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Black hole entropy corrections in the grand canonical ensemble

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We study entropy corrections due to thermal fluctuations for asymptotically anti-de Sitter black holes in the grand canonical ensemble. To leading order, these can be expressed in terms of the black hole response coefficients via fluctuation moments. We also analyze entropy corrections due to mass and charge fluctuations of R-charged black holes, and our results indicate an universality in the logarithmic corrections to charged anti-de Sitter black hole entropy in various dimensions.

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I. INTRODUCTION

Black holes are probably the most mysterious objects in the General Theory of Relativity that are still far from being fully understood. Whereas a quantum description of the nature of black hole spacetimes has remained elusive as of now apart from certain specific examples in loop quantum gravity (LQG), a lot of progress has nevertheless been made in the understanding of black hole thermodynamics (see, e.g [1,2]). Although the origin of entropy in black holes is quantum in nature unlike ordinary thermodynamic systems, semiclassical analyses have often provided insights into the rich phase structure possessed by black hole systems. For example, it is well known by now that, whereas asymptotically flat black holes are thermally unstable, asymptotically anti-de Sitter (AdS) ones can be in thermal equilibrium with their own radiation. These often exhibit critical phenomena akin to phase transitions in ordinary liquid gas systems. Phase structures for AdS black holes have been extensively studied in the literature [3–5], and, for many cases, critical exponents have been computed [6–8] revealing universality in a class of examples.

In the context of black hole thermodynamics, it has been established that black hole entropy, as given by the celebrated Beckenstein-Hawking area law $S = \frac{A}{4}$ (in units of the Planck area), acquires logarithmic corrections due to thermal fluctuations in the black hole parameters.[1] This has been extensively investigated by several authors over the past decade (see, e.g [10–12]). Indeed, considered as a microcanonical ensemble, where the black hole is isolated, it has a fixed horizon area and an entropy proportional to a quarter of this area. Allowing for fluctuations of the black hole extensive parameters give rise to logarithmic corrections[2] to the area law. These corrections, which are classical in nature thus arise if one allows the black hole to be in an equilibrium canonical (or grand canonical) ensemble where the black hole is allowed to exchange energy (and charge, angular momentum) with its surroundings. Inclusion of fluctuations in black hole thermodynamics is nontrivial and these often provide ensemble dependent phase behavior. For example, it is well known that whereas, in the canonical ensemble, electrically charged AdS black holes exhibit liquid gas like phase coexistence between a small black hole and a large black hole phase culminating at a critical point, in the grand canonical ensemble the situation is entirely different, and one only has a Hawking-Page-like phase transition from a thermal AdS space time to a nonextremal Reissner-Nordstrom (RN)-AdS black hole at low temperatures [3,4]. In the context of black hole entropy, corrections to the same are also found to be ensemble dependent. Although some specific examples have been considered so far, a general treatment of such corrections in the grand canonical ensemble is however lacking. It is this issue that we set out to address in this paper.

Corrections to black hole entropy can be computed from fluctuation theory by considering the entropy of black hole in an appropriate equilibrium ensemble. In the canonical ensemble, where the system is in equilibrium with a heat bath and exchanges energy with the surrounding, the corrections to the Beckenstein-Hawking area law can be simply related to the specific heat of the black hole [10]. This follows from standard analyses in thermodynamic fluctuation theory. In the grand canonical ensemble, where one allows for fluctuations in all the extensive thermodynamic parameters (i.e. *charges*) of the black hole while keeping the corresponding *potentials* fixed, the situation is more intricate. Electrically charged black holes in the grand canonical ensemble have been studied for the four dimensional RN-AdS black hole in [12], via the grand

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[1]These are *classical* fluctuations, where we tacitly assume that the area of the black hole is large. This is the regime that we will be interested in here. The entirely different issue of quantum corrections to black hole entropy has been studied in the framework of LQG in [9].

[2]We will adhere to standard terminology. By ''corrections'' to black hole entropy we mean the terms that appear in the expression of the entropy in addition to the Beckenstein-Hawking area law, in the canonical or grand canonical ensemble.

canonical partition sum, by postulating a power law relation between the energy, area, and charge. More generic examples, where the black hole is allowed to exchange angular momenta as well, have not been considered in the literature. This is mainly due to the fact that the "angular momentum spectrum" is hitherto not completely known in the formalism of LQG. However, these are interesting and important to study, as they serve to directly elucidate the effect of charge fluctuations on the black hole entropy, *vis à vis* fluctuations in the angular momentum. We will in fact show in the sequel that the nature of these fluctuations are completely different.

