PHYSICAL REVIEW D **84**, 046006 (2011)

Analytic nonintegrability in string theory

Pallab Basu*

Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506, USA

Leopoldo A. Pando Zayas†

Michigan Center for Theoretical Physics, University of Michigan, Ann Arbor, Michigan 48109, USA

(Received 13 May 2011; published 11 August 2011)

Using analytic techniques developed for Hamiltonian dynamical systems, we show that a certain classical string configuration in $AdS_5 \times X_5$ with X_5 in a large class of Einstein spaces is nonintegrable. This answers the question of integrability of string on such backgrounds in the negative. We consider a string localized in the center of AdS_5 that winds around two circles in the manifold X_5.

DOI: 10.1103/PhysRevD.84.046006 PACS numbers: 11.25.Tq, 05.45.−a, 11.30.Na

I. INTRODUCTION

Chaotic motion has been one of the most studied aspects of nonlinear dynamical systems as its application extends to many areas [1]. Although its mathematical roots date back to Poincarè and the three-body problem, it was really during the last part of the twentieth century when chaotic motion study flourished, largely thanks to new advances in computing power. Naturally, under the shadow of quantum mechanics, it is logical to try to understand the quantum properties in systems whose classical limit is chaotic; this area has become known as quantum chaos [2]. In the context of the AdS/CFT correspondence [3], there is a particularly special chance to understand some of these questions as we have a setting in which the classical regime of a theory is dual to the highly quantum regime of another. Understanding classical chaos and the corresponding quantization in the context of string theory provides a new framework with enhanced interpretational opportunities.

The simplest version of the AdS/CFT correspondence [3] states a complete equivalence between strings on $AdS_5 \times S^5$ with Ramond-Ramond fluxes and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) with gauge group $SU(N)$. Chaotic behavior of some classical configurations of strings in the context of the AdS/CFT has been recently established for several interesting string theory backgrounds: ring strings in the Schwarzschild black hole in asymptotically AdS_5 backgrounds [4], strings in the AdS soliton background [5] and in $AdS_5 \times T^{1,1}$ [6], which is a coset but not a maximally symmetric one.

In general the question of integrability is settled through a numerical analysis of the system [1]. Over the last decades an analytical approach has been developed to determine whether a system is integrable. Some powerful results due to Ziglin [7,8] and further refined by Morales-Ruiz and Ramis [9] turn the question of integrability of some simple systems into an algorithmic process. In this paper we study a large class of systems that appear in string theory. We generalize some of our previous results for classical strings on $AdS_5 \times T^{1,1}$ [6] to include more general backgrounds of the form $AdS_5 \times X_5$, where X_5 is in a general class of five-dimensional Einstein spaces admitting a $U(1)$ fibration.

II. ANALYTIC NONINTEGRABILITY

The general basis for proving nonintegrability of a system of differential equations $\dot{\vec{x}} = \vec{f}(\vec{x})$ is the analysis of the variational equation around a particular solution $\bar{x} = \bar{x}(t)$ [9,10]. The variational equation around $\bar{x}(t)$ is a linear system obtained by linearizing the vector field around $\bar{x}(t)$. If the nonlinear system admits some first integrals, so does the variational equation. Thus, proving that the variational equation does not admit any first integral within a given class of functions implies that the original nonlinear system is nonintegrable. In particular, one works in the analytic setting where, inverting the solution $\bar{x}(t)$, one obtains a (noncompact) Riemann surface Γ given by integrating $dt = dw/\dot{\bar{x}}(w)$ with the appropriate limits. Linearizing the system of differential equations around the straight line solution yields the normal variational equation (NVE), which is the component of the linearized system which describes the variational normal to the surface Γ.

Given a Hamiltonian system, the main statement of Ziglin's theorems is to relate the existence of a first integral of motion with the monodromy matrices around the straight line solution [7,8]. The simplest way to compute such monodromies is by changing coordinates to bring the normal variational equation into a known form (hypergeometric, Lamé, Bessel, Heun, etc.).

