Noncommutative spectral geometry, algebra doubling, and the seeds of quantization

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A physical interpretation of the two-sheeted space, the most fundamental ingredient of noncommutative spectral geometry proposed by Connes as an approach to unification, is presented. It is shown that the doubling of the algebra is related to dissipation and to the gauge structure of the theory, the gauge field acting as a reservoir for the matter field. In a regime of completely deterministic dynamics, dissipation appears to play a key role in the quantization of the theory, according to the 't Hooft's conjecture. It is thus argued that the noncommutative spectral geometry classical construction carries the seeds of quantization, implicit in its feature of the doubling of the algebra.

I. INTRODUCTION AND MOTIVATION

Noncommutative spectral geometry [1,2] (NCSG) is a rich mathematical theory that combines notions of noncommutative geometry with spectral triples, a mathematical tool conceived by Alain Connes. Within this context, Connes and collaborators built a model which offers a purely geometric explanation for the standard model (SM) of electroweak and strong interactions—the most successful model of particle physics at hand today—compatible with right-handed neutrinos and neutrino masses [3]. This model succeeds at finding a way to merge the diffeomorphism invariance which governs general relativity, with the local gauge invariance which governs gauge theories upon which the SM is based. The NCSG model has also been used to derive supersymmetric extensions to the SM [4].

This unification model lives by construction at high energy scales (namely at unification scale), thus providing a natural environment to address unresolved issues of early universe cosmology [5–12]. Various criticisms have however been raised. One may for instance argue that since the model is at present purely classical, strictly speaking one cannot employ it within the context of the early universe since then the energy scales were so high that quantum corrections could no longer be neglected. Or one may argue that since the action functional is obtained through a perturbative approach in inverse powers of the cutoff scale, it ceases to be valid at lower energy scales relevant for astrophysical studies. Note that the original approach may *a priori* also be treated nonperturbatively, however it is very difficult to compute exactly the spectral action in its nonperturbative form. Another criticism that a physicist may have is that while the model is naturally developed in

the Euclidean signature, any physical studies must be performed in the Lorentzian signature.

The aim of this paper is twofold: first, to address the connection between gauge theories and the algebra doubling and offer a simple physical insight to this rich mathematical theory; second, to reply to some of the above mentioned criticisms.

In what follows, after a short introduction to noncommutative spectral geometry in Sec. II, we show in Sec. III how the algebra doubling, which is an essential mathematical feature of the NCSG construction, is related to the gauge structure of the theory. We introduce the notion of dissipation within this context, which in Sec. IV will lead to the quantum aspect of the noncommutative spectral geometry and the notion of temperature. In our discussion we will resort to the proposal by 't Hooft [13–15], according to which quantum features and behaviors in a theory would result from a more fundamental deterministic scenario due to a process of information loss. In other words, according to 't Hooft's proposal, quantum mechanics emerges from an underlying deterministic classical dynamics acting at an energy scale much higher than the one of our observations, provided information loss (dissipation) has occurred. This means that the NCSG "classical" construction, holding at high energy scales, may carry in itself the seeds for quantum behavior, provided in the same construction there is room for dissipation. In the following, we argue that this is indeed the case, since, as we show, the characterizing feature of the algebra doubling is related to dissipation, which in turn can be described in terms of gauge fields. Thus the two-sheeted space selected in NCSG is related to gauge theories, as well as to dissipation and to quantization. We summarize our physical interpretation of the NCSG purely gravitational approach to unification in our conclusions.

Before proceeding, it is important to clarify in which sense we talk of dissipation and of temperature in what

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follows. This is necessary because the standard model, as is well known, is a zero temperature quantum field theory (QFT) model describing a closed (nondissipative) system. It is however equally well known that even in high energy physics there are many cases in which the zero temperature approximation is inadequate, as, for example, in the quark-gluon plasma physics, unified gauge theory scenarios, astrophysics, cosmology and in general in all cases where one studies critical phenomena in symmetry breaking phase transition processes in the early universe (see e.g., Ref. [16]). When we mention temperature we think therefore to such circumstances where use of a thermal field theory is unavoidable. It is in fact a fortunate case that the doubling of the algebra in NCSG also implies finite temperature features, as we will discuss in Secs. IV and VI. We talk of dissipation and open systems in the same specific sense one observes in a system of electromagnetically interacting matter field, that neither the energy-momentum tensor of the matter field nor that of the gauge field are conserved. It is, however, $\partial_\mu T^{\mu\nu}_{\text{matter}} = eF^{\mu\nu}j_\mu = -\partial_\mu T^{\mu\nu}_{\text{gaugefield}}$, so that what is conserved [17,18] is the *total* $T^{\mu\nu}_{\text{total}} = T^{\mu\nu}_{\text{matter}} + T^{\mu\nu}_{\text{gaugefield}}$, which is the energy-momentum tensor of the *closed* system {matter field, electromagnetic field}: each element of the couple is *open* (dissipating) on the other one, although the *closedness* of the total system is ensured. It is in this sense that dissipation enters our discussion of the implications of the algebra doubling without spoiling the closedness of the SM.

II. ELEMENTS OF NONCOMMUTATIVE SPECTRAL GEOMETRY

For the reader's convenience we summarize briefly some of the basic features and ingredients of NCSG in the present section.

At low energy scales, the laws of physics can be described by the action functional $S = S_{\text{E-H}} + S_{\text{SM}}$ which is the sum of the Einstein-Hilbert action $S_{\text{E-H}}$ and the SM action S_{SM}. While the former depends on the geometry of the underlying manifold $\mathcal{M}$, the latter is governed by internal symmetries of a gauge group. The symmetries of the two parts are also different: general relativity is governed by diffeomorphism invariance (outer automorphisms), while gauge symmetries are based on local gauge invariance (inner automorphisms). Near the Planck scale, this sum fails to capture the correct description of physics, and one may argue that the distinct feature between the underlying symmetries of the two parts of S may be at the origin of the unsuccessful search for a unified theory of all interactions including gravity. The full group of invariance of the total (including gravity and matter) action functional S is the semidirect product $\mathcal{U} = \mathcal{G} \rtimes \text{Diff}(\mathcal{M})$ of the group $\mathcal{G}$ of gauge transformations of the matter sector (the standard model) and the group $\text{Diff}(\mathcal{M})$ of diffeomorphisms of the manifold $\mathcal{M}$. One thus tries to find a space whose group

of diffeomorphisms is $\mathcal{U}$; this can be achieved within noncommutative geometry.[1]

To capture the effect of the SM on the continuous four-dimensional manifold, Connes considered a model of a two-sheeted space, made from the product of a four-dimensional smooth compact Riemannian manifold $\mathcal{M}$ with a fixed spin structure, by a discrete noncommutative space $\mathcal{F}$ composed by only two points. In this approach, the SM of electroweak and strong interactions is seen as a phenomenological model, which however specifies the geometry of space-time in such a way so that the Maxwell-Dirac action functional leads to the SM action. Following this proposal, the geometric space is defined as the tensor product of a continuous geometry $\mathcal{M}$ for space-time by an internal geometry $\mathcal{F}$ for the SM.

An essential step for this proposal is to adapt the notion of metric for spaces which do not require commutativity of the coordinates. We will thus sketch the transition from the Riemannian $g_{\mu\nu}$ paradigm, based on the Taylor expansion in local coordinates x^μ of the square of the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, to the NCSG paradigm.

The noncommutative nature of the discrete space $\mathcal{F}$ is given by the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A}$ is an involution of operators on the finite-dimensional Hilbert space $\mathcal{H}$ of Euclidean fermions, and $\mathcal{D}$ is a self-adjoint unbounded operator in $\mathcal{H}$. Spectral triples are analogous to Fourier transform in commutative spaces and are introduced in order to create a link with experimental data, which are all of a spectral nature. It is worth noting that the spectral nature approach is intrinsic to the noncommutative spectral geometry.

