Central charges of para-Liouville and Toda theories from M5-branes

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We propose that N M5-branes, put on $\mathbb{R}^4/\mathbb{Z}_m$ with deformation parameters $\epsilon_{1,2}$, realize two-dimensional theory with $\widehat{\mathrm{SU}}(m)_N$ symmetry and mth para-W_N symmetry. This includes the standard W_N symmetry for $m = 1$ and super-Viraroro symmetry for $m = N = 2$. We provide a small check of this proposal by calculating the central charge of the 2D theory from the anomaly polynomial of the 6D theory.

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I. INTRODUCTION

N M5-branes, put on $\mathbb{R}^4$ with Nekrasov's deformation parameters $\epsilon_{1,2}$, are now believed to realize two-dimensional theory with W_N symmetry [1,2]. One check of this statement was given in [3] following the observation made in [4]. Namely, the equivariant integral on $\mathbb{R}^4$ of the anomaly polynomial of N M5-branes determined in [5] gives the anomaly polynomial of the 2D theory, from which the central charge of the Toda theory with W_N symmetry can be reproduced. The same analysis can be performed for 6D $\mathcal{N} = (2, 0)$ theory of type $G = A, D, E$ whose anomaly polynomial is also known [6,7]; and it correctly reproduces the central charge of the Toda theory of type G.

In a recent paper [8], it was proposed that two M5-branes on $\mathbb{R}^4/\mathbb{Z}_2$ with deformations $\epsilon_{1,2}$ give rise to a system with the $\widehat{\mathrm{SU}}(2)_2$ symmetry and the super-Virasoro symmetry. As a generalization, we propose that N M5-branes on $\mathbb{R}^4/\mathbb{Z}_m$ realize a 2D system with a free boson, $\widehat{\mathrm{SU}}(m)_N$, and the mth para-W_N symmetry.[1] Here $\mathbb{Z}_m$ acts as $(z, w) \mapsto (e^{2\pi i/m}z, e^{-2\pi i/m}w)$ on $(z, w) \in \mathbb{C}^2 \simeq \mathbb{R}^4$. We give a small piece of supporting evidence by calculating the central charge from the 6D anomaly polynomial. We will also speculate what happens if the 6D $\mathcal{N} = (2, 0)$ theory of type G is used instead. In the following G stands for one of A_n, D_n, or E_n; r_G, h_G, and d_G are the rank, the (dual) Coxeter number, and the dimension of G, respectively. They satisfy $d_G = r_G(h_G + 1)$.

II. PARA-W SYMMETRY AND PARA-TODA THEORY

One way to realize the $W(\hat{G})$ symmetry [10–15] is to consider the chiral algebra of the coset

$$\hat{G}_k \times \hat{G}_m/\hat{G}_{k+m}, \tag{1}$$

for $m = 1$. The mth para-$W(\hat{G})$ symmetry is obtained by taking an arbitrary positive integer m in this coset; in particular, it reduces to the super-Virasoro algebra when $m = N = 2$. Generalization of Neveu-Schwarz-Ramond superstrings using these algebras were explored, e.g., in [16,17].

The mth para-$W(\hat{G})$ algebra is the symmetry of the mth para-Toda model of type G, which has the following action [18]:

$$S = S\left(\frac{\hat{G}_m}{\hat{\mathrm{U}}(1)^{r_G}}\right) + \int d^2x\left[\partial_\mu\Phi\partial_\mu\Phi + \sum_{i=1}^{r_G}\Psi_i\bar{\Psi}_i\exp\left(\frac{b}{\sqrt{m}}\alpha_i\cdot\Phi\right)\right]. \tag{2}$$

Here, $\hat{G}_m/\hat{\mathrm{U}}(1)^{r_G}$ describes the generalized parafermions Ψ_i of type G [19], α_i are simple roots of G, Φ are r_G free bosons with background charge $(b + 1/b)\rho/\sqrt{m}$ with the Weyl vector ρ. The central charge is given by

$$c = c\left(\frac{\hat{G}_m}{\hat{\mathrm{U}}(1)^{r_G}}\right) + r_G + \frac{h_G d_G}{m}\left(b + \frac{1}{b}\right)^2 = \frac{m d_G}{m + h_G} + \frac{h_G d_G}{m}\left(b + \frac{1}{b}\right)^2. \tag{3}$$

Note that the parafermion Ψ_i has dimension $1 - 1/m$, and the exponential of bosons has dimension $1/m$, so that the interaction terms are marginal. For $m = 1$ this is the usual affine Toda theory, and for $m = N = 2$ this is the $\mathcal{N} = 1$ super-Liouville theory.