The main idea in this paper is to exploit the Smarr type relations between the black hole parameters to calculate corrections to its entropy, via fluctuation theory in the grand canonical ensemble. This also takes into account fluctuations of the angular momentum for rotating black holes, which, to the best of our knowledge, have not been studied so far. Our method elucidates the general procedure to evaluate these corrections in the grand canonical ensemble in the presence of arbitrary fluctuating parameters *in any dimension*, using standard thermodynamics. Using this, we also calculate the entropy corrections in the *mixed ensembles* [5] of Kerr-Newman (KN)-AdS black holes in four dimensions (where one fixes a charge and a potential), which allow for interesting phase coexistence behavior. We further relate these corrections to black hole response coefficients, thus generalizing the result of [10]. Our methods are applicable to string theoretic black holes as well, and we show that there exists a certain universality in the correction terms for the entropy of electrically charged AdS black holes.

This paper is organized as follows. In Sec. II, we review the general method to calculate fluctuation induced corrections to black hole area law. A formula for such corrections, involving the black hole response coefficients, is also written down, generalizing the known canonical ensemble result, using the properties of the generalized Massieu transform of the entropy. This is illustrated for the example of the Banados-Teitelboim-Zanelli (BTZ) black hole in Sec. III. In Sec. IV, we evaluate, in the grand canonical ensemble, corrections to the Beckenstein-Hawking entropy of charged, rotating AdS black holes in four dimensions. In addition, we calculate these in the mixed ensembles alluded to in [5]. Section V is devoted to the calculation of the entropy of class of string theoretic black holes in the grand canonical ensemble. Finally, we end with our conclusions in Sec. VI.

II. BLACK HOLE ENTROPY AND THERMAL FLUCTUATIONS

In order to calculate corrections to black hole entropy from thermal fluctuations, one starts from the expression for the partition function in the grand canonical ensemble, which is given, for continuous values of the energy and the charges by

$$Z(\mu_i,\beta)=\int_0^\infty\int_0^\infty\cdots\int_0^\infty \rho(E,N_i)\times\exp[-\beta(E-\mu_iN_i)]dEdN_1dN_2\cdots dN_i, \quad (1)$$

where ρ is the density of states, N_i are the "particle numbers" or "charges," with μ_i being the corresponding chemical potentials. β is the inverse temperature. In the canonical ensemble, the expression simplifies as ρ is a function of only the energy of the system, and the integrations over the N_i's are absent. For charged rotating black holes, $i=2$ with N_1 and N_2 being the electric charge and the angular momentum, and μ_1, μ_2 are the electric potential and angular velocity. Note that this expression is valid only for a continuous distribution of the eigenvalues of the Hamiltonian, the charge operator, the angular momentum operator, etc., in the corresponding quantum theory. Equation (1) is then inverted [13] by using an inverse Laplace transform to obtain the density of states, which, in the saddle point approximation, is

$$\rho(E,N_i)=\frac{Z(\beta_0,\lambda_{i0})\exp(\beta_0E+\lambda_{i0}N_i)}{(2\pi)^{3/2}\Delta_0^{1/2}}, \quad (2)$$

where $\beta=\frac{1}{T}$ and $\lambda_i=-\frac{\mu_i}{T}$. By definition, $\lambda_i=(\frac{\partial S}{\partial N_i})$, where S is expressed as a function of (E,N_i). Δ_0 is the determinant of the matrix

$$D=\begin{pmatrix} \frac{\partial^2 \ln Z}{\partial\lambda_1^2} & \frac{\partial^2 \ln Z}{\partial\lambda_1\partial\lambda_2} & -\frac{\partial^2 \ln Z}{\partial\lambda_1\partial\beta} \\ \frac{\partial^2 \ln Z}{\partial\lambda_2\partial\lambda_1} & \frac{\partial^2 \ln Z}{\partial\lambda_2^2} & -\frac{\partial^2 \ln Z}{\partial\lambda_2\partial\beta} \\ -\frac{\partial^2 \ln Z}{\partial\beta\partial\lambda_1} & -\frac{\partial^2 \ln Z}{\partial\beta\partial\lambda_2} & \frac{\partial^2 \ln Z}{\partial\beta^2} \end{pmatrix} \quad (3)$$

evaluated at the equilibrium values of the temperature and the chemical potentials. It is to be noted (see, e.g [14]) that the logarithm of the (grand canonical) partition function appearing in Eq. (3) is the generalized Massieu transform of the entropy,

$$\ln Z=S-\beta E-\lambda_iN_i, \quad (4)$$

where the left hand side of Eq. (4) should be thought of as a function of β and λ.

