Thermodynamics of the Lee-Wick partners: An alternative approach

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It was pointed out some time ago that there can be two variations in which the divergences of a quantum field theory can be tamed using the ideas presented by Lee and Wick. In one variation the Lee-Wick partners of the normal fields live in an indefinite metric Hilbert space but have positive energy and in the other variation the Lee-Wick partners can live in a normal Hilbert space but carry negative energy. Quantum mechanically the two variations mainly differ in the way the fields are quantized. In this article the second variation of Lee and Wick's idea is discussed. Using statistical mechanical methods the energy density, pressure and entropy density of the negative energy Lee-Wick fields have been calculated. The results exactly match with the thermodynamic results of the conventional, positive energy Lee-Wick fields. The result sheds some light on the second variation of Lee-Wick's idea. The result seems to say that the thermodynamics of the theories do not care about the way they are quantized.

I. INTRODUCTION

The Lee-Wick field theories [1,2] originated in an attempt to address the problem related with the infinities in quantum field theories. Recently some authors have tried to implement Lee-Wick's idea in a higher-derivative version of a quantum field theory [3,4]. All these theories assumed the existence of some partners of the standard model particles. The main ideas of the Lee-Wick standard model in Ref. [3] have been extended in Ref. [5] where the authors use two Lee-Wick partners for each standard model field: one with negative and the other with positive norm. Later, this idea has been used in Ref. [6] to improve gauge coupling unification without introducing additional fields in the higher-derivative theory. The Higgs sector of the Lee-Wick standard model has also been constrained in Ref. [7].

There has been at least one attempt [8] to use the concepts of these Lee-Wick constructions in cosmology where the authors were able to show the bouncing nature of the universe whose energy is dominated by the energies of a scalar field and its Lee-Wick partner. In Ref. [9] the authors tried to formulate a possible thermodynamic theory of particles which includes the Lee-Wick partners using a method of statistical field theory previously formulated by Dashen, Ma and Bernstein in Ref. [10].

The present article mainly focusses on a variant of the original Lee-Wick idea,[1] which was concerned about the taming of the divergences in a quantum field theory. In

1984 Boulware and Gross [12] tried to show that the original proposal of Lee and Wick was related to a complex implementation of the Pauli-Villars regularization scheme [13]. The complexity of the idea arises from the fact that the Pauli-Villars regulator fields in the Lee-Wick theories are not just *ad hoc* regulator fields, they also have dynamics. To implement the Pauli-Villars idea Lee and Wick introduced massive partner fields for all the normal fields in the theory. The scheme becomes involved when one tries to quantize the partner fields. It turns out that the partner fields can be quantized in two ways. In one way the norm of the states of the partner fields on the underlying Hilbert space remains definite and in the other case the norm of the states of the partner fields on the Hilbert space becomes indefinite. In the former case the energy of the Lee-Wick partner fields turns out to be negative and in the later case the energy of the Lee-Wick partner fields remain positive.

Both the options, as stated above, have their merits and demerits. In the first option, where the partner field states have positive definite norms but negative energy, the theory remains quantum mechanically understandable but does not have a proper ground state. There are run-away solutions. In the other option, where the Lee-Wick partner field states live on an indefinite metric Hilbert space but carry positive energy, there are zero-norm states which can grow indefinitely. This option also gives rise to run-away solutions. Historically Lee and Wick preferred to work with the theory defined on an indefinite metric. The difficulties of the run-away solutions were addressed by applying future boundary conditions which again made the theory noncausal.

The present article deals with the option which Lee and Wick discarded, a quantum field theory of Lee-Wick fields whose states do live on a definite metric Hilbert space but has negative energy. The motivation for such an unconventional work comes from a very interesting result related to

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[1]Some work in this direction was started much before by Pauli in Ref. [11].

the thermodynamics of the standard Lee-Wick theory as given in Ref. [9]. It turns out that the thermodynamics of the indefinite metric, positive energy Lee-Wick partners is exactly the same as definite metric but negative energy Lee-Wick fields. This similarity of the thermodynamics of the two different scenarios gives us a glimpse of the path which Lee and Wick did not take historically.

There are a plethora of problems related with the option which is presented in this article, the most important of them being that the theory is energetically unstable. Presently we do not give all the pathological properties of the alternative Lee-Wick prescription, nor do we know the cures of all the formal (pathological) diseases of the theory. The formal aspects of the negative energy Lee-Wick sector remains mostly open for further investigation in the near future. In spite of all the conceptual difficulties related to run-away solutions the result presented is too strong to be taken as a coincidence. Interestingly, the energy instability of the model plays an important role in the thermodynamics of the unusual fields and indirectly affects the results presented in this article which match surprisingly with the thermodynamics of the positive energy Lee-Wick fields. Readers who are purely interested in the formal aspects of the alternative Lee-Wick prescription can go through Ref. [12] for a more lucid and formal development of the basic ideas.

