

Effective field theory calculation of conservative binary dynamics at third post-Newtonian order

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We reproduce the two-body gravitational conservative dynamics at third post-Newtonian order for spinless sources by using the effective field theory methods for the gravitationally bound two-body system, proposed by Goldberger and Rothstein. This result has been obtained by automatizing the computation of Feynman amplitudes within a MATHEMATICA algorithm, paving the way for higher-order computations not yet performed by traditional methods.

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I. INTRODUCTION

The problem of finding the equations of motion of a two-body system subject to gravitational interaction has been intensively studied since the advent of general relativity. Because no exact solution is known, different frameworks and approximations have been developed out in the past: the present work lies within the post-Newtonian (PN) approximation of general relativity, see e.g. [1,2] for reviews and the current situation of this approach.

In particular the computation of the effective two-body dynamics within the PN approximation has been the subject of intensive research in the last three decades, both in the conservative and in the dissipative sector. At present the Arnowitt-Deser-Misner Hamiltonian ruling the conservative dynamics of a gravitationally bound, spinless binary system, has been computed up to third PN order in [3], whose calculations have been later finalized and confirmed by [4–6] in harmonic coordinates, and also by [7], where a different resolution of the source singularity has been adopted.

From the 2.5PN order onwards, the dynamics of binary systems is modified by dissipative effects [8], associated with gravitational radiation. At present the energy flux emitted by a spinless binary system on quasicircular orbits has been computed up to 3.5PN order with respect to the leading order contribution [9].

The main result of this paper is the recomputation of the two-body effective action for the conservative dynamics at 3PN order for spinless objects via an *automatized algorithm*, making use of a different method than those implemented so far to obtain such a result: we performed the computation within the framework of the effective field theory methods (EFT) for nonrelativistic particles introduced in [10]. This method has already been applied with

success on the conservative sector of the theory to determine the two-body 1PN [10] and 2PN [11] Hamiltonian (see also [12] for some investigations beyond 2PN), as well as to the study of the n-body 2PN [13] Hamiltonian (for 3-body systems, first obtained in [14]). On the radiative side, [15] showed an application of EFT methods to the calculation of some terms of the gravitational wave energy flux for spinless systems up to 3PN order, and in [16] it is shown how present gravitational wave observations can set bounds on the fundamental parameters of the theory. The EFT approach has lead not only to rederivation of old and established results, but also contributed to the computation of new spin effects in the 3PN conservative [17] (also obtained in [18] with more traditional methods) and non-conservative dynamics [19] (first obtained in [20]).

An EFT method possesses several features making it suitable for computing physical quantities in problems involving well-separated scales. This is the case for gravitationally bound two-body systems, where the different scales can be identified in the size of the compact objects r_s , the orbital separation r and the gravitational wavelength λ with hierarchy $r_s < r \sim r_s/v^2 < \lambda \sim r/v$, being v the relative velocity between the two bodies (we posit the speed of light $c = 1$). We neglect here radiation effects and consider pointlike sources, comforted by the effacement principle [21], which guarantees that finite size-effects come into play from 5PN order, for spinless objects. The power-counting scheme outlined in [10] enables to clearly separate scales and to frame divergences arising from zero-size sources within a well under control field theory setting. Moreover EFT methods make systematic use of Feynman diagrams and exhibits manifest power-counting rules, enabling to implement PN computations within an algorithm automatizing the intricate computations involving hundreds of Feynman diagrams. Such an algorithm is completely general and it can be generalized to the calculation of higher-order dynamics.

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A key simplification enabled by EFT methods with respect to the standard ones is the use of perturbative techniques to integrate out the degrees of freedom responsible of mediating the conservative gravitational interaction (the so-called *potential gravitons* in the language of [10]) to directly obtain the 2-body action, without the need to solve for the metric.

This field of studies is nowadays of great phenomenological impact as the large interferometers LIGO and Virgo have been operating until recently (fall 2010) at unprecedented sensitivity and a further increase in sensitivity is scheduled for their *advanced* runs, due to start in the year 2014, see e.g. [22] for a recent publication of coalescing binary signal search. As reported in [23], reasonable astrophysical estimates make plausible the detection of gravitational wave signals from coalescing binaries by the advanced interferometers.

In the search for signals from coalescing binaries the detector output is processed via standard match-filtering methods [24], where the experimental data are confronted against banks of *template* waveforms. The result is particularly sensitive to the time varying phase of the oscillating gravitational wave signal [25]. In order to compute such a phase with $O(1)$ accuracy it is necessary to take into account PN corrections to both the energy and the emitted energy flux up to 3PN order at least.

The frequency region accessible to the advanced versions of LIGO/Virgo (roughly from 10 Hz to 10 kHz) corresponds to the last stages (whose duration is at most few tens of minutes) of the coalescence of astrophysical objects like black holes and neutron stars and by then the orbit will have circularized, as eccentricity decay faster than orbital separation [26], enabling the energy and flux functions to depend on the single parameter v .

The inspiral phase of the coalescence is bound to end when the two constituents of the binary system merge into a single object and the perturbative PN analysis cannot be followed beyond the merger. Here however numerical techniques have been proven useful, see e.g. [27] for reviews, to construct complete analytical waveforms from inspiral to merger and the subsequent *ring down*, as it is conventionally named the last phase in which the final object produced by the merger undergoes damped oscillation before settling down to quietness. The ring-down phase also admits an analytical perturbative description [28].

The present work represents a decisive step towards the computation of the Hamiltonian dynamics at fourth PN order, which is a key ingredient in the construction of template waveforms to be used in actual experimental searches, like the effective one body waveforms [29] which have been used in the recent experimental search published in [30].

The paper is organized as follows. In Sec. II we give an overview of the effective field theory methods for gravity, which are applied in Sec. III to the determination of the

2-body effective action at order 3PN. We summarize and conclude in Sec. IV.

II. MAIN

A. Lagrangian and Feynman rules

The relevant physical scales for the 2-body problem in gravity are the total mass M , the separation r and the relative velocity v . By nPN correction it is conventionally meant corrections of order $v^{2n} \sim (G_N M/r)^n$, where G_N is the standard Newton constant and the virial relation has been applied, showing that an expansion in v^2 is at the same time an expansion in the strength of the gravitational field.

At 3PN order, the EFT description of massive compact objects in binary systems takes them as nondynamical, background pointlike sources: quantitatively this corresponds to particle worldlines interacting with gravitons. The action S we consider is then given by

$$S = S_{\text{EH}} + S_{\text{GF}} + S_{pp}, \quad (1)$$

where the first and the third terms are, respectively, the d -dimensional Einstein-Hilbert action and the worldline point particle action

$$\begin{aligned} S_{\text{EH}} &= 2\Lambda^2 \int d^{d+1}x \sqrt{-g} R(g), \\ S_{pp} &= - \sum_{i=1,2} m_i \int d\tau_i = - \sum_{i=1,2} m_i \int \sqrt{-g_{\mu\nu}(x_i^\mu)} dx_i^\mu dx_i^\nu, \end{aligned} \quad (2)$$

with $\Lambda^{-2} \equiv 32\pi G$, being G the d -dimensional gravitational constant [31]: as dimensional regularization will be needed, it is necessary to keep d generic until the actual value of Feynman integrals will have been computed. The goal is to compute the effective action S_{eff} for particles alone, with gravitons mediating interactions being integrated out by standard perturbative methods, i.e. by the aid of Feynman diagrams. For the gauge fixing term, we follow [11]:

$$S_{\text{GF}} = -\Lambda^2 \int d^{d+1}x \sqrt{-g} \Gamma_\mu \Gamma^\mu \quad (3)$$

with $\Gamma^\mu \equiv \Gamma_{\alpha\beta}^\mu g^{\alpha\beta}$, corresponding to the same harmonic gauge adopted in [1]. Still following [11], we adopt the standard Kaluza-Klein (KK) parametrization of the metric (first applied to this framework in [32]), suitable for a nonrelativistic expansion around a Minkowski metric:

$$g_{\mu\nu} = e^{2\phi/\Lambda} \begin{pmatrix} -1 & A_j/\Lambda \\ A_i/\Lambda & e^{-c_d \phi/\Lambda} \gamma_{ij} - A_i A_j / \Lambda^2 \end{pmatrix}, \quad (4)$$

with $\gamma_{ij} = \delta_{ij} + \sigma_{ij}/\Lambda$, $c_d = 2 \frac{(d-1)}{(d-2)}$ and i, j running over the d spatial dimensions.

In terms of the metric parametrization (4), each worldline coupling to the gravitational degrees of freedom ϕ, A_i, σ_{ij} reads

$$S_{pp} = -m \int d\tau = -m \int dt e^{\phi/\Lambda} \sqrt{\left(1 - \frac{A_i}{\Lambda} v^i\right)^2 - e^{-c_d \phi/\Lambda} \left(v^2 + \frac{\sigma_{ij}}{\Lambda} v^i v^j\right)}, \quad (5)$$

and its Taylor expansion provides the various particle-gravity vertices of the EFT.