To be more precise, let $\mathcal{H} = L^2(\mathcal{M}, S)$ be the Hilbert space of square integrable sections of the spinor bundle, $\mathcal{A} = C^\infty(\mathcal{M})$ be the algebra of smooth functions on $\mathcal{M}$ acting on $\mathcal{H}$ as simple multiplication operators

$$(f\xi)(x) = f(x)\xi(x), \quad \forall f \in C^\infty(\mathcal{M}) \text{ and } \forall \xi \in L^2(\mathcal{M}, S),$$

and $D = \partial\!\!\!/_\mathcal{M} = \sqrt{-1}\gamma^\mu\nabla^s_\mu$ (where ∇^s_μ is the spin connection[2]

The algebra $\mathcal{A}$, related to the gauge group of local gauge transformations, is the algebra of coordinates; all

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[1]A main difference between noncommutative spectral geometry and other approaches of quantizing gravity is that here one is searching for a hidden signature of space-time geometry within the functional of gravity coupled to the SM at present energy scales, instead of postulating the geometry around the Planck scale which necessitates an extrapolation by many orders of magnitude.

[2]The spin connection ∇^s_μ is expressed in a vierbein e. Let e^a_μ be defined as $g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}$, so that $\gamma_a = e^\mu_a \gamma_\mu$ satisfy the anticommutation $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$. Setting $\gamma_{ab} = (1/4)[\gamma_a, \gamma_b]$, one then gets

$$\nabla^s_\mu = \partial_\mu + \frac{1}{2}\omega^{ab}_\mu\gamma_{ab}$$

to be the Dirac operator on the spin Riemannian manifold $\mathcal{M}$.

information about space is encoded in $\mathcal{A}$. In the product noncommutative space $\mathcal{M} \times \mathcal{F}$, the algebra $\mathcal{A}$ is Abelian, whereas the derivative in the discrete direction is a finite difference quotient. For spaces whose coordinates do not commute, geometry is defined by specifying the Dirac operator, thus distance is described through D and not through the metric tensor $g_{\mu\nu}$. The familiar geodesic formula

$$d(x, y) = \inf \int_{\gamma} ds, \tag{1}$$

where the infimum is taken over all possible paths connecting x to y, used to determine the distance $d(x, y)$ between two points x and y within Riemannian geometry, is then replaced by

$$d(x, y) = \sup\{|f(x) - f(y)|: f \in \mathcal{A}, \|[D, f]\| \leq 1\}. \tag{2}$$

Within the noncommutative spectral geometry D plays the role of the inverse of the line element ds. The operator D corresponds to the inverse of the Euclidean propagator of fermions, and is given by the Yukawa coupling matrix which encodes the masses of the elementary fermions and the Kobayashi-Maskawa mixing parameters. In conclusion, NCSG is given by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, in other words by an involutive algebra $\mathcal{A}$ represented as operators in Hilbert space $\mathcal{H}$ and the line element $ds = 1/D$.

The product geometry is specified by the rules

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2, \qquad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \tag{3}$$

and hence for $\mathcal{M} \times \mathcal{F}$ the rules read

$$\begin{aligned} \mathcal{A} &= C^{\infty}(\mathcal{M}) \otimes \mathcal{A}_{\mathcal{F}} = C^{\infty}(\mathcal{M}, \mathcal{A}_{\mathcal{F}}), \\ \mathcal{H} &= L^2(\mathcal{M}, S) \otimes \mathcal{H}_{\mathcal{F}} = L^2(\mathcal{M}, S \otimes \mathcal{H}_{\mathcal{F}}), \\ D &= \partial\!\!\!/_{\mathcal{M}} \otimes 1 + \gamma_5 \otimes D_{\mathcal{F}}; \end{aligned} \tag{4}$$

where γ_5 is the chirality operator in the four-dimensional case.

Assuming the algebra $\mathcal{A}$ constructed in the geometry $\mathcal{M} \times \mathcal{F}$ is symplectic-unitary, it must be of the form [19]

$$\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}), \tag{5}$$

with $k = 2a$ and $\mathbb{H}$ being the algebra of quaternions, which has a basis $\{1, i\sigma^{\alpha}\}$, where $\sigma^{\alpha}(\alpha = 1, 2, 3)$ are the Pauli matrices. The field of quaternions $\mathbb{H}$ plays an important role in this construction and its choice remains to be explained. To obtain the SM one assumes quaternion linearity. The first possible value for the even number k is 2, corresponding to a Hilbert space of four fermions, but this choice is ruled out from the existence of quarks. The next possible value is $k = 4$ leading to the correct number of $k^2 = 16$ fermions in each of the three generations. Note that if new particles are discovered at the Large Hadron Collider (LHC), one may be able to accommodate them by considering a higher value for the even number k.

Another basic ingredient of the NCSG approach is to consider the Dixmier trace—a noncommutative analogue of integration on a compact n-dimensional Riemannian spin manifold—as the fundamental functional to define the action of the theory. The Dixmier trace is then connected with residues of zeta functions.

The noncommutative spectral geometry model is based upon the spectral action principle stating that, within the context of a product noncommutative geometry, the bare bosonic Euclidean action is given by the trace of the heat kernel associated with the square of the noncommutative Dirac operator and is of the form

$$\mathrm{Tr}\,(f(D/\Lambda)), \tag{6}$$

where f is a cutoff function and Λ fixes the energy scale; D and Λ have physical dimensions of a mass and there is no absolute scale on which they can be measured. This action can be seen *à la* Wilson as the bare action at the mass scale Λ. The fermionic term can be included in the action functional by adding $(1/2)\langle J\psi, D\psi \rangle$, where J is the real structure on the spectral triple and ψ is a spinor in the Hilbert space $\mathcal{H}$ of the quarks and leptons.

Dealing within noncommutative spaces, one obtains the group $\mathcal{U}$ of symmetries of gravity and matter as the group of automorphisms, and the $S = S_{\text{E-H}} + S_{\text{SM}}$ as the spectral action. Moreover, the fermions of the standard model provide the Hilbert space of a spectral triple for the algebra, while the bosons (including the Higgs boson) are obtained through inner fluctuations of the Dirac operator of the product geometry.

For the four-dimensional Riemannian geometry, the trace $\mathrm{Tr}(f(D/\Lambda))$ is expressed perturbatively in terms of the geometrical Seeley-deWitt coefficients a_n, which are known for any second order elliptic differential operator, as [20–23]

$$\begin{aligned} \mathrm{Tr}(f(D/\Lambda)) \sim\ & 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 + \cdots \\ & + \Lambda^{-2k} f_{-2k} a_{4+2k} + \cdots, \end{aligned} \tag{7}$$

where the smooth even cutoff function f, which decays fast at infinity, appears through its momenta f_k given by

$$f_0 \equiv f(0) \qquad f_k \equiv \int_0^{\infty} f(u)u^{k-1}du, \qquad \text{for } k > 0,$$

$$f_{-2k} = (-1)^k \frac{k!}{(2k)!} f^{(2k)}(0).$$

Moreover, since its Taylor expansion at zero vanishes, the asymptotic expansion Eq. (7) reduces to

$$\mathrm{Tr}\,(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4. \tag{8}$$

In conclusion, the cutoff function f plays a role only through its three momenta f_0, f_2, f_4, which are three real parameters in the model; they are intimately related to the coupling constants at unification, the gravitational constant, and the cosmological constant. In this four-dimensional

Riemannian manifold (one brane of the two-sheeted space), the term in Λ^4 gives a cosmological term, the term in Λ^2 gives the Einstein-Hilbert action functional, and the Λ-independent term yields the Yang-Mills action for the gauge fields corresponding to the internal degrees of freedom of the metric.