Morales-Ruiz and Ramis proposed a major improvement on Ziglin's theory by introducing techniques of differential Galois theory [11–13]. The key observation is to change the formulation of integrability from a question of monodromy to a question of the nature of the Galois group of the NVE. Intuitively, if we go back to Kovalevskaya, we

*pallab.basu@uky.edu
†lpandoz@umich.edu

are interested in understanding whether the Kolmogorov-Arnold-Moser (KAM) tori are resonant or not resonant, or, in simpler terms, if their characteristic frequencies in the action-angle formalism are rational or irrational. (See the pedagogical introductions provided in [9,14].) This statement turns out to be dealt with most efficiently in terms of the Galois group of the NVE. The key result is now stated as this: If the differential Galois group of the NVE is nonvirtually Abelian, that is, if the identity-connected component is a non-Abelian group, then the Hamiltonian system is nonintegrable. The calculation of the Galois group is rather intricate, but the key simplification comes through the application of Kovacic's algorithm [15]. Kovacic's algorithm implements Picard-Vessiot theory for second order homogeneous linear differential equations with polynomial coefficients, giving a constructive answer to the existence of integrability by quadratures. Fortunately, Kovacic's algorithm is implemented in most computer algebra software including MAPLE and MATHEMATICA. It is a little tedious but straightforward to check the algorithm manually. So, once we write down our NVE in a suitable linear form with polynomial coefficients, it becomes a simple task to check their solvability in quadratures. An important property of the Kovacic's algorithm is that it works if and only if the system is integrable; thus a failure of completing the algorithm equates to a proof of nonintegrability. This route of declaring systems nonintegrable has been successfully applied to various situations. Some interesting examples include general homogeneous potentials [16], cosmological models [17], fluid dynamics [18], generalizations of the Hénon-Heiles system [14] and various others [9].

III. WRAPPED STRINGS IN GENERAL $\mathrm{AdS}_5 \times X_5$

The methods of analytic nonintegrability can be applied to a large class of spaces in string theory. Let us start by considering a five-dimensional Einstein space X_5, with $R_{ij} \sim g_{ij}$. Any such Einstein space furnishes a solution to the type IIB supergravity equations known as a Freund-Rubin compactification [19]. The solution takes the form

$$ds^2 = ds^2(\mathrm{AdS}_5) + ds^2(X_5), \qquad F_5 = (1 + \star)\mathrm{vol}(\mathrm{AdS}_5),$$

where vol is the volume five-form and $\star$ is the Hodge dual operator. Of particular interest in string theory is the case when X_5 is Sasaki-Einstein, that is, on top of being Einstein, it admits a spinor satisfying $\nabla_\mu \epsilon \sim \Gamma_\mu \epsilon$.

The configuration that we are interested in exploring is a string sitting at the center of AdS_5 and winding in the *circles* provided by the base space. More explicitly, consider the AdS_5 metric in global coordinates: $ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2$. Then, our solution is localized at $\rho = 0$. Largely inspired by the Sasaki-Einstein class, we consider spaces X_5 that are a $U(1)$ fiber over a four-dimensional manifold. In the case of topologically trivial fibration, we are precluded from applying our argument; those manifolds can be considered separately. The general local structure of Sasaki-Eintein metrics is

$$ds^2_{X^5_{S-E}} = \left(d\psi + \frac{i}{2}(K_{,i}dz^i - K_{,\bar{i}}d\bar{z}^i)\right)^2 + K_{,i\bar{j}}dz^i d\bar{z}^j, \quad (1)$$

where K is a Kähler potential on the complex base with coordinates z_i with $i = 1, 2$. This is the general structure that will serve as our guiding principle, but we will not be limited to it. Roughly, our ansatz for the classical string configuration is $z_i = r_i(\tau)e^{i\alpha_i\sigma}$, where τ and σ are the worldsheet coordinates of the string. Crucially, we have introduced winding of the strings characterized by the constants α_i. The goal is to solve for the functions $r_i(\tau)$.

