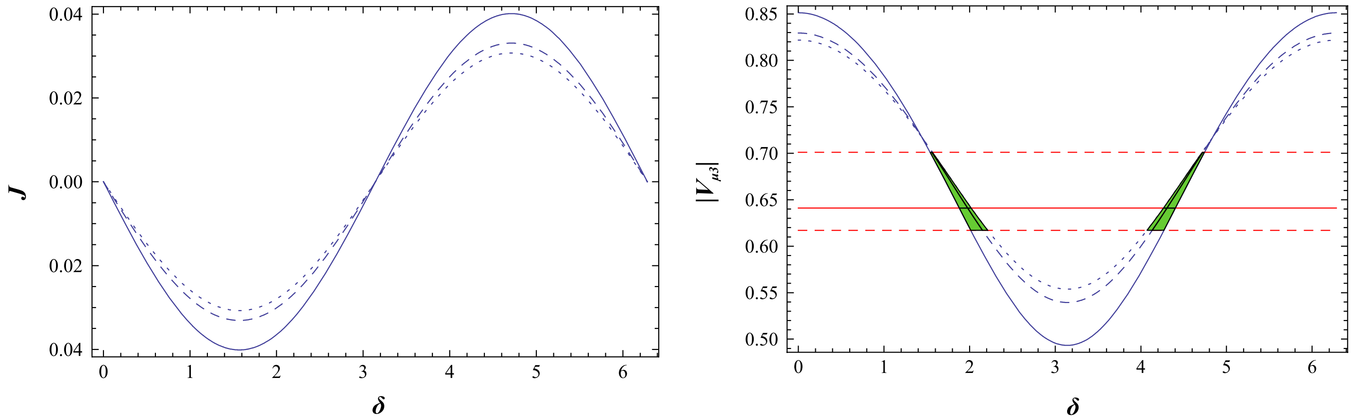

FIG. 5 (color online). Case V^c. Left: J as a function of δ for $|V_{e3}|$ equal to 0.179 (solid line), 0.145 (dashed line), and 0.134 (dotted line) for case V^c. Right: $|V_{\mu3}|$ as a function of δ for $|V_{e3}|$ equal to 0.179 (solid line), 0.145 (dashed line), and 0.134 (dotted line). The solid and dashed horizontal lines are for the best value and the 1σ bounds of $|V_{\mu3}|$, respectively.

The results are shown in Figs. 7–9. Now $|V_{e2}|$ is playing the role of $|V_{\mu3}|$ for the cases V^b and V^c. From Fig. 7 and the right panel in Fig. 8, we see that the allowed range for δ is constrained to be even closer to $\pi/2$ and $3\pi/2$. But there are still regions consistent with data at the slightly larger than 1σ level. J can be as large as 0.04 when $|V_{e3}|$ is equal to its 1σ upper bound value of 0.179. The correlations of J, $V_{\mu3}$, and V_{e3} are shown in Fig. 9.

Case W^c: In this case the third row is left unchanged from tribimaximal mixing [35] and the mixing matrix W^c is given by

$$W^c = \begin{pmatrix} \frac{2c}{\sqrt{6}} - \frac{se^{i\delta}}{\sqrt{6}} & \frac{c}{\sqrt{3}} + \frac{se^{i\delta}}{\sqrt{3}} & \frac{se^{i\delta}}{\sqrt{2}} \\ -\frac{c}{\sqrt{6}} - \frac{2se^{-i\delta}}{\sqrt{6}} & \frac{c}{\sqrt{3}} - \frac{se^{-i\delta}}{\sqrt{3}} & \frac{c}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \quad (16)$$

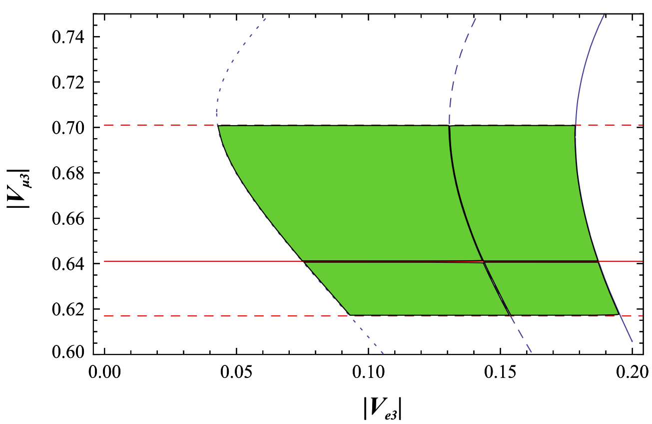

FIG. 6 (color online). Case V^c. Contours of $|V_{e3}|$ and $|V_{\mu3}|$ for different values of J. The curves are for J equal to ±0.04 (solid line), ±0.03 (dashed line), and ±0.01 (dotted line). The solid and dashed horizontal lines are for the best value and the 1σ bounds of $|V_{\mu3}|$, respectively.