The asymptotic expression for the spectral action functional leads to the full Lagrangian for the standard model minimally coupled to gravity, with neutrino mixing and Majorana mass terms. Thus, this approach leads to a geometric explanation of the SM; in particular, the vacuum expectation value of the Higgs field is related to the noncommutative distance between the two sheets. The Higgs field is found to be conformally coupled to the Ricci scalar. The generalized Einstein-Hilbert action contains in addition a minimally coupled massless scalar field σ related to the distance d between the two sheets by $d \propto e^{-\sigma(y)}$, with y an element of the product noncommutative space.

III. NONCOMMUTATIVE SPECTRAL GEOMETRY, THE ALGEBRA DOUBLING AND THE GAUGE STRUCTURE

In Ref. [1] Alain Connes considers the work of Heisenberg establishing, in the early years of quantum mechanics (QM), the *matrix mechanics*—where physical quantities are governed by noncommutative algebra—and he discusses how close such a discovery is to experimental reality and how strict is its relation to the observed discretization of the energy of the atomic levels and of angular momentum. In this section our aim is twofold: first, we show that one central ingredient in NCSG, namely, the "doubling" of the algebra $\mathcal{A} \to \mathcal{A}_1 \otimes \mathcal{A}_2$ acting on the "doubled" space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ [cf. Eq. (3)], is also present in the standard QM formalism of the density matrix and Wigner function. We then show that the doubling of the algebra is implicit even in the classical theory when considering the Brownian motion of a particle, which had such an important role in the development of the atomistic view of matter, and that it is related to dissipation. Second, in Sec. we show that the doubling of the algebra and the dissipation are related to the gauge structure of the theory.

For the first part of our discussion, we start by considering the standard expression [24] of the Wigner function

$$W(p, x, t) = \frac{1}{2\pi\hbar} \int \psi^*\left(x - \frac{1}{2}y, t\right)\psi\left(x + \frac{1}{2}y, t\right)e^{-i(py/\hbar)}dy.$$

(9)

By putting

$$x_\pm = x \pm \frac{1}{2}y,$$

(10)

the associated density matrix is

$$W(x_+, x_-, t) \equiv \langle x_+|\rho(t)|x_-\rangle = \psi^*(x_-, t)\psi(x_+, t),$$

(11)

and the mean value of a quantum operator A is given by

$$\bar{A}(t) = \langle \psi(t)|A|\psi(t)\rangle$$

$$= \iint \psi^*(x_-, t)\langle x_-|A|x_+\rangle\psi(x_+, t)dx_+dx_-$$

$$= \iint \langle x_+|\rho(t)|x_-\rangle\langle x_-|A|x_+\rangle dx_+ dx_-.$$

(12)

In the formalism of the density matrix and the Wigner function, the coordinate $x(t)$ of a quantum particle is thus split into two coordinates $x_+(t)$ (going forward in time) and $x_-(t)$ (going backward in time). The forward and the backward in time evolution of the density matrix $W(x_+, x_-, t)$ is then described by "two copies" of the Schrödinger equation,

$$i\hbar\frac{\partial \psi(x_+, t)}{\partial t} = H_+ \psi(x_+, t),$$

(13)

$$-i\hbar\frac{\partial \psi^*(x_-, t)}{\partial t} = H_- \psi^*(x_-, t),$$

(14)

respectively, i.e.,

$$i\hbar\frac{\partial\langle x_+|\rho(t)|x_-\rangle}{\partial t} = H\langle x_+|\rho(t)|x_-\rangle,$$

(15)

where H is given in terms of the two Hamiltonian operators $H_\pm$ as

$$H = H_+ - H_-.$$

(16)

The connection with Alain Connes' discussion of spectroscopic experiments, noncommutative matrix algebra, energy level discretization and the algebra doubling is thus evident: the density matrix and the Wigner function *require* the introduction of a doubled set of coordinates $(x_\pm, p_\pm)$ [or (x, p_x) and (y, p_y)] and of their respective algebras. Using the two copies of the Hamiltonian $H_\pm$ operating on the outer product of two Hilbert spaces $\mathcal{H}_+ \otimes \mathcal{H}_-$ has been implicitly required in QM since the very beginning of the theory. Use of Eqs. (15) and (16), shows immediately that the eigenvalues of H are directly the Bohr transition frequencies $h\nu_{nm} = E_n - E_m$, which was the first hint toward an explanation of spectroscopic structure.

We now show that the need to double the degrees of freedom is implicit even in the classical theory when considering the Brownian motion. We closely follow Ref. [25], where the results here summarized are derived.

We recall that in the classical Brownian theory one has the equation of motion

$$m\ddot{x}(t) + \gamma\dot{x}(t) = f(t),$$

(17)

where $f(t)$ is a random (Gaussian distributed) force obeying

$$\langle f(t)f(t')\rangle_{\text{noise}} = 2\gamma k_B T\delta(t - t').$$

(18)

Equation (17) can be derived from a Lagrangian in a canonical procedure by employing a delta functional classical constraint representation as a functional integral. By

averaging over the fluctuating force f, one indeed obtains [25]

$$\langle \delta[m\ddot{x} + \gamma\dot{x} - f]\rangle_{\text{noise}} = \int \mathcal{D}y \left\langle \exp\left[\frac{i}{\hbar} \int dt L_f(\dot{x}, \dot{y}, x, y)\right]\right\rangle_{\text{noise}},$$
(19)

where

$$L_f(\dot{x}, \dot{y}, x, y) = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - y\dot{x}) + fy.$$
(20)

Note that $\hbar$ is introduced solely for dimensional reasons. We thus see that the constraint condition at the classical level introduced a new coordinate y, and the standard Euler-Lagrange equations are obtained, namely

$$\frac{d}{dt}\frac{\partial L_f}{\partial \dot{y}} = \frac{\partial L_f}{\partial y}; \qquad \frac{d}{dt}\frac{\partial L_f}{\partial \dot{x}} = \frac{\partial L_f}{\partial x},$$
(21)

i.e.,

$$m\ddot{x} + \gamma\dot{x} = f, \qquad m\ddot{y} - \gamma\dot{y} = 0.$$
(22)

We remark that the Lagrangian system equations (20)–(22) were obtained in a completely classical context in a search aimed to build up a canonical formalism for the dissipative system [26–28]. The x-system is an *open* system. In order to set up the canonical formalism it is required to *close* the system; this is the role of the y-system, which is the time-reversed copy of the x-system. The $\{x\text{-}y\}$ system is thus a closed system.

We also remark that the exact expression for the imaginary part of the action reads [29,30]

$$\text{Im}\,\mathcal{S}[x, y] = \frac{1}{2\hbar} \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt ds N(t - s)y(t)y(s),$$
(23)

where $N(t - s)$ denotes the quantum noise in the fluctuating random force given by the Nyquist theorem [30].

The meaning of Eq. (23) is that nonzero y yields an "unlikely process" in the classical limit "$\hbar \to 0$", in view of the large imaginary part of the action. At quantum level, instead, nonzero y may allow quantum noise effects arising from the imaginary part of the action [30]. This sheds some light on the role played by the doubled degrees of freedom in the interplay between classical and quantum. We thus see that the second sheet cannot be neglected: in the perturbative approach one may drop higher order terms in the action functional expansion, since they correspond to unlikely processes at the classical level. However, these terms may be responsible for quantum noise corrections and therefore, in order to not preclude the quantization effects, one should keep them.

The gauge structure

Let us now show how the doubling of the degrees of freedom is related to the gauge structure of the theory. Our subsequent discussion will thus unveil the relation between the two-sheeted space in the NCSG construction and the gauge structure of the theory.

We consider the equation of the classical one-dimensional damped harmonic oscillator

$$m\ddot{x} + \gamma\dot{x} + kx = 0,$$
(24)

with time independent m, γ and k, which is a simple prototype of open systems.