It follows from Eq. (4) that the numerator of Eq. (2) is the exponential of the grand canonical entropy (as a function of E and N_i), and, hence, we have, apart from irrelevant constants,

$$S_g=\ln\rho+\frac{1}{2}\ln\Delta_0. \quad (5)$$

If we define the microcanonical entropy as the logarithm of the density of states (as a function of only energy), then the above formula gives a correction to the microcanonical entropy in the grand canonical (or canonical) ensemble.[3]

The expression for Δ_0 can be computed via the response coefficients of the black hole. For example, in the canonical ensemble where the system is in equilibrium with a heat bath, Eq. (5) simplifies to

$$S_c = \ln\rho + \frac{1}{2}\ln CT^2, \tag{6}$$

where C is the specific heat and T the equilibrium temperature. In the grand canonical ensemble, where we allow for the fluctuations in the particle numbers, the elements of the matrix D, which are directly related to the moments of fluctuation of the black hole parameters (see, eg. [14]), get related to the black hole response coefficients. For example, in a grand canonical ensemble with the mass M and electric charge Q being the fluctuating parameters, it can be shown that the determinant Δ_0 can be written as

$$\Delta_0 = \langle\Delta M^2\rangle\langle\Delta Q^2\rangle - (\langle\Delta M\Delta Q\rangle)^2. \tag{7}$$

A similar formula holds for angular momentum fluctuations. Simple algebraic manipulations then show that when the black hole is in a grand canonical ensemble with the energy and a single charge (electric charge or angular momentum) being allowed to fluctuate, the determinant Δ_0 can be expressed as[4]

$$\Delta_0 = T^3 C_\lambda \kappa + T^4\lambda\alpha\kappa - T^4\alpha^2, \tag{8}$$

where we have defined the heat capacity at constant λ, and the response coefficients κ and α (analogous to the "compressibility" and the "expansivity") with $N = \{Q, J\}$, as

$$C_\lambda = \frac{1}{\beta}\left(\frac{\partial S}{\partial T}\right)_\lambda, \qquad \kappa = -\frac{1}{T}\left(\frac{\partial N}{\partial \lambda}\right)_T, \qquad \alpha = \left(\frac{\partial N}{\partial T}\right)_{\lambda T}. \tag{9}$$

Note that $\Delta_0 > 0$ gives the general condition for stability in the grand canonical ensemble. If one allows for fluctuations in the energy and a single charge (with chemical potential μ and $\lambda = -\frac{\mu}{T}$), this yields

$$C_\lambda > T\alpha\left(\frac{\alpha}{\kappa} - \lambda\right). \tag{10}$$

The reader will notice that Eq. (10), which is applicable in the grand canonical ensemble, generalizes the well known stability condition of the positivity of the specific heat in the canonical ensemble. A similar formula can be easily written down for the case when both the black hole charges are allowed to fluctuate.

On physical grounds, one would expect that Eq. (10) holds in the regions where the black hole is globally stable (i.e. the Gibbs free energy is negative). This is indeed the case for the BTZ black hole in the grand canonical ensemble, as this is locally as well as globally stable everywhere.[5] However, for other black holes, as we will elucidate in the sequel, the region of global stability is more constrained than the region of parameter space for which Eq. (10) is valid. The latter equation can be shown to be valid when the black hole is locally stable.

We also mention here that for the grand canonical ensemble, it is more appropriate to deal with (fixed) thermodynamic potentials than the charges. In this sense, Eq. (8) should be thought of as being expressed in terms of the appropriate potentials. The results that we will obtain will be in terms of the fixed potentials, which can then be used to deduce leading order corrections to black hole entropy in the limit of small potentials. We should again remind the reader that our analysis is strictly valid only in the classical regime, i.e. for large black holes, away from extremality.

Finally, note that Eq. (5) is based on the assumption that the spectrum of energy and the charges are continuous parameters. The situation changes markedly if this is not the case (as is expected from LQG). We should, therefore, include an appropriate Jacobian in going to the continuum limit in Eq. (1), as advocated in [11]. The Jacobian factor[6] K modifies the result for the entropy correction, and the corrected entropy is given by [11]

$$S_g = \ln\rho + \ln K + \frac{1}{2}\ln\Delta_0. \tag{11}$$

Evaluation of the Jacobian factor requires knowledge about the quantum spectrum of the black hole parameters. This is not fully understood, and we will proceed with the assumption that the area, charge and angular momentum spectra are linear in the quantum numbers that they are measured in [12].

Let us also mention that an equivalent approach is to calculate the partition function of Eq. (1) directly, in the Gaussian approximation [9]. It can be checked that this gives the same result for the correction to the entropy as that outlined above.

We now illustrate this procedure for the case of the BTZ black hole in the grand canonical ensemble.

III. BTZ BLACK HOLE IN THE GRAND CANONICAL ENSEMBLE

The entropy of the BTZ black hole is given, in terms of its mass and angular momentum, as

[3]Such corrections to the microcanonical entropy appear for ordinary thermodynamic systems as well, but for such systems, in the infinite volume limit, when one talks about thermodynamic quantities per unit volume, these can be neglected.