The present article is presented in the following manner. A brief introduction on the unusual regulator fields and their properties is presented in the next section. Section III discusses the technique to find out the thermal distribution function of the regulator fields. In Sec. IV the energy density, pressure and entropy density of a gas comprising of elementary particles and their unusual field partners are calculated using the thermal distribution functions. The last section (Sec. V) summarizes the important results obtained in this article.

II. CANONICAL QUANTIZATION OF THE POSITIVE ENERGY AND NEGATIVE ENERGY LEE-WICK FIELDS

In the paper written by Grinstein, O'Connell and Wise, [3] on Lee-Wick standard model, the authors proposed a higher-derivative field theory as the underlying theory of nature. The quadratic kinetic terms of the normal field theories, both bosonic and fermionic, are regained by introducing new degrees of freedom. It turns out that the Lagrangians of the new fields have wrong signs. In their work Grinstein, O'Connell and Wise [3] did not give any prescription for the canonical quantization of the new fields. In a later work by Fornal, Grinstein and Wise [9] the authors derived the thermodynamics of the unusual new degrees of freedom using methods of statistical field theory. In this section we first canonically quantize these new partner fields. This exercise will immediately show that these fields do carry positive energy. In the next step

we give the prescription for obtaining negative energy fields in the Lee-Wick paradigm.

A. Indefinite metric positive energy case

We do the quantization for the scalar field with the understanding that the other bosons in the theory do follow the same quantization rules. If the Lagrangian of the Lee-Wick partner of an usual scalar field be represented as ξ, then according to [3], the noninteractive part of its Lagrangian is given as

$$\mathcal{L}_\xi = -\tfrac{1}{2}\partial_\mu\xi\partial^\mu\xi + \tfrac{1}{2}m_\xi^2\xi^2, \tag{1}$$

where m_ξ is the mass of the ξ field. The field ξ can be expanded, in the Fourier space, in the same fashion as a standard scalar field:

$$\xi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2\epsilon(\mathbf{p})}}[a(\mathbf{p})e^{-ip\cdot x} + \bar{a}(\mathbf{p})e^{ip\cdot x}], \tag{2}$$

where $\epsilon(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_\xi^2}$ is the dispersion relation of the Lee-Wick partner excitation. The partner field can be quantized by the following condition:

$$[\xi(t, \mathbf{x}), \pi_\xi(t, \mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y}), \tag{3}$$

where $\pi_\xi \equiv \delta L_\xi/\delta\dot{\xi} = -\dot{\xi}$. The quantization condition for the partner fields yields the following unusual commutation relation

$$[a(\mathbf{p}), \bar{a}(\mathbf{q})] = -\delta^3(\mathbf{p} - \mathbf{q}), \tag{4}$$

while the other commutators involving $a(\mathbf{p})$ and $\bar{a}(\mathbf{q})$ are all zero. The equation above predicts that the field excitations of the partner fields will have indefinite norm in the Hilbert space. The unusual commutation relation between the creation and annihilation operators of the ξ field excitations is related to the negative sign of the canonically conjugate momentum corresponding to the ξ field.

If eigenstates of the number operator,

$$N(\mathbf{p}) = -\bar{a}(\mathbf{p})a(\mathbf{p}), \tag{5}$$

are defined by the following way

$$N(\mathbf{p})|n(\mathbf{p})\rangle = n(\mathbf{p})|n(\mathbf{p})\rangle, \tag{6}$$

where $n(\mathbf{p})$ is a positive integer interpreted as the number of particles with momentum $\mathbf{p}$ then the Hamiltonian of the field configuration is

$$H = \int \epsilon(\mathbf{p})N(\mathbf{p})d^3p, \tag{7}$$

where we have dropped the zero-point contribution in the Hamiltonian. The important point to note is the negative sign in the definition of the number operator. In this case if we stick to the conventional definition of the number operator (the same operator without the negative sign) then it can be shown that it will have negative eigenvalues.

The quantity $-\bar{a}(\mathbf{p})a(\mathbf{p})$ has a positive spectrum. This unconventional behavior of the number operator originates from the unconventional commutation relation of the creation and annihilation operators of the field excitations as given in Eq. (4). The negative sign of the original Lagrangian of the ξ field is balanced by the negative sign of the number operator and consequently the Hamiltonian of the field turns out to be positive.