As we have seen, to set up the canonical formalism for open systems, the doubling of the degrees of freedom is required in such a way as to complement the given open system with its time-reversed image, thus obtaining a globally closed system for which the Lagrangian formalism is well defined. The doubling of the x degree of freedom leads one to consider the oscillator in the *doubled* y coordinate

$$m\ddot{y} - \gamma\dot{y} + ky = 0.$$
(25)

The system of the oscillator Eq. (24) and its time-reversed ($\gamma \to -\gamma$) image Eq. (25) is then a closed system described by the Lagrangian density, Eq. (20), where we put $f = kx$. The canonically conjugate momenta p_x and p_y can now be introduced as customary in the Lagrangian formalism,

$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{y} - \frac{\gamma}{2}y, \qquad p_y \equiv \frac{\partial L}{\partial \dot{y}} = m\dot{x} + \frac{\gamma}{2}x, \quad (26)$$

and the dynamical variables $\{x, p_x; y, p_y\}$ span the new phase space.

It is convenient to use the coordinates $x_1(t)$ and $x_2(t)$ obtained through the (canonical) transformation

$$x_1(t) = \frac{x(t) + y(t)}{\sqrt{2}}, \qquad x_2(t) = \frac{x(t) - y(t)}{\sqrt{2}},$$
(27)

in terms of which the motion equations can be rewritten as

$$m\ddot{x}_1 + \gamma\dot{x}_2 + kx_1 \& = 0,$$
(28a)
$$m\ddot{x}_2 + \gamma\dot{x}_1 + kx_2 = 0$$
(28b)

and $p_1 = m\dot{x}_1 + (1/2)\gamma x_2$; $p_2 = -m\dot{x}_2 - (1/2)\gamma x_1$. The Hamiltonian is then found to be

$$H = H_1 - H_2$$
$$= \frac{1}{2m}\left(p_1 - \frac{\gamma}{2}x_2\right)^2 + \frac{k}{2}x_1^2 - \frac{1}{2m}\left(p_2 + \frac{\gamma}{2}x_1\right)^2 - \frac{k}{2}x_2^2.$$
(29)

Following Refs. [31–34] we can now introduce the vector potential as

$$A_i = \frac{B}{2}\epsilon_{ij}x_j \qquad (i, j = 1, 2),$$
(30)

with

$$B \equiv \frac{c}{e}\gamma, \qquad \epsilon_{ii} = 0, \qquad \epsilon_{12} = -\epsilon_{21} = 1.$$
(31)

We realize that H_i (with $i = 1, 2$) in Eq. (29) describe two particles with opposite charges $e_1 = -e_2 = e$ in the

(oscillator) potential $\Phi \equiv (k/2/e)(x_1^2 - x_2^2) \equiv \Phi_1 - \Phi_2$ with $\Phi_i \equiv (k/2/e)x_i^2$ and in the constant magnetic field $\boldsymbol{B}$ defined as $\boldsymbol{B} = \nabla \times \boldsymbol{A} = -B\hat{\boldsymbol{3}}$, namely,

$$H = H_1 - H_2 = \frac{1}{2m}\left(p_1 - \frac{e_1}{c}A_1\right)^2 + e_1\Phi_1$$
$$-\frac{1}{2m}\left(p_2 + \frac{e_2}{c}A_2\right)^2 + e_2\Phi_2. \quad (32)$$

Using Eq. (30) the Lagrangian of the system can be written in the familiar form

$$L = \frac{1}{2m}\left(m\dot{x}_1 + \frac{e_1}{c}A_1\right)^2 - \frac{1}{2m}\left(m\dot{x}_2 + \frac{e_2}{c}A_2\right)^2$$
$$-\frac{e^2}{2mc^2}(A_1^2 + A_2^2) - e\Phi$$
$$= \frac{m}{2}(\dot{x}_1^2 - \dot{x}_2^2) + \frac{e}{c}(\dot{x}_1 A_1 + \dot{x}_2 A_2) - e\Phi. \quad (33)$$

Remarkably, we have the Lorentzian-like (pseudo-Euclidean) metric in Eq. (32) [cf. also Eqs. (16), (32), and (41) below]. The minus sign, not imposed by hand, but required by the doubling of the degrees of freedom, is crucial in our derivation (and in the NCSG construction).

In conclusion, the doubled coordinate, e.g. x_2, acts as the gauge field component A_1 to which the x_1 coordinate is coupled, and vice versa. The energy dissipated by one of the two systems is gained by the other one and vice versa, in analogy to what happens in standard electrodynamics as observed at the end of Sec. I. The interpretation is recovered of the gauge field as the bath or reservoir in which the system is embedded [33,34]. The gauge structure thus appears intrinsic to the doubling procedure.

Let us see then how such a conclusion can be also reached in the case of a fermion field.

For brevity we discuss the simple case of the massless fermion[3] and the U(1) local gauge transformation group. We will see how in this case the doubling of the algebra $\mathcal{A} \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2$ acting on the outer product space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is related with the gauge structure of the theory.

We consider the classical (prequantum) theory. The Lagrangian of the massless free Dirac field is

$$L = -\bar{\psi}\gamma^\mu\partial_\mu\psi. \quad (34)$$

Under the U(1) local gauge transformation,

$$\psi(x) \rightarrow \exp[ig\alpha(x)]\psi(x), \quad (35)$$

L transforms as

$$L \rightarrow L' = L - ig\partial^\mu\alpha(x)\bar{\psi}(x)\gamma_\mu\psi(x). \quad (36)$$

It is well known that in order to make L invariant under the local gauge transformation, Eq. (35), the coupling of the

current $j_\mu = i\bar{\psi}\gamma_\mu\psi$ with the gauge vector field A_μ has to be introduced in L in such a way that, when $\psi(x)$ transforms as in Eq. (35), $j^\mu(x)A_\mu(x)$ transforms as

$$j^\mu(x)A_\mu(x) \rightarrow j^\mu(x)A_\mu(x) + j^\mu(x)\partial_\mu\alpha(x), \quad (37)$$

i.e.,

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\alpha(x). \quad (38)$$

The Lagrangian L modified by the coupling $gj^\mu A_\mu$ leads to the lagrangian L_g defined as

$$L_g = -\bar{\psi}\gamma^\mu\partial_\mu\psi + ig\bar{\psi}\gamma^\mu\psi A_\mu, \quad (39)$$

which is by construction invariant under the U(1) local gauge transformations, Eqs. (35) and (38), namely $L_g \rightarrow L_g' = L_g$. As usual, in order for A_μ to be a dynamical field, the term $-(1/4)F^{\mu\nu}F_{\mu\nu}$ has to be added to the modified Lagrangian L_g. Moreover, the Lorentz gauge condition

$$\partial^\mu A_\mu(x) = 0, \quad (40)$$

has to be adopted in order to ensure that only transverse modes of the A_μ field enter physical states. As said in the introduction, these are represented by square integrable (spinor) functions in the Hilbert space $\mathcal{H} = L^2(\mathcal{M}, S)$, where the algebra acts by multiplication operators. Equation (40) expresses the restriction to the physical states in $\mathcal{H}$, where the gauge constraint is satisfied, which we will denote by $\langle\partial^\mu A_\mu(x)\rangle = 0$, where $\langle\ldots\rangle$ stands for expectation values in the physical states $\langle\text{phys}|\ldots|\text{phys}\rangle$.

Now, let us go back to the Lagrangian equation (33) for a classical fermion field and show how the doubling of the fermion degrees of freedom is related, under convenient constraints, to the local gauge invariance.