[4]A similar relation can be readily obtained for more than two fluctuating parameters. The result however is lengthy and not particularly illuminating.

[5]We allude to the standard definitions that a thermodynamic system is locally stable if the Hessian of its entropy does not develop negative eigenvalues. Global stability implies that the Gibbs free energy is negative.

[6]We denote the Jacobian by K in this paper, so as not to confuse with the notation for angular momentum.

$$S = \frac{4\pi}{\sqrt{2}}\left[Ml^2\left(1 + \left[1 - \frac{J^2}{M^2 l^2}\right]^{1/2}\right)\right]^{1/2}, \qquad (12)$$

where l is related to the cosmological constant $\Lambda = -\frac{1}{l^2}$. A useful quantity for the thermodynamic description is the angular velocity, which is given by the expression

$$\Omega = 8\pi^2 \frac{J}{S^2}, \qquad (13)$$

in terms of which the Hawking temperature of the black hole is

$$T = \frac{1}{8\pi^2 l^2} S(1 - \Omega^2 l^2), \qquad (14)$$

which implies that non extremal BTZ black holes with non zero temperature exist only for $\Omega l < 1$ [15]. The correction to the entropy can be evaluated from Eq. (3) by noting that

$$\beta = \frac{1}{T}; \qquad \lambda = \frac{8\pi^2 \Omega l^2}{S(\Omega^2 l^2 - 1)}. \qquad (15)$$

The determinant Δ_0 can be calculated from Eq. (3) [or equivalently Eq. (8)]:

$$\Delta_0 = \frac{1}{4096} \frac{S^6(1 - \Omega^2 l^2)^3}{\pi^8 l^6}. \qquad (16)$$

For a continuous energy spectrum, the final result is, apart from unimportant constants,

$$S_g = \ln\rho + 3\ln S_{\rm bh}, \qquad (17)$$

where $S_{\rm bh}$ is identified with the area of the BTZ black hole. Hence we see that in the grand canonical ensemble, the entropy is proportional to $\ln S$ but with a prefactor of 3, instead of $\frac{3}{2}$ in the canonical ensemble, calculated in [9,10]. Inclusion of the Jacobian factor in Eq. (11) alters the result. Following [12], for the BTZ black hole, this is given by

$$K^{-1} = \left(\frac{\partial E}{\partial x}\right)\left(\frac{\partial J}{\partial y}\right) = \left(\frac{\partial E}{\partial A}\right)\left(\frac{\partial A}{\partial x}\right)\left(\frac{\partial J}{\partial y}\right). \qquad (18)$$

Assuming that the spectra of the area and the angular momentum are linear in the quantum numbers x and y, we obtain, apart from multiplicative constants,[7]

$$K = \frac{8 l^2 \pi^2}{S(1 - \Omega^2 l^2)}. \qquad (19)$$

Inclusion of this factor thus implies that the final form of the corrected entropy for the BTZ black hole is

$$S_g = \ln\rho + 2\ln S_{\rm bh}. \qquad (20)$$

It is also to be noted that for the BTZ black hole,

$$\Delta_0 = -\frac{G^3}{\pi^2}, \qquad (21)$$

where G is the Gibbs free energy given by

$$G = M - TS - \Omega J. \qquad (22)$$

This ensures that the determinant Δ_0 is always positive in the region of global stability where G is negative.

Finally, a few words about the fluctuation moments. It follows from Eq. (3) that the relative fluctuations are

$$\begin{aligned} \frac{\langle \Delta M^2 \rangle}{M^2} &= \frac{4(1 + 3\Omega^2 l^2)}{S(1 + \Omega^2 l^2)} \\ \frac{\langle \Delta J^2 \rangle}{J^2} &= \frac{1 + 3\Omega^2 l^2}{S\Omega^2 l^2} \\ \frac{\langle \Delta M \Delta J \rangle}{MJ} &= \frac{2(3 + \Omega^2 l^2)}{S(1 + \Omega^2 l^2)}. \end{aligned} \qquad (23)$$

Equation (23) shows the qualitative difference between mass and angular momentum fluctuations for BTZ black holes. In the limit of small Ωl, the relative fluctuation of the mass $\frac{\Delta M^2}{M^2} \sim S^{-1}$, as does the cross correlation term. The relative angular momentum fluctuation however goes as $\frac{\Delta J^2}{J^2} \sim S^{-1}(\Omega l)^{-2}$. This illustrates the difference in the behavior of relative fluctuations of the energy and the angular momentum.