We have

$$|V_{e2}| = \frac{1}{\sqrt{3}}\left[\left(\sqrt{2}|V_{e3}|\cos\delta + \sqrt{1-2|V_{e3}|^2}\right)^2 + 2|V_{e3}|^2\sin^2\delta\right]^{1/2},$$
$$J = -\frac{1}{3\sqrt{2}}\sin\delta|V_{e3}|\sqrt{1-2|V_{e3}|^2}. \quad (17)$$

The expression for $|V_{e2}|$ is the same as for case W^b, but the sign of J is opposite to those for case W^b when reading Figs. 8 and 9, respectively. One can easily read off the constraints on δ and J from Figs. 7 and 8. *A crucial*

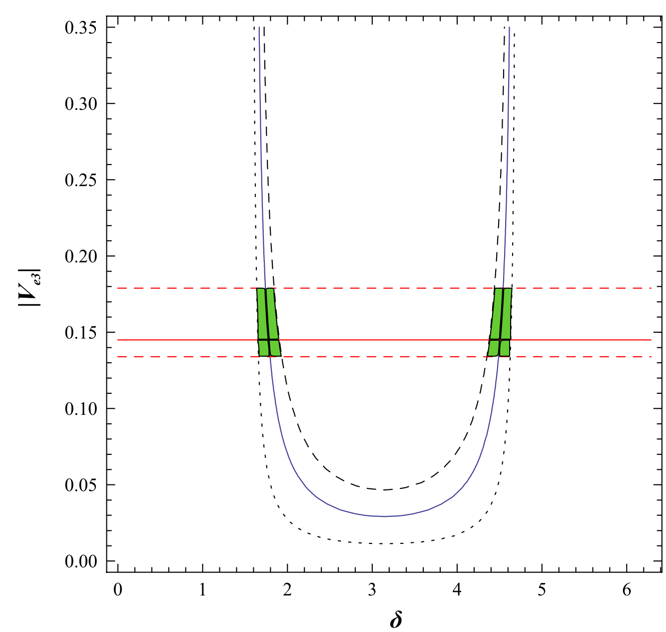

FIG. 7 (color online). Case W^b. $|V_{e3}|$ as a function of δ for $|V_{e2}|$ equal to 0.538 (dashed line), 0.553 (solid line), and 0.568 (dotted line). The solid and dashed horizontal lines are for the best value and 1σ bounds of $|V_{e3}|$, respectively.