To apply the tools of analytic nonintegrability to the class of solutions above we will: (1) Select a particular solution, that is, define the *straight line solution*. (2) Write the normal variational equation (NVE). (3) Check if the identity component of the differential Galois group of the NVE is Abelian, that is, apply the Kovacic's algorithm to determine if the NVE is integrable by quadrature.

Given this ansatz above, we can now summarize the general results. We prove that the corresponding effective Hamiltonian systems have 2 degrees of freedom and admit an invariant plane $\Gamma = \{r_2 = \dot{r}_2 = 0\}$ whose normal variational equation around integral curves in Γ we study explicitly.

A. $T^{p,q}$

These 5-manifolds are not necessarily Sasaki-Einstein; however, some of them are Einstein which allow for consistent string backgrounds. More importantly, some of these spaces provide exact conformal sigma models and are thus exact string backgrounds in all orders in α' [20]. However, they are never maximally symmetric, and the integrability discussed for $\mathrm{AdS}_5 \times S^5$ is not applicable. In this section, we provide a unified treatment of this class for generic values of p and q. The metric has the form

$$\begin{aligned} ds^2 &= a^2(d\psi + p\cos\theta_1 d\phi_1 + q\cos\theta_2 d\phi_2)^2 \\ &\quad + b^2(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + c^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2). \end{aligned}$$

The classical string configuration we are interested is $\theta_1 = \theta_1(\tau)$, $\theta_2 = \theta_2(\tau)$, $\psi = \psi(\tau)$, $t = t(\tau)$, $\phi_1 = \alpha_1\sigma$, $\phi_2 = \alpha_2\sigma$, where α_i are constants quantifying how the string winds along the ϕ_i directions. Recall that t is from AdS_5. The Polyakov Lagrangian is

$$\begin{aligned} \mathcal{L} = -\frac{1}{2\pi\alpha'}[&\dot{t}^2 - b^2\dot{\theta}_1^2 - c^2\dot{\theta}_2^2 - a^2\dot{\psi}^2 \\ &+ \alpha_1^2(b^2 - a^2p^2)\sin^2\theta_1 + \alpha_2^2(c^2 - a^2q^2)\sin^2\theta_2 \\ &+ 2\alpha_1\alpha_2 pqa^2\cos\theta_1\cos\theta_2]. \end{aligned} \quad (2)$$

There are several conserved quantities; the corresponding nontrivial equations are

$$\ddot{\theta}_1 + \frac{\alpha_1}{b^2}\sin\theta_1[\alpha_1(b^2 - a^2p^2)\cos\theta_1 - a^2\alpha_2 pq\cos\theta_2] = 0,$$
$$\ddot{\theta}_2 + \frac{\alpha_2}{c^2}\sin\theta_2[\alpha_2(c^2 - a^2q^2)\cos\theta_2 - a^2\alpha_1 pq\cos\theta_2] = 0.$$

There is immediately some insight into the role of the fibration structure. Note that the topological winding in the space which is described by p and q intertwines with the wrapping of the strings α_1 and α_2. The effective numbers that appear in the interaction part of the equations are $\alpha_1 p$ and $\alpha_2 q$. For example, from the point of view of the interactions terms, taking $p = 0$ or $q = 0$ is equivalent to taking one of the $\alpha_i = 0$ which leads to an integrable system of two noninteracting gravitational pendulums.