For the fermionic case one can take the Lagrangian of the Lee-Wick partner field to be

$$\mathcal{L} = -\psi^{\dagger}\gamma^0(i\slashed{\partial} - m_{\psi})\psi, \tag{8}$$

where m_{ψ} is the mass of ψ field excitations.[2] The negative sign of the Lagrangian in Eq. (8) gives the unusual sign in conjugate momentum. The fermionic field can be expanded in the Fourier basis as

$$\psi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2\epsilon(\mathbf{p})}}$$
$$\times \sum_{s=1,2}[a_s(\mathbf{p})u_s(\mathbf{p})e^{-ip\cdot x} + \bar{b}_s(\mathbf{p})v_s(\mathbf{p})e^{ip\cdot x}], \tag{9}$$

where $a_s(\mathbf{p})$ and $b_s(\mathbf{p})$ are the annihilation operators of the fermionic and antifermionic excitations of the ψ field. Quantizing the fermion field ψ in the conventional sense

$$\{\psi(t, \mathbf{x}), \pi_{\psi}(t, \mathbf{y})\} = \delta^3(\mathbf{x} - \mathbf{y}), \tag{10}$$

one gets

$$a_s^2(\mathbf{p}) = \bar{a}_s^2(\mathbf{p}) = 0, \quad \{a_s(\mathbf{p}), \bar{a}_{s'}(\mathbf{k})\} = -\delta_{s,s'}\delta^3(\mathbf{p} - \mathbf{k}). \tag{11}$$

In the fermionic case one can define the number operator for the fermions and antifermions exactly in the same way as given in Eq. (6) and the form of the Hamiltonian will be similar to that in Eq. (7). In this case also the Lee-Wick excitations will have positive energy but indefinite norm. An excellent exposition of the relationship between the unusual commutation relations of the creation and annihilation operators (of the bosonic/fermionic fields) and the indefinite metric it induces in the Hilbert space is presented in the first two appendices of Ref. [2].

In Ref. [9] the authors tried to envisage the Lee-Wick partners as intermediate resonances. According to them the normal scalar particles scatter with each other as $\phi(\mathbf{p}_1) + \phi(\mathbf{p}_2) \rightarrow \phi(\mathbf{p}'_1) + \phi(\mathbf{p}'_2)$ through two Lee-Wick resonances. The scattering in reality happens like $\phi(\mathbf{p}_1) + \phi(\mathbf{p}_2) \rightarrow \xi(\tilde{\mathbf{p}}_1) + \xi(\tilde{\mathbf{p}}_2)$ and then $\xi(\tilde{\mathbf{p}}_1) + \xi(\tilde{\mathbf{p}}_2) \rightarrow \phi(\mathbf{p}'_1) + \phi(\mathbf{p}'_2)$ where $\xi(\tilde{\mathbf{p}}_1)$ and $\xi(\tilde{\mathbf{p}}_2)$ stands for the Lee-Wick fields. In this formalism the Lee-Wick partners are unstable resonances, with a negative decay width, and their existence is ephemeral. Writing the S matrix as $S = 1 - i\mathcal{T}$ one can write the $\mathcal{T}$ matrix amplitude for $\phi(\mathbf{p}_1) + \phi(\mathbf{p}_2) \rightarrow \xi(\tilde{\mathbf{p}}_1) + \xi(\tilde{\mathbf{p}}_2)$ as

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[2]The interested reader can consult Refs. [14,15] related to fermions leaving in an indefinite metric.

$$\langle \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 | \mathcal{T}(E) | \mathbf{p}_1, \mathbf{p}_2 \rangle = 2\pi\delta(E_1 + E_2 - \epsilon_1 - \epsilon_2)$$
$$\times \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_2)\mathcal{M}(E), \tag{12}$$

where the energy $E = E_1 + E_2$ and momentum $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$ are such that $(p_1 + p_2)^2 = m_{\xi}^2$ which sets the threshold value for the creation of the Lee-Wick resonances. Here ϵ_i stands for the energy of the Lee-Wick fields and m_{ξ} is the mass of the Lee-Wick excitations. The important ingredient in Ref. [9] lies in the prescription used for writing $\mathcal{M}(E)$:

$$\mathcal{M}(E) = -\frac{1}{2}\frac{g^2}{E^2 - \mathbf{p}^2 - m_{\xi}^2 + im_{\xi}\Gamma}, \tag{13}$$

where g specifies the coupling of the normal fields with the Lee-Wick fields. The interesting part of the above prescription lies in the overall minus sign of $\mathcal{M}(E)$ and the negative sign in front of the decay width Γ in the denominator. The authors of Ref. [9] then utilize a conventional relation between the scattering matrix elements and the grand partition function of a thermodynamic system to predict the various thermodynamic parameters. Thermodynamics of ephemeral resonance fields with negative decay widths is highly nontrivial but in principle one can find it out as done in Ref. [9].