Also the pure gravity sector $S_{\text{bulk}} = S_{\text{EH}} + S_{\text{GF}}$ can be explicitly written in terms of the KK variables, S_{EH} in KK variable has been first derived in [33]. We report here only a part of S_{bulk} , including all the terms that are needed for the 3PN calculation:

$$\begin{aligned} S_{\text{bulk}} \supset \Lambda^2 \int d^{d+1}x \sqrt{-\gamma} \left\{ \frac{1}{4} [(\nabla\sigma)^2 - 2(\nabla\sigma_{ij})^2 - (\dot{\sigma}^2 - 2(\dot{\sigma}_{ij})^2)e^{-c_d \phi/\Lambda}] - c_d [(\nabla\phi)^2 - \dot{\phi}^2 e^{-(c_d \phi/\Lambda)}] \right. \\ \left. + \left[\frac{F_{ij}^2}{2} + (\nabla \cdot \mathbf{A})^2 - \dot{\mathbf{A}}^2 e^{-(c_d \phi/\Lambda)} \right] e^{c_d \phi/\Lambda} + 2c_d (\dot{\phi} \nabla \cdot \mathbf{A} - \dot{\mathbf{A}} \cdot \nabla \phi) \right. \\ \left. + \frac{1}{\Lambda} [2(F_{ij} A^i \dot{A}^j + \mathbf{A} \cdot \dot{\mathbf{A}} (\nabla \cdot \mathbf{A})) e^{c_d \phi/\Lambda} - c_d \dot{\phi} [2\mathbf{A} \cdot \nabla \phi + \dot{\phi} A^2]] \right\}, \quad (6) \end{aligned}$$

where $F_{ij} \equiv A_{j,i} - A_{i,j}$ and indices are raised and contracted by means the d -dimensional metric tensor γ (although in most terms one can just use δ_{ij} because the neglected parts are not needed at 3PN) [34]. All spatial derivatives are meant to be simple (not covariant) [35] and, when ambiguities might raise, gradients are always meant to act on contravariant fields (so that, for instance, $\nabla \cdot \mathbf{A} \equiv \gamma^{ij} A_{i,j}$ and $F_{ij}^2 \equiv \gamma^{ik} \gamma^{jl} F_{ij} F_{kl}$).

All the Feynman rules needed for our calculation can be read off Eq. (6). In particular: the first line gives the σ and ϕ propagators, as well as all the $\phi^n \sigma^m$ interaction vertices; the second line gives the \mathbf{A} propagator and, together with the third, contains enough information to compute all the other vertices needed for the 3PN calculation.

Borrowing the definitions from [10], the two bodies exchange *potential* gravitons, responsible for binding the system as they mediate instantaneous interactions: their characteristic four-momentum k_μ scales thus as $k_\mu \sim (k^0 \sim v/r, \mathbf{k} \sim r^{-1})$ and we neglect altogether the emission of gravitational waves because we are interested in the conservative part of the binary system dynamics. When a compact object emits a single graviton, momentum is

actually *not* conserved and the nonrelativistic particle recoils of a fractional amount roughly given by $|\delta p|/|p| \simeq |k|/|p| \simeq \hbar/J$, where $J \sim mvr$ is the angular momentum of the system: however, for macroscopic systems such a quantity is negligibly small.

In order to allow manifest scaling it is necessary to work with the space-Fourier transformed fields

$$\begin{aligned} \phi_{\mathbf{k}}(t) &\equiv \int d^d x \phi(t, \mathbf{x}) e^{-i\mathbf{k}\mathbf{x}}, \\ A_{i,\mathbf{k}}(t) &\equiv \int d^d x A_i(t, \mathbf{x}) e^{-i\mathbf{k}\mathbf{x}}, \\ \sigma_{ij,\mathbf{k}}(t) &\equiv \int d^d x \sigma_{ij}(t, \mathbf{x}) e^{-i\mathbf{k}\mathbf{x}}. \end{aligned} \quad (7)$$

The fields defined above are the fundamental variables in terms of which the Feynman graphs are going to be constructed; the action governing their dynamics can be found from Eqs. (5) and (6) by direct substitution and replacement of spatial derivatives by the appropriate $i\mathbf{k}$ factors.

From the quadratic part of the Lagrangian the propagators can be written as

$$\left. \begin{aligned} P[\phi_{\mathbf{k}}(t_a) \phi_{\mathbf{k}'}(t_b)] &= -\frac{1}{2c_d} \\ P[A_{i,\mathbf{k}}(t_a) A_{j,\mathbf{k}'}(t_b)] &= \frac{1}{2} \delta_{ij} \\ P[\sigma_{ij,\mathbf{k}}(t_a) \sigma_{kl,\mathbf{k}'}(t_b)] &= \frac{1}{2} P_{ij,kl} \end{aligned} \right\} \times (2\pi)^d \delta^d(\mathbf{k} + \mathbf{k}') \mathcal{P}(\mathbf{k}^2, t_a, t_b) \delta(t_a - t_b), \quad (8)$$

where $P_{ij,kl} \equiv -[\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} + (2 - c_d) \delta_{ij} \delta_{kl}]$ and

$$\begin{aligned} \mathcal{P}(\mathbf{k}^2, t_a, t_b) &= \frac{i}{\mathbf{k}^2 - \partial_{t_a} \partial_{t_b}} \\ &\simeq \frac{i}{\mathbf{k}^2} \left(1 + \frac{\partial_{t_a} \partial_{t_b}}{\mathbf{k}^2} + \frac{\partial_{t_a}^2 \partial_{t_b}^2}{\mathbf{k}^4} + \dots \right) \quad (9) \end{aligned}$$

is the full relativistic propagator, which can be thought as an instantaneous nonrelativistic part plus insertion terms involving time derivatives. Only few of these time derivative terms need to be included in a Feynman diagram at a given PN order.

B. Feynman diagrams

We have exposed all the ingredients needed to obtain the 2-body effective action S_{eff} with manifest power counting in G and v at the third post-Newtonian order. This can be done by integrating out the graviton fields from the full action derived above

$$S_{\text{eff}} = \int D\phi D\sigma_{ij} DA_k \exp[i(S_{\text{EH}} + S_{\text{GF}} + S_{pp})]. \quad (10)$$

We can now proceed to lay down all the Feynman diagrams relevant for our computation, then by applying the

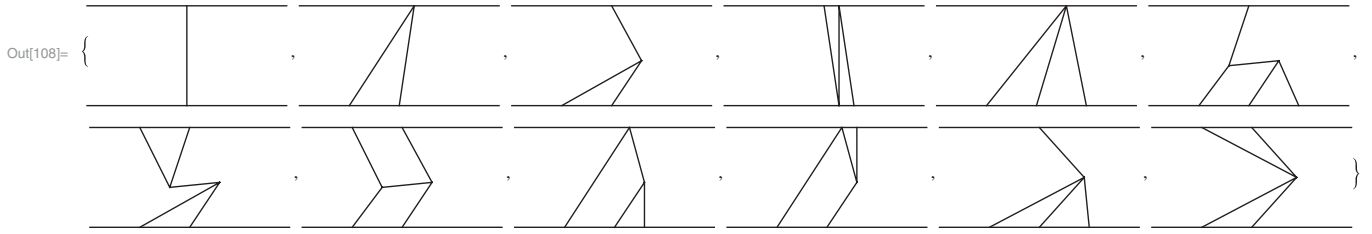


FIG. 1. The 12 topologies up to G^3 as they are produced by a Mathematica code. The last 4 start contributing at 3PN order. Here and in the following figures, we count them from left to right, from top to bottom.

Feynman rules derived from the above interactions the amplitudes will be computed, collecting all results belonging to the same PN order.

1. Topologies

Following the procedure and the terminology of [11], we first select the topologies having the appropriate power of G for being potentially relevant from a given PN order on, that is $G^{(n+1)}$ for n th-PN order. The attribution of a power of G to a topology is made according to the following rule: each vertex involving n gravitational fields gives a factor $G^{n/2-1}$ if it is a bulk one, and a factor $G^{n/2}$ if it is attached to an external particle.

Concretely, there is just one topology contributing at the Newtonian order G : the one representing the single graviton exchange. At $\mathcal{O}(G^2)$ order two other topologies (modulo switch of the external particles) enter the game but actually

just one of these contributes at 1PN since the other involves an extra v^2 factor, which is equivalent to increasing by one unity the PN order. As underlined in [11], this is a very welcome effect of using the KK parametrization, also with respect to the Arnowitt-Deser-Misner one, see [36].

At 2PN order, 5 new G^3 topologies have to be added, while 4 additional ones come into play only from 3PN on because of v^2 suppressions. All these 12 topologies contributing to G^3 order have been already displayed in [11], and they are also reported here in Fig. 1.

