Aharonov-Bohm effect in a class of noncommutative theories

Ashok Das[*]

Department of Physics and Astronomy, University of Rochester, Rochester, NY, USA
and Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Calcutta-700 064, India

H. Falomir[†] and M. Nieto

IFLP/CONICET—Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de La Plata,
CC 67, (1900) La Plata, Argentina

J. Gamboa[‡] and F. Méndez[§]

Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago, Chile

The Aharonov-Bohm effect including spin-noncommutative effects is considered. At linear order in θ, the magnetic field is gauge invariant although spatially strongly anisotropic. Despite this anisotropy, the Schrödinger-Pauli equation is separable through successive unitary transformations and the exact solution is found. The scattering amplitude is calculated and compared with the usual case. In the noncommutative Aharonov-Bohm case the differential cross section is independent of θ.

PACS numbers: 03.65.−w

I. INTRODUCTION

The discovery of noncommutative geometry has allowed the exploration of new directions in theoretical physics [1]. For example, the noncommutative constructions of quantum field theory [2], extensions beyond the relativistic symmetry [3], and implications on condensed matter physics have been widely discussed [4]. This research has also stimulated the construction of new models in quantum mechanics [5], which have a very natural transcription and—at the same time—has opened new windows and roads to explore (for example superconductivity [6]).

In this context the algebra [7] (we set $\hbar = 1$)

$$[\hat{x}_i, \hat{x}_j] = i\theta^2 \epsilon_{ijk}\hat{s}_k, \qquad [\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \qquad [\hat{p}_i, \hat{p}_j] = 0,$$
$$[\hat{x}_i, \hat{s}_j] = i\theta\epsilon_{ijk}\hat{s}_k, \qquad [\hat{s}_i, \hat{s}_j] = i\epsilon_{ijk}\hat{s}_k, \qquad (1)$$

where $i, j, k = 1, 2, 3$ and θ is a parameter with dimension of length, corresponds just to a deformation of the Heisenberg $\otimes$ spin algebra, which naturally induces spin-dipolar and higher order interactions as a result of long-range spin interactions. The algebra (1) is a nonrelativistic version of the Snyder algebra [8], which is rich enough to explore interesting consequences or simplifications in real physical systems.

Remarkably, the operators $(\hat{x}_i, \hat{p}_i)$ in (1) can be realized in terms of the conventional dynamical variables satisfying the Heisenberg algebra of the usual coordinate and momentum operators (x_i, p_i) through the relations

$$\hat{x}_i = x_i + \theta s_i, \qquad \hat{p}_i = p_i := -\iota\partial_i, \qquad \hat{s}_i = s_i, \quad (2)$$

with

$$[x_i, x_j] = 0 = [p_i, p_j], \qquad [x_i, p_j] = i\delta_{ij}, \qquad (3)$$

where the matrices s_i, which commute with (x_i, p_i), provide a $(2s + 1)$-dimensional irreducible representation (with s integer or half-integer) of the $su(2)$ Lie algebra,

$$[s_i, s_j] = i\epsilon_{ijk}s_k.$$

Therefore, with the realization in (2), the Schrödinger equation associated with the Hamiltonian $H(\hat{\mathbf{p}}, \hat{\mathbf{x}})$ can be written as

$$\iota\partial_t\psi(\mathbf{x}, t) = H(\mathbf{p}, \mathbf{x} + \theta\mathbf{s})\psi(\mathbf{x}, t), \qquad (4)$$

where $\psi(\mathbf{x}, t)$ is a spinor with $(2s + 1)$ components.

This simple observation, however, has nontrivial and unexpected consequences as can be seen by studying, for example, the isotropic harmonic oscillator in this noncommutative space, given by the potential $V(\hat{\mathbf{x}}) = \frac{\omega^2}{2}\hat{\mathbf{x}}^2$. Expressed in the basis of the normal Heisenberg algebra (2), it turns into

$$V = \frac{\omega^2}{2}(\mathbf{x} + \theta\mathbf{s})^2 = \frac{1}{2}\omega^2\mathbf{x}^2 + \frac{1}{2}\omega^2\theta^2\mathbf{s}^2 + \omega^2\theta\mathbf{x} \cdot \mathbf{s}, \quad (5)$$

where the term $\mathbf{x} \cdot \mathbf{s}$ becomes responsible—at least for particles with spin 1/2—of the infinite degeneracy of the ground state and the spontaneous breaking of rotational symmetry, as shown in [7].

From these considerations as well as from the study of the effects induced by the algebra (1) on other physical systems one can speculate about the order of magnitude of θ. For example, the present approach could be connected to

---

[*]das@pas.rochester.edu
[†]falomir@fisica.unlp.edu.ar
[‡]jgamboa55@gmail.com
[§]fernando.mendez.f@gmail.com

recent experiments carried out using ^{52}Cr condensates [9–11], where one uses the fact that, at large distances, the spins interact via the potential

$$V = \mu\left(\frac{\mathbf{s}_1 \cdot \mathbf{s}_2 - 3(\mathbf{s}_1 \cdot \hat{\mathbf{r}})(\mathbf{s}_2 \cdot \hat{\mathbf{r}})}{r^3}\right), \qquad (6)$$

where $\mathbf{r}$ is the relative position vector, $\hat{\mathbf{r}} = \mathbf{r}/r$, and μ is the interaction strength (see e.g. [12]). This potential can, in fact, be completely rederived from the present approach to noncommutative quantum mechanics with [13]

$$\mu = \theta^2. \qquad (7)$$

Following Ref. [11], one can relate θ with the parameters of the experiment involving a gas of ^{52}Cr with total spin 3 to find

$$\theta^2 = \frac{C_{dd}}{4\pi} = \frac{48a_0\hbar^2}{m},$$

where a_0 represents Bohr's radius while m is the mass of the ^{52}Cr isotope. This leads to the value [13]

$$\theta \sim 10^{-11} \text{ cm,} \qquad (8)$$

which would not change significantly for spin $1/2$ atoms, even though the experiments involving these kinds of particles are, in principle, much more complicated.