B. When the states have positive definite norm but negative energy

In the present case we will try to find out the thermodynamics of the negative energy Lee-Wick fields which live in a normal Hilbert space. To do this one can start from the same Lagrangian for the real scalars as used in Ref. [3] and as given in Eq. (1). The field expansion of the ξ field can be kept exactly the same as in Eq. (2). To have a definite metric starting with a negative Lagrangian one has to quantize the fields with the wrong sign as

$$[\xi(t, \mathbf{x}), \pi_{\xi}(t, \mathbf{y})] = -i\delta^3(\mathbf{x} - \mathbf{y}). \tag{14}$$

This quantization condition yields the usual commutation relation between the creation and annihilation operators of the ξ field excitations. The ξ field excitations will now lie in a normal Hilbert space (i.e. the states having definite norm). The price one has to pay for a cross over from the indefinite norm landscape to the positive norm landscape is the ground state. If we stick to the definition of the number operator as defined in Eq. (5) then $N(\mathbf{p})$ will have negative eigenvalues. Consequently the Hamiltonian of the field configuration can still be written as

$$H = \int \epsilon(\mathbf{p})N(\mathbf{p})d^3p,$$

but unlike the previous case the Hamiltonian will be having negative eigenvalues. Although this Hamiltonian will have zero as an eigenvalue, but unlike the indefinite norm case,

this eigenvalue is not the least but the maximum of the eigenspectrum. Consequently the definite norm Lee-Wick field excitations do not have a proper ground state. If we interpret Eq. (5), for the indefinite metric case, as the sum of positive energy excitations then Eq. (5) has to be interpreted as a sum of positive energy de-excitations of the definite metric field.

From a very similar analysis it can be shown that if one changes the quantization condition for the fermions from that given in Eq. (10) to

$$\{\psi(t, \mathbf{x}), \pi_\psi(t, \mathbf{y})\} = -i\delta^3(\mathbf{x} - \mathbf{y}), \qquad (15)$$

and defines the number operator for the particles as $N(\mathbf{p}) = \bar{a}(\mathbf{p})a(\mathbf{p})$, which has positive eigenvalues, one gets a Hamiltonian as

$$H = -\int \epsilon(\mathbf{p})N(\mathbf{p})d^3p. \qquad (16)$$

The negative sign in the Hamiltonian arises because the Lagrangian of the fermionic field as given in Eq. (8) carries a negative sign. Similarly the antifermions also carry negative energy. In this article we are following the convention of Lee and Wick as given in the first appendix in Ref. [2]. In their convention the fermionic algebra can be set by a set of two anticommutation relations (one for the positive definite norm, the other for the indefinite norm) for the creation and annihilation operators of the fermions and correspondingly there will be two expressions of the number operator. According to Lee and Wick a fermionic system can be quantized by the following set of rules:

$$a_s^2(\mathbf{p}) = \bar{a}_s^2(\mathbf{p}) = 0,$$
$$\{a_s(\mathbf{p}), \bar{a}_{s'}(\mathbf{k})\} = \pm\delta_{s,s'}\delta^3(\mathbf{p} - \mathbf{k}), \qquad (17)$$
$$N(\mathbf{p}) = \pm\bar{a}_s(\mathbf{p})a_s(\mathbf{p}).$$

When dealing with the indefinite metric theory we chose the minus sign in the anticommutator and the minus sign in the number operator. On the other hand when we are dealing with fermion fields whose states live in a normal Hilbert space we choose the positive sign of the anticommutator and the positive signed number operator.

III. THERMAL DISTRIBUTION FUNCTIONS OF THE REGULATOR FIELDS

If some one assumes the existence of such exotic negative energy Lee-Wick fields, may be during the earliest phases of the universe, then one can calculate the thermodynamics of the negative energy Lee-Wick fields. The calculation of the thermodynamics of the negative energy fields starts with the prediction of a thermal distribution function of such fields. In this section the thermal distribution function of both the bosonic and the fermionic Lee-Wick partners are calculated.

For the bosonic excitations of the negative energy Lee-Wick fields we know that the number operator and

Hamiltonian has the form as given in Eq. (5) and (7). In the present case the Hamiltonian of the fields as given in Eq. (7) has negative eigenvalues because the fields are assumed to be quantized with the wrong sign as in Eq. (14). Because of the presence of on-shell excitations the thermal vacuum becomes $|\Omega\rangle \equiv |n(\mathbf{p_1}), n(\mathbf{p_2}), \cdots\rangle$ where $n(\mathbf{p_1})$ is the number of excitations carrying momentum $\mathbf{p_1}$. The action of the number operator on such a vacuum is

$$N(\mathbf{p})|\Omega\rangle = n(\mathbf{p})|\Omega\rangle. \qquad (18)$$

As the value of $n(\mathbf{p})$ actually turns out to be negative for the kind of Lee-Wick theory we are considering so in reality $|n(\mathbf{p})|$ gives the number of particles with momentum $\mathbf{p}$ and energy $\epsilon(\mathbf{p})$ which are missing from the thermal vacuum. In this analysis we will consider a noninteracting real scalar field for which the chemical potential $\mu = 0$. In the present case with a Hamiltonian of the form as given in Eq. (7), where the number operator has negative eigenvalues, the single particle partition function will be,

$$z_{\text{LW}}^B = \text{Tr}e^{-\beta H} = \sum_{|n(\mathbf{p})|=0}^{\infty} e^{\beta|n(\mathbf{p})|\epsilon(\mathbf{p})}, \qquad (19)$$

where $\beta = \frac{1}{T}$. From the last equation it is seen that the series representing the single particle partition function for the Lee-Wick partner of a standard model boson does not converge for $\beta > 0$. Consequently we regularize the last expression by cutting off the summation for a finite value of $n(\mathbf{p})$ as:

$$z_{\text{LW}}^B = \sum_{|n(\mathbf{p})|=0}^{M-1} e^{\beta|n(\mathbf{p})|\epsilon(\mathbf{p})} = \frac{1 - e^{\beta\epsilon(\mathbf{p})M}}{1 - e^{\beta\epsilon(\mathbf{p})}}, \qquad (20)$$

where M is a dimensionless cutoff which can be made indefinitely big at the end of the calculation.