The field algebra is doubled by introducing the fermion tilde-field $\tilde{\psi}(x)$. The tilde-system is a "copy" (with the same spectrum and couplings) of the ψ-system. The Lagrangian is written now as

$$\hat{L} = L - \tilde{L} = -\bar{\psi}\gamma^\mu\partial_\mu\psi + \bar{\tilde{\psi}}\gamma^\mu\partial_\mu\tilde{\psi}. \quad (41)$$

We assume, for simplicity, that in $\hat{L}$ there is no coupling term of the field $\psi(x)$ with the tilde-field $\tilde{\psi}(x)$. The Hamiltonian for the system is of the form $\hat{H} = H - \tilde{H}$ [to be compared with Eq. (16)], which in terms of creation and annihilation operators of the $\psi(x)$ and $\tilde{\psi}(x)$ fields is given by $\hat{H} = \sum_{\mathbf{k}}\hbar\omega_{\mathbf{k}}(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - \tilde{a}_{\mathbf{k}}^\dagger\tilde{a}_{\mathbf{k}})$. Let the zero energy eigenstate of $\hat{H}$ be denoted by $|0(\theta)\rangle$.[4] The space of states $\hat{\mathcal{H}} = \mathcal{H} \otimes \tilde{\mathcal{H}}$ is constructed out of $|0(\theta)\rangle$ by repeated applications of creation operators of $\psi(\theta; x)$ and $\tilde{\psi}(\theta; x)$ and is called the θ-representation $\{|0(\theta)\rangle\}$ [29,33,34].

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[3]Extension to the massive fermion case, the boson case and non-Abelian gauge transformation groups is possible; see Refs. [33,34].

[4]In other words, $|0(\theta)\rangle$ is the vacuum with respect to the fields $\psi(\theta; x)$ and $\tilde{\psi}(\theta; x)$ obtained from $\psi(x)$ and $\tilde{\psi}(x)$, respectively, by means of the Bogoliubov transformation [29,35].

In the following we consider the subspace $\mathcal{H}_\theta \subset \{|0(\theta)\rangle\}$ made of all the states $|a\rangle_\theta$, including $|0(\theta)\rangle$, such that the θ—*state condition*

$$[a_\mathbf{k}^\dagger a_\mathbf{k} - \tilde{a}_\mathbf{k}^\dagger \tilde{a}_\mathbf{k}]|a\rangle_\theta = 0, \qquad \text{for any } \mathbf{k}, \qquad (42)$$

holds for any $|a\rangle_\theta$ in $\mathcal{H}_\theta$. This condition can be shown to be the realization in $\mathcal{H}_\theta$ of the Lorentz gauge condition Eq. (40).[5] We have

$$\langle j_\mu(x)\rangle_\theta = \langle \tilde{j}_\mu(x)\rangle_\theta, \qquad (43)$$

where $\langle \ldots \rangle_\theta$ denotes matrix elements in $\mathcal{H}_\theta$. We will denote equalities between matrix elements in $\mathcal{H}_\theta$, say $\langle A\rangle_\theta = \langle B\rangle_\theta$, by $A \cong B$ and call them θ-*weak equalities*. Since they are equalities among c-numbers, they are classical equalities.

Now, the key point is that, due to Eq. (43), the matrix elements in $\mathcal{H}_\theta$ of the Lagrangian equation (41) (as well as of a more general Lagrangian than the simple one presently considered) are invariant under the simultaneous local gauge transformations of ψ and $\tilde{\psi}$ given by Eq. (35) and

$$\tilde{\psi}(x) \rightarrow \exp[ig\alpha(x)]\tilde{\psi}(x), \qquad (44)$$

respectively, i.e.,

$$\langle \hat{L}\rangle_\theta \rightarrow \langle \hat{L}'\rangle_\theta = \langle \hat{L}\rangle_\theta, \quad \text{in } \mathcal{H}_\theta, \qquad (45)$$

under the gauge transformations in Eqs. (35) and (44).

We thus realize that a crucial role in the θ-weak gauge invariance of $\hat{L}$ under Eqs. (34) and (44) is played by the tilde- term $\tilde{\psi}\gamma^\mu \partial_\mu \tilde{\psi}$ since it transforms in such a way as to compensate the local gauge transformation of the ψ kinematic term, i.e.,

$$\tilde{\psi}(x)\gamma^\mu \partial_\mu \tilde{\psi}(x) \rightarrow \tilde{\psi}(x)\gamma^\mu \partial_\mu \tilde{\psi}(x) + g\partial^\mu\alpha(x)\tilde{j}_\mu(x). \quad (46)$$

This suggests the introduction of the vector field A'_μ by

$$g j^{\bar{\mu}}(x)A'_{\bar{\mu}}(x) \cong \tilde{\psi}(x)\gamma^{\bar{\mu}} \partial_{\bar{\mu}} \tilde{\psi}(x), \qquad \bar{\mu} = 0, 1, 2, 3. \quad (47)$$

Here and in the following, the bar over μ means no summation over repeated indices. Equation (46) implies that A'_μ transforms as

$$A'_\mu(x) \rightarrow A'_\mu(x) + \partial_\mu\alpha(x), \qquad (48)$$

when Eqs. (35) and (44) are implemented. This suggests identifying, in $\mathcal{H}_\theta$, A'_μ with the conventional U(1) gauge vector field and introducing it in the original Lagrangian through the usual coupling term $ig\bar{\psi}\gamma^\mu\psi A'_\mu$.

We remark that, provided we restrict ourselves to the θ-weak equalities, i.e., to matrix elements in $\mathcal{H}_\theta$, matrix elements of physical observables, which are solely functions of the $\psi(x)$ field, are not changed by Eq. (47).

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[5]Equation (42) turns out to be equivalent to the Gupta-Bleurer condition in quantum electrodynamics [29,33,34].

Moreover, observables turn out to be invariant under gauge transformations. Next, one can show that the conservation laws derivable from $\hat{L}$, namely, in the simple case of Eq. (41) the current conservation laws,

$$\partial^\mu j_\mu(x) = 0, \qquad \partial^\mu \tilde{j}_\mu(x) = 0, \qquad (49)$$

are also preserved as θ-weak equalities when Eq. (47) is adopted. One may also show that

$$\partial^\nu F'_{\mu\nu}(x) \cong -g j_\mu(x), \qquad \partial^\nu F'_{\mu\nu}(x) \cong -g\tilde{j}_\mu(x), \quad (50)$$

in $\mathcal{H}_\theta$. In the Lorentz gauge, from Eq. (50) we also obtain the θ-weak relations

$$\partial^\mu A'_\mu(x) \cong 0, \qquad \partial^2 A'_\mu(x) \cong g j_\mu(x). \qquad (51)$$

In conclusion, our discussion shows the intrinsic gauge properties of the doubling procedure: we have obtained that in $\mathcal{H}_\theta$ the doubled algebra Lagrangian Eq. (41) for the field ψ and its double $\tilde{\psi}$ can be substituted by the Lagrangian

$$\hat{L}g \cong -\frac{1}{4}F'^{\mu\nu}F'_{\mu\nu} - \bar{\psi}\gamma^\mu \partial_\mu \psi + ig\bar{\psi}\gamma^\mu\psi A'_\mu, \quad (52)$$

which is indeed the standard $U(1)$ local gauge invariant Lagrangian for the fermion field ψ. Remarkably, the tilde-kinematical term is replaced, in a θ-weak sense, by the gauge field-current coupling. The second equation in Eq. (50), shows that the variations of the gauge field tensor $F'_{\mu\nu}$ have their source in the current $\tilde{j}_\mu$, which suggests that the tilde-field plays the role of a "reservoir." Such an interpretation in terms of a reservoir may thus be extended also to the gauge field A'_μ, which indeed acts in a way to compensate the changes in the matter field configurations due to the local gauge freedom.

Finally, in the case an interaction term is present in the Lagrangian equation (41), $\hat{L}_{tot} = \hat{L} + \hat{L}_I, \hat{L}_I = L_I - \tilde{L}_I$, the above conclusions still hold provided $\mathcal{H}_\theta$ is an invariant subspace under the dynamics described by $\hat{L}_{tot}$.