Taking into account these results, a natural question arises about the implications of algebra (1) on the topological nature of systems of physical interest. This is a nontrivial question since the spin degrees of freedom now appear in the commutation relations (1) of the coordinates. The Aharonov-Bohm effect turns out to be the paradigm in the study of such issues. Here one considers a charged particle in a magnetic field which is zero everywhere except inside an infinitely long and impenetrable solenoid of essentially zero radius. The dynamics, in this case, reduces to the motion in a plane perpendicular to the external magnetic field.

The goal of this paper is to generalize the study of such a system to a particle moving in a noncommutative space defined by (1). We determine the Schödinger-Pauli equation of a particle moving under the action of an external magnetic field, study its properties, and evaluate the scattering amplitude. For simplicity, we will restrict ourselves to a plane orthogonal to the magnetic field and consider the motion of a charged spin $1/2$ particle. The calculations will be done to the leading order in θ in perturbation theory because of the mathematical difficulties posed by this problem. The paper is organized as follows. In Sec. II, the gauge potential in the noncommutative space is discussed and, in Sec. III, the Schrödinger-Pauli equation is determined. In Sec. IV, the scattering amplitude is evaluated and, in Sec. V, we discuss briefly the physical implications of these results.

II. THE AHARONOV-BOHM POTENTIAL IN THE NONCOMMUTATIVE PLANE

In the context of the approach outlined above, in this section we calculate the vector potential for the Aharonov-Bohm effect when the realization (2) is taken into account. Let us start with the commutative case where we assume that a charged spin $1/2$ particle is interacting with a background magnetic field along the x_3-axis. The coordinates of the plane normal to the magnetic field are denoted by x_i, $i = 1, 2$ with $r^2 = x_1^2 + x_2^2$. The problem has cylindrical symmetry and the vector potential, which gives rise to a zero magnetic field everywhere except at the origin $r = 0$ with a finite flux, is given by

$$A_i = -\frac{\alpha}{2}\epsilon_{ij}\partial_j(\ln r^2) = -\alpha\epsilon_{ij}\frac{x_j}{r^2}, \qquad A_3 = 0, \quad (9)$$

where α is a constant proportional to the magnetic flux $\Phi_B = 2\pi\alpha$. The potential (9) is defined in the Coulomb gauge satisfying $\partial_i A_i = 0$, and the magnetic field along the z-axis has the form

$$B_3(r) = \epsilon_{ij}\partial_i A_j = -\alpha\epsilon_{ij}\partial_i\epsilon_{jk}\partial_k(\ln r) = \alpha\nabla^2(\ln r)$$
$$= 2\pi\alpha\delta(r) = \Phi_B\delta^2(\mathbf{x}). \qquad (10)$$

In the following we consider the corresponding potential in the noncommutative plane,

$$A_i^{\text{NC}} = -\frac{\alpha}{2}\epsilon_{ij}\partial_j(\ln\hat{r}^2), \qquad i, j = 1, 2, \qquad A_3^{\text{NC}} = 0,$$
$$(11)$$

and consequently the two component vector potential (for a magnetic field along the 3-axis) is independent of the noncommutative coordinate $\hat{x}_3$, namely $A_i^{\text{NC}} = A_i^{\text{NC}}(\hat{x}_1, \hat{x}_2)$.

Note that the underlying noncommutative algebra is a deformation of the standard Heisenberg and spin algebras (spin is an independent variable) and it follows from (1) that

$$[\hat{x}_1, \hat{x}_2] = i\theta^2\hat{s}_3 = \frac{i\theta^2}{2}\sigma_3. \qquad (12)$$

It is worth recognizing that σ_3 is an element of the underlying noncommutative algebra independent of the noncommutative coordinate $\hat{x}_3$. Once this is recognized, one can work with the commutative basis (Bopp shift), where the realization of the deformed algebra in (2) leads to the replacement

$$\hat{r}^2 = r^2 + \theta\mathbf{x}\cdot\boldsymbol{\sigma} + \frac{1}{2}\theta^2, \qquad (13)$$

where for spin $1/2$ we have identified $\mathbf{s} = \frac{1}{2}\boldsymbol{\sigma}$ with σ_i, $i = 1, 2$ denoting the two Pauli matrices. Therefore the Cartesian components of the gauge potential in the noncommutative plane are defined as (matrices)

$$A_i^{NC} = -\frac{\alpha}{2}\epsilon_{ij}\partial_j(\ln\hat{r}^2)$$

$$= -\frac{\alpha}{2}\epsilon_{ij}\partial_j\left[\ln\left(r^2 + \frac{\theta^2}{2}\right)\mathbf{1}_2 + \ln\left(\mathbf{1}_2 + \frac{2\theta\mathbf{x}\cdot\boldsymbol{\sigma}}{2r^2 + \theta^2}\right)\right]. \quad (14)$$