Next we calculate the thermal distribution function of the field excitations from the expression of the single cell partition function of the field excitations as given in Eq. (20). In conventional statistical mechanics we can find the single cell distribution function via

$$f(\mathbf{p}) = \frac{1}{\beta}\left(\frac{\partial \ln z}{\partial \mu}\right)_{V,\beta}, \qquad (21)$$

where μ is an auxiliary chemical potential whose exact nature is not important for our purpose. In presence of an auxiliary chemical potential the single particle partition function can be written as

$$z_{\text{LW}}^B = \sum_{|n(\mathbf{p})|=0}^{M-1} e^{\beta|n(\mathbf{p})|\{\epsilon(\mathbf{p})-\mu\}} = \frac{1 - e^{\beta\{\epsilon(\mathbf{p})-\mu\}M}}{1 - e^{\beta\{\epsilon(\mathbf{p})-\mu\}}}. \qquad (22)$$

Applying conventional methods, the distribution function can also be written as

$$f_{\text{B}}(\mathbf{p}) = \frac{1}{\beta}\left(\frac{\partial \ln z_{\text{LW}}^B}{\partial \mu}\right)_{V,\beta}, \qquad (23)$$

which comes out to be,

$$f_{\rm B}(\mathbf{p}) = -\frac{e^{\beta\{\epsilon(\mathbf{p})-\mu\}}}{1 - e^{\beta\{\epsilon(\mathbf{p})-\mu\}}} + M\frac{e^{\beta\{\epsilon(\mathbf{p})-\mu\}M}}{1 - e^{\beta\{\epsilon(\mathbf{p})-\mu\}M}}. \quad (24)$$

Now setting the auxiliary chemical potential to be zero we get the distribution function of the fields as:

$$f_{\rm B}(\mathbf{p}) = -\frac{1}{e^{-\beta\epsilon(\mathbf{p})} - 1} + \frac{M}{e^{-\beta\epsilon(\mathbf{p})M} - 1}. \quad (25)$$

This is the distribution function of the fields whose Lagrangian is as given in Eq. (1). These fields are quantized via Eq. (14). These are not the fields which appear in the standard model of particle physics. The distribution function as plotted in Fig. 1 shows that the average excitation per energy level is negative definite. Obviously these systems describe a physical theory which is nontrivial and the negative sign of the distribution is only meaningful when compared with the positive definite distribution of the normal standard model bosons. In general these kind of distributions will produce negative energy density and pressure but once these energy density and pressure is added with the positive energy density and pressure of the standard model bosons we get a net positive energy density and pressure. The important point to notice about the distributions is that there is no pile up of quantas near $\epsilon(\mathbf{p}) = 0$ as is in the case of the Bose-Einstein distribution. The reason being that the spectra of the Lee-Wick excitations with negative energy, as given in Eq. (25), has two infinite spikes as the energy tends to zero and they cancel each other near the origin.

If we take the Lagrangian of the fermionic fields as given in Eq. (8) and quantize them via Eq. (15) then the anticommutators of the creation and the annihilation operators, which define the fermionic excitations of the ψ field discussed in the last section, is given as

$$a_s^2(\mathbf{p}) = \bar{a}_s^2(\mathbf{p}) = 0, \quad \{a_s(\mathbf{p}), \bar{a}_{s'}(\mathbf{k})\} = \delta_{s,s'}\delta^3(\mathbf{p} - \mathbf{k}), \quad (26)$$

where s, s' may be some internal quantum numbers. We can proceed in a similar way as done before and calculate the thermal distribution of these excitations. If we use the conventional number operator $N(\mathbf{p}) \equiv \bar{a}_s(\mathbf{p})a_s(\mathbf{p})$, which has positive eigenvalues as 0 and 1, then the Hamiltonian of a single oscillator is

$$H(\mathbf{p}) = -\frac{1}{2}\epsilon(\mathbf{p})[\bar{a}_s(\mathbf{p})a_s(\mathbf{p}) - a_s(\mathbf{p})\bar{a}_s(\mathbf{p})]$$
$$= -\epsilon(\mathbf{p})\left[N(\mathbf{p}) - \frac{1}{2}\delta^3(\mathbf{0})\right], \quad (27)$$