We take the straight line solution to be $\theta_2 = \dot{\theta}_2 = 0$. The equation for θ_1 becomes

$$\ddot{\theta}_1 + \frac{\alpha_1}{b^2}[\alpha_1(b^2 - a^2p^2)\cos\theta_1 - a^2\alpha_2 pq]\sin\theta_1 = 0. \quad (3)$$

Let us denote the solution to this equation $\bar{\theta}_1$; it can be given explicitly but we will not need the precise form. This solution also defines the Riemann surface Γ introduced before. The NVE is obtained by considering small fluctuations in θ_2 around the above solutions and takes the form:

$$\ddot{\eta} + \frac{\alpha_2}{c^2}[\alpha_2(c^2 - a^2q^2) - \alpha_1 pq\cos\bar{\theta}_1]\eta = 0. \quad (4)$$

Our goal is to study the NVE. To make the equation amenable to the Kovacic's algorithm, we introduce the following substitution: $\cos(\bar{\theta}_1) = z$. In this variable, the NVE takes a form similar to the Lamé equation:

$$f(z)\eta''(z) + \frac{1}{2}f'(z)\eta'(z) + \frac{\alpha_2}{c^2}[\alpha_2(c^2 - a^2q^2) - \alpha_1 pqz]\eta(z) = 0, \quad (5)$$

where prime now denotes differentiation with respect to z and $f(z) = \dot{\bar{\theta}}_1^2\sin^2(\bar{\theta}_1) = (6E^2 - \frac{1}{3}(4\alpha_1\alpha_2 z + \alpha_2^2(1-z^2)))(1-z^2)$. Equation (5) is a second order homogeneous linear differential equation with polynomial coefficients, and it is, therefore, ready for the application of Kovacic's algorithm. For generic values of the above parameters, the Kovacic's algorithm does not produce a solution, meaning the system defined in Eq. (3) is not integrable.

The case of $T^{1,1}$ is particularly interesting because for this case the supergravity solution is supersymmetric, and a lot of attention has been paid to extending configurations of $AdS_5 \times S^5$ to the case of $AdS_5 \times T^{1,1}$ [21–26].

B. $Y^{p,q}$

These spaces have played a central role in the development of the AdS/CFT correspondence as they provide an infinite class of dualities. These spaces are Sasaki-Einstein, but they are not coset spaces as was the case for the $T^{p,q}$ discussed above. Following the general discussion of Sasaki-Einstein spaces above, we write the metric on these spaces as

$$ds^2 = \frac{1}{9}(d\psi - (1-cy)\cos\theta d\phi + yd\beta)^2 + \frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{p(y)}{6}(d\beta + c\cos\theta d\phi)^2,$$

and $p(y) = [a - 3y^2 + 2cy^3]/[3(1-cy)]$. The classical string configuration is described by the ansatz $\theta = \theta(\tau)$, $y = y(\tau)$, $\phi = \alpha_1\sigma$, $\beta = \alpha_2\sigma$. The Polyakov Lagrangian is simply

$$\mathcal{L} = -\frac{1}{2\pi\alpha'}\bigg[\dot{t}^2 - \frac{1-cy}{6}\dot{\theta}^2 - \frac{1}{6p(y)}\dot{y}^2 - \frac{1}{9}\dot{\psi}^2 + \frac{1-cy}{6}\alpha_1^2\sin^2\theta + \frac{p(y)}{6}(\alpha_2 + c\alpha_1\cos\theta)^2 + \frac{1}{9}(\alpha_2 y - \alpha_1(1-cy)\cos\theta)^2\bigg]. \quad (6)$$

As in previous cases, the equations of motion for t and ψ are integrated immediately, leaving only two nontrivial equations for θ and y. The straight line solution can be taken to be $\theta = \dot{\theta} = 0$. Then the equation for y is simplified to

$$\ddot{y} - \frac{p'}{p}\dot{y}^2 + \frac{pp'}{2}(\alpha_2 + c\alpha_1)^2 + \frac{2}{3}p(\alpha_2 + c\alpha_1)(y(\alpha_2 + c\alpha_1) - \alpha_1) = 0.$$