It is possible to factor out all the spin dependence in the previous expression by defining

$$\sigma_r = \frac{\mathbf{x}\cdot\boldsymbol{\sigma}}{|\mathbf{x}|} = \hat{\mathbf{u}}_r\cdot\boldsymbol{\sigma} = \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}, \quad (15)$$

and its derivative

$$\sigma_\phi := \frac{\partial\sigma_r}{\partial\phi} = \hat{\mathbf{u}}_\phi\cdot\boldsymbol{\sigma} = i\begin{pmatrix} 0 & -e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}, \quad (16)$$

with ϕ denoting the polar angle. Here $\mathbf{u}_r$, $\mathbf{u}_\phi$ denote, respectively, the unit vectors along the radial and the azimuthal angle directions. These matrices obviously satisfy

$$\sigma_r^2 = \mathbf{1}_2, \qquad \sigma_\phi^2 = \mathbf{1}_2, \qquad \sigma_r\sigma_\phi = i\sigma_3 = -\sigma_\phi\sigma_r. \quad (17)$$

Using the Taylor series expansion for the logarithm, a straightforward calculation leads to

$$A_i^{NC} = A_i(r)f(\mathbf{x},\theta) - \frac{\alpha}{4}\frac{x_i}{r^2}\ln\left(\frac{r^2+(r+\theta)^2}{r^2+(r-\theta)^2}\right)\sigma_\phi, \quad (18)$$

where $A_i(r)$ is the vector potential in the commutative plane defined in Eq. (9), and the function $f(\mathbf{x},\theta)$ is given by

$$f(\mathbf{x},\theta) = r\left\{\left(\frac{(2r+\theta)}{r^2+(r+\theta)^2}\right)\left(\frac{\mathbf{1}_2+\sigma_r}{2}\right)\right.$$
$$\left. +\left(\frac{(2r-\theta)}{r^2+(r-\theta)^2}\right)\left(\frac{\mathbf{1}_2-\sigma_r}{2}\right)\right\}$$
$$= \frac{4r^4}{4r^4+\theta^4}\mathbf{1}_2 - r\theta\frac{2r^2-\theta^2}{4r^4+\theta^4}\sigma_r \xrightarrow{\theta\to 0}\mathbf{1}_2, \quad (19)$$

so that A_i^{NC} reduces to A_i (as a diagonal matrix) when $\theta = 0$. On the other hand, for $\theta \neq 0$, $f(\mathbf{x},\theta)$ is a continuous function of $\mathbf{x}$ vanishing at the origin.

We see from (18) that the vector potential in the noncommutative plane has a component in the radial direction ($\mathbf{u}_r$), which is not present in the standard case. In fact, in polar coordinates we can write

$$\mathbf{A}^{NC} = \alpha\left[\frac{4r^3\mathbf{1}_2 - \theta(2r^2-\theta^2)\sigma_r}{4r^4+\theta^4}\right]\hat{\mathbf{u}}_\phi$$
$$-\frac{\alpha}{4r}\ln\left(\frac{2r^2+2r\theta+\theta^2}{2r^2-2r\theta+\theta^2}\right)\sigma_\phi\hat{\mathbf{u}}_r. \quad (20)$$

For large r with a finite θ, this vector potential vanishes as

$$\mathbf{A}^{NC}\to\frac{\alpha}{r}\left\{\left(\mathbf{1}_2-\frac{\theta}{2r}\sigma_r\right)\mathbf{u}_\phi - \frac{\theta}{2r}\sigma_\phi\mathbf{u}_r + O\left(\left(\frac{\theta}{r}\right)^3\right)\right\}, \quad (21)$$

while near the origin we find a significant departure from Eq. (9),

$$\mathbf{A}^{NC}\xrightarrow{r\to 0}\frac{\alpha}{\theta}\left\{\left(1-\frac{2r^2}{\theta^2}\right)\sigma_r\mathbf{u}_\phi - \left(1-\frac{2r^2}{3\theta^2}\right)\sigma_\phi\mathbf{u}_r + O\left(\left(\frac{r}{\theta}\right)^3\right)\right\}. \quad (22)$$

Some comments are in order here. First we note from its definition in Eq. (14) that this vector potential satisfies the Coulomb gauge condition, $\partial_i A_i^{NC} = 0$ (to all orders in θ). In the noncommutative case, the magnetic field is defined as the (gauge covariant) commutator of the covariant derivatives [2]

$$B^{NC} = F_{12}^{NC} = \frac{i}{e}[\partial_1 - ieA_1^{NC}, \partial_2 - ieA_2^{NC}]$$
$$= \partial_1 A_2^{NC} - \partial_2 A_1^{NC} - ie[A_1^{NC}, A_2^{NC}]. \quad (23)$$

The independent component of the magnetic field F_{12}^{NC} does not depend on the $\hat{x}_3$ coordinate because of the assumption (11). It is the commutator in the definition of the field strength tensor (23), which brings in the dependence on the other element σ_3 of the noncommutative algebra through (12) and leads to an $O(\theta^2)$ contribution. In fact, a straightforward calculation leads to

$$B_{(1)}^{NC}(\mathbf{x}) = \boldsymbol{\nabla}\times\mathbf{A}^{NC}$$
$$= \alpha\left(\frac{16r^2\theta^4}{(4r^4+\theta^4)^2}\right)\mathbf{1}_2$$
$$+ \alpha\left\{\theta\frac{(2r^2+\theta^2)(4r^4-8r^2\theta^2+\theta^4)}{r(4r^4+\theta^4)^2}\right.$$
$$\left. + \frac{1}{4r^2}\ln\left(\frac{r^2+(r+\theta)^2}{r^2+(r-\theta)^2}\right)\right\}\sigma_r, \quad (24)$$

while for the commutator in Eq. (23)

$$B_{(2)}^{NC}(\mathbf{x}) = -ie[A_1^{NC}, A_2^{NC}]$$
$$= \left(\frac{e}{c}\right)\frac{\alpha^2}{16r}\partial_r\ln^2\left(\frac{r^2+(r+\theta)^2}{r^2+(r-\theta)^2}\right)\sigma_3 = \mathcal{O}(\theta^2). \quad (25)$$

It can be checked in a straightforward manner that the magnetic field $B^{NC}(r,\phi) = B_{(1)}^{NC} + B_{(2)}^{NC}$ is invariant under rotations along the z-axis generated by

$$U(\gamma) := e^{i\gamma J} = e^{i\gamma(L+(\sigma_3/2))}, \quad (26)$$

where J is the total angular momentum along the z-axis on the noncommutative plane while $L = \mathbf{x}\times(-i\boldsymbol{\nabla})$ corresponds to the orbital angular momentum.