where $\frac{1}{2}\epsilon(\mathbf{p})\delta^3(\mathbf{0})$ is the zero-point energy. The negative sign of the Hamiltonian is due to the negative sign of the Lagrangian of the ψ field. A similar analysis can be done for the antifermions also. If we use this oscillator Hamiltonian to calculate the single particle partition function there will be no problem related to the convergence of the series. The single particle partition function for the anticommuting fields turns out to be

$$z_{\rm LW}^F = \sum_{n(\mathbf{p})=0}^{1} e^{\beta n(\mathbf{p})\{\epsilon(\mathbf{p})-\mu\}} = 1 + e^{\beta\{\epsilon(\mathbf{p})-\mu\}}, \quad (28)$$

where μ is an auxiliary chemical potential. Now applying the formula in Eq. (23) and setting $\mu = 0$ at the end we get the distribution function for the definite metric Lee-Wick partners of the standard model fermions as

$$f_{\rm F}(\mathbf{p}) = -\frac{1}{e^{-\beta\epsilon(\mathbf{p})} + 1}. \quad (29)$$

Unlike the previous case, in the present scenario the distribution function has no dependence on the dimensionless regulator M.

The distribution functions of these unusual fermionic fields at different temperatures are plotted as a function of energy in Fig. 2. The negative sign of the distribution
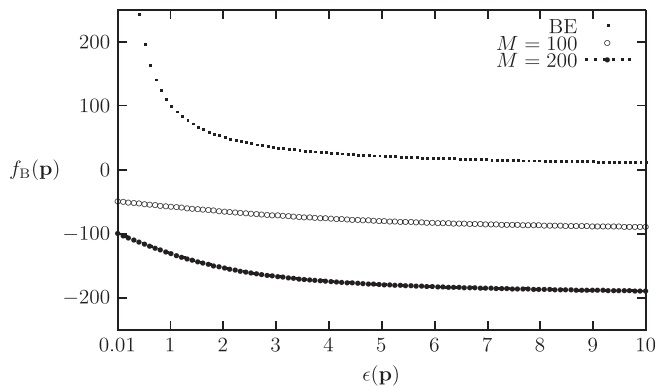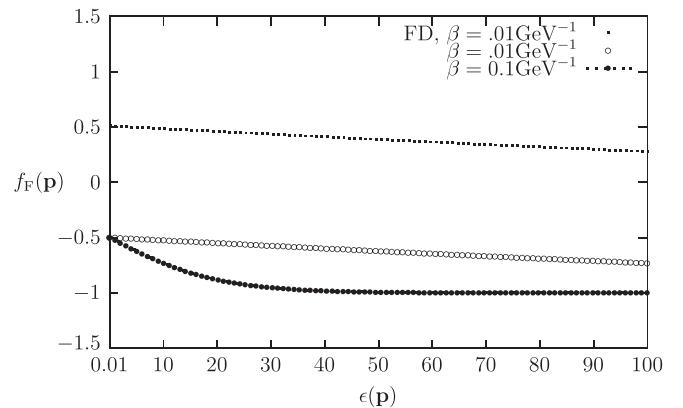

FIG. 1. The plot of the distribution function as given in Eq. (25). The topmost curve is for a normal Bose-Einstein distribution and the lower two curves correspond for $f_{\rm B}(\mathbf{p})$ for the ultraviolet cutoff $M = 100$ and 200. The inverse temperature in all the cases is .01 GeV^{-1} and the energy $\epsilon(\mathbf{p})$ is in GeV.

FIG. 2. The plot of the distribution function as given in Eq. (29). The topmost curve is for a normal Fermi-Dirac distribution at $\beta = .01$ GeV^{-1} and the lower two curves correspond for $f_{\rm F}(\mathbf{p})$ for β values .01 GeV^{-1} and .1 GeV^{-1} and the energy $\epsilon(\mathbf{p})$ is in GeV.

function signifies that the present field configurations arises due to a de-excitation or loss of positive energy particles. The vacuum defined is not stable and there exists much less energetic states than the vacuum itself. These kind of fields are unstable. The maximum energy of the field configurations is zero.

IV. ENERGY DENSITY, PRESSURE AND ENTROPY DENSITY FROM THE DISTRIBUTION FUNCTION

A. The bosonic case

To calculate the relevant thermodynamic quantities for the bosonic field from a statistical mechanical point of view we will employ Eq. (25). The energy density can be calculated using the following known equation

$$\rho = \frac{g}{(2\pi)^3} \int \epsilon(\mathbf{p}) f_B(\epsilon) d^3$$

$$p = \frac{g}{2\pi^2} \int_0^\infty \epsilon(\mathbf{p}) f_B(\epsilon) |\mathbf{p}|^2 d|\mathbf{p}|, \tag{30}$$

where g stands for any intrinsic degree of freedom of the particle. For a relativistic excitation $\epsilon^2 = \mathbf{p}^2 + m^2$ where m is the mass of the bosonic excitations. Changing the integration variable from $|\mathbf{p}|$ to ϵ one gets

$$\rho = -\frac{g}{2\pi^2} \int_0^\infty \left(\epsilon^3 - \frac{m^2}{2} \epsilon \right) \frac{d\epsilon}{e^{-\beta\epsilon} - 1}$$

$$+ \frac{Mg}{2\pi^2} \int_0^\infty \left(\epsilon^3 - \frac{m^2}{2} \epsilon \right) \frac{d\varepsilon}{e^{-\beta\epsilon M} - 1}. \tag{31}$$