To complete the 3PN order the introduction of G^4 topologies is required. The generation of the new topologies at any given order n is done iteratively from the ones at the previous order $n - 1$: to any G^{n-1} topology a new propagator is attached in all possible ways, provided that one extremum ends on an external particle, and the second extremum is placed in one of the following three locations: (i) an internal vertex, (ii) a vertex attached to the other

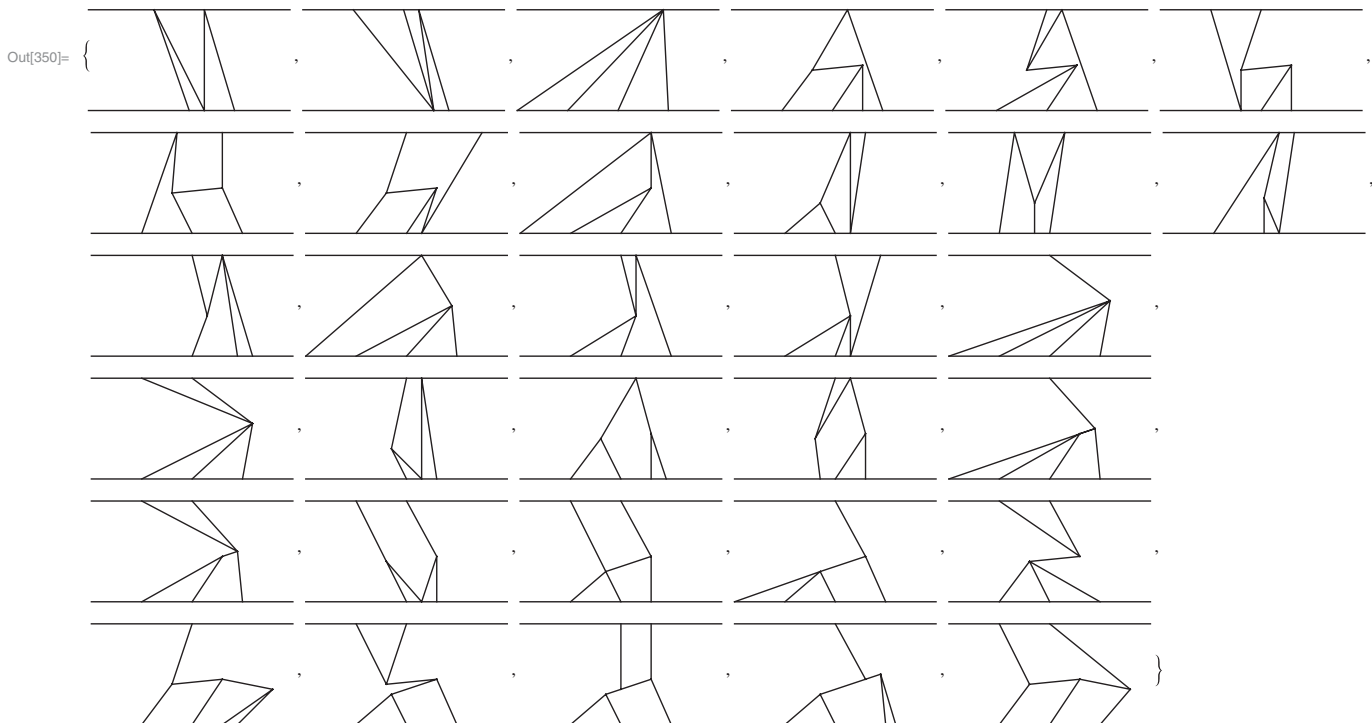


FIG. 2. The 32 G^4 topologies. Only the first 8 contribute at 3PN order, and they are all factorizable into simpler subtopologies.

TABLE I. On the top: number of topologies entering at a given PN order. On the bottom: number of diagrams that start to contribute to the effective action at a given G and v power. The 63 diagrams whose contribution starts at 3PN are written in boldface characters.

	0PN	1PN	2PN	3PN	4PN
G	1				
G^2		1	1		
G^3			5	4	
G^4				8	21
G^5					50

	v^0	v^2	v^4	v^6
G	1	1	1	
G^2	1	8	7	7
G^3	5	48	159	...
G^4	8	299	...	
G^4	50	...		

external particle, (iii) a propagator, so to “break” it into two propagators ending into the same newly-born 3-legged internal vertex; barring in all cases the possibility that internal graviton loops are created, as they would give rise to uninteresting quantum corrections (i.e. proportional to $\hbar/J \ll 1$). In this way one actually obtains a redundant set of G^n topologies because many of them are equivalent, i.e. they are skeletons of the same Feynman graphs, then the algorithm to eliminate equivalent graphs can be straightforwardly implemented on a computer program.

We have indeed implemented the whole procedure into a MATHEMATICA code [37] and 32 G^4 topologies have been found, only 8 of which turned out to be relevant at the 3PN level of accuracy, as the remaining 24 contribute at higher PN orders because of v^2 suppressions. Moreover, as Fig. 2 shows, none of the 8 3PN-relevant topologies is really new, since they are actually factorizable in lower order subtopologies. We have also worked out, for the sake of future developments, the situation at 4PN and summarized the number of topologies in the top part of Table I.

That makes a total of 20 topologies that must be taken into account for the 3PN calculation, and only 2 of them (the two G^3 ones involving 4-legged vertices, the last two ones of Fig. 1) are genuinely new ones, i.e. nonfactorizable into subtopologies.

2. Graphs

The next step is to generate Feynman graphs from all topologies by replacing all the not-yet-specified propagators by the proper ϕ , A or σ ones. This is the point where powers of v (traced by time derivatives in the Lagrangian) enter the game, both in external and bulk vertices and in the relativistic propagators. The lowest power of v^2 appearing in a graph, together with its G order, determines the lowest PN order at which such a graph must be taken into account. As different vertices carry different time derivative structures, see for instance Eq. (6), topologies of the same G order, or even the same topology, can generate Feynman graphs whose lowest PN orders differ. In particular some topologies allow only graphs containing v^2 corrections at least, thus entering the game at higher PN order than one would have inferred by a naive G power counting; this is the origin of the above mentioned simplification introduced by the use of KK variables, which reduces the number of topologies (and thus of graphs) to be taken into account at any given PN order. This welcome feature becomes more and more important at higher PN orders.

The generation and classification of all the Feynman graphs needed at a given PN level is an automatizable procedure, as it suffices to try all possibilities (although one can envisage shortcuts towards more efficient algorithms) and find out the lowest power of v^2 of each graph using the Feynman rules implicit in the Lagrangians (5) and (6). Our MATHEMATICA code has thus generated all the 80 diagrams giving a contribution at 3PN (as well as other 515 for the 4PN case): 63 of them are new ones, while the other 17 have been computed at 2PN order by Gilmore and Ross [11,38]. The bottom part of Table I shows the classification of the new diagrams according to their G and lowest v^2 powers.

III. THE RESULTS

We use a MATHEMATICA [37] code together with the EINS [39] and FEYNALC [40] software to generate and to compute all the diagrams. Here we show the results concerning the 3PN part only [41].

A. $\mathcal{O}(G)$ diagrams

They are shown in Fig. 3 and they can all be done by means of the dimensionally regularized Fourier formula

$$\text{Out}_{[210]} = \left\{ \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \left[\begin{array}{c} | \\ \text{---} \\ | \end{array} \right] \frac{1}{2}, \quad \left[\begin{array}{c} | \\ \text{---} \\ | \end{array} \right] \frac{1}{2}, \quad \left[\begin{array}{c} | \\ \text{---} \\ | \end{array} \right] \frac{1}{2} \right\}$$

FIG. 3 (color online). The three $\mathcal{O}(G)$ diagrams. The ϕ , A and σ propagators are represented, respectively, by blue dashed, red dotted and green solid lines. From the left to the right, they contain v^6 , v^4 and v^2 corrections under the form of vertices expansion and/or double time derivative insertions in the propagators. The number in the right part of each diagram is its multiplicity factor.

$$\int_{\mathbf{p}} \frac{e^{-i\mathbf{p}\mathbf{r}}}{p^{2\alpha}} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(d/2 - \alpha)}{\Gamma(\alpha)} \left(\frac{2}{r}\right)^{d-2\alpha} \quad (11)$$

(with $\int_{\mathbf{p}} \equiv \int \frac{d^d p}{(2\pi)^d}$ and $\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2$), as well as its generalization to the case where factors of p^μ 's appear in the integrand; the corresponding formulae can be obtained by taking the appropriate number of \mathbf{r} gradients of Eq. (11).

The result is then multiplied by a symmetry factor accounting for:

- (i) factors coming from the development of the exponential function contained in (10);
- (ii) a factor 1/2 for diagrams symmetric under particles exchange, in order not to overcount when

performing symmetrization (i.e. $+1 \leftrightarrow 2$) in the very final stage of the calculation;

- (iii) the number of possible Wick contractions giving rise to the same diagram, with the convention that equivalent legs attached to an *internal* vertex are indistinguishable (because we chose to incorporate the corresponding factors already in the internal vertices definitions).