We close this section by remarking that it also can be shown that in the formalism of the algebra doubling a relevant role is played by the noncommutative q-deformed Hopf algebra [36], pointing to a deep physical meaning of the noncommutativity in this construction. Indeed, the map $\mathcal{A} \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2$ in Eq. (3) is just the Hopf coproduct map $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{A} \equiv \mathcal{A}_1 \otimes \mathcal{A}_2$ which duplicates the algebra. The Bogoliubov transformation of "angle" θ relating the fields $\psi(\theta; x)$ and $\tilde{\psi}(\theta; x)$ to $\psi(x)$ and $\tilde{\psi}(x)$, is known to be obtained by convenient combinations of the *deformed* coproduct operation of the form $\Delta a_q^\dagger = a_q^\dagger \otimes q^{1/2} + q^{-1/2} \otimes a_q^\dagger$, where $q \equiv q(\theta)$ is the deformation parameters and $a_q^\dagger$ are the creation operators in the q-deformed Hopf algebra [36]. These deformed coproduct maps are noncommutative and the deformation parameter is related to the condensate content of $|0(\theta)\rangle$ [constrained by the θ-state condition

Eq. (42)]. In this connection it is interesting to observe that the q-derivative is a finite difference derivative, which has to be compared with the fact that in the NCSG construction the derivative in the discrete direction is a finite difference quotient, as mentioned in Sec. II.

A relevant point is that the deformation parameter *labels* the θ-representations $\{|0(\theta)\rangle\}$ and, for $\theta \neq \theta'$, $\{|0(\theta)\rangle\}$ and $\{|0(\theta')\rangle\}$ are unitarily inequivalent representations of the canonical (anti)commutation rules. This is a characteristic feature of quantum field theory [29,35]. Its physical meaning is that an order parameter exists, which assumes different θ-dependent values in each of the representations. In other words, the deformed Hopf algebra structure induces the *foliation* of the whole Hilbert space into physically inequivalent subspaces.

IV. ALGEBRA DOUBLING, DISSIPATION, AND QUANTIZATION

We have considered till now the doubling of the algebra such as the one occurring in the NCSG construction and have shown that such a doubling is related to the gauge structure of the theory. We have done this by considering essentially classical systems and have mentioned in several points features of such systems at a quantum level. We have also stressed that the doubling of the system degrees of freedom, say x, amounts to considering the fact that the system is embedded in some environment, which is indeed described by the doubled y coordinate.

In a series of papers [13–15] 't Hooft has discussed classical, deterministic models and has conjectured that, provided some specific energy conditions are met and some constraints are imposed, loss of information might lead to a quantum evolution. In this section, following Refs. [37,38], we show that in agreement with 't Hooft's conjecture, loss of information (dissipation) in a regime of completely deterministic dynamics appears to be responsible for the system's quantum mechanical evolution. Our conjecture is then that the NCSG classical construction carries the seeds of quantization *implicit* in its feature of the doubling of the algebra.

In order to be specific, we consider the classical damped harmonic x-oscillator described by Eq. (24) and its time-reversed image, the y-oscillator, Eq. (25). It is also convenient to put [32] $x_1 = r \cosh u$, $x_2 = r \sinh u$, and

$$\mathcal{C} = \frac{1}{4\Omega m}[(p_1^2 - p_2^2) + m^2\Omega^2(x_1^2 - x_2^2)], \quad (53)$$

$$J_2 = \frac{m}{2}[(\dot{x}_1 x_2 - \dot{x}_2 x_1) - \Gamma r^2], \quad (54)$$

where $\mathcal{C}$ is taken to be positive and

$$\Gamma = \frac{\gamma}{2m} \quad , \Omega = \sqrt{\frac{1}{m}\left(\kappa - \frac{\gamma^2}{4m}\right)}, \quad \text{with } \kappa > \frac{\gamma^2}{4m}.$$

Using $z = r^2$ and the canonical transformation

$$q_1 = \int \frac{dz m\Omega}{\sqrt{4J_2^2 + 4m\Omega\mathcal{C}z - m^2\Omega^2 z^2}},$$

$$q_2 = 2u + \int \frac{dz}{z} \frac{2J_2}{\sqrt{4J_2^2 + 4m\Omega\mathcal{C}z - m^2\Omega^2 z^2}}, \quad (55)$$

$$p_1 = \mathcal{C}, \qquad p_2 = J_2,$$

the system's Hamiltonian Eq. (29) can be rewritten as

$$H = \sum_{i=1}^{2} p_i f_i(q), \quad (56)$$

with $f_1(q) = 2\Omega$, $f_2(q) = -2\Gamma$. Note that $\{q_i, p_i\} = 1$, and the other Poisson brackets are vanishing.

Equation (56) belongs to the class of Hamiltonians considered by 't Hooft. There, the $f_i(q)$ are nonsingular functions of the canonical coordinates q_i and the equations for the q's, namely $\dot{q}_i = \{q_i, H\} = f_i(q)$), are decoupled from the conjugate momenta p_i. A complete set of observables, called *beables*, then exists, which Poisson commute at all times. The meaning of this is that the system admits a deterministic description even when expressed in terms of operators acting on some functional space of states $|\psi\rangle$, such as the Hilbert space [14]. We stress that such a description, in terms of operators and Hilbert space, does not imply *per se* quantization of the system. As we will see, quantization is achieved only as a consequence of dissipation.

Thus we see that J_2 and $\mathcal{C}$ are beables [it can be seen from the Hamiltonian Eq. (56) that q_1 and q_2 are also beables]. Next we put $H = H_{\mathrm{I}} - H_{\mathrm{II}}$, with

$$H_{\mathrm{I}} = \frac{1}{2\Omega\mathcal{C}}(2\Omega\mathcal{C} - \Gamma J_2)^2, \qquad H_{\mathrm{II}} = \frac{\Gamma^2}{2\Omega\mathcal{C}}J_2^2 \quad (57)$$

and impose the constraint

$$J_2|\psi\rangle = 0, \quad (58)$$

which defines physical states and guarantees that H is bounded from below.

Because of the constraint equation (58) we can then write

$$H|\psi\rangle = H_{\mathrm{I}}|\psi\rangle = 2\Omega\mathcal{C}|\psi\rangle = \left(\frac{1}{2m}p_r^2 + \frac{K}{2}r^2\right)|\psi\rangle, \quad (59)$$

with $K \equiv m\Omega^2$. We thus realize that H_{I} reduces to the Hamiltonian for the two-dimensional "isotropic" (or "radial") harmonic oscillator $\ddot{r} + \Omega^2 r = 0$.

The physical states are invariant under time-reversal ($|\psi(t)\rangle = |\psi(-t)\rangle$) and periodical with period $\tau = 2\pi/\Omega$.

The generic state $|\psi(t)\rangle_H$ can be written as

$$|\psi(t)\rangle_H = \hat{T}\left[\exp\left(\frac{i}{\hbar}\int_{t_0}^{t} 2\Gamma J_2 dt'\right)\right]|\psi(t)\rangle_{H_{\mathrm{I}}}, \quad (60)$$

where $\hat{T}$ denotes time-ordering and the constant $\hbar$, with dimension of an action, is needed for dimensional reasons. The states $|\psi(t)\rangle_H$ and $|\psi(t)\rangle_{H_I}$ satisfy the equations

$$i\hbar \frac{d}{dt}|\psi(t)\rangle_H = H|\psi(t)\rangle_H, \tag{61}$$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle_{H_I} = 2\Omega \mathcal{C}|\psi(t)\rangle_{H_I}. \tag{62}$$

Note that $H_I = 2\Omega \mathcal{C}$ has the spectrum $\mathcal{H}_I^n = \hbar\Omega n$, $n = 0, \pm 1, \pm 2, \ldots$; since our choice has been that $\mathcal{C}$ is positive, only positive values of n will be considered.