In the above integral it is assumed that $|\mathbf{p}| \gg m$ and to have a closed integral the lower limit of the integral is assumed to be zero. In the extreme relativistic limit the system temperature $T \gg m$. Both of the integrals can only be done when $\beta < 0$, and in that case the result of the last integral is

$$\rho = -\frac{g}{2\pi^2} \left(\frac{\pi^4 T^4}{15} - \frac{m^2 \pi^2 T^2}{12} \right)$$

$$+ \frac{g}{2\pi^2} \left(\frac{\pi^4 T^4}{15 M^3} - \frac{m^2 \pi^2 T^2}{12 M} \right). \tag{32}$$

Analytically continuing the above result for $\beta > 0$ and taking $M \to \infty$ we see that for normal temperatures the energy density for extreme relativistic excitations of the bosonic fields is of the following form

$$\rho = -g \left(\frac{\pi^2 T^4}{30} - \frac{m^2 T^2}{24} \right). \tag{33}$$

As expected, the energy density turns out to be negative for the excitations in this case. The pressure of the bosonic field excitations can be found out from

$$p = \frac{g}{(2\pi)^3} \int \frac{|\mathbf{p}|^2}{3\epsilon} f_B(\epsilon) d^3 p = \frac{g}{2\pi^2} \int_0^\infty \frac{|\mathbf{p}|^4}{3\epsilon} f_B(\epsilon) d|\mathbf{p}|. \tag{34}$$

Following similar steps as in the case of the energy density, it is seen that the pressure of extremely relativistic excitations of the bosonic fields turns out to be

$$p = -g \left(\frac{\pi^2 T^4}{90} - \frac{m^2 T^2}{24} \right). \tag{35}$$

The entropy density of the bosonic field is simply given by

$$s = \frac{\rho + p}{T} = -g \left(\frac{2\pi^2 T^3}{45} - \frac{m^2 T}{12} \right). \tag{36}$$

These values of the energy density, pressure and entropy density exactly match the corresponding values calculated for the conventional, positive energy Lee-Wick partners. In [9] the authors were trying to formulate thermodynamics for a higher-derivative theory. The higher-derivative theory was converted into standard theory (theory up to a second derivative) with the introduction of Lee-Wick partners whose states have indefinite norms. The authors in the previous work did not quantize the system explicitly but were working with the form of the propagators of the Lee-Wick partners.

If we assume that in the early universe for each bosonic degrees of freedom in the standard model there exists a corresponding Lee-Wick bosonic degree of freedom whose field configuration has negative energy then the net energy density, pressure and entropy density of the early universe turns out to be

$$\rho_B = \rho_{SM} + \rho = \frac{gm^2 T^2}{24}, \qquad p_B = p_{SM} + p = \frac{gm^2 T^2}{24},$$

$$s_B = s_{SM} + s = \frac{gm^2 T}{12}, \tag{37}$$

which are all positive as expected. Here the energy density, pressure and entropy density for the standard model bosonic particles are $\rho_{SM} = g\frac{\pi^2 T^4}{30}$, $p_{SM} = g\frac{\pi^2 T^4}{90}$ and $s_{SM} = g\frac{2\pi^2 T^3}{45}$ respectively [16].

B. The fermionic case

In this subsection we apply Eq. (29) to find the energy density, pressure and entropy density of the fermionic excitations. In this case the distribution function do not have any dependence on the regulator M. For relativistic excitations the integrals which give the energy density and pressure for the fermionic case are exactly similar with the bosonic case except that now we have to use the distribution for the fermions. The integrals can be easily done, granted $\beta < 0$, but the results can be analytically continued for positive temperatures. The results in this case are listed below. The energy density, pressure and entropy density of the Lee-Wick partners are as follows:

$$\rho = -g \left(\frac{7\pi^2 T^4}{240} - \frac{m^2 T^2}{48} \right), \tag{38}$$

$$p = -g\left(\frac{7\pi^2 T^4}{720} - \frac{m^2 T^2}{48}\right), \qquad (39)$$

$$s = -g\left(\frac{7\pi^2 T^3}{180} - \frac{m^2 T}{24}\right). \qquad (40)$$

The energy density and pressure quoted above are equivalent to the energy density and pressure for the positive energy Lee-Wick partners as calculated in [9] for the special case of $g = 2$. If we assume that to each unusual fermionic degree of freedom there corresponds one standard fermionic degree from the standard model, then the total fermionic contribution is

$$\rho_F = \rho_{SM} + \rho = \frac{gm^2 T^2}{48}, \qquad p_F = p_{SM} + p = \frac{gm^2 T^2}{48},$$

$$s_F = s_{SM} + s = \frac{gm^2 T}{24} \qquad (41)$$

which are all positive. Here the energy density, pressure density and entropy density for the standard model fermionic particles are $\rho_{SM} = g\frac{7\pi^2 T^4}{240}$, $p_{SM} = g\frac{7\pi^2 T^4}{720}$ and $s_{SM} = g\frac{7\pi^2 T^3}{180}$, respectively [16].