The results of the integration have been evaluated at $d = 3$ and multiplied by the imaginary unit i to give the following potential terms for the 3PN effective Lagrangian (G_b^a being the value of the b -th G^a diagram, listed as in Figs. 3–6):

$$\begin{aligned} G_1^1 = & -\frac{G_N m_1 m_2}{32r} \left\{ \frac{\dot{\mathbf{a}}_1 \cdot \dot{\mathbf{a}}_2 + 3\mathbf{n} \cdot \dot{\mathbf{a}}_1 \mathbf{n} \cdot \dot{\mathbf{a}}_2}{9} r^4 + [\mathbf{a}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{a}_2 - 2\mathbf{a}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{a}_1 - \mathbf{a}_2 \cdot \dot{\mathbf{a}}_1 \mathbf{r} \cdot \mathbf{v}_1 - \mathbf{a}_2 \cdot \dot{\mathbf{a}}_1 \mathbf{r} \cdot \mathbf{v}_2 \right. \\ & - 2\mathbf{r} \cdot \dot{\mathbf{a}}_1 \mathbf{v}_1 \cdot \mathbf{a}_2 - 18\mathbf{v}_1 \cdot \mathbf{a}_1 \mathbf{v}_1 \cdot \mathbf{a}_2 - 2\mathbf{a}_1 \cdot \mathbf{a}_2 v_1^2 + 9a_2^2 v_1^2 - 2\mathbf{a}_1 \cdot \mathbf{a}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 - \mathbf{v}_1 \cdot \mathbf{a}_2 \mathbf{v}_2 \cdot \mathbf{a}_1 - 14\mathbf{r} \cdot \dot{\mathbf{a}}_1 \mathbf{v}_2 \cdot \mathbf{a}_2 \\ & - 111\mathbf{v}_1 \cdot \mathbf{a}_1 \mathbf{v}_2 \cdot \mathbf{a}_2 - 9a_1^2 v_2^2 - 7\mathbf{a}_1 \cdot \mathbf{a}_2 v_2^2] r^2 + 22v_1^6 + 2(\mathbf{v}_1 \cdot \mathbf{v}_2)^3 - 2\mathbf{a}_1 \cdot \mathbf{a}_2 (\mathbf{r} \cdot \mathbf{v}_1)^2 - \mathbf{a}_1 \cdot \mathbf{a}_2 (\mathbf{r} \cdot \mathbf{v}_2)^2 \\ & + 16v_1^2 (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - (\mathbf{r} \cdot \mathbf{a}_1)^2 \mathbf{r} \cdot \mathbf{a}_2 - \mathbf{r} \cdot \mathbf{a}_2 \mathbf{r} \cdot \dot{\mathbf{a}}_1 \mathbf{r} \cdot \mathbf{v}_1 - \mathbf{r} \cdot \mathbf{a}_2 \mathbf{r} \cdot \dot{\mathbf{a}}_1 \mathbf{r} \cdot \mathbf{v}_2 - 2\mathbf{a}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{v}_1 \mathbf{r} \cdot \mathbf{v}_2 \\ & - 7\mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_1 \mathbf{v}_1 \cdot \mathbf{a}_2 - 3\mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{a}_2 - 3(\mathbf{r} \cdot \mathbf{a}_2)^2 v_1^2 - 2\mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{a}_2 v_1^2 + 88\mathbf{r} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{a}_1 v_1^2 \\ & + 22v_1^4 \mathbf{v}_1 \cdot \mathbf{v}_2 - 2\mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{a}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 32\mathbf{r} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{a}_1 \mathbf{v}_1 \cdot \mathbf{v}_2 + 16\mathbf{r} \cdot \mathbf{v}_2 v_1^2 \mathbf{v}_2 \cdot \mathbf{a}_1 + 8\mathbf{r} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 \mathbf{v}_2 \cdot \mathbf{a}_1 \\ & - 41\mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_1 \mathbf{v}_2 \cdot \mathbf{a}_2 - 17\mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2 \mathbf{v}_2 \cdot \mathbf{a}_2 + 3(\mathbf{r} \cdot \mathbf{a}_1)^2 v_2^2 + 34v_1^4 v_2^2 - 7\mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{a}_2 v_2^2 + 124\mathbf{r} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{a}_1 v_2^2 \\ & + 23v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 v_2^2 + 20\mathbf{r} \cdot \mathbf{v}_2 \mathbf{v}_2 \cdot \mathbf{a}_1 v_2^2 + 2\mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{a}_2 (\mathbf{n} \cdot \mathbf{v}_1)^2 + \mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{a}_2 (\mathbf{n} \cdot \mathbf{v}_2)^2 + 8(\mathbf{n} \cdot \mathbf{v}_2)^2 v_1^4 \\ & - 22\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 v_1^4 - 6\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + 2\mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{a}_2 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 4\mathbf{r} \cdot \mathbf{a}_1 (\mathbf{n} \cdot \mathbf{v}_2)^2 \mathbf{v}_1 \cdot \mathbf{v}_2 \\ & + 12(\mathbf{n} \cdot \mathbf{v}_2)^2 v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 16\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 8(\mathbf{n} \cdot \mathbf{v}_2)^3 \mathbf{v}_2 \cdot \mathbf{a}_1 r/3 - 4\mathbf{r} \cdot \mathbf{v}_1 (\mathbf{n} \cdot \mathbf{v}_2)^2 \mathbf{v}_2 \cdot \mathbf{a}_1 \\ & - 20\mathbf{r} \cdot \mathbf{a}_1 (\mathbf{n} \cdot \mathbf{v}_1)^2 v_2^2 - 10\mathbf{r} \cdot \mathbf{a}_1 (\mathbf{n} \cdot \mathbf{v}_2)^2 v_2^2 - 16\mathbf{r} \cdot \mathbf{a}_1 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 v_2^2 - 20(\mathbf{n} \cdot \mathbf{v}_1)^2 v_1^2 v_2^2 \\ & - 39\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 v_1^2 v_2^2 + 2\mathbf{r} \cdot \mathbf{a}_1 (\mathbf{n} \cdot \mathbf{v}_2)^4 + 4\mathbf{r} \cdot \mathbf{a}_1 \mathbf{n} \cdot \mathbf{v}_1 (\mathbf{n} \cdot \mathbf{v}_2)^3 + 20(\mathbf{n} \cdot \mathbf{v}_2)^4 v_1^2 + 4\mathbf{n} \cdot \mathbf{v}_1 (\mathbf{n} \cdot \mathbf{v}_2)^3 v_1^2 \\ & \left. + 9(\mathbf{n} \cdot \mathbf{v}_1)^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 5(\mathbf{n} \cdot \mathbf{v}_1)^3 (\mathbf{n} \cdot \mathbf{v}_2)^3 \right\}, \\ G_2^1 = & \frac{G_N m_1 m_2}{4r} \left\{ r^4 \frac{\dot{\mathbf{a}}_1 \cdot \dot{\mathbf{a}}_2}{3} + [\mathbf{a}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{a}_2 - 2\mathbf{a}_2 \cdot \dot{\mathbf{a}}_1 \mathbf{r} \cdot \mathbf{v}_1 - 2\mathbf{v}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \dot{\mathbf{a}}_1 - 2\mathbf{a}_2 \cdot \dot{\mathbf{a}}_1 \mathbf{r} \cdot \mathbf{v}_2 \right. \\ & - 3\mathbf{a}_1 \cdot \mathbf{a}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 2\mathbf{v}_2 \cdot \mathbf{a}_1 \mathbf{v}_1 \cdot \mathbf{a}_2 - 14\mathbf{v}_1 \cdot \mathbf{a}_1 \mathbf{v}_1 \cdot \mathbf{a}_2 - 3v_1^2 \mathbf{a}_1 \cdot \mathbf{a}_2 - 5v_2^2 \mathbf{a}_1 \cdot \mathbf{a}_2 - 3\mathbf{a}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{a}_1] r^2 \\ & - 2\mathbf{a}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{v}_1 \mathbf{r} \cdot \mathbf{v}_2 - 2\mathbf{r} \cdot \mathbf{a}_1 \mathbf{v}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{v}_2 - \mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{a}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 3\mathbf{a}_1 \cdot \mathbf{a}_2 (\mathbf{r} \cdot \mathbf{v}_1)^2 - 6\mathbf{r} \cdot \mathbf{a}_1 \mathbf{v}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{v}_1 \\ & - \mathbf{a}_1 \cdot \mathbf{a}_2 (\mathbf{r} \cdot \mathbf{v}_2)^2 + 12\mathbf{v}_1 \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 6v_1^2 \mathbf{v}_2 \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2 + 8\mathbf{v}_2 \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 10v_2^2 \mathbf{v}_2 \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2 \\ & - 2\mathbf{r} \cdot \mathbf{a}_1 (\mathbf{n} \cdot \mathbf{v}_2)^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 2\mathbf{v}_2 \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_1 (\mathbf{n} \cdot \mathbf{v}_2)^2 - 2r\mathbf{v}_2 \cdot \mathbf{a}_1 (\mathbf{n} \cdot \mathbf{v}_2)^3 + 2(\mathbf{v}_1 \cdot \mathbf{v}_2)^3 \\ & \left. + 6v_1^2 (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + 6v_1^4 \mathbf{v}_1 \cdot \mathbf{v}_2 + v_1^2 v_2^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 4\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - 6v_1^2 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 3(\mathbf{n} \cdot \mathbf{v}_1)^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 \mathbf{v}_1 \cdot \mathbf{v}_2 \right\}, \\ G_3^1 = & \frac{G_N m_1 m_2}{2r} \{ [2\mathbf{a}_1 \cdot \mathbf{a}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 2\mathbf{v}_2 \cdot \mathbf{a}_1 \mathbf{v}_1 \cdot \mathbf{a}_2 - 4\mathbf{v}_1 \cdot \mathbf{a}_1 \mathbf{v}_2 \cdot \mathbf{a}_2] r^2 \\ & - 4\mathbf{v}_2 \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 4v_2^2 \mathbf{v}_1 \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^3 - 2v_1^2 (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + 2v_1^4 v_2^2 \\ & + v_1^2 v_2^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - v_1^2 v_2^2 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \}, \end{aligned}$$

where $\mathbf{n} \equiv \mathbf{r}/r$ and the Newton's G_N constant is related to the d -dimensional gravitational constant G by the following relation involving the arbitrary subtraction length scale L :