III. N M5-BRANES ON $\mathbb{R}^4/\mathbb{Z}_m$

The anomaly polynomial I_8 of 6D $\mathcal{N} = (2, 0)$ is given by the general form

$$I_8 = \mathcal{A} I_8(1) + \mathcal{B} p_2(NW)/24, \tag{4}$$

where $I_8(1)$ is the anomaly polynomial of a single M5-brane, NW is the normal bundle to the world volume W of the 6D theory, and $\mathcal{A}$, $\mathcal{B}$ are integers determined by the type of the 6D theory. For N M5-branes, $\mathcal{A} = N$ and $\mathcal{B} = N^3 - N$. When compactified on a four-manifold X_4 with a suitable twist, it was determined in [3] that the resulting 2D theory has the central charge

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[1]See also [9] for a related work.

$$c = \chi(X_4)\mathcal{A} + (P_1(X_4) + 2\chi(X_4))\mathcal{B}. \quad (5)$$

Here $\chi(X_4)$ and $P_1(X_4)$ are the Euler number and 3 times the signature of X_4, respectively. We let X_4 be $\mathbb{R}^4/\mathbb{Z}_m$ with the deformations $\epsilon_{1,2}$. Then χ and P_1 are to be taken in the equivariant sense, and are given by[2]

$$\chi(X_4) = m, \qquad P_1(X_4) = \frac{1}{m}\frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1\epsilon_2} - 2m. \quad (6)$$

Therefore, N M5-branes on $\mathbb{R}^4/\mathbb{Z}_m$ give rise to a 2D system with the central charge

$$c = Nm + \frac{N^3 - N}{m}\left(b + \frac{1}{b}\right)^2, \quad (7)$$

where we used the standard identification $\epsilon_1/\epsilon_2 = b^2$.

According to the general lore, the Hilbert space of the 2D theory comes from the BPS states of the supersymmetric quantum mechanics on the moduli space of instantons on X_4. In our case, we consider U(N) instantons on $\mathbb{R}^4/\mathbb{Z}_m$, for which it is known that there is an action of a free boson and of $\widehat{\mathrm{SU}}(m)_N$; this was found in [20,21] and string theory interpretation was later given, e.g., in [22]. Then, the central charge (7) needs to be subdivided to

$$c = 1 + c(\widehat{\mathrm{SU}}(m)_N) + \left[\frac{m(N^2 - 1)}{m + N} + \frac{N^3 - N}{m}\left(b + \frac{1}{b}\right)^2\right]. \quad (8)$$

The third term is the central charge of the mth para-Toda theory of type SU(N) we found in Eq. (3).

We interpret this calculation as a check to the proposal that N M5-branes on $\mathbb{R}^4/\mathbb{Z}_m$ with deformations $\epsilon_{1,2}$ give rise to a 2D system with actions of a free boson, $\widehat{\mathrm{SU}}(m)_N$ and mth para-W_N algebra with central charge (3). This statement reduces to the now-standard relations in [1,2] when $m = 1$, and to the proposal in [8] when $m = N = 2$.