It is worth pointing out here that there is some confusion regarding higher-derivative theories of fermions. The confusion is regarding the number of Lee-Wick partners (one left-handed and the other right-handed) of the chiral fermions. The authors of Ref. [9] claim that there will be two positive energy Lee-Wick partners of a chiral fermion which are interrelated. Where as in Ref. [17] the author claims that the two positive energy Lee-Wick partners of the chiral fermion may not be interdependent. In that case the Lee-Wick degrees of freedom exceeds the one of its standard model partner yielding negative energy, pressure and entropy density. For multiple Lee-Wick partners for each standard model fermion the Lee-Wick degrees of freedom exceeds the one of its standard model partner yielding negative energy and negative pressure.

V. DISCUSSION AND CONCLUSION

Initially it was pointed out that Lee-Wick's idea of implementing the Pauli-Villars regularization scheme can be implemented in two ways. This idea was presented by Boulware and Gross [12] way back in 1984. In one way the regulator fields live in a indefinite metric space but carry positive energy and in the other way the regulator fields live in a definite metric space but carry negative energy. Lee and Wick took the first option and tried to redress the issue of indefinite norm in such a way that unitarity is preserved in the theory. The second option remained uncultivated. In this article we explored the second option with limited means. No cures for the energy instability of these kind of theories are known to the present authors. The results presented in the article show the dubious nature of

the energy instability of the fields, but instead of making the theory meaningless the same instabilities conspire to produce a result which matches with the thermodynamics of the Lee-Wick partners living in the indefinite metric space.

In this article we have studied a system of bosonic and fermionic fields, whose Lagrangians have the wrong sign and, which are quantized with the wrong sign of the commutators and the anticommutators. These fields are Lee-Wick partners who live in a normal Hilbert space but have negative energy excitations. The negative energy of the field configuration is not due to any particular form of the potential but solely an outcome of the negative sign of the Lagrangian and the modified quantization process. The vacuum of the theory is not the state with the lowest energy, it is rather the state with the maximal energy making the field configuration unstable. The bosonic and fermionic degrees of freedom do still follow commutation and anticommutation relations and specifically the fermionic fields still follow the Pauli exclusion principle. In this article the emphasize had been on the calculation of energy density, pressure and entropy density of the unusual field configurations.

To calculate the above mentioned thermodynamic quantities one requires to have a statistical mechanics of the field excitations. One encounters the difficulty of a diverging sum when calculating the single particle partition function of the bosonic fields. Keeping to conventional ways, where the temperature of the system is positive definite, the partition function can only be summed when one uses an ultraviolet cutoff. The distribution function calculated from the partition function turns out to be negative definite, which is a nontrivial result. The negative nature of the distribution function implies that there must be an average loss of particles in any energy level.

The energy density, pressure calculated from the distribution functions of the unusual fields discussed in this article match exactly with the results calculated by Fornal, Grinstein and Wise in [9]. The derivation of the new distribution functions and the connection between the regulator field thermodynamics as presented in this article and the thermodynamics of the Lee-Wick partners, as presented in Ref. [9], is one of the main motivations for this work. The main emphasize of the present article has not been to recalculate the results obtained in Ref. [9] as the theory presented in this article is not the same as that of Ref. [9]. The two theories are quantized differently. The similarity of the thermodynamic results of the two variants of the Lee-Wick model implies that the different kind of instabilities plaguing the theories are related in presence of a thermal bath. From the main analysis of this article it can be inferred that the conventional Lee-Wick prescription is equivalent to its variant in the thermodynamic sector. The analysis of the thermodynamics of the positive definite metric Lee-Wick partners gives a clear idea about the

origin of the negative energy density and pressure of these field configurations. The theory presented in the article is amenable to the standard techniques of finite-temperature field theory. One can utilize the results of finite-temperature quantum field theory to calculate various thermal effects, like the thermal mass of the standard model particles in presence of the thermalized Lee-Wick partners, in the early universe. The thermal distribution functions of the Lee-Wick partners calculated in the present article can be used to write the propagators of the unusual fields in the real-time formalism [18].

The fact that the thermodynamics of the indefinite metric, positive energy Lee-Wick fields and the thermodynamics of the definite metric but negative energy Lee-Wick partners turns out to be the same remains an interesting result which invites further work in these fields in the future.

ACKNOWLEDGMENTS

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