Let us now exploit the periodicity of the physical states $|\psi\rangle$. Following Ref. [39], one may generally write

$$|\psi(\tau)\rangle = \exp\left(i\phi - \frac{i}{\hbar}\int_0^\tau \langle\psi(t)|H|\psi(t)\rangle dt\right)|\psi(0)\rangle$$
$$= \exp(-i2\pi n)|\psi(0)\rangle, \tag{63}$$

i.e.,

$$\frac{\langle\psi(\tau)|H|\psi(\tau)\rangle}{\hbar}\tau - \phi = 2\pi n, \qquad n = 0, 1, 2, \ldots.$$

Using $\tau = 2\pi/\Omega$ and $\phi = \alpha\pi$ leads to

$$\mathcal{H}_{I,\text{eff}}^n \equiv \langle\psi_n(\tau)|H|\psi_n(\tau)\rangle = \hbar\Omega\left(n + \frac{\alpha}{2}\right). \tag{64}$$

The index n has been introduced to exhibit the n dependence of the state and the corresponding energy. We see that $\mathcal{H}_{I,\text{eff}}^n$ gives the effective nth energy level of the physical system, namely, the energy given by $\mathcal{H}_I^n$ corrected by its interaction with the environment. We conclude that the dissipation term J_2 of the Hamiltonian is responsible for the zero point ($n = 0$) energy: $E_0 = (\hbar/2)\Omega\alpha$.

We remark that in quantum mechanics the zero point energy is formally due to the nonzero commutator of the canonically conjugate q and p operators: the zero point energy *is* the "signature" of quantization. Our discussion thus shows that dissipation manifests itself as "quantization." In other words, the (zero point) "quantum contribution" E_0 to the spectrum of physical states signals the underlying dissipative dynamics.

Let us consider further the dynamical role of J_2. Using $u(t) = -\Gamma t$, Eq. (60) can be rewritten as

$$|\psi(t)\rangle_H = \hat{T}\left[\exp\left(i\frac{1}{\hbar}\int_{u(t_0)}^{u(t)} 2J_2 du'\right)\right]|\psi(t)\rangle_{H_I}, \tag{65}$$

and we have that

$$-i\hbar \frac{\partial}{\partial u}|\psi(t)\rangle_H = 2J_2|\psi(t)\rangle_H. \tag{66}$$

Thus, $2J_2$ induces translations in the u variable and in operatorial notation one can write $p_u = -i\hbar(\partial/\partial u)$. Equation (58) thus defines families of physical states,

representing stable, periodic trajectories. Note that $2J_2$ implements transitions from family to family, according to Eq. (66). Equation (61) can then be rewritten as

$$i\hbar \frac{d}{dt}|\psi(t)\rangle_H = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle_H + i\hbar \frac{du}{dt}\frac{\partial}{\partial u}|\psi(t)\rangle_H. \tag{67}$$

The contribution to the energy due to dissipation is thus described by "translations" in the u variable.

Consider the defining relation for temperature in thermodynamics (with $k_B = 1$)

$$\frac{\partial S}{\partial U} = \frac{1}{T}. \tag{68}$$

Using $S \equiv (2J_2/\hbar)$ and $U \equiv 2\Omega\mathcal{C}$, Eq. (56) gives $T = \hbar\Gamma$. Provided S is identified with the entropy, $\hbar\Gamma$ can be regarded as the temperature. Thus, the "full Hamiltonian" Eq. (56) plays the role of the free energy $\mathcal{F}$, and $2\Gamma J_2$ represents the heat contribution in H (or $\mathcal{F}$). Note that the statement that $2J_2/\hbar$ behaves as the entropy is not surprising since it controls the dissipative (thus irreversible loss of information) part of the dynamics.

It is worth noting that the thermodynamical picture outlined above is also consistent with the results on the canonical quantization of open systems in quantum field theory [28].

V. NONCOMMUTATIVE GEOMETRY AND THE DISSIPATIVE INTERFERENCE PHASE

We have seen that doubling of the algebra amounts to considering the system, its environment, and their reciprocal interaction. The relation which exists in the NCSG construction between the doubling of the algebra and the noncommutative geometry finds a realization in the relation between dissipation and noncommutative geometry in the plane of the doubled coordinates (x_1, x_2). The reason is that dissipation implies the appearance of a "dissipative interference phase" a notion which we will clarify in the present section.

Although in the following we consider the example of the damped harmonic oscillator and of its time-reversed image, our conclusions also apply to more general cases.

Since we will consider paths in the doubled coordinate plane, it is convenient to work with the (x_+, x_-) coordinates, introduced in Sec. III [which slightly differ in their definition from the (x_1, x_2) coordinates].

We remark that H given by Eq. (29) does not change its form when x_1, x_2, p_1, p_2 are replaced by x_+, x_-, p_+, p_-, respectively. The components in the (x_+, x_-) plane of forward and backward in time velocity $v_\pm = \dot{x}_\pm$ are then obtained as

$$v_\pm = \frac{\partial H}{\partial p_\pm} = \pm\frac{1}{m}\left(p_\pm \mp \frac{\gamma}{2}x_\mp\right), \tag{69}$$

and they do not commute

$$[v_+, v_-] = i\hbar \frac{\gamma}{m^2}. \tag{70}$$

It is thus impossible to fix these velocities v_+ and v_- as being identical [40]. By putting $mv_\pm = \hbar K_\pm$, Eq. (70) gives

$$[K_+, K_-] = \frac{i\gamma}{\hbar} \equiv \frac{i}{L^2}, \tag{71}$$

and a canonical set of conjugate position coordinates (ξ_+, ξ_-) may be defined by $\xi_\pm = \mp L^2 K_\mp$ so that

$$[\xi_+, \xi_-] = iL^2. \tag{72}$$

The commutation relation Eq. (72) characterizes the non-commutative geometry in the plane (x_+, x_-).

We now show that an Aharanov–Bohm-type phase interference can always be associated with the noncommutative (X, Y) plane where

$$[X, Y] = iL^2; \tag{73}$$

L denotes the geometric length scale in the plane [40].

Consider a particle moving in the plane along two paths, $\mathcal{P}_1$ and $\mathcal{P}_2$, starting and finishing at the same point, in a forward and in a backward direction, respectively. Let $\mathcal{A}$ denote the resulting area enclosed by the paths. We will show that the phase interference ϑ may be written as

$$\vartheta = \frac{\mathcal{A}}{L^2}. \tag{74}$$

A phase-space action integral

$$\mathcal{S}(\mathcal{P}) = \int_{\mathcal{P}} p_i dq^i, \tag{75}$$

may be associated with each path $\mathcal{P}$ (in phase space) for motion at fixed energy. The phase interference ϑ between the two paths $\mathcal{P}_1$ and $\mathcal{P}_2$ is given by the difference

$$\vartheta = \frac{1}{\hbar} \int_{\mathcal{P}_1} p_i dq^i - \frac{1}{\hbar} \int_{\mathcal{P}_2} p_i dq^i = \frac{1}{\hbar} \oint_{\mathcal{P}=\partial\Omega} p_i dq^i, \tag{76}$$

with $\mathcal{P}$ the closed path going from the initial point to the final point via path $\mathcal{P}_1$ and returning back to the initial point via $\mathcal{P}_2$. It constitutes the boundary of a two-dimensional surface Ω: $\mathcal{P} = \partial\Omega$. Then, due to Stokes theorem, i.e.,

$$\vartheta = \frac{1}{\hbar} \oint_{\mathcal{P}=\partial\Omega} p_i dq^i = \frac{1}{\hbar} \int_\Omega (dp_i \wedge dq^i), \tag{77}$$

the phase interference ϑ between two alternative paths turns out to be proportional to the area $\mathcal{A}$ of the surface Ω in phase space $(p_1, \ldots, p_f; q^1, \ldots, q^f)$.

Equation (73) in the noncommutative plane can be written as

$$[X, P_X] = i\hbar \quad \text{where} \quad P_X = \left(\frac{\hbar Y}{L^2}\right), \tag{78}$$

and Eq. (77) then reads

$$\vartheta = \frac{1}{\hbar} \int_\Omega (dP_X \wedge dX) = \frac{1}{L^2} \int_\Omega (dY \wedge dX), \tag{79}$$

which proves Eq. (74), i.e., the quantum phase interference between two alternative paths in the plane is determined by the noncommutative length scale L and the enclosed area $\mathcal{A}$.