We note that to leading order in θ the magnetic field has been *smeared out* by the noncommutativity of coordinates,

resulting in a well behaved function near the origin (for $\theta \neq 0$),

$$B^{NC}(r, \phi) = \frac{2e\alpha}{\theta^2}\sigma_3 - \frac{16\alpha}{3\theta^3}r\sigma_r + \frac{16\alpha}{3\theta^4}r^2(3\mathbf{1}_2 - e\sigma_3)$$
$$+ O(r^3). \quad (27)$$

The flux of the magnetic field through a circle of radius r can be easily evaluated. To first order in the noncommutativity parameter θ, the vector potential in Eq. (20) gives rise to a magnetic field vanishing everywhere except at the origin, with a magnetic flux as in the conventional case,

$$\int_0^{2\pi} rd\phi \mathbf{u}_\phi \cdot \mathbf{A}^{NC} = \int_0^{2\pi} rd\phi \mathbf{u}_\phi \cdot \frac{\alpha}{r}\left\{\left(\mathbf{1}_2 - \frac{\theta}{2r}\sigma_r\right)\mathbf{u}_\phi\right.$$
$$\left. - \frac{\theta}{2r}\sigma_\phi \mathbf{u}_r\right\} = 2\pi\alpha\mathbf{1}_2, \quad (28)$$

where we have used the fact that the integral of σ_r over the angular coordinate ϕ vanishes. This suggests that, to linear order in the noncommutative parameter θ, one should not expect any significant departure in the interference pattern from that of the usual Aharonov-Bohm effect in the normal plane.

In the following section we will analyze the equation of motion of the electron in the aforementioned magnetic field, retaining modifications due to noncommutativity up to first order in the parameter θ.

III. THE SCHRÖDINGER-PAULI EQUATION

In this section we write down the explicit form of the Schrödinger-Pauli equation in the presence of the gauge field (20). Let us consider the Hamiltonian

$$H = \frac{1}{2}(\hat{\mathbf{p}} - e\mathbf{A}^{NC})^2 = \frac{1}{2}(-\iota\boldsymbol{\nabla} - e\mathbf{A}^{NC})^2, \quad (29)$$

corresponding to a nonrelativistic spinorial particle ($m = 1$). The Hamiltonian H is invariant under a rotation by an angle γ around the z-axis under which the vector potential transforms as

$$\mathbf{A}^{(NC)} \rightarrow U(\gamma)\mathbf{A}^{(NC)}U^\dagger(\gamma), \quad (30)$$

with $U(\gamma) := e^{\iota\gamma J}$ defined in (26). As a result, H commutes with $J := L + \sigma_3/2$ and it leaves invariant the subspaces of the form

$$\mathcal{H}_l := \left\{\begin{pmatrix} e^{il\phi}\psi_l(r) \\ e^{i(l+1)\phi}\chi_l(r) \end{pmatrix}: \psi_l(r), \chi_l(r) \in \mathbf{L}_2(\mathbb{R}^+; rdr)\right\}, \quad (31)$$

for all $l \in \mathbb{Z}$. The eigenvalue equation for H in this subspace gives rise to a system of coupled differential equations which, in the leading order in the perturbation parameter θ, gives a good description of our system at large r. As we will show, it is possible to get an exact expression for this first-order correction.

For small θ, eigenvalue equation for H reduces to

$$-\psi_l''(r) - \frac{1}{r}\psi_l'(r) + \left\{\frac{(l-\beta)^2}{r^2} - E\right\}\psi_l(r)$$
$$= -\theta\beta\left\{-\frac{1}{r^2}\chi_l'(r) + \left(\frac{l+1-\beta}{r^3}\right)\chi_l(r)\right\} + O(\theta^2),$$

$$-\chi_l''(r) - \frac{1}{r}\chi_l'(r) + \left\{\frac{(l+1-\beta)^2}{r^2} - E\right\}\chi_l(r)$$
$$= -\theta\beta\left\{\frac{1}{r^2}\psi_l'(r) + \frac{l-\beta}{r^3}\psi_l(r)\right\} + O(\theta^2), \quad (32)$$

with $\beta := e\alpha$. Let us note that, although for small r the coefficients in the eigenvalue equation are all regular, the small-θ expansion introduces singular terms (at $r = 0$) which we here treat as perturbations on the solutions of the usual Aharonov-Bohm problem.

It is worthwhile to point out that the first-order perturbed Hamiltonian can be constructed through successive unitary transformations from that of the standard Aharonov-Bohm Hamiltonian, inheriting therefore the spectrum and topological properties of the last one. The noncommutative effects in the present approach include anisotropic ones that could—n principle—be measured in experiments.