To be able to write the NVE in a form conducive to the application of Kovacic's algorithm, we substitute $y_s(t) = y$, and the NVE takes the form

$$(1-cy)\dot{y}^2(t)\frac{d^2\eta}{dy^2} + q(y)\eta(y) \quad (7)$$

$$+ (\ddot{y}(t)(1-cy) - c\dot{y}^2(t))\frac{d\eta}{dy} = 0, \quad (8)$$

where $\dot{y}$ and $\ddot{y}$ can be written in terms of y as

$$\dot{y}^2(t) = 6(E + p(y)V(y,0)) = 6p(y)\Big(\frac{p(y)}{6}(\alpha_2 + c\alpha_1)^2 + \frac{1}{9}(\alpha_2 y - \alpha_1(1-cy))\Big)$$

and

$$q(y) = \alpha_1(1-cy)\bigg[5/3\alpha_1 - \frac{c(a - 3y^2 + 2cy^3)(\alpha_2 + c\alpha_1)}{(3-3cy)(1-cy)} - 2/3(\alpha_2 + c\alpha_1)y\bigg].$$

With these identifications we have rewritten the NVE as a homogeneous second order linear differential equation with polynomial coefficients. The Kovacic's algorithm again fails to yield a solution, pointing to the fact that the system is generically nonintegrable.

IV. THE EXCEPTIONAL CASE: S^5

In this section we provide an integrable example where the Kovacic's algorithm should succeed. To expose the Sasaki-Einstein structure of S^5, it is convenient to write the metric as a $U(1)$ fiber over $\mathbb{P}^2$. The round metrics on S^5 may be elegantly expressed in terms of the left-invariant one-forms of $SU(2)$. The left-invariant one-forms can be written as $\sigma_1 = \frac{1}{2}(\cos(d\psi)d\theta + \sin(\psi)\sin(\theta)d\phi)$, $\sigma_2 \frac{1}{2} \times (\sin(\psi)d\theta - \cos(\psi)\sin(\theta)d\phi)$, $\sigma_3 = \frac{1}{2}(d\psi + \cos(\theta)d\phi)$. In terms of these one-forms, the metrics on $\mathbb{P}^2$ and S^5 may be written as follows:

$$\begin{aligned} ds^2_{\mathbb{P}^2} &= d\mu^2 + \sin^2(\mu)(\sigma_1^2 + \sigma_2^2 + \cos^2(\mu)\sigma_3^2), \\ ds^2_{S^5} &= ds^2_{\mathbb{P}^2} + (d\chi + \sin^2(\mu)\sigma_3)^2, \end{aligned} \tag{9}$$

where χ is the local coordinate on the Hopf fiber and $A = \sin^2(\mu)\sigma_3 = \sin^2(\mu)(d\psi + \cos(\theta)d\phi)/2$ is the one-form potential for the Kähler form on $\mathbb{P}^2$.

The classical string configuration is $\theta = \theta(\tau)$, $\mu = \mu(\tau)$, $\chi = \chi(\tau)$, $\phi = \alpha_1\sigma$, $\psi = \alpha_2\sigma$. The Lagrangian is

$$\begin{aligned} \mathcal{L} = -\frac{1}{2\pi\alpha'}\Big[\dot{t}^2 - \dot{\mu}^2 - \frac{1}{4}\sin^2\mu\dot{\theta}^2 - \dot{\chi}^2 \\ + \frac{1}{4}\sin^2\mu(\alpha_1^2\sin^2\theta + (\alpha_2 + \alpha_1\cos\theta)^2)\Big]. \end{aligned} \tag{10}$$

The nontrivial equations of motion are

$$\begin{aligned} &\ddot{\mu} + \frac{1}{8}\sin(2\mu)[\dot{\theta}^2 - 2\alpha_1\alpha_2\cos\theta - \alpha_1^2 - \alpha_2^2] = 0, \\ &\ddot{\theta} + 2\dot{\mu}\,\dot{\theta}\cot(\mu) + \alpha_1\alpha_2\sin\theta = 0. \end{aligned} \tag{11}$$

Inspection of the above system shows that we have various natural choices. We discuss the two natural choices of straight line solutions in what follows.