IV. $D = 4$ ADS BLACK HOLES IN VARIOUS ENSEMBLES

We now turn to the example of AdS black holes in four dimensions. The generalized Smarr formula for KN-AdS black holes is well known [16]

$$m = \frac{\sqrt{s^2\pi^2 + 4\pi^4 j^2 + \pi^4 q^4 + 2q^2\pi^3 s + 4j^2\pi^3 s + 2q^2 s^2 \pi^2 + 2s^3\pi + s^4}}{2\pi^{3/2}\sqrt{s}}. \qquad (24)$$

[7]For determining leading order corrections, it is enough for us to use the equilibrium values of the energy and the area.

Here, we have conveniently rescaled the entropy, mass, charge and angular momentum of the black hole by the AdS radius as

$$s = \frac{S}{l^2}, \qquad m = \frac{M}{l}, \qquad q = \frac{Q}{l}, \qquad j = \frac{J}{l^2}. \tag{25}$$

Equivalently, one could set the AdS radius l to be unity and work with the unscaled parameters.[8]

Let us begin with the Kerr-AdS black hole, obtained by setting $q = 0$ in Eq. (24). In this case, it is convenient to express the angular momentum in terms of the angular velocity[9] as

$$j = \frac{\omega s^{3/2}\sqrt{s+\pi}}{2\pi^{3/2}\sqrt{\pi + s - \omega^2 s}}. \tag{26}$$

In terms of the angular velocity, the temperature can be expressed as

$$t = \frac{\pi^2 + 4s\pi - 2\pi\omega^2 s + 3s^2 - 3s^2\omega^2}{4\pi^{3/2}\sqrt{s(\pi+s)(\pi+s-\omega^2 s)}}. \tag{27}$$

The Massieu transform of the entropy is obtained as

$$\ln Z = \frac{s(s^2 - s^2\omega^2 - \pi^2)}{\pi^2 + 4s\pi - 2\pi\omega^2 s + 3s^2 - 3s^2\omega^2}, \tag{28}$$

where, as usual, the right hand side of Eq. (28) has to be thought of as a function of β and $\lambda = -\beta\omega$, where

$$\beta = \frac{1}{t}, \qquad \lambda = -\frac{4\pi^{3/2}\omega\sqrt{s(s+\pi)[s(1-\omega^2)+\pi]}}{3s^2(1-\omega^2) + 2s\pi(2-\omega^2) + \pi^2} \tag{29}$$

and ω is the angular velocity given by

$$\omega = \frac{2\pi^{3/2} j\sqrt{s+\pi}}{\sqrt{s}\sqrt{s^3 + s^2\pi + 4j^2\pi^3}}. \tag{30}$$

The exact expression for the determinant in Eq. (5) [or equivalently Eq. (8)] is somewhat lengthy and will not be reproduced here. We simply state that in the limit of small angular velocities, we obtain the result

$$\Delta_0 = \frac{s(\pi + 3s)^4}{64\pi^6(3s - \pi)}. \tag{31}$$

Hence, the corrected grand canonical entropy is, in this case, for continuous distributions of the area and angular momentum,

$$s_g = \ln\rho + 2\ln s_{\rm bh}. \tag{32}$$

The relative fluctuations can be calculated for the Kerr-AdS (KAdS) black hole as in the BTZ example, and we find that similar to the latter, $\frac{\langle\Delta m^2\rangle}{m^2}$ and $\frac{\langle\Delta m\Delta j\rangle}{mj}$ goes as s^{-1}, whereas $\frac{\langle\Delta j^2\rangle}{j^2} \sim s^{-1}\omega^{-2}$.

Before moving on, a word about the stability of the system for different values of the black hole parameters. Calculation of the Gibbs free energy in this case shows that for a given value of ω, the free energy becomes negative for $s = \pi(1-\omega^2)^{-(1/2)}$. For $\omega = \frac{1}{2}$, this implies that $s = 3.63$. This signals the Hawking-Page phase transition, with $t_{hp} = 0.297$. On the other hand, the Davies temperature t_d of the black hole (where the specific heat diverges) can be calculated for $\omega = \frac{1}{2}$ to be at $t_d = 0.257$ (for $s = 1.208$). This is the temperature for which Δ_0 diverges as well, via a change of sign. Thus, positive regions of Δ_0 denotes a locally stable black hole, which remains metastable till the Hawking-Page point before becoming globally stable.[10] These results are graphically depicted in Fig. (1), where the blue, green, and red curves are the plots of the temperature, the Gibbs free energy, and Δ_0 for various values of the entropy.