$$G = G_N L^{d-3}, \quad (12)$$

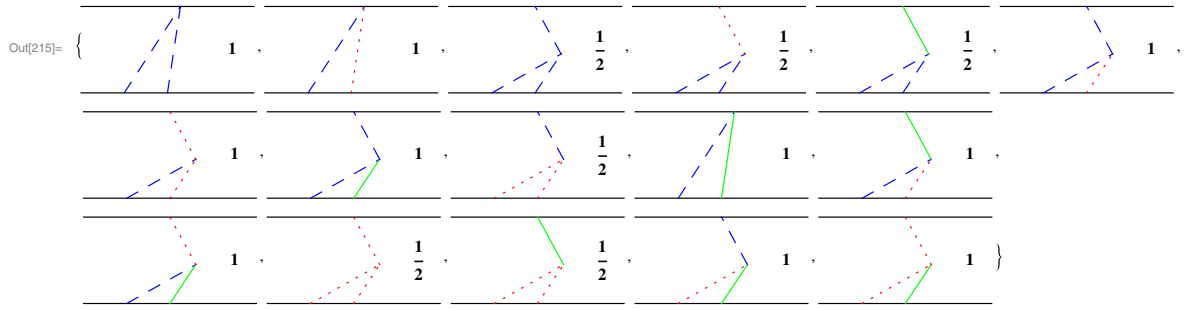


FIG. 4 (color online). The 16 $\mathcal{O}(G^2)$ diagrams. Diagrams from 2 to 9 actually start contributing at 2PN order, and must be consequently evaluated at their next-to leading order in v^2 (which involves an appropriate vertices expansion or one double time derivative insertion in the propagators) to catch their 3PN contribution. Analogously, the first diagram enters already at 1PN order, so it must be expanded up to v^4 . As in Fig. 3, the ϕ , A and σ propagators are represented, respectively, by blue dashed, red dotted and green solid lines and the number in the right part of each diagram is its multiplicity factor.

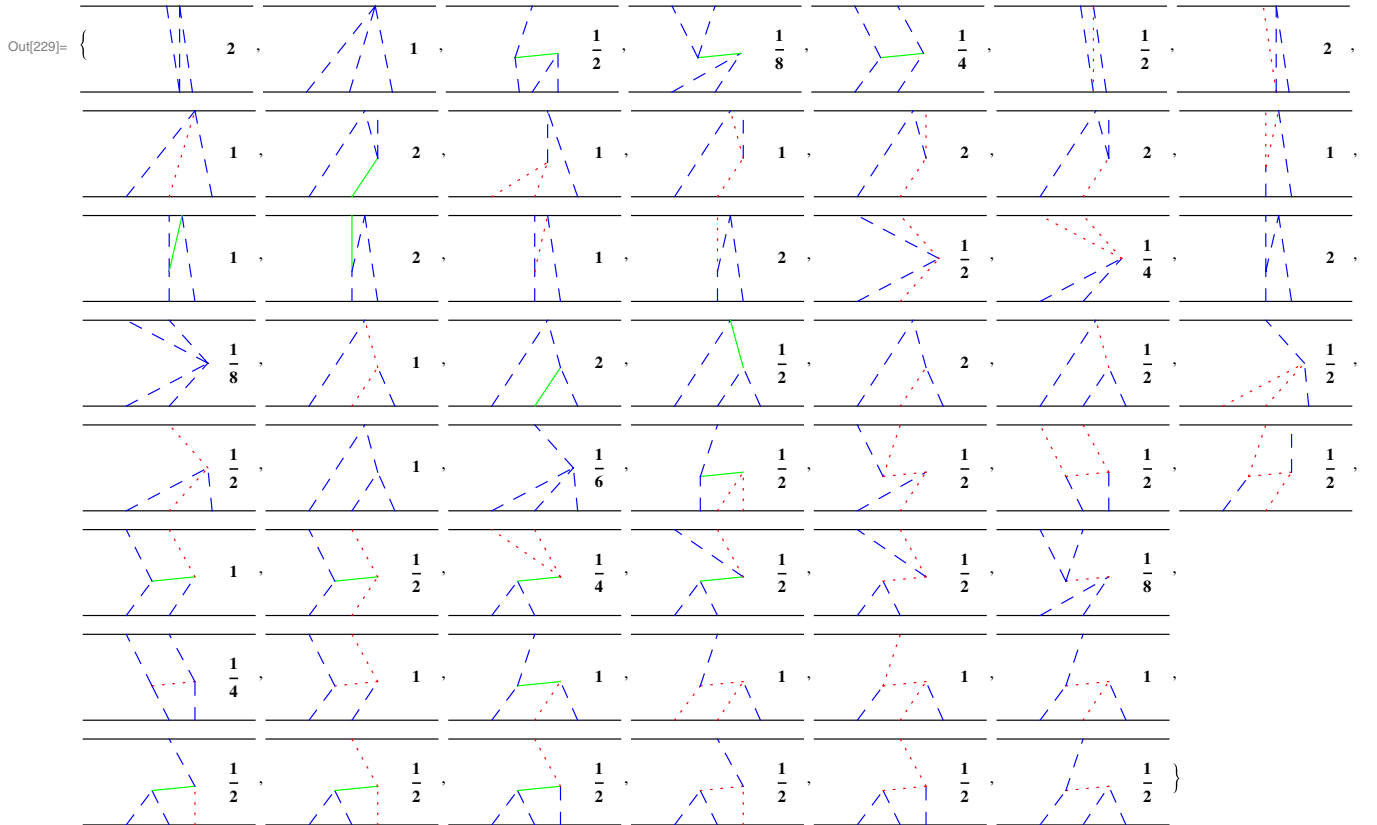


FIG. 5 (color online). The 53 $\mathcal{O}(G^3)$ diagrams. The first 5 enter at 2PN order, and must be consequently evaluated at their next-to leading order in v^2 , which involve an appropriate vertices expansion or one double time derivative insertion in the propagators. As in Figs. 3 and 4, the ϕ , A and σ propagators are represented, respectively, by blue dashed, red dotted and green solid lines and the number in the right part of each diagram is its multiplicity factor.

which gives trivially $G \rightarrow G_N$ in this case, but gives also a nontrivial contribution in presence of divergences, see next section.

B. $\mathcal{O}(G^2)$ diagrams

The first, second, and tenth diagram of Fig. 4 can be factorized in terms of lowest order ones, all the others need the use of the well known formula

$$\int_{\mathbf{p}_1} \frac{1}{p_1^{2\alpha} (\mathbf{p} - \mathbf{p}_1)^{2\beta}} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(d/2 - \beta)\Gamma(d/2 - \alpha)\Gamma(\alpha + \beta - d/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d - \alpha - \beta)} p^{d-2\alpha-2\beta}, \quad (13)$$

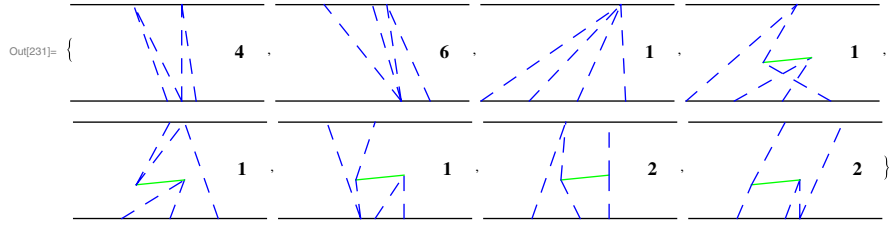


FIG. 6 (color online). The 8 $\mathcal{O}(G^4)$ diagrams. As in Figs. 3–5, the ϕ , A and σ propagators are represented, respectively, by blue dashed, red dotted and green solid lines. The number in the right part of each diagram is its multiplicity factor.

and of its generalization to inclusion of p_1^μ 's factors in the integrand (up to four different indices are needed at 3PN), which can be worked out by means of appropriate contractions with p^μ 's. The result is

$$G_1^2 = -\frac{G_N^2 m_1^2 m_2}{16r^2} [2r^2 \mathbf{a}_1 \cdot \mathbf{a}_2 + 36v_2^2 \mathbf{r} \cdot \mathbf{a}_1 + 8\mathbf{v}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{v}_1 - 8\mathbf{r} \cdot \mathbf{a}_2 (\mathbf{n} \cdot \mathbf{v}_1)^2 - 8(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - 24v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 144v_1^2 v_2^2 - 32v_1^4 + 49v_2^4 + 48v_1^2 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 24(\mathbf{n} \cdot \mathbf{v}_1)^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 32\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 28v_2^2 (\mathbf{n} \cdot \mathbf{v}_1)^2 - 24v_1^2 (\mathbf{n} \cdot \mathbf{v}_1)^2 - 32(\mathbf{n} \cdot \mathbf{v}_1)^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 + 32(\mathbf{n} \cdot \mathbf{v}_1)^3 \mathbf{n} \cdot \mathbf{v}_2 - 8(\mathbf{n} \cdot \mathbf{v}_1)^4],$$

$$G_2^2 = \frac{2G_N^2 m_1^2 m_2}{r^2} [r^2 \mathbf{a}_1 \cdot \mathbf{a}_2 + 2\mathbf{v}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{v}_1 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - 4v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 3v_2^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 2(\mathbf{n} \cdot \mathbf{v}_1)^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 4\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2],$$