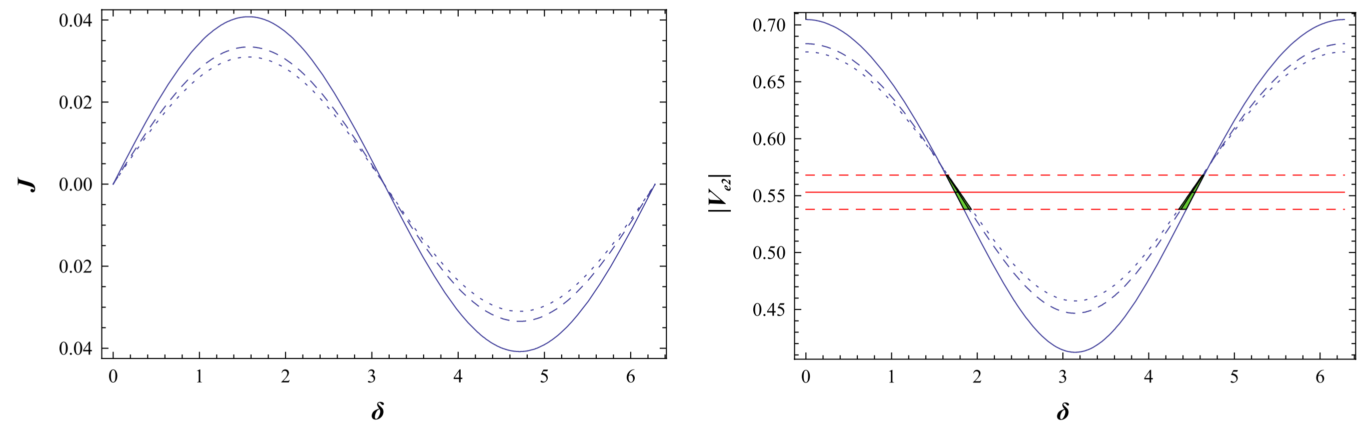

FIG. 8 (color online). Case W^b. Left: J as a function of δ for $|V_{e3}|$ equal to 0.179 (solid line), 0.145 (dashed line), and 0.134 (dotted line). Right: $|V_{e2}|$ as a function of δ for $|V_{e3}|$ equals to 0.179 (solid line), 0.145 (dashed line), and 0.134 (dotted line). The solid and dashed horizontal lines are for the best value and the 1σ bounds of $|V_{e2}|$, respectively.

difference is that $V_{\mu 3}$ *is no longer* $1/\sqrt{2}$, *but* $c/\sqrt{2}$. $|V_{\mu 3}|$ *can vary from* 0.684–0.694 (0.670–0.703) *at the* 1σ (3σ) *level*. *This case is consistent with data at* 1σ *level*. Precise measurement of $|V_{\mu 3}|$ can be used to distinguish these two cases.

IV. SUMMARY

Recent data from T2K and MINOS show evidence of a nonzero V_{e3} at 90% C.L.. There may be the need for modifications to the tribimaximal mixing with which V_{e3} is equal to zero resulting in a CP conserving mixing. We have studied several possible ways to minimally modify the tribimaximal mixing by keeping one of the columns or one of the rows in the tribimaximal mixing unchanged. Six cases were studied. Two of the cases have $V_{e3} = 0$. These two cases are in tension with data at the 3σ level. Also for these two cases, the CP violating Jarlskog parameter J is identically zero. CP violation in neutrino oscillation can provide new tests for these two cases. For the other four cases, all have two parameters in the mixing matrix. Current data on neutrino oscillation can put constraints on the parameters, but remain consistent with data within 2σ for case V^b, and within 1σ for the other three cases. The allowed ranges for CP violation are also constrained. Future experiments can test the predictions to rule out these models.

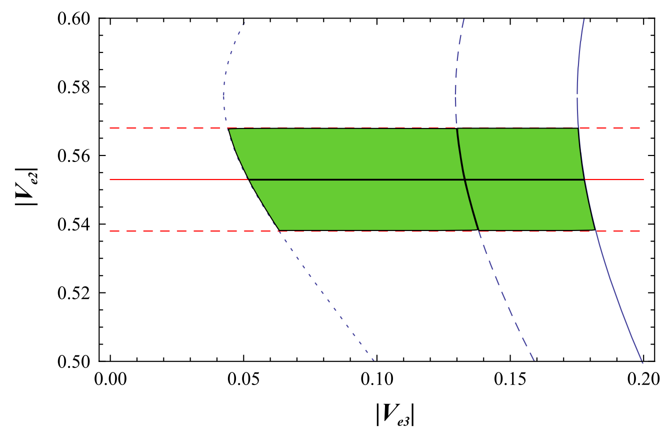

FIG. 9 (color online). Case W^b. Contours of $|V_{e3}|$ and $|V_{e2}|$ for different values of J. The curves are for J equal to ± 0.04 (solid line), ± 0.03 (dashed line), and ± 0.01 (dotted line). The solid and dashed horizontal lines are for the best value and the 1σ bounds of $|V_{e2}|$, respectively.

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APPENDIX A

Here we review some features of the family group A_4 [24] that led us to favor V^b, which we analyzed in [23]. A key point to achieve the tribimaximal mixing in models based on A_4 symmetry is to obtain the matrices U_l and U_ν which diagonalize the charged lepton and neutrino mass matrices M_l and M_ν, $U_l^\dagger M_l U_r = D_l$ and $U_\nu^T M_\nu U_\nu = D_\nu$ in the following forms [23]:

$$U_l = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \omega^2 & \omega \\ 1 & 1 & 1 \\ 1 & \omega & \omega^2 \end{pmatrix}, \qquad U_\nu = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}, \tag{A1}$$

where $\omega^3 = 1$. The mixing matrix V is given by $U_l^\dagger U_\nu$. Note that U_r which plays a role in diagonalizing the charged lepton mass does not show up in V. With suitable choices of phase conventions, V can be written in the form in Eq. (2). In general the elements in the diagonal matrix D_ν have phases, the Majorana phases. We provide some details in Appendix B for obtaining the relevant mass matrix.