We now calculate the Jacobian factor of Eq. (11). Assuming again that the spectra of the area and the angular momentum are linear in their respective quantum numbers, we obtain

$$K = \frac{4\pi^{3/2}\sqrt{s(\pi+s)(\pi+s(1-\omega^2))}}{3s^2(1-\omega^2) + 2\pi s(2-\omega^2) + \pi^2}. \tag{33}$$

For small values of $\frac{\omega}{s}$, $K \sim s^{-(1/2)}$, and the final form of the corrected entropy in the grand canonical ensemble is

$$s_g = \ln\rho + \frac{3}{2}\ln s_{\rm bh}. \tag{34}$$

A similar analysis can be carried out for RN-AdS black holes, obtained by setting $j = 0$ in Eq. (24). These have been previously studied in the grand canonical ensemble in [18] (see also [19]). We will simply state the main result here. Expressing the charge in terms of the potential as $q = \frac{\phi\sqrt{s}}{\sqrt{\pi}}$, we obtain

$$\lambda = -\frac{4\phi\pi^{3/2}\sqrt{s}}{3s + \pi(1-\phi^2)}. \tag{35}$$

The Massieu transform of the entropy, given by the logarithm of the grand canonical partition function, is

[8]In this section and the next, we will use lower case letters to denote the thermodynamic parameters, with the understanding that these are scaled parameters in the sense just mentioned. This is done to simplify the algebra, and factors of the AdS radius can be included at any stage.

[9]Note that the angular velocity that enters in the formulae is the one measured with respect to a nonrotating frame at infinity [16,17], i.e. it is the difference of the angular velocity at the horizon and that at the boundary of spacetime.

[10]A similar conclusion can be reached by calculating the eigenvalues of the Hessian of the entropy, as in standard thermodynamics.

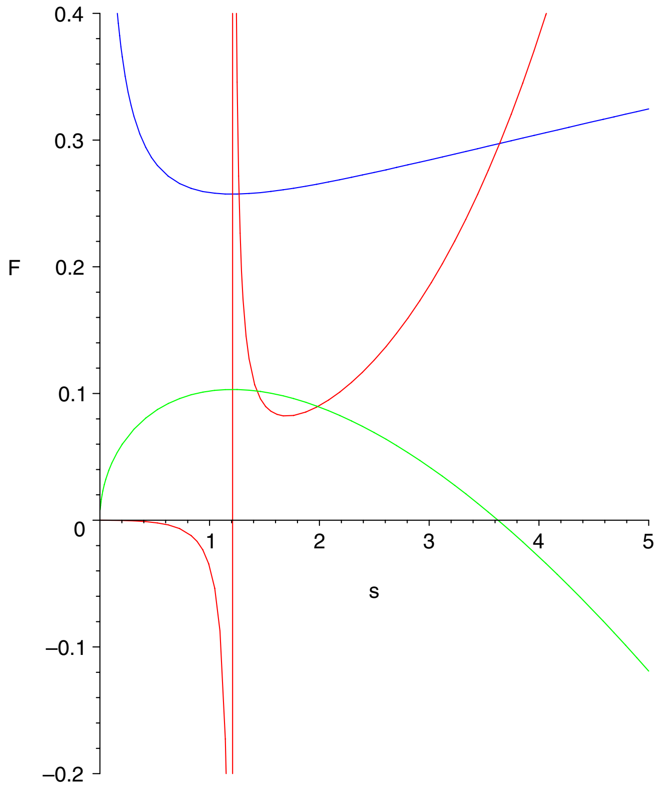

FIG. 1 (color online). Plots of various parameters (denoted by F) of the Kerr-AdS black hole with entropy for $\omega = \frac{1}{2}$. The blue curve denotes the temperature, the green denotes the Gibbs free energy, and the red curve is for Δ_0.

$$\ln Z = \frac{s[s - \pi(1 - \phi^2)]}{3s + \pi(1 - \phi^2)}, \tag{36}$$

where, as usual, the right-hand side (rhs) of Eq. (36) has to be considered as a function of β and λ. Finally, Δ_0 is given by

$$\Delta_0 = \frac{1}{32}\frac{(3s + \pi(1 - \phi^2))^4}{\pi^5(3s - \pi(1 - \phi^2))}. \tag{37}$$

In the limit $s \gg \phi$, the correction to the entropy is

$$s_g = \ln\rho + \frac{3}{2}\ln s_{\mathrm{bh}}. \tag{38}$$

Whereas this is true for the case of continuous energy and charge spectra, inclusion of the Jacobian factor of Eq. (11) modifies the result to

$$s_g = \ln\rho + \ln s_{\mathrm{bh}}, \tag{39}$$

as can be straightforwardly deduced from an analogue of Eq. (18). This is in agreement with the result obtained in [11].