$$G_3^2 = -\frac{G_N^2 m_1^2 m_2}{3(d-3)} [2\mathbf{a}_1 \cdot \mathbf{a}_2 + a_1^2] + \frac{G_N^2 m_1^2 m_2}{36r^2} \left[24r^2 \log\left(\frac{r}{L_0}\right) (2\mathbf{a}_1 \cdot \mathbf{a}_2 + a_1^2) - r^2 (32\mathbf{a}_1 \cdot \mathbf{a}_2 + 22a_1^2) + 48\mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{a}_2 - 180v_1^2 \mathbf{r} \cdot \mathbf{a}_1 - 96\mathbf{v}_2 \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2 - 156v_2^2 \mathbf{r} \cdot \mathbf{a}_1 - 12(\mathbf{r} \cdot \mathbf{a}_1)^2 + 96\mathbf{r} \cdot \mathbf{a}_1 (\mathbf{n} \cdot \mathbf{v}_2)^2 + 54\mathbf{v}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{v}_1 + 288v_1^2 \mathbf{r} \cdot \mathbf{a}_2 - 54\mathbf{r} \cdot \mathbf{a}_2 (\mathbf{n} \cdot \mathbf{v}_1)^2 - 54(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + 150v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 369v_1^2 v_2^2 - 78v_1^4 + 576v_1^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 - 300v_1^2 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 42(\mathbf{n} \cdot \mathbf{v}_1)^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 216\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 243v_2^2 (\mathbf{n} \cdot \mathbf{v}_1)^2 + 222v_1^2 (\mathbf{n} \cdot \mathbf{v}_1)^2 - 216(\mathbf{n} \cdot \mathbf{v}_1)^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 - 56(\mathbf{n} \cdot \mathbf{v}_1)^3 \mathbf{n} \cdot \mathbf{v}_2 + 20(\mathbf{n} \cdot \mathbf{v}_1)^4 \right],$$

$$G_4^2 = \frac{G_N^2 m_1^2 m_2}{6r^2} [r^2 \mathbf{a}_1 \cdot \mathbf{a}_2 + 3\mathbf{r} \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{a}_2 + 12\mathbf{r} \cdot \mathbf{a}_1 \mathbf{v}_1 \cdot \mathbf{v}_2 - 3\mathbf{v}_2 \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2 - 3v_2^2 \mathbf{r} \cdot \mathbf{a}_1 + 6\mathbf{r} \cdot \mathbf{a}_1 (\mathbf{n} \cdot \mathbf{v}_2)^2 + 11\mathbf{v}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{v}_1 + 25v_1^2 \mathbf{r} \cdot \mathbf{a}_2 - 2\mathbf{r} \cdot \mathbf{a}_2 (\mathbf{n} \cdot \mathbf{v}_1)^2 - 11(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + 6v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 25v_1^2 v_2^2 - 3v_2^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 50v_1^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 - 24v_1^2 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 12(\mathbf{n} \cdot \mathbf{v}_1)^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 26\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 2v_2^2 (\mathbf{n} \cdot \mathbf{v}_1)^2 + 3v_2^2 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 8(\mathbf{n} \cdot \mathbf{v}_1)^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 - 4(\mathbf{n} \cdot \mathbf{v}_1)^3 \mathbf{n} \cdot \mathbf{v}_2],$$

$$G_5^2 = \frac{G_N^2 m_1^2 m_2}{12r^2} [10\mathbf{v}_2 \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2 + 2v_2^2 \mathbf{r} \cdot \mathbf{a}_1 - 4\mathbf{r} \cdot \mathbf{a}_1 (\mathbf{n} \cdot \mathbf{v}_2)^2 + 6(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + 18v_1^2 v_2^2 + 3v_2^4 - 24v_1^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 - 12\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 3v_2^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 + 12(\mathbf{n} \cdot \mathbf{v}_1)^2 (\mathbf{n} \cdot \mathbf{v}_2)^2],$$

$$\begin{aligned}
G_6^2 &= -\frac{G_N^2 m_1^2 m_2}{3r^2} [r^2(\mathbf{a}_1 \cdot \mathbf{a}_2 + 2a_1^2) + 27v_1^2 \mathbf{r} \cdot \mathbf{a}_1 - 9v_2^2 \mathbf{r} \cdot \mathbf{a}_1 + 14\mathbf{v}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{v}_1 + 4v_1^2 \mathbf{r} \cdot \mathbf{a}_2 - 14\mathbf{r} \cdot \mathbf{a}_2 (\mathbf{n} \cdot \mathbf{v}_1)^2 - 14(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 \\
&\quad - 39v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 22v_1^2 v_2^2 + 9v_1^4 + 8v_1^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 + 78v_1^2 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 56\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 50v_2^2 (\mathbf{n} \cdot \mathbf{v}_1)^2 \\
&\quad - 24v_1^2 (\mathbf{n} \cdot \mathbf{v}_1)^2 - 56(\mathbf{n} \cdot \mathbf{v}_1)^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 + 8(\mathbf{n} \cdot \mathbf{v}_1)^4], \\
G_7^2 &= \frac{8G_N^2 m_1^2 m_2}{d-3} \mathbf{a}_1 \cdot \mathbf{a}_2 + \frac{4G_N^2 m_1^2 m_2}{r^2} \left[r^2 \mathbf{a}_1 \cdot \mathbf{a}_2 \left(1 - 4 \log \left(\frac{r}{L_0} \right) \right) - 2\mathbf{r} \cdot \mathbf{a}_1 \mathbf{v}_1 \cdot \mathbf{v}_2 + 4\mathbf{v}_2 \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2 - 4\mathbf{v}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{v}_1 \right. \\
&\quad - v_1^2 \mathbf{r} \cdot \mathbf{a}_2 + 4(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + 3v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + v_1^2 v_2^2 + v_2^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 2v_1^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 + 2v_1^2 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \\
&\quad \left. + 2(\mathbf{n} \cdot \mathbf{v}_1)^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 8\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2 \right], \\
G_8^2 &= \frac{G_N^2 m_1^2 m_2}{9r^2} [9v_1^2 \mathbf{r} \cdot \mathbf{a}_1 - 21v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 27v_1^2 v_2^2 - 33v_1^4 + 42v_1^2 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 96(\mathbf{n} \cdot \mathbf{v}_1)^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 54v_2^2 (\mathbf{n} \cdot \mathbf{v}_1)^2 \\
&\quad + 24v_1^2 (\mathbf{n} \cdot \mathbf{v}_1)^2 - 128(\mathbf{n} \cdot \mathbf{v}_1)^3 \mathbf{n} \cdot \mathbf{v}_2 + 56(\mathbf{n} \cdot \mathbf{v}_1)^4], \\
G_9^2 &= \frac{4G_N^2 m_1^2 m_2}{d-3} a_1^2 + \frac{2G_N^2 m_1^2 m_2}{r^2} \left[4r^2 a_1^2 \left(1 - \log \left(\frac{r}{L_0} \right) \right) + 8v_1^2 \mathbf{r} \cdot \mathbf{a}_1 - 6v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 3v_1^2 v_2^2 + v_1^4 \right. \\
&\quad \left. + 12v_1^2 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 9v_1^2 (\mathbf{n} \cdot \mathbf{v}_1)^2 \right], \\
G_{10}^2 &= -\frac{6G_N^2 m_1^2 m_2}{r^2} [(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - v_1^2 v_2^2], \\
G_{11}^2 &= -\frac{8G_N^2 m_1^2 m_2}{r^2} \mathbf{v}_2 \cdot \mathbf{a}_1 \mathbf{r} \cdot \mathbf{v}_2, \\
G_{12}^2 &= -\frac{8G_N^2 m_1^2 m_2}{r^2} [\mathbf{v}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{v}_1 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 - 2(\mathbf{n} \cdot \mathbf{v}_1)^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 2\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2], \\
G_{13}^2 &= \frac{4G_N^2 m_1^2 m_2}{r^2} [4\mathbf{r} \cdot \mathbf{a}_1 \mathbf{v}_1 \cdot \mathbf{v}_2 + 6\mathbf{v}_1 \cdot \mathbf{a}_2 \mathbf{r} \cdot \mathbf{v}_1 - v_1^2 \mathbf{r} \cdot \mathbf{a}_2 - 6(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + 2v_1^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + v_1^2 v_2^2 - 2v_1^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 \\
&\quad + v_1^2 \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 4(\mathbf{n} \cdot \mathbf{v}_1)^2 \mathbf{v}_1 \cdot \mathbf{v}_2 + 12\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \mathbf{v}_1 \cdot \mathbf{v}_2], \\
G_{14}^2 &= \frac{2G_N^2 m_1^2 m_2}{r^2} [v_1^2 v_2^2 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + v_1^2 (\mathbf{n} \cdot \mathbf{v}_2)^2], \\
G_{15}^2 &= -\frac{8G_N^2 m_1^2 m_2}{r^2} v_1^2 \mathbf{r} \cdot \mathbf{a}_1, \\
G_{16}^2 &= \frac{8G_N^2 m_1^2 m_2}{r^2} [v_1^2 - 2(\mathbf{n} \cdot \mathbf{v}_1)^2] \mathbf{v}_1 \cdot \mathbf{v}_2.
\end{aligned}$$

Divergences appear at this order from the terms G_3^2 , G_7^2 and G_9^2 , under the shape of $(d-3)$ simple poles, giving rise also to logarithmic factors involving the subtraction scale $L_0 \equiv L e^{-\gamma/2} / \sqrt{4\pi}$, where $\gamma = .577216\dots$ is the Euler-Mascheroni constant and L is the length scale written in Eq. (12). Following [1], in Sec. III E we will get rid of the poles by means of a shift of the world-line parameters; as to the logarithms involving the arbitrary subtraction scale, they are known to remain in the Lagrangian and also in the equation of motion for \mathbf{r} , which is however a gauge-dependent variable. They however drop out in the expression of observables, like the energy E of the system as a function of the orbital frequency ω , which are gauge-independent quantities as both E and ω can be measured asymptotically far from the binary system.