Indeed, up to first order in θ, the Hamiltonian in Eq. (29) can be written as

$$H = -\frac{1}{2}\left[\left\{\left(\frac{\partial}{\partial r} + \frac{1}{2r}\right)^2 + \frac{1}{4r^2} + \frac{1}{r^2}\left(\frac{\partial}{\partial\phi} - i\beta\right)^2\right\}\mathbf{1}_2\right.$$
$$\left. + \frac{i\beta\theta}{r^2}\left\{\sigma_\phi\frac{\partial}{\partial r} + \frac{\sigma_r}{r}\left(\frac{\partial}{\partial\phi} - i\beta\right)\right\}\right], \quad (33)$$

and using the identities

$$r^{-1/2}\frac{\partial}{\partial r}[r^{1/2}] = \frac{\partial}{\partial r} + \frac{1}{2r},$$
$$e^{\iota\beta\phi}\frac{\partial}{\partial\phi}[e^{-\iota\beta\phi}] = \frac{\partial}{\partial\phi} - \iota\beta, \quad (34)$$

it can also be factorized as

$$H = -\frac{1}{2}r^{-1/2}e^{\iota\beta\phi}\left[\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{4r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right\}\mathbf{1}_2\right.$$
$$\left. + \frac{\iota\beta\theta}{r^2}\left\{\sigma_\phi\left(\frac{\partial}{\partial r} - \frac{1}{2r}\right) + \sigma_r\frac{1}{r}\frac{\partial}{\partial\phi}\right\}\right]r^{1/2}e^{-\iota\beta\phi}$$
$$\equiv (r^{-1/2}e^{\iota\beta\phi})\tilde{H}(r^{1/2}e^{-\iota\beta\phi}). \quad (35)$$

Here $\tilde{H}$ is

$$\tilde{H} = \tilde{H}_0 - \theta\frac{\iota\beta}{2r^2}\left[\sigma_\phi\left(\frac{\partial}{\partial r} - \frac{1}{2r}\right) + \sigma_r\frac{1}{r}\frac{\partial}{\partial\phi}\right], \quad (36)$$

with

$$\tilde{H}_0 = -\frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{4r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right]\mathbf{1}_2. \quad (37)$$

The Hamiltonian $\tilde{H}$ can be factorized again by using the following identity:

$$\left[\tilde{H}_0, \frac{i\beta\theta}{2r}\sigma_\phi\right] = \frac{i\beta\theta}{2r^2}\left(\sigma_\phi\left[\frac{\partial}{\partial r} - \frac{1}{2r}\right] + \sigma_r\frac{1}{r}\frac{\partial}{\partial\phi}\right); \quad (38)$$

the relation between $\tilde{H}$ and $\tilde{H}_0$ becomes now

$$\tilde{H} = \tilde{H}_0 - \left[\tilde{H}_0, \frac{i\beta\theta}{2r}\sigma_\phi\right]$$
$$= \left(1 + \frac{i\beta\theta}{2r}\sigma_\phi\right)\tilde{H}_0\left(1 - \frac{i\beta\theta}{2r}\sigma_\phi\right) = U\tilde{H}_0U^\dagger, \quad (39)$$

up to first order in θ.

Observe that the multiplicative unitary operator $U = (1 + \frac{i\beta\theta}{2r}\sigma_\phi)$ (which encloses all the dependence on θ) factorizes out the nontrivial matrix dependence due to spin, leaving $\tilde{H}_0$ which is diagonal. Moreover, it commutes with the factor $r^{1/2}e^{-ie\alpha\phi}$ in the similarity transformation in (35) so that up to first order in θ, the Hamiltonian $\hat{H}$ describing the modified Aharonov-Bohm effect can be written as

$$H = U[(r^{-1/2}e^{i\beta\phi})\tilde{H}_0(r^{1/2}e^{-i\beta\phi})]U^\dagger, \quad (40)$$

where the term in parentheses is just the standard Aharonov-Bohm Hamiltonian, which acts as a diagonal operator on the spinor components.

Having achieved this result, it is easy to relate the eigenfunctions $\psi_E(r, \phi)$ of the perturbed Hamiltonian in Eq. (33),

$$H\psi_E(r, \phi) = E\psi_E(r, \phi), \quad (41)$$

with the corresponding eigenfunction of the usual Aharonov-Bohm scalar particle on the usual commutative plane, through the above discussed unitary transformation. In particular, notice that the spectrum of the Hamiltonian in

the noncommutative case is the same as that of the standard Aharonov-Bohm effect.

Indeed, if

$$\tilde{H}_0\tilde{\chi}_E^{(0)}(r, \phi) = E\tilde{\chi}_E^{(0)}(r, \phi), \quad (42)$$

then from Eq. (40) we get

$$\psi_E(r, \phi) = r^{-1/2}e^{i\beta\phi}U\tilde{\chi}_E^{(0)}(r, \phi). \quad (43)$$

Notice that the separability of variables in each component of the solution is ensured by the previously discussed rotational symmetry of the Hamiltonian.

It is straightforward to find solutions of (42): each component of the spinor $\tilde{\chi}_E^{(0)}(r, \phi)$ (which is not coupled) can be expressed as

$$\tilde{\chi}_j(r, \phi) = e^{i\nu\phi}\chi_j^\nu(r), \qquad j = 1, 2,$$

where the parameters ν are not necessarily integers, since it is the function $\psi_E(r, \phi)$ in Eq. (43) which must be single-valued. The radial functions $\chi_j^\nu(r)$ satisfy the equation

$$-\frac{1}{2}\left[\frac{d^2}{dr^2} + \frac{1 - 4\nu^2}{4r^2}\right]\chi_j^\nu(r) = E\chi_j^\nu(r), \quad (44)$$

whose solutions are expressed in terms of Bessel functions as

$$\chi_j^\nu(r) = \sqrt{r}[A_jJ_{|\nu|}(kr) + B_jY_{|\nu|}(kr)], \quad (45)$$

where $k = \sqrt{2E}$ and A_j, B_j ($j \in \{1, 2\}$) are integration constants which must be determined according to the boundary condition the function $\psi_E(r, \phi)$ must satisfy.