One can easily find that the neutrino mass matrix must be the following form:

$$M_\nu = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \gamma & 0 \\ \beta & 0 & \alpha \end{pmatrix}, \tag{A2}$$

to obtain the right form for U_ν. With appropriate redefinition of phases and mixing angles, one obtained the form $V_{\rm TB}$ for V.

To achieve the above with specific models, there are some requirements for Higgs boson fields' vacuum expectations values as can be seen in Appendix B. In general, M_ν above will be modified. If one keeps the charged lepton mass matrix unchanged, the most general Higgs potential may not respect the conditions leading to modification resulting modifying the "11" entry to $\alpha - \epsilon$ and the "33" entry to $\alpha + \epsilon$ for the neutrino mass matrix M_ν given by Eq. (B4) in Appendix B. With appropriate redefinition of phases and angles shown in Appendix B, one obtains V^b where we have redefined $\delta = \eta + \pi/2$ to the form in Eq. (B8).

Note that in taking the form of V^b in Eq. (3), without Majorana phases in V, we have absorbed possible Majorana phases in the masses $\tilde m_i = m_i e^{i2\kappa_i}$. Here m_i are real and positive. Without loss of generality, one can always choose one of the κ_i to be zero, for example $\kappa_2 = 0$. The above form belongs to the minimal modification to the $V_{\rm TB}$ mixing pattern specified earlier with the elements in the second column remaining as $V_{\alpha 2} = 1/\sqrt{3}$.

We have arrived at this minimal modification from a specific model. In fact, the above parametrization is the most general one for $V_{\alpha 2} = 1/\sqrt{3}$ up to phase conventions. One can understand this by starting with a most general parametrization used by the Particle Data Group, $V_{\rm PDG}$, and then setting certain angles to some particular values to make sure that $V_{\alpha 2} = 1/\sqrt{3}$. Since we want all of the elements in the second column to be $1/\sqrt{3}$, for convenience we exchange the second and the third columns, and move the Dirac phase at different locations, by redefining the phase of charged leptons and neutrinos, according to the following:

$$V^b = P_L V_{\rm PDG} E P_R, \tag{A3}$$

where $P_{L,R}$ are diagonal phase matrices with elements, $P_L = \mathrm{diag}(e^{i\delta}, 1, 1)$, $P_R = \mathrm{diag}(e^{-i\delta}, 1, 1)$, and E is a matrix switching the second and third columns with elements $E_{ij} = \delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2}$.

We have

$$V^b = \begin{pmatrix} c_{13}c_{12} & s_{13} & c_{13}s_{12}e^{i\delta} \\ -s_{12}c_{23}e^{-i\delta} - c_{12}s_{23}s_{13} & s_{23}c_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} \\ s_{12}s_{23}e^{-i\delta} - c_{12}c_{23}s_{13} & c_{23}c_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} \end{pmatrix}. \tag{A4}$$

To have all of the elements in the second column in the above matrix to be $1/\sqrt{3}$, one just needs to set $s_{13} = 1/\sqrt{3}$, $c_{13} = \sqrt{2/3}$, and $s_{23} = c_{23} = 1/\sqrt{2}$. We then obtain Eq. (3) with $c = c_{12}$ and $s = s_{12}$.

In this basis, one can reconstruct the elements in the neutrino mass matrix in terms of the mixing angle τ, phase δ, and masses $\tilde m_i$ by requiring $M_\nu = U_l^* V D_\nu V^T U_l^\dagger$. We have

$$\begin{aligned} \alpha &= \frac{1}{2}((c^2 - s^2 e^{-2i\delta})\tilde m_1 - (c^2 - s^2 e^{2i\delta})\tilde m_3), \\ \beta &= \frac{1}{2}((c^2 + s^2 e^{-2i\delta})\tilde m_1 + (c^2 + s^2 e^{2i\delta})\tilde m_3), \\ \epsilon &= ics(e^{-i\delta}\tilde m_1 - e^{i\delta}\tilde m_3), \\ \gamma &= \tilde m_2. \end{aligned} \tag{A5}$$

The α, β, and ϵ here are equivalent to those given in Appendix B with a different basis. The important thing is that the number of independent parameters is the same total of six.