C. $\mathcal{O}(G^3)$ diagrams

Most of the diagrams of Fig. 5 can be computed via recursive application of Eq. (13), with the exception of the 5th diagram and all the ones sharing the same H-shaped topology. In this case one has to work on the integrand by parts to obtain the following recursive relationship

$$\begin{aligned}
I(\alpha, \beta, \gamma, \delta, \epsilon) &\equiv \int_{\mathbf{p}_1, \mathbf{p}_2} \frac{1}{p_1^{2\alpha} (\mathbf{p} - \mathbf{p}_1)^{2\beta} p_2^{2\gamma} (\mathbf{p} - \mathbf{p}_2)^{2\delta} (\mathbf{p}_1 - \mathbf{p}_2)^{2\epsilon}} \\
&= [\gamma(I(\alpha-1, \beta, \gamma+1, \delta, \epsilon) - I(\alpha, \beta, \gamma+1, \delta, \epsilon-1)) \\
&\quad + \delta(I(\alpha, \beta-1, \gamma, \delta+1, \epsilon) \\
&\quad - I(\alpha, \beta, \gamma, \delta+1, \epsilon-1))] / (2\epsilon + \gamma + \delta - d), \quad (14)
\end{aligned}$$

and its generalization to the inclusion of $p_{1,2}^\mu$ factors in the integrand. The above formula (14) also holds exchanging in the right hand side ($\{\alpha, \beta\} \leftrightarrow \{\gamma, \delta\}$) or ($\{\alpha, \gamma\} \leftrightarrow \{\beta, \delta\}$) and it can be iterated to obtain an integral of the type Eq. (13).

As in the G^2 case, here also poles and logarithmic terms are obtained, which again are non physical, see Sec. III E. The output of our code for these diagrams is

$$\begin{aligned}
G_1^3 &= \frac{G_N^3 m_1^2 m_2^2}{4r^3} [12v_1^2 - 3\mathbf{v}_1 \cdot \mathbf{v}_2 + 7\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 4(\mathbf{n} \cdot \mathbf{v}_1)^2], \\
G_2^3 &= -\frac{G_N^3 m_1^3 m_2}{4r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 + 3v_1^2 + 9v_2^2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 2(\mathbf{n} \cdot \mathbf{v}_1)^2], \\
G_3^3 &= \frac{G_N^3 m_1^3 m_2}{30(d-3)r^3} [10\mathbf{v}_1 \cdot \mathbf{v}_2 - 13v_1^2 - 30\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 39(\mathbf{n} \cdot \mathbf{v}_1)^2] \\
&\quad + \frac{G_N^3 m_1^3 m_2}{900r^3} \left[90 \log\left(\frac{r}{L_0}\right) (13v_1^2 - 10\mathbf{v}_1 \cdot \mathbf{v}_2 + 30\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 39(\mathbf{n} \cdot \mathbf{v}_1)^2) \right. \\
&\quad \left. + 850\mathbf{v}_1 \cdot \mathbf{v}_2 - 2401v_1^2 - 450v_2^2 - 3450\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 3243(\mathbf{n} \cdot \mathbf{v}_1)^2 \right], \\
G_4^3 &= -\frac{\pi^2 G_N^3 m_1^2 m_2^2}{128r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 + 2v_1^2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 6(\mathbf{n} \cdot \mathbf{v}_1)^2], \\
G_5^3 &= \frac{G_N^3 m_1^2 m_2^2}{72r^3} [9\pi^2 (v_1^2 - \mathbf{v}_1 \cdot \mathbf{v}_2 + 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2) + 50\mathbf{v}_1 \cdot \mathbf{v}_2 \\
&\quad - 644v_1^2 - 294\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 348(\mathbf{n} \cdot \mathbf{v}_1)^2], \\
G_6^3 &= \frac{G_7^3}{2} = -\frac{G_{11}^3}{4} = -\frac{G_{12}^3}{4} = \frac{2G_N^3 m_1^2 m_2^2}{r^3} \mathbf{v}_1 \cdot \mathbf{v}_2, \\
G_8^3 &= -\frac{G_{23}^3}{4} = -\frac{3G_{29}^3}{16} = \frac{3G_{46}^3}{32} = \frac{2G_N^3 m_1^3 m_2}{r^3} \mathbf{v}_1 \cdot \mathbf{v}_2, \\
G_9^3 &= \frac{G_{34}^3}{8} = -\frac{G_N^3 m_1^2 m_2^2}{r^3} [v_1^2 - (\mathbf{n} \cdot \mathbf{v}_1)^2], \\
G_{10}^3 &= \frac{3G_{28}^3}{4} = -\frac{3G_{45}^3}{8} = \frac{4G_N^3 m_1^3 m_2}{r^3} v_1^2, \\
G_{13}^3 &= \frac{2G_N^3 m_1^2 m_2^2}{r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 - 2\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + (\mathbf{n} \cdot \mathbf{v}_1)^2], \\
G_{14}^3 &= \frac{8G_N^3 m_1^2 m_2^2}{r^3} v_1^2, \\
G_{15}^3 &= -3G_{16}^3 = -\frac{6G_N^3 m_1^2 m_2^2}{r^3} [v_1^2 - 2(\mathbf{n} \cdot \mathbf{v}_1)^2], \\
G_{17}^3 &= -\frac{2G_N^3 m_1^2 m_2^2}{r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 + v_1^2 - 2\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 2(\mathbf{n} \cdot \mathbf{v}_1)^2], \\
G_{18}^3 &= \frac{2G_N^3 m_1^2 m_2^2}{r^3} [4(\mathbf{n} \cdot \mathbf{v}_1)^2 - \mathbf{v}_1 \cdot \mathbf{v}_2 - v_1^2], \\
G_{19}^3 &= G_{20}^3 = G_{33}^3 = G_{39}^3 = 0, \\
G_{21}^3 &= -\frac{G_N^3 m_1^2 m_2^2}{r^3} [v_1^2 - 4\mathbf{v}_1 \cdot \mathbf{v}_2 + 14\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 7(\mathbf{n} \cdot \mathbf{v}_1)^2], \\
G_{22}^3 &= \frac{\pi^2 G_N^3 m_1^2 m_2^2}{8r^3} [v_1^2 - 4\mathbf{v}_1 \cdot \mathbf{v}_2 + 12\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2],
\end{aligned}$$

$$G_{24}^3 = \frac{2G_N^3 m_1^3 m_2}{r^3} [v_1^2 - 2(\mathbf{n} \cdot \mathbf{v}_1)^2]$$

$$G_{25}^3 = \frac{3G_N^3 m_1^3 m_2}{2r^3} [v_2^2 - (\mathbf{n} \cdot \mathbf{v}_2)^2],$$

$$G_{26}^3 = -\frac{2G_N^3 m_1^3 m_2}{r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 + v_1^2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - (\mathbf{n} \cdot \mathbf{v}_1)^2],$$

$$G_{27}^3 = \frac{G_N^3 m_1^3 m_2}{r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 - \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2],$$

$$G_{30}^3 = -\frac{G_N^3 m_1^3 m_2}{2r^3} [v_1^2 - 4\mathbf{v}_1 \cdot \mathbf{v}_2 + 12\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 5(\mathbf{n} \cdot \mathbf{v}_1)^2],$$

$$G_{31}^3 = -\frac{4G_N^3 m_1^3 m_2}{15(d-3)r^3} [v_1^2 - 5\mathbf{v}_1 \cdot \mathbf{v}_2 + 15\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2] - \frac{2G_N^3 m_1^3 m_2}{75r^3} \\ \times \left[30 \log\left(\frac{r}{L_0}\right) (5\mathbf{v}_1 \cdot \mathbf{v}_2 - v_1^2 - 15\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 3(\mathbf{n} \cdot \mathbf{v}_1)^2) - 25\mathbf{v}_1 \cdot \mathbf{v}_2 - (v_1^2) + 225\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 57(\mathbf{n} \cdot \mathbf{v}_1)^2 \right],$$

$$G_{32}^3 = \frac{2G_N^3 m_1^3 m_2}{r^3} [v_1^2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2] \left[\frac{2}{3(d-3)} - 2 \log\left(\frac{r}{L_0}\right) + 1 \right],$$

$$G_{35}^3 = \frac{8G_N^3 m_1^2 m_2^2}{r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 - \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2],$$

$$G_{36}^3 = -\frac{G_N^3 m_1^2 m_2^2}{2r^3} [3\pi^2 (v_1^2 - \mathbf{v}_1 \cdot \mathbf{v}_2 + 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2) + 60\mathbf{v}_1 \cdot \mathbf{v}_2 - 60v_1^2 - 148\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 148(\mathbf{n} \cdot \mathbf{v}_1)^2],$$