Comparison with Eq. (31) shows that the parameter ν must be chosen so as

$$\psi_E^\ell(r, \phi) = \left(1 + i\frac{\beta\theta}{2r}\sigma_\phi\right)\begin{pmatrix} e^{i\ell\phi}[A_1^\ell J_{|\ell-\beta|}(kr) + B_1^\ell Y_{|\ell-\beta|}(kr)] \\ e^{i(\ell+1)\phi}[A_2^\ell J_{|\ell+1-\beta|}(kr) + B_2^\ell Y_{|\ell+1-\beta|}(kr)] \end{pmatrix}. \quad (46)$$

The coefficients in Eq. (46) must be determined by imposing suitable boundary conditions. In particular, this first-order correction in perturbation theory must be square-integrable in a neighborhood of the origin. It can be straightforwardly seen that this condition requires that $B_j^\ell = 0$, $j = 1, 2$, $\forall \ell \in \mathbf{Z}$.

Employing the recurrence relation for Bessel functions,

$$Z_{n+1}(x) + Z_{n-1}(x) = \frac{2n}{x}Z_n(x), \quad (47)$$

the solution in Eq. (46) can also be written as

$$\psi_E^\ell(r, \phi) = \begin{pmatrix} e^{i\ell\phi}\left\{A_1^\ell J_{|\ell-\beta|}(kr) + \frac{\theta\beta kA_2^\ell}{4|\ell-\beta+1|}[J_{|\ell-\beta+1|-1}(kr) + J_{|\ell-\beta+1|+1}(kr)]\right\} \\ e^{i(\ell+1)\phi}\left\{A_2^\ell J_{|\ell-\beta+1|}(kr) - \frac{\theta\beta kA_1^\ell}{4|\ell-\beta|}[J_{|\ell-\beta|-1}(kr) + J_{|\ell-\beta|+1}(kr)]\right\} \end{pmatrix}. \quad (48)$$

The general solution of the Schrödinger-Pauli equation in the noncommutative plane, up to first order in θ, is then constructed as the combination[1]

$$\psi_E(r, \phi) = \sum_{\ell=-\infty}^{\infty} \psi_E^\ell(r, \phi), \qquad (49)$$

with $\psi_E^\ell(r, \phi)$ given in Eq. (46) or (48).

With this solution, our aim is now to evaluate the scattering amplitude, which could be of interest for an experimental test of this kind of system. This will be done in the next section.

$$\psi_E(r, \phi) \approx \begin{pmatrix} \sum_{\ell=-\infty}^{\infty} e^{i\ell\phi} A_1^\ell \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{\pi}{2}|\ell - \beta| - \pi/4\right) \\ \sum_{\ell=-\infty}^{\infty} e^{i(\ell+1)\phi} A_2^\ell \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{\pi}{2}|\ell - \beta + 1| - \pi/4\right) \end{pmatrix}, \qquad (50)$$

and we observe that there are no terms proportional to θ. This is completely different than in conventional noncommutative quantum mechanics, where the differential cross section [15] may depend on theta in the case of a small angle.

Therefore, the differential cross section for our problem is the same as in the Aharonov-Bohm effect for spin 1/2 particles. As a check, consider a polarized beam with $A_2^\ell = 0$. It is direct to check that the previous expression has the shape

$$\psi_E(r, \phi) = \begin{pmatrix} e^{ikr\cos(\phi)} \\ 0 \end{pmatrix} + \begin{pmatrix} F_1(\phi)\frac{e^{ikr}}{\sqrt{ir}} \\ 0 \end{pmatrix},$$

by choosing

$$A_1^\ell = e^{(i\pi/2)(2\ell - |\ell - \beta|)}, \qquad (51)$$

and then

$$F_1 = \frac{1}{\sqrt{2k\pi}} \sum_{\ell=-\infty}^{\infty} e^{i\ell\phi}(e^{i\pi(\ell - |\ell - \beta|)} - 1), \qquad (52)$$

which is the standard result for the Aharonov-Bohm effect (see for example [15]). Therefore, the differential cross section, for a beam polarized parallel to the solenoid direction, turns out to be

IV. SCATTERING AMPLITUDE

We are now interested in a situation in which an incident particle reaches the center at $r = 0$ and is scattered out by the Aharonov-Bohm flux in the noncommutative plane we are considering. In this case, the wave function constructed in Eq. (49) equals the sum of the plane wave and a spherical outgoing wave.

The asymptotic form of the solution (49), for $kr \gg 1$, turns out to be

$$\frac{d\sigma}{d\phi} = \frac{\sin^2(\pi\beta)}{2k\pi\sin^2(\phi/2)}. \qquad (53)$$

Opposite polarization, namely $A_1^\ell = 0$, gives same result, as is expected from the general result [16,17]

$$\frac{d\sigma}{d\phi} = (1 - (\hat{\mathbf{n}} \times \hat{\mathbf{z}})^2\sin^2(\phi/2))\left(\frac{d\sigma}{d\phi}\right)_{\text{unpol}}, \qquad (54)$$

where $\hat{\mathbf{n}}$ is the polarization direction and $\hat{\mathbf{z}}$, the direction defined by the solenoid.

Anyway, a comment is in order. As previously stated, we are employing perturbation theory, which introduces potentials which are singular at the origin, even though the noncommutativity of this plane smears out the field intensity as discussed in Sec. II, leading to smooth coefficients in the Hamiltonian.

V. CONCLUSIONS

In this paper we have studied the Aharonov-Bohm effect with spin-noncommutativity, namely, for the case when the noncommutative coordinates involve spin. Although the Aharonov-Bohm effect, in this case, retains some important properties of the conventional case, such as topological properties, spin induces a strong anisotropy which is completely different from the conventional Aharonov-Bohm effect. From the analysis of the scattering problem one sees that the anisotropy is an effect that occurs close to $\theta \sim \sqrt{r}$ so that long-range scattering effects may not be affected.