Notice that the existence of a phase interference is connected to the zero point fluctuations in the coordinates; indeed Eq. (73) implies a zero point uncertainty relation $(\Delta X)(\Delta Y) \geq L^2/2$.

For Eq. (71) in the dissipative case, i.e.,

$$L^2 = \frac{\hbar}{\gamma}, \tag{80}$$

we then conclude that, provided $x_+ \neq x_-$, the quantum dissipative phase interference $\vartheta = \mathcal{A}/L^2 = \mathcal{A}\gamma/\hbar$ is associated with the two paths $\mathcal{P}_1$ and $\mathcal{P}_2$ in the noncommutative plane.

VI. CONCLUSIONS

We have considered the implications of the central ingredient in the NCSG, namely, the doubling of the algebra $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ acting on the space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. First, we have shown that the doubling of the algebra is related to dissipation (in the sense above specified) and the gauge field structure. As a result, the two-sheeted geometry must not be considered as just a simple almost commutative space, which is the simplest generalization beyond commutative geometries but, instead, the construction which can lead to gauge fields, required to explain the standard model. Second, by exploiting 't Hooft's conjecture, according to which loss of information within the framework of completely deterministic dynamics might lead to a quantum evolution, we have argued that dissipation, implied by the algebra doubling, may lead to quantum features. We have thus suggested that the NCSG classical construction carries the seeds of quantization *implicit* in the doubling of the algebra.

We have shown that in Alain Connes' two-sheeted construction, the doubled degree of freedom is associated with unlikely processes in the classical limit. Thus, in the perturbative approach one may drop higher order terms in the expansion, since they correspond to unlikely processes at the classical level. However, since the higher order terms in the expansion are the ones responsible for quantum corrections, the second sheet cannot be neglected at the classical level, if one does not want to preclude quantization effects. Put it differently, the second sheet—representing gauge fields—cannot be neglected once the universe entered the radiation dominated era. However, at the grand unified theories scale, when inflation took place, the effect of gauge fields, in other words the discrete space of two points, is fairly shielded.

At the end of Sec. III we have mentioned that the deformed Hopf algebra plays a relevant role in the algebra doubling and it induces the *foliation* of the Hilbert space into physically inequivalent subspaces. These describe different *phases* of the system, the ground state associated to each of them being, as known [29,35], a broken symmetry vacuum characterized by a different (θ-dependent) value of the order parameter. Variations in the order parameter (derivatives in the deformation parameter, or, in the language of Sec. IV, translations in the u parameter classifying 't Hooft families of states) thus describe phase transitions in the system evolution. In this connection, it is also interesting to observe that the state $|0(\theta)\rangle$ can be shown to be a finite temperature state, which means that the algebra doubling leads to a thermal field theory [28,29] (also consistently with the remarks at the end of Sec. IV). We will not comment more on this point. However, we remark that in the SM, although born and formulated as a zero temperature QFT model, one cannot avoid considering finite temperature effects as, for example, in the quark-gluon plasma studies, when studying phase transition processes in relation to cosmological scenarios in the early universe evolution, and/or in discussing unification models. Thermal aspects appear thus to be a valuable feature of the NCSG construction and of our discussion, in which they are implicit.

[1] A. Connes, *Noncommutative Geometry* (Academic, New York, 1994).

[2] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives* (Hindustan Book Agency, India, 2008).

[3] A. H. Chamseddine, A. Connes, and M. Marcolli, Adv. Theor. Math. Phys. **11**, 991 (2007).

[4] T. v. d. Broek and W. D. van Suijlekom, Commun. Math. Phys. **303**, 149 (2011).

[5] W. Nelson and M. Sakellariadou, Phys. Rev. D **81**, 085038 (2010).

[6] W. Nelson and M. Sakellariadou, Phys. Lett. B **680**, 263 (2009).

[7] M. Marcolli and E. Pierpaoli, arXiv:0908.3683.

[8] M. Buck, M. Fairbairn, and M. Sakellariadou, Phys. Rev. D **82**, 043509 (2010).

[9] W. Nelson, J. Ochoa, and M. Sakellariadou, Phys. Rev. Lett. **105**, 101602 (2010).

[10] W. Nelson, J. Ochoa, and M. Sakellariadou, Phys. Rev. D **82**, 085021 (2010).

[11] M. Sakellariadou, Int. J. Mod. Phys. D **20**, 785 (2011).

[12] M. Sakellariadou, Proc. Sci., CNCFG2010 (2010) 028.

[13] G. 't Hooft, Classical Quantum Gravity **16**, 3263 (1999).

[14] G. 't Hooft, arXiv:hep-th/0003005.

[15] G. 't Hooft, J. Phys. Conf. Ser. **67**, 012015 (2007).

[16] Y. M. Bunkov and H. Godfrin, *Topological Defects and the Non-Equilibrium Dynamics of Symmetry-Breaking Phase Transitions* (Kluver, Dordrecht, 2000).

[17] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Butterworth-Heinemann, Oxford, 1980).

[18] L. Lopez, *Gauge Field Theory: An Introduction* (Pergamon, New York, 1981).

[19] A. H. Chamseddine and A. Connes, Phys. Rev. Lett. **99**, 191601 (2007).

[20] A. H. Chamseddine and A. Connes, J. Math. Phys. (N.Y.) **47**, 063504 (2006).

[21] A. H. Chamseddine and A. Connes, Phys. Rev. Lett. **77**, 4868 (1996).

[22] A. H. Chamseddine and A. Connes, Commun. Math. Phys. **186**, 731 (1997).

[23] A. H. Chamseddine and A. Connes, Commun. Math. Phys. **293**, 867 (2009).

[24] R. P. Feynman, *Statistical Mechanics: A Set of Lectures*, edited by W. A. Benjamin (Westview, Reading, MA, 1972).

[25] M. Blasone, Y. N. Srivastava, G. Vitiello, and A. Widom, Ann. Phys. (N.Y.) **267**, 61 (1998).

[26] H. Bateman, Phys. Rev. **38**, 815 (1931).

[27] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).

[28] E. Celeghini, M. Rasetti, and G. Vitiello, Ann. Phys. (N.Y.) **215**, 156 (1992).

[29] M. Blasone, P. Jizba, and G. Vitiello, *Quantum Field Theory and Its Macroscopic Manifestations* (Imperial College, London, England, 2011).

[30] Y. N. Srivastava, G. Vitiello, and A. Widom, Ann. Phys. (N.Y.) **238**, 200 (1995).

[31] Y. Tsue, A. Kuriyama, and M. Yamamura, Prog. Theor. Phys. **91**, 469 (1994).

[32] M. Blasone, E. Graziano, O. K. Pashaev, and G. Vitiello, Ann. Phys. (N.Y.) **252**, 115 (1996).

[33] E. Celeghini, E. Graziano, K. Nakamura, and G. Vitiello, Phys. Lett. B **285**, 98 (1992).

[34] E. Celeghini, E. Graziano, K. Nakamura, and G. Vitiello, Phys. Lett. B **304**, 121 (1993).

[35] H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North-Holland, Amsterdam, 1982).

[36] E. Celeghini, S. De Martino, S. De Siena, A. Iorio, M. Rasetti, and G. Vitiello, Phys. Lett. A **244**, 455 (1998).

[37] M. Blasone, P. Jizba, and G. Vitiello, Phys. Lett. A **287**, 205 (2001).

[38] M. Blasone, E. Celeghini, P. Jizba, and G. Vitiello, Phys. Lett. A **310**, 393 (2003)

[39] M. V. Berry, Proc. R. Soc. A **392**, 45 (1984).

[40] S. Sivasubramanian, Y. Srivastava, G. Vitiello, and A. Widom, Phys. Lett. A **311**, 97 (2003).