For the RN-AdS black hole in the grand canonical ensemble, one can calculate the relative fluctuations of the mass and charge as before. Here, we find that whereas the relative fluctuation in the mass and the cross correlation term between the mass and the charge goes as s^{-1}, as for rotating black holes, the relative fluctuation $\frac{\langle\Delta q^2\rangle}{q^2} \sim \phi^{-2}$. This is different from the result obtained in [12] and needs to be investigated further.[11]

The entropy of the general KN-AdS black hole in four dimensions in the grand canonical ensemble can be similarly determined, although the formulae are too long to reproduce here. We state the final result that in the limit of small potentials (electric potential or angular velocity), the entropy of the KN-AdS black hole is given by

$$s_g = \ln\rho + \frac{5}{2}\ln s_{\mathrm{bh}} \tag{40}$$

for the case of continuous energy electric charge and angular momentum distributions. Including the Jacobian factor modifies the coefficient $\frac{5}{2}$ to 2. The relative fluctuations of the angular momentum and the charge follow the same behavior as in the KAdS and RN-AdS cases. All the cross correlation terms fall off as s^{-1}. The behavior of the relative charge fluctuation, which again goes as $\frac{1}{\phi^2}$ needs to be studied further.[12]

It is interesting to further study the KN-AdS black hole in the mixed ensembles [8] alluded to in the introduction. These are defined to be ensembles in which one thermodynamic charge (i.e. the electric charge or the angular momentum) and the potential conjugate to the other (the electric potential or the angular velocity) are held fixed. We start with the fixed (electric) charge ensemble. In this case, the angular momentum can be solved in terms of the angular velocity as

$$j = \frac{\omega(\pi s + s^2 + \pi^2 q^2)\sqrt{s}}{2\pi^{3/2}\sqrt{(\pi + s)(\pi + s - \omega^2 s)}}. \tag{41}$$

The Massieu transform of the entropy yields the logarithm of the partition function in the fixed charge ensemble

$$\ln Z = \frac{s(s^4 - s^4\omega^2 + \pi s^3 - \pi\omega^2 s^3 + 3\pi^2\omega^2 s^2 - 4\pi^2 s^2 - 7\pi^3 s + 2\pi^3\omega^2 s - 3\pi^4)}{3s^4 - 3s^4\omega^2 + 7s^3\pi - 5\pi\omega^2 s^3 + 4s^2\pi^2 - \pi^2\omega^2 s^2 - \pi^3 s - \pi^4}. \tag{42}$$

[11] Using $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$ in Eq. (27) of [12], we seem to reach the same conclusion, i.e. the relative fluctuation of the charge $\sim\phi^{-2}$

[12] We postpone further discussions on this to the final section.

The determinant Δ_0 (for the case $q = 1$) is given, in the limit of small ω, by

$$\Delta_{0,q=1} = \frac{(s^2 + s\pi + \pi^2)(3s^2 - s\pi - \pi^2)^4}{64\pi^6 s^3 (s + \pi)(3s^2 - s\pi + 3\pi^2)}, \tag{43}$$

from which it can be seen that the correction to the entropy follows the same equation as the Kerr-AdS black hole of Eq. (34), as expected. The fixed angular momentum case can be similarly studied and the expression for the entropy is found to be the same as in Eq. (39).

We end this section with a couple of comments about the mixed ensembles. It is known that in the fixed charge ensemble, the black hole exhibits phase coexistence of large and small black hole branches and also shows first order phase transitions akin to van der Waals systems (which is also seen in the canonical ensemble of the RN-AdS or Kerr-AdS black holes) [8]. This happens below a critical value of the (fixed) charge, i.e. $q = \frac{1}{6}$.[13] Although we are mainly interested in the large black hole regime, it is nevertheless interesting to explore the behavior of Δ_0 in the regions of phase coexistence. We find that Δ_0 remains positive in the physical regions of an isotherm (where the specific heat c_ω is positive), remains negative in the unphysical region (where the specific heat is negative), changing sign through divergences at the turning points of the isotherm (where the specific heat diverges). For $q > \frac{1}{6}$, no phase coexistence exist (for $\omega < 1$) and Δ_0 is always positive, starting from zero at extremality. A similar analysis can be done for the fixed j ensemble as well.

V. R-CHARGED BLACK HOLES IN VARIOUS DIMENSIONS

Finally, as an illustration of the generality of our method, we briefly address the issue of entropy of string theoretic black holes in dimensions $D = 5$, 7, and 4, corresponding to rotating $D3$, $M5$, and $M2$ branes in the grand canonical ensemble. These are familiar examples of R-charged black holes [20,21]. For example, the $D = 5$ case corresponds to a spinning $D3$-brane configuration in which rotations in planes orthogonal to the brane is characterized by the group $SO(6)$. Upon a Kaluza Klein reduction of the spinning $D3$-brane on S^5, the three independent spins on the $D3$-brane world volume (which are the three independent Cartan generators of $SO(6)$) reduce to three $U(1)$ gauge charges of the corresponding AdS_5 black hole. We will only deal with the compact horizon case here, and our notations will follow [21] (see also [22]). The analysis proceeds entirely in the same manner as that outlined previously. For the single R-charged black hole in $D = 5$ (case 1 of [22]), the mass is given by