A. θ straight line

Let us assume $\theta = \dot{\theta} = 0$; then the equation for μ becomes

$$\ddot{\mu} - \frac{1}{8}(\alpha_1 + \alpha_2)^2\sin(2\mu) = 0. \tag{12}$$

We call the solution of this equation μ_s. The NVE is

$$\ddot{\eta} + 2\cot(\mu_s)\dot{\mu}_s\dot{\eta} + \alpha_1\alpha_2\eta = 0. \tag{13}$$

With $\sin(\mu) = z$, the NVE may be written as

$$r(z)\frac{d^2}{dz^2}\eta(z) + q(z)\frac{d}{dz}\eta(z) + \alpha_1\alpha_2 z^2\eta(z) = 0, \tag{14}$$

with

$$r(z) = z^2(2E + 1/8(\alpha_1 + \alpha_2)^2(1 - 2z^2))(1 - z^2)$$

and

$$\begin{aligned} q(z) = -1/8z(&-32E + 48Ez^2 - 2\alpha_1^2 + 9\alpha_1^2z^2 - 8\alpha_1^2z^4 \\ &- 4\alpha_1\alpha_2 + 18\alpha_1\alpha_2z^2 - 16\alpha_1\alpha_2z^4 - 2\alpha_2^2 \\ &+ 9\alpha_2^2z^2 - 8\alpha_2^2z^4). \end{aligned}$$

This equation is now in the form conducive to Kovacic's algorithm which succeeds and gives a solution. Since the above approach obscures the nature of integrability of $AdS_5 \times S^5$, we now consider another example which leaves no doubt about the integrability.

B. μ straight line

Let us assume the straight line is now given by $\mu = \pi/2$, $\dot{\mu} = 0$. The equation for θ becomes

$$\ddot{\theta} + \alpha_1\alpha_2\sin\theta = 0. \tag{15}$$

Let us call the solution to this equation θ_s. Then the NVE is

$$\ddot{\eta} + \frac{1}{4}(\dot{\theta}_s^2 - 2\alpha_1\alpha_2\cos(\theta_s) - \alpha_1^2 - \alpha_2^2)\eta = 0. \tag{16}$$

Note that the equation of motion for θ_s implies

$$\begin{aligned} \ddot{\theta} + \alpha_1\alpha_2\sin\theta = 0 \to \frac{d}{d\tau}(\dot{\theta}_s^2 - 2\alpha_1\alpha_2\cos\theta_s) = 0, \\ \to \dot{\theta}_s^2 - 2\alpha_1\alpha_2\cos\theta_s = C_0. \end{aligned} \tag{17}$$

Thus the NVE equation can be written as a simple harmonic equation:

$$\ddot{\eta} + \frac{1}{4}(C_0 - \alpha_1^2 - \alpha_2^2)\eta = 0. \tag{18}$$

We do not require Kovacic's algorithm to tell us that there is an analytic solution for this equation. The power of differential Galois theory also guarantees that the result is really independent of the straight line solution (Riemann surface) that one chooses. We conclude this subsection with the jovial comment that we now know a very precise sense in which *string theory in* $AdS_5 \times S^5$ *is like a harmonic oscillator*.

V. CONCLUSIONS

In this paper we have shown that certain classical string configurations corresponding to a string winding along two of the angles of a general class of five-dimensional Einstein manifolds X_5, realized as a nontrivial S^1 fibration over a 4D base, are nonintegrable. The result highlights the limit of integrability within the AdS/CFT correspondence. Integrability has been one of the main areas of study for almost ten years. The paper forces the AdS/CFT to expand with the newly discovered fact that most configurations beyond $AdS_5 \times S^5$ are nonintegrable; this requires a new dictionary.

In all the previous examples in the literature, homogeneity of the potential played a crucial role in the proof [9,16,17]. A mathematical curiosity arises from the fact that traditionally, due to the works of Hadamard and later of Anosov, chaos has been associated with the motion of particles in negatively curved spaces through the Jacobi equation. The class of five-dimensional Einstein spaces used here has positive curvature. The main mechanism for nonintegrability is provided by the winding of the strings which is a property unique to strings and therefore not well understood. More precisely, we found an interesting interplay between topology $c_1 = \int dA$ and dynamics as the Chern class determines the possibility of an interaction term in the dynamical system. As pointed out in the main text, in various cases, the interaction, and therefore nonintegrability, appears as the product of the Chern number and the winding number of the string.