However, this anisotropy opens also the interesting possibility of studying other systems that behave very similarly to the problem studied here. Indeed, in the case of cold atoms with nonzero total spin using suitable magnetic traps it is possible to confine the atoms to a plane and, therefore, such as in the case of an anyon gas, one could have a gas of cold atoms, where each atom would have an

---

[1]Strictly speaking, we should consider also linearly independent (order θ) square-integrable solutions in the $\ell = -1, 0, 1$ invariant subspaces, which have a nonregular behavior at the origin and are related to the existence of nontrivial self-adjoint extensions of the Aharonov-Bohm Hamiltonian [14] (see the Appendix). For simplicity, we impose a regularity condition at the origin on the solutions of the Aharonov-Bohm problem from which we construct the eigenfunctions we consider in the following.

attached magnetic flux as in the conventional anyon gas [18]. The experiments measuring spin effects of cold atoms in gases have so far been done only for the case of total spin 3. More accurate measurements in other systems are likely to come in the near future, where predictions of spin-noncommutativity can possibly be checked.

ACKNOWLEDGMENTS

APPENDIX: SOLUTIONS IN THE CRITICAL SUBSPACE

In the *critical* $\ell = -1, 0, 1$ invariant subspaces, we can find linearly independent $O(\theta)$ solutions (eigenfunctions of

the Aharonov-Bohm Hamiltonian), whose existence is related to the existence of nontrivial self-adjoint extensions of this Hamiltonian [14]. Indeed, the functions

$$
\zeta_E^{-1} = \begin{pmatrix} 0 \\ \theta B_2^{-1} Y_\beta(kr) \end{pmatrix},
$$
$$
\zeta_E^0 = \begin{pmatrix} \theta B_1^0 Y_\beta(kr) \\ \theta B_2^0 e^{i\phi} Y_{1-\beta}(kr) \end{pmatrix}, \tag{A1}
$$
$$
\zeta_E^1 = \begin{pmatrix} \theta B_2^{-1} e^{i\phi} Y_{1-\beta}(kr) \\ 0 \end{pmatrix}
$$

are square-integrable near the origin (with respect to the measure $r dr d\phi$) and satisfy (up to order θ) the eigenvalue equation (41) with $2E = k^2$. Then, in these critical subspaces, one should take those linear combinations of functions in Eqs. (48) and (A1) which belong to the domain of the selected Hamiltonian self-adjoint extension. They are of the form

$$
\psi_E^{-1}(r, \phi) = \begin{pmatrix} e^{-i\phi}\left\{A_1^{-1} J_{1+\beta}(kr) + \frac{\theta\beta k A_2^{-1}}{4\beta}[J_{\beta-1}(kr) + J_{\beta+1}(kr)]\right\} \\ \left\{A_2^{-1} J_\beta(kr) - \frac{\theta\beta k A_1^{-1}}{4(1+\beta)}[J_\beta(kr) + J_{2+\beta}(kr)] + \theta B_2^{-1} Y_\beta(kr)\right\} \end{pmatrix}
$$
$$
\simeq \begin{pmatrix} r^{\beta-1}\left[\theta\frac{2^{-\beta-1}A_2^{-1}k^\beta}{\Gamma(\beta)} + O(r^2)\right] \\ r^{-\beta}\left[-\theta\frac{2^\beta k^{-\beta}\theta B_2^{-1}\Gamma(\beta)}{\pi} + O(r^2)\right] + O(r^\beta) \end{pmatrix}, \tag{A2}
$$

$$
\psi_E^0(r, \phi) = \begin{pmatrix} \left\{A_1^0 J_\beta(kr) + \frac{\theta\beta k A_2^0}{4(1-\beta)}[J_{-\beta}(kr) + J_{\beta+1}(kr)] + \theta B_1^0 Y_\beta(kr)\right\} \\ e^{i\phi}\left\{A_2^0 J_{1-\beta}(kr) - \frac{\theta\beta k A_1^0}{4\beta}[J_{\beta-1}(kr) + J_{\beta+1}(kr)] + \theta B_2^0 Y_{1-\beta}(kr)\right\} \end{pmatrix}
$$
$$
\simeq \begin{pmatrix} r^{-\beta}\left[\theta 2^\beta k^{-\beta}\left(\frac{k\beta A_2^0}{(4-4\beta)\Gamma(1-\beta)} - \frac{B_1^0\Gamma(\beta)}{\pi}\right) + O(r^2)\right] + O(r^\beta) \\ r^{1-\beta}\left[\frac{2^{\beta-1}k^{1-\beta}(\pi A_2^0 + \theta B_2^0\cos(\pi\beta)\Gamma(2-\beta)\Gamma(\beta-1))}{\pi\Gamma(2-\beta)} + O(r^2)\right] + r^{\beta-1}\left(-\frac{2^{-\beta-1}k^{\beta-1}\theta(k\pi A_1^0 + 4B_2^0\Gamma(1-\beta)\Gamma(\beta))}{\pi\Gamma(\beta)} + O(r^2)\right) \end{pmatrix}, \tag{A3}
$$

$$
\psi_E^1(r, \phi) = \begin{pmatrix} e^{i\phi}\left\{A_1^1 J_{1-\beta}(kr) + \frac{\theta\beta k A_2^1}{4(2-\beta)}[J_{1-\beta}(kr) + J_{3-\beta}(kr)]\right\} \\ e^{i2\phi}\left\{A_2^1 J_{2-\beta}(kr) - \frac{\theta\beta k A_1^1}{4(1-\beta)}[J_{-\beta}(kr) + J_{-\beta}(kr)]\right\} \end{pmatrix}
$$
$$
\simeq \begin{pmatrix} r^{\beta-1}\left[-\theta\frac{2^{1-\beta}B_1^1 k^{\beta-1}\Gamma(1-\beta)}{\pi} + O(r^2)\right] + O(r^{1-\beta}) \\ r^{-\beta}\left[\theta\frac{2^{\beta-2}\beta A_1^1 k^{1-\beta}}{(\beta-1)\Gamma(1-\beta)} + O(r^2)\right] \end{pmatrix}, \tag{A4}
$$

where the coefficients are to be fixed by imposing the conditions the functions in the domain of the operator satisfy near the origin.