APPENDIX B

In Appendix B, we briefly outline how V^b can be obtained from models based on A_4 family symmetry [24] with the standard model (SM) gauge symmetry.

A way to obtain U_l in Eq. (A1) is to assign the three left-handed SM lepton doublet $l_L = (l_{L1}, l_{L2}, l_{L3})$ and three right-handed neutrino (l_{R1}, l_{R2}, l_{R3}) representations into a triplet, and the three singlets 1, $1'$, and $1''$ of A_4 group, respectively. A SM doublet and A_4 triplet Higgs representation $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ then leads to the Yukawa coupling terms,

$$\begin{aligned} L_l = \lambda_e(\bar l_L \Phi_1) l_{R1} + \lambda_\mu \omega^2 (\bar l_L \Phi)_{1'} l_{R2} + \lambda_\tau \omega (\bar l_L \Phi)_{1''} l_{R3} \\ + \mathrm{H.c.} \end{aligned} \tag{B1}$$

After Φ develops a vacuum expectation value (VEV) of the form $\langle\Phi\rangle = (v_\Phi, v_\Phi, v_\Phi)$, the charged lepton mass matrix is given by $M_l = U_l D_l$ with $D_l = \mathrm{Diag}(m_e, m_\mu, m_\tau)$ and $m_i = \sqrt{3} v_\Phi \lambda_i$. This gives the right U_l with $U_r = I$.

To obtain the right form of the neutrino mass matrix, three right-handed neutrinos $\nu_R = (\nu_{R1}, \nu_{R2}, \nu_{R3})$, a SM doublet Higgs ϕ, and a SM singlet $\chi = (\chi_1, \chi_2, \chi_3)$, transforming under the A_4 group as triplet, singlet, and triplet, respectively. The Yukawa coupling terms relevant are

$$L_\nu = \lambda_\nu(\bar{l}_L \nu_R)_1 + m(\bar{\nu}_R \nu_R^C)_1 + \lambda_\chi((\nu_R \nu_R^C)_3 \chi)_1 + \text{H.c.} \tag{B2}$$

With the VEV structure $\langle\phi\rangle = v_\phi$ and $\langle\chi\rangle = (0, v_\chi, 0)$, the neutrino mass matrix is given by

$$M = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix}, \qquad M_R = \begin{pmatrix} m & 0 & m_\chi \\ 0 & m & 0 \\ m_\chi & 0 & m \end{pmatrix}, \tag{B3}$$

where $M_D = \text{Diag}(1, 1, 1)\lambda_\nu v_\phi$ and $m_\chi = \lambda_\chi v_\chi$. The light neutrino mass matrix is $M_\nu = -M_D M_R^{-1} M_D^T$ which is the desired form giving U_ν in Eq. (A1).

The charged lepton masses are controlled by the VEV of Φ which requires the components of Φ to have the same VEV. This choice leaves a Z_3 unbroken symmetry in the theory. The neutrino mass matrix is, on the other hand, controlled by the VEV of ϕ and χ with only $\langle\chi_2\rangle$ nonzero. This preserves a Z_2 unbroken symmetry in the theory. If there is no communication between these two sectors, the residual Z_3 and Z_2 symmetries are left unbroken. But in general these two sectors cannot be completely sequestered and can interact which complicates the situation [26,28]. *For example, the VEV structure with only $\langle\chi_2\rangle$ as nonzero may not be maintained*. One of the consequences that concern us is that the mass matrix M_ν will be modified to [23]

$$M_\nu = \begin{pmatrix} \alpha - \epsilon & 0 & \beta \\ 0 & \gamma & 0 \\ \beta & 0 & \alpha + \epsilon \end{pmatrix}. \tag{B4}$$