The direct connection between analytic nonintegrability and chaotic behavior is still open. This question has been discussed in the literature, and we refer the reader to [9] for further details. For the sake of disclosure, we note that we have not directly proved that the systems discussed here are chaotic. However, together with our previous publication [6], we believe the case for outright chaotic behavior is overwhelmingly strong.

Our work now opens the door to the interpretation of many chaotic quantities in the context of the AdS/CFT correspondence. For example, the meaning of Lyapunov spectrum, Kolmogorov-Sinai entropy and fractal dimension are but a few of the quantities expecting their quantum analogs in this context.

ACKNOWLEDGMENTS

P. B. thanks D. Das, S. Das, A. Ghosh and A. Shapere. This work is partially supported by the U.S. Department of Energy under Grant No. DE-FG02-95ER40899 and the National Science Foundation under Grant Nos. NSF-PHY-0855614 and PHY-0970069.

[1] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, England, 2002), 2nd ed.
[2] M. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, Berlin, 1990).
[3] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[4] L. A. Pando Zayas and C. A. Terrero-Escalante, J. High Energy Phys. 09 (2010) 094.
[5] P. Basu, D. Das, A. Ghosh, Phys. Lett. B **699**, 388 (2011).
[6] P. Basu and L. A. Pando Zayas, Phys. Lett. B **700**, 243 (2011).
[7] S. L. Ziglin, Funct. Anal. Appl. **16**, 181 (1983).
[8] S. L. Ziglin, Funct. Anal. Appl. **17**, 6 (1983).
[9] J. Morales Ruiz, *Differential Galois Theory and Nonintegrability of Hamiltonian Systems* (Birkhäuser, Basel, 1999).
[10] A. Goriely, *Integrability and Nonintegrability of Dynamical Systems* (World Scientific, Singapore, 2001).
[11] J. J. Morales-Ruiz and C. Simó, J. Diff. Equ. **107**, 140 (1994).
[12] J. J. Morales-Ruiz and J. P. Ramis, Methods Appl. Anal. **8**, 33 (2001).
[13] J. J. Morales-Ruiz, J.-P. Ramis, and C. Simò, Annali della Scuola normale superiore di Pisa, Classe di scienze **40**, 845 (2007).
[14] J. J. Morales-Ruiz, Reg. Chaotic Dyn. **5**, 251 (2000).
[15] J. J. Kovacic, J. Symb. Comput. **2**, 3 (1986).
[16] J. J. Morales-Ruiz and J. P. Ramis, Methods Appl. Anal. **8**, 113 (2001).
[17] A. J. Maciejewski and M. Szydlowski, J. Nonlinear Math. Phys. Supplement, 200206 Proceedings: NEEDS99 **8**, 200 (2001).
[18] S. L. Ziglin, Funct. Anal. Appl. **37**, 225 (2003).
[19] P. G. O. Freund and M. A. Rubin, Phys. Lett. **97B**, 233 (1980).
[20] L. A. Pando Zayas and A. A. Tseytlin, Classical Quantum Gravity **17**, 5125 (2000).
[21] L. A. Pando Zayas and J. Sonnenschein, J. High Energy Phys. 05 (2002) 010.
[22] N. Itzhaki, I. R. Klebanov, and S. Mukhi, J. High Energy Phys. 03 (2002) 048.
[23] N. Kim, Phys. Rev. D **69**, 126002 (2004).
[24] X.-J. Wang, Phys. Rev. D **72**, 086006 (2005).
[25] S. Benvenuti and M. Kruczenski, J. High Energy Phys. 10 (2006) 051.
[26] S. Benvenuti and E. Tonni, J. High Energy Phys. 02 (2008) 022.