As previously mentioned, in the present article we discard such contributions and impose just regularity at the origin on the solutions of the Aharonov-Bohm problem from which we construct the eigenfunctions of the present one. But it is worth mentioning that these additional contributions open the possibility of a spin-flip in the critical subspaces. This will be considered elsewhere.

[1] For a review see I. Hinchliffe, N. Kersting, and Y. L. Ma, Int. J. Mod. Phys. A **19,** 179 (2004).

[2] See e.g. M. R. Douglas and N. Nekrasov, Rev. Mod. Phys. **73,** 977 (2001).

[3] See e.g. G. Amelino-Camelia, J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and S. Sarkar, Nature (London) **393,** 763 (1998); S. M. Carroll, J. A. Harvey, V. Alan Kostelecky, Charles D. Lane, and Takemi Okamoto, Phys. Rev. Lett. **87,** 141601 (2001).

[4] See e.g. L. Susskind, arXiv:hep-th/0101029; A. P. Polychronakos, J. High Energy Phys. 06 (2001) 070.

[5] The literature is very extensive; some papers are: V. P. Nair and A. P. Polychronakos, Phys. Lett. B **505,** 267 (2001); G. V. Dunne, J. Jackiw, and C. Trugenberger, Phys. Rev. D **41,** 661 (1990); J. Gamboa, M. Loewe, and J. Rojas, Phys. Rev. D **64,** 067901 (2001); J. Gamboa, M. Loewe, J. Rojas, and F. Méndez, Int. J. Mod. Phys. A **17,** 2555 (2002); Mod. Phys. Lett. A **16,** 2075 (2001); F. Delduc, Q. Duret, F. Gieres, and M. Lefrancois, J. Phys. Conf. Ser. **103,** 012020 (2008); A. Berard and H. Mohrbach, Phys. Lett. A **352,** 190 (2006); K. Bolonek and P. Kosinski, arXiv:0704.2538; F. S. Bemfica and H. O. Girotti, J. Phys. A **38,** L539 (2005); A. Kijanka and P. Kosinski, Phys. Rev. D **70,** 127702 (2004); L. Mezincescu, arXiv:hep-th/0007046; C. Duvall and P. A. Horvathy, Phys. Lett. B **547,** 306 (2002); **479,** 284 (2000); C. Acatrinei, J. High Energy Phys. 09 (2001) 007; M. Gomes and V. G. Kupriyanov, Phys. Rev. D **79,** 125011 (2009); F. G. Scholtz, B. Chakraborty, S. Gangopadya, and A. Hazra, Phys. Rev. D **71,** 085005 (2005); O. Bertolami and C. Zarro, Phys. Rev. D **81,** 025005 (2010); C. Bastos, O. Bertolami, N. Dias, and J. Prata, Phys. Rev. D **78,** 023516 (2008); O. Bertolami, J. G. Rosa, C. M. L. de Aragao, P. Castorina, and D. Zappala, Phys. Rev. D **72,** 025010 (2005); M. Chaichian, A. Demichev, P. Presnajder, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Lett. B **527,** 149 (2002); M. Gomes, V. G. Kupriyanov, and A. J. da Silva, Phys. Rev. D **81,** 085024 (2010); A. F. Ferrari, M. Gomes, and C. A. Stechhahn, Phys. Rev. D **76,** 085008 (2007).

[6] A. Das, J. Gamboa, F. Méndez, and F. Torres, Phys. Lett. A **375,** 1756 (2011).

[7] H. Falomir, J. Gamboa, J. Lopez-Sarrion, F. Mendez, and P. Pisani, Phys. Lett. B **680,** 384 (2009).

[8] H. S. Snyder, Phys. Rev. **71,** 38 (1947); **72,** 68 (1947).

[9] For a review see V. A. Zagrebnov and J.-B. Bru, Phys. Rep. **350,** 291 (2001).

[10] J. Stuhler, A. Griesmaier, T. Koch, T. Pfau, S. Giovanazzi, P. Pedri, and L. Santos, Phys. Rev. Lett. **95,** 150406 (2005).

[11] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, Rep. Prog. Phys. **72,** 126401 (2009).

[12] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Butterworth, Washington, DC, 2003).

[13] J. Gamboa and F. Mendez, "Bose-Einstein Condensation Theory for Any Integer Spin: Approach Based in Noncommutative Quantum Mechanics" (to be published).

[14] See e.g. H. Falomir and P. A. G. Pisani, J. Phys. A **34,** 4143 (2001).

[15] H. Falomir, J. Gamboa, M. Loewe, F. Mendez, and J. C. Rojas, Phys. Rev. D **66,** 045018 (2002).

[16] C. R. Hagen, Phys. Rev. Lett. **64,** 503 (1990).

[17] F. Vera and I. Schmidt, Phys. Rev. D **42,** 3591 (1990).

[18] See e.g. F. Wilczek, Phys. Rev. Lett. **49,** 957 (1982).