The above will lead to a different form for U_ν from that in Eq. (A1) which can be written as

$$U_\nu = \begin{pmatrix} -c_\theta & 0 & -is_\theta \\ 0 & 1 & 0 \\ -s_\theta e^{i\rho} & 0 & ic_\theta e^{i\rho} \end{pmatrix}, \tag{B5}$$

where $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$ with

$$\begin{aligned} \tan^2(2\theta) &= \frac{4|\beta|^2}{(|\alpha+\epsilon| - |\alpha-\epsilon|)^2} \\ &\quad \times \left(1 - \frac{4|\alpha^2 - \epsilon^2|}{(|\alpha+\epsilon| + |\alpha-\epsilon|)^2}\sin^2\sigma\right), \\ \sigma &= \arg\left(\frac{\beta}{\sqrt{\alpha^2-\epsilon^2}}\right), \\ \rho &= \tilde{\delta} + \arg\left(\sqrt{\frac{\alpha-\epsilon}{\alpha+\epsilon}}\right), \\ \tan\tilde{\delta} &= \frac{|\alpha+\epsilon| - |\alpha-\epsilon|}{|\alpha+\epsilon| + |\alpha-\epsilon|}\tan\sigma. \end{aligned} \tag{B6}$$

The phase conventions are chosen such that setting $\epsilon = 0$, the resulting mixing matrix V goes to V_{TB}.

In this basis, V is given by

$$V = \frac{1}{\sqrt{3}}\begin{pmatrix} -(c_\theta + s_\theta e^{i\rho}) & 1 & i(c_\theta e^{i\rho} - s_\theta) \\ -(\omega c_\theta + \omega^2 s_\theta e^{i\rho}) & 1 & i(\omega^2 c_\theta e^{i\rho} - \omega s_\theta) \\ -(\omega^2 c_\theta + \omega s_\theta e^{i\rho}) & 1 & i(\omega c_\theta e^{i\rho} - \omega^2 s_\theta) \end{pmatrix}, \tag{B7}$$

which can be further rewritten as

$$V = V_{\text{TB}}\begin{pmatrix} \cos\tau & 0 & i\sin\tau e^{i\eta} \\ 0 & 1 & 0 \\ i\sin\tau e^{-i\eta} & 0 & \cos\tau \end{pmatrix} V_p, \tag{B8}$$

where V_p is a diagonal phase matrix $V_p = (e^{i(\xi+\rho/2)}, 1, e^{i(-\xi+\rho/2)})$ multiplied from the right. τ, η, and ξ are given by

$$\begin{aligned} \sin^2\tau &= \frac{1}{2}(1 - \sin(2\theta)\cos\rho), \\ \tan\xi &= -\frac{1-\tan\theta}{1+\tan\theta}\tan(\rho/2), \\ \tan\eta &= \tan(2\theta)\sin\rho. \end{aligned} \tag{B9}$$

Absorbing the phases in V_p into the neutrino masses, the total Majorana phases for $\tilde{m}_1$, $\tilde{m}_2$, and $\tilde{m}_3$ are $\kappa_1 = -(2\xi + \rho + 2\alpha_1)$, $\kappa_2 = \alpha_2$, and $\kappa_3 = -(-2\xi + \rho + 2\alpha_3)$ with

$$\begin{aligned} \alpha_1 &= -\frac{1}{2}[\arg(c_\theta^2|\alpha-\epsilon| + 2s_\theta c_\theta|\beta|e^{i(\tilde{\delta}+\sigma)} + s_\theta^2|\alpha+\epsilon|e^{2i\tilde{\delta}})] \\ &\quad - \arg(-\sqrt{\alpha-\epsilon}), \\ \alpha_2 &= -\frac{1}{2}\arg(\gamma), \\ \alpha_3 &= -\frac{1}{2}[\arg(c_\theta^2|\alpha-\epsilon| - 2s_\theta c_\theta|\beta|e^{i(\tilde{\delta}+\sigma)} + s_\theta^2|\alpha+\epsilon|e^{2i\tilde{\delta}})] \\ &\quad - \arg(i\sqrt{\alpha+\epsilon}). \end{aligned} \tag{B10}$$

In this basis, V is given by Eq. (B8), but with V_p removed.

The absolute masses squared are given by

$$\begin{aligned} m_1^2 &= |\alpha|^2 + |\epsilon|^2 + |\beta|^2 - \frac{2\,\text{Re}(\alpha\epsilon^*)}{\cos(2\theta)}, \\ m_2^2 &= |\gamma|^2, \\ m_3^2 &= |\alpha|^2 + |\epsilon|^2 + |\beta|^2 + \frac{2\,\text{Re}(\alpha\epsilon^*)}{\cos(2\theta)}. \end{aligned} \tag{B11}$$

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