IV. SPECULATION CONCERNING 6D THEORY OF GENERAL TYPE ON $\mathbb{R}^4/\mathbb{Z}_m$

First, let us consider what happens if we start from 6D $\mathcal{N} = (2, 0)$ theory of type A_{N-1}, instead of N M5-branes. One needs to decouple the center-of-mass mode, which changes $\mathcal{A}$, $\mathcal{B}$ in Eqs. (4) and (5) to $\mathcal{A} = N - 1$ and $\mathcal{B} = N^3 - N$. The resulting 2D theory has the central charge of the form

$$c = c\left(\frac{\widehat{\mathrm{SU}}(m)_N}{\hat{\mathrm{U}}(1)^{m-1}}\right) + \left[c\left(\frac{\widehat{\mathrm{SU}}(N)_m}{\hat{\mathrm{U}}(1)^{N-1}}\right) + (N-1) + \frac{N^3 - N}{m}\left(b + \frac{1}{b}\right)^2\right]. \quad (9)$$

We see two cosets realizing generalized parafermions, known also as RCFT$[A_{m-1}, A_{N-1}]$ and RCFT$[A_{N-1}, A_{m-1}]$ in the terminology of [23], respectively. In general, RCFT$[\Gamma, G]$ for $\Gamma, G = A, D, E$ is a rational CFT with central charge

$$c(\mathrm{RCFT}[\Gamma, G]) = \frac{h_\Gamma r_\Gamma r_G}{h_\Gamma + h_G}. \quad (10)$$

Note that RCFT$[G, A_{m-1}]$ is the generalized parafermion of type G, but that RCFT$[\Gamma, G]$ is not yet constructed for a general pair of G, Γ. We have

$$c(\mathrm{RCFT}[\Gamma, G]) + c(\mathrm{RCFT}[G, \Gamma]) = r_\Gamma r_G, \quad (11)$$

and it is believed they comprise a "level-rank-dual" pair of RCFTs.

The 6D theory of type G has $\mathcal{A} = r_G$ and $\mathcal{B} = d_G h_G$ in Eqs. (4) and (5) [6,7]. Using $d_G = r_G(h_G + 1)$, we find that the 6D theory on $\mathbb{R}^4/\mathbb{Z}_m$ has the central charge

$$c = c(\mathrm{RCFT}[A_{m-1}, G]) + \left[c(\mathrm{RCFT}[G, A_{m-1}]) + r_G + \frac{d_G h_G}{m}\left(b + \frac{1}{b}\right)^2\right], \quad (12)$$

where the second term is the central charge of the mth para-Toda theory of type G we saw in Eq. (3). This suggests that the resulting 2D theory has the symmetry RCFT$[A_{m-1}, G]$ and the mth para-$W(\hat{G})$ symmetry. Note that RCFT$[A_{m-1}, G]$ is not yet constructed when $G \neq A$.

We can further generalize the system by considering instantons of gauge group $G = A, D, E$ on the ALE orbifold of type Γ. Nekrasov's deformation cannot be performed when $\Gamma \neq A$, because the ALE orbifold of type D and E does not have U$(1)^2$ isometry. We can still expect that this construction might naturally give us RCFT$[\Gamma, G]$ + RCFT$[G, \Gamma]$. The symmetry under G and Γ can be understood once one realizes that the 6D theory of type G is type IIB string on the ALE orbifold of type G. Then, the 2D system is type IIB string on the ALE orbifold of type G times the ALE orbifold of type Γ, which is manifestly symmetric under the exchange of G and Γ.

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[2]The calculation is done as follows. Let us parametrize $\mathbb{R}^4 \simeq \mathbb{C}^2$ by (z, w), on which two rotations act via $(z, w) \mapsto (e^{\epsilon_1}z, e^{\epsilon_2}w)$. On the blowup of $\mathbb{R}^4/\mathbb{Z}_m$, we have m fixed points of U$(1)^2$ actions, whose local coordinates are given by $(z_j, w_j) = (z^{m-j+1}w^{1-j}, z^{j-m}w^j)$ for $j = 1, \ldots, m$. Let us define $\epsilon_{1,2}(j)$ by the U$(1)^2$ action at the fixed points: $(z_j, w_j) \mapsto (e^{\epsilon_1(j)}z_j, e^{\epsilon_2(j)}w_j)$. Then the topological numbers are given by the fixed point formula: $\chi = \sum_j 1$ and $P_1 = \sum_j(\epsilon_1(j)^2 + \epsilon_2(j)^2)/(\epsilon_1(j)\epsilon_2(j))$.

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