$$G_{37}^3 = -\frac{G_N^3 m_1^2 m_2^2}{2r^3} [\pi^2 (\mathbf{v}_1 \cdot \mathbf{v}_2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2) - 32\mathbf{v}_1 \cdot \mathbf{v}_2 + 32\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2],$$

$$G_{38}^3 = \frac{\pi^2 G_N^3 m_1^2 m_2^2}{8r^3} [v_1^2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2],$$

$$G_{40}^3 = -\frac{32G_{41}^3}{3} = -\frac{\pi^2 G_N^3 m_1^2 m_2^2}{4r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2],$$

$$G_{42}^3 = \frac{G_N^3 m_1^2 m_2^2}{4r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 + 2v_1^2 + \mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 2(\mathbf{n} \cdot \mathbf{v}_1)^2],$$

$$G_{43}^3 = \frac{G_N^3 m_1^2 m_2^2}{2r^3} [\pi^2 (\mathbf{v}_1 \cdot \mathbf{v}_2 + v_1^2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2) + 4\mathbf{v}_1 \cdot \mathbf{v}_2 - 4v_1^2 + 4\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 4(\mathbf{n} \cdot \mathbf{v}_1)^2],$$

$$G_{44}^3 = \frac{4G_N^3 m_1^3 m_2}{(d-3)r^3} [v_1^2 - \mathbf{v}_1 \cdot \mathbf{v}_2 + 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2] + \frac{2G_N^3 m_1^3 m_2}{r^3} \left[6 \log\left(\frac{r}{L_0}\right) (\mathbf{v}_1 \cdot \mathbf{v}_2 - v_1^2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 3(\mathbf{n} \cdot \mathbf{v}_1)^2) \right. \\ \left. + 5v_1^2 - 5\mathbf{v}_1 \cdot \mathbf{v}_2 + 21\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 21(\mathbf{n} \cdot \mathbf{v}_1)^2 \right],$$

$$G_{47}^3 = -\frac{4G_N^3 m_1^3 m_2}{3(d-3)r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 + v_1^2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2] + \frac{2G_N^3 m_1^3 m_2}{3r^3} \left[6 \log\left(\frac{r}{L_0}\right) (v_1^2 + \mathbf{v}_1 \cdot \mathbf{v}_2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \right. \\ \left. - 3(\mathbf{n} \cdot \mathbf{v}_1)^2) - \mathbf{v}_1 \cdot \mathbf{v}_2 - v_1^2 + 9\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 9(\mathbf{n} \cdot \mathbf{v}_1)^2 \right],$$

$$G_{48}^3 = \frac{2G_N^3 m_1^3 m_2}{5(d-3)r^3} [v_1^2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2] + \frac{2G_N^3 m_1^3 m_2}{25r^3} \left[17v_1^2 - 56(\mathbf{n} \cdot \mathbf{v}_1)^2 - 15 \log\left(\frac{r}{L_0}\right) (v_1^2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2) \right],$$

$$G_{49}^3 = -\frac{4G_N^3 m_1^3 m_2}{15(d-3)r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2] - \frac{4G_N^3 m_1^3 m_2}{75r^3} [12\mathbf{v}_1 \cdot \mathbf{v}_2 - 66\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 15 \log\left(\frac{r}{L_0}\right) (\mathbf{v}_1 \cdot \mathbf{v}_2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2)],$$

$$\begin{aligned}
G_{50}^3 &= -\frac{2G_N^3 m_1^3 m_2}{5(d-3)r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2] - \frac{2G_N^3 m_1^3 m_2}{25r^3} \left[17\mathbf{v}_1 \cdot \mathbf{v}_2 - 56\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 - 15 \log\left(\frac{r}{L_0}\right) (\mathbf{v}_1 \cdot \mathbf{v}_2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2) \right], \\
G_{51}^3 &= -\frac{2G_N^3 m_1^3 m_2}{15(d-3)r^3} [v_1^2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2] + \frac{2G_N^3 m_1^3 m_2}{75r^3} \left[23v_1^2 + 36(\mathbf{n} \cdot \mathbf{v}_1)^2 + 15 \log\left(\frac{r}{L_0}\right) (v_1^2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2) \right], \\
G_{52}^3 &= \frac{2G_N^3 m_1^3 m_2}{3(d-3)r^3} [\mathbf{v}_1 \cdot \mathbf{v}_2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2] - \frac{2G_N^3 m_1^3 m_2}{3r^3} \left[\mathbf{v}_1 \cdot \mathbf{v}_2 + 6\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 + 3 \log\left(\frac{r}{L_0}\right) (\mathbf{v}_1 \cdot \mathbf{v}_2 - 3\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2) \right], \\
G_{53}^3 &= \frac{G_N^3 m_1^3 m_2}{10(d-3)r^3} [v_1^2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2] + \frac{G_N^3 m_1^3 m_2}{300r^3} \left[17v_1^2 - 100\mathbf{v}_1 \cdot \mathbf{v}_2 - 81(\mathbf{n} \cdot \mathbf{v}_1)^2 + 300\mathbf{n} \cdot \mathbf{v}_1 \mathbf{n} \cdot \mathbf{v}_2 \right. \\
&\quad \left. - 90 \log\left(\frac{r}{L_0}\right) (v_1^2 - 3(\mathbf{n} \cdot \mathbf{v}_1)^2) \right].
\end{aligned}$$

D. $\mathcal{O}(G^4)$ diagrams

The G^4 topology set shown in Fig. 6 does not present any new difficulty, and the integrations can be done straightforwardly:

$$\begin{aligned}
G_1^4 &= G_2^4 = \frac{3G_6^4}{2} = \frac{G_7^4}{8} = \frac{3G_8^4}{4} = \frac{G_N^4 m_1^3 m_2^2}{2r^4}, \\
G_3^4 &= \frac{G_4^4}{8} = \frac{G_N^4 m_1^4 m_2}{24r^4}, \\
G_5^4 &= 0.
\end{aligned}$$

E. Reproducing the 3PN Lagrangian

Adding up all the terms shown so far a Lagrangian containing poles and logarithmic terms is obtained. The poles can be removed by performing the shift of the world-line parameters as given by Eq. (1.13) of [6], acting over the source positions $\mathbf{x}_{1,2}$ according to $\mathbf{x}_1 \rightarrow \mathbf{x}'_1 = \mathbf{x}_1 + \xi_1$ with

$$\xi_1 \equiv \frac{11}{3} G^2 m_1^2 \left[\frac{1}{d-3} - 2 \log\left(\frac{r}{L_0}\right) - \frac{327}{1540} \right] \mathbf{a}_1, \quad (15)$$

and analogously for ξ_2 [42].

Finally, in order to get the exact form of the 3PN Lagrangian as it is written in Eq. (174) of [1] (and originally derived in [5]), one has to act on factors like $\mathbf{a}_1 \cdot \mathbf{a}_2$, $\dot{\mathbf{a}}_1 \cdot \dot{\mathbf{a}}_2$ and similar by means of the standard *double-zero* technique, see e.g. [43],

$$\begin{aligned}
\mathbf{a}_1 \cdot \mathbf{a}_2 &= \left(\mathbf{a}_1 + \frac{Gm_2}{r^3} \mathbf{r} - \frac{Gm_2}{r^3} \mathbf{r} \right) \cdot \left(\mathbf{a}_2 - \frac{Gm_1}{r^3} \mathbf{r} + \frac{Gm_1}{r^3} \mathbf{r} \right) \\
&\simeq \mathbf{a}_1 \cdot \frac{Gm_1}{r^3} \mathbf{r} - \mathbf{a}_2 \cdot \frac{Gm_2}{r^3} \mathbf{r} + \frac{G^2 m_1 m_2}{r^4}, \quad (16)
\end{aligned}$$

which makes use of the lowest order equations of motion to get rid of the 3PN Lagrangian terms quadratic in the accelerations. This transformation should be iterated (as in the case of the $\dot{\mathbf{a}}_1 \cdot \dot{\mathbf{a}}_2$ term) and combined with deriva-

tions by parts until only terms linear in the accelerations remain.

Moreover, as terms like $\mathbf{a}_1 \cdot \mathbf{a}_2$ appear also in the 2PN effective Lagrangian, the same technique must be applied at 2PN, but employing the equation of motion up to 1PN accuracy rather than just the Newtonian part written above; this gives a further contribution to the 3PN Lagrangian, and all these terms nicely fit with all the previous ones to give Eq. (174) of [1].

IV. CONCLUSIONS

We have computed the conservative dynamics of a gravitationally bound binary system within the framework of the PN approximation to general relativity to third post-Newtonian order. By a systematic use of the effective field theory methods for general relativity proposed by Goldberger and Rothstein it has been possible to automatize the computation and compute the effective Lagrangian by summing the contributions of 80 Feynman diagrams. We have computed the effective Lagrangian reproducing known results [3] and paving the way for the yet uncomputed 4PN dynamics. Beside the clear theoretical interest of this calculation, the 4PN Hamiltonian has a direct phenomenological impact as it enters the determination of the templates waveforms used in analyzing the data from the large interferometers LIGO and Virgo like the effective one body ones [29].

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compute, confirming our results, a subset of the diagrams presented in this section (more precisely, all the diagrams except for $G_{5,34-37,42,43}^3$).

- [42] Since the transformation is of $\mathcal{O}(3\text{PN})$, it should be implemented on the Newtonian part of the Lagrangian only, with the caution that also the $\mathcal{O}(d-3)$ part of

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