$$m = \frac{3}{2}r_+^4 + \frac{3}{2}r_+^2 + \frac{3}{2}r_+^2 a + a \tag{44}$$

and the entropy is

$$s = 2\pi r_+^2\sqrt{r_+^2 + a}, \tag{45}$$

where r_+ denotes the position of the horizon, the charge parameter a is related to the physical charge by

$$q = \sqrt{a(r_+^2 + a)(r_+^2 + 1)}. \tag{46}$$

It is useful to solve for the charge parameter a in terms of the electric potential [22],

$$a = \frac{r_+^2\phi^2}{r_+^2 + (1 - \phi^2)}. \tag{47}$$

Then, the calculation of the Massieu transform of the entropy is standard, and this is given by

$$\ln Z = \frac{\pi r_+^3 (r_+^4 - 1 + \phi^2)}{(2r_+^2 + 1 - \phi^2)\sqrt{r_+^2 + 1}\sqrt{r_+^2 + 1 - \phi^2}}; \tag{48}$$

from this, we can calculate, as before,

$$\Delta_0 = \frac{r_+^2(r_+^2 + 1)^3(2r_+^2 + 1 - \phi^2)^4(3r_+^4 + 6r_+^2 + 3 + \phi^2)}{4\pi^2(r_+^2 + 1 - \phi^2)^3[2r_+^6 + 3r_+^4(1 - \phi^2) - 1 + 2\phi^2 - \phi^4]}. \tag{49}$$

In the grand canonical ensemble, where ϕ is fixed to a small value compared to the horizon radius, $s \sim r_+^3$, and $\Delta_0 \sim r_+^8$, and hence

$$s_g = \ln\rho + \ln s_{\rm bh}, \tag{50}$$

where we have used the fact that the Jacobian of Eq. (8) goes, in this case, as $K \sim s^{-(1/3)}$. For single R-charged black holes in $D = 4$ and 7, we find that the correction to the entropy follows the same rule as in Eq. (50). Note further that this is the same as Eq. (39). This observation is indicative of the fact that charged AdS black holes in any dimension might have a universal logarithmic correction to the entropy in the grand canonical ensemble, with unit coefficient. The evidence presented in this paper towards this is, however, not exhaustive, and it would be interesting to study this issue further, to see if it can be conclusively established. For rotating AdS black holes, our results indicate that there is possibly no universal nature of such entropy corrections.

[13] The reader is referred to Fig. (31) of [8] for a quick reference.

VI. DISCUSSIONS AND CONCLUSIONS

In this paper, we have studied the effect of thermal fluctuations on the entropy for a wide class of AdS black holes in the grand canonical ensemble and a couple of mixed ensembles. We have argued that the Smarr formula can be effectively used to calculate corrections to the Beckenstein-Hawking area law in these ensembles, by using standard tools of thermodynamics, and are expressible entirely in terms of the black hole response coefficients. Further, we have derived a generalized stability condition for black holes in the grand canonical ensemble, where all the black hole extensive parameters are allowed to fluctuate. The generalized Massieu transform of the entropy has also been used to calculate moments of charge, mass and angular momentum fluctuations. Our results highlight the difference between charge and angular momentum fluctuations. Such differences are also seen in the study of the state space scalar curvature of thermodynamic geometries [8]. As mentioned in the text, the case of relative charge fluctuations in the grand canonical ensemble needs to be investigated further. For the RN-AdS as well as the KN-AdS black hole, we find that this is proportional to the inverse square of the potential.

For R-charged black holes, a similar analysis can be done. Here, we find that $\frac{\langle\Delta m^2\rangle}{m^2}$ and $\frac{\langle\Delta m\Delta q\rangle}{mq} \sim s^{-1}$ in $D = 5$, 7. and 4. Interestingly, whereas for $D = 5$ and 7 the relative fluctuation $\frac{\langle\Delta q^2\rangle}{q^2}$ falls off as a fractional power of s, in $D = 4$, we find $\frac{\langle\Delta q^2\rangle}{q^2} \sim \frac{1}{\phi^2}$, similar to the RN-AdS and KN-AdS cases. This merits further discussion which we leave for a future work.

In this work, we have ignored the quantum corrections to the black hole entropy [23] as appears in the microcanonical ensemble. It should be possible, however, to incorporate these in our analysis, at least in a class of examples. It would also be interesting to understand the role of thermal fluctuations for multiply charged string theoretic black holes in the canonical ensemble, especially since the latter exhibit rich phase structure and liquid gas like first order phase transitions. Finally, the role of thermal fluctuations in string theoretic black holes should lead to interesting results in the dual gauge theory side initially studied in [24]. This is left for a future investigation.

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