Euclidean dilaton black hole vortex and Dirac fermions

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We considered the behavior of Dirac fermion modes in the background of a Euclidean dilaton black hole with an Abelian Higgs vortex passing through it. Fermions were coupled to the fields due to the superconducting string model. The case of nonextremal and extremal charged black holes in the theory with an arbitrary coupling constant between the dilaton field and $U(1)$-gauge field was considered. We elaborated the cases of zero and nonzero Dirac fermion modes. One finds evidence that the system under consideration can support fermion fields acting like a superconducting hair on a black hole in the sense that a nontrivial spinor field configuration can be carried by a Euclidean spherically symmetric charged dilaton black hole. It was revealed that the localization of Dirac fermion modes depended on the cosmic string winding number and the value of black hole surface gravity.

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I. INTRODUCTION

In recent years, studies of a much more realistic case than scalar fields have attracted more attention. In particular, the solution of field equations describing fermions in a curved geometry is one of the theoretical tools of investigating the underlying structure of spacetime. A better understanding of the properties of black holes also requires examination of the behavior of matter fields in the vicinity of them [1]. Dirac fermions' behavior was studied in the context of Einstein-Yang-Mills background [2]. Fermion fields were analyzed in the near-horizon limit of an extreme Kerr black hole [3] as well as in the extreme Reissner-Nordström (RN) case [4]. It was also revealed [5,6] that the only black hole solution of four-spinor Einstein-dilaton-Yang-Mills equations was those for which the spinors vanished identically outside a black hole. Dirac fields were considered in Bertotti-Robinson spacetime [7,8] and in the context of a cosmological solution with homogeneous Yang-Mills fields acting as an energy source [9].

The late-time decay of fermion fields in the background of various kinds of black holes is an important problem from the point of view of the uniqueness theorem for them. The late-time behavior of massless and massive Dirac fermion fields was widely studied in spacetimes of static as well as stationary black holes [10–16].

Brane models in which our Universe is represented as a (3 + 1)-dimensional submanifold living in higher dimensional spacetime also attract the attention to brane black holes. The decay of massive Dirac hair on a brane black hole was considered in [17].

It could have happened that at the beginning our Universe underwent several phase transitions. The mechanism of spontaneous symmetry breaking involved in the early Universe phase transitions might have produced stable topological defects like cosmic strings, monopoles, and domain walls [18]. Among them cosmic strings and cosmic string black hole systems have acquired much interest. Assuming a distributional mass source, the metric of this system was derived in [19] (the so-called thin string limit). In Ref. [20] the numerical and analytic evidence for the existence of an Abelian Higgs vortex on a Schwarzschild black hole was given, while in Refs. [21,22] the extensions of the aforementioned arguments to the case of a charged RN black hole and dilaton black holes were performed. It was also found that an analog of the Meissner effect (i.e., the expulsion of the vortex fields from the black hole) could take place. It happened that this phenomenon occurs for some range of black hole parameters [23]. On the contrary, extremal dilaton black holes always expel vortex Higgs fields from their interior [24]. A very similar situation takes place in the case of the other topological defect, a domain wall which can be expelled from various kinds of black holes [25].

There are some cosmic strings which may become superconducting by the implementation of fermions. They may be responsible for various exotic astrophysical phenomena. For instance, closed superconducting loops, the so-called vortons [26], may constitute a fraction of cold dark matter in the galactic halo, and their slow quantum decays may be connected with the ultrahigh energy cosmic rays [27]. To everyone's dismay, it turned out that also the high-redshift gamma-ray bursts could be a reasonable way to test the superconducting string model [28].

In Ref. [29] it was revealed that the Euclidean vortex solution in the spacetime of a black hole led to the nonperturbative exponential decay of an electric field outside the event horizon of a Schwarzschild black hole. It was shown [30] that a Euclidean Schwarzschild black hole could support a vortex solution at the event horizon. In Ref. [31] the generalization of the above problem to the case of Einstein-Maxwell-dilaton gravity was proposed.

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On the other hand, in Ref. [32] it was observed that the Dirac operator in the spacetime of the system composed of a Euclidean magnetic RN black hole and a vortex in the theories containing superconducting cosmic strings [33] possessed zero modes. In turn, the aforementioned zero modes caused the fermion condensate around the magnetic RN black hole.

The motivation of our paper was to provide some continuity of the research presented in Ref. [32] and to generalize it to the theory constituting the modification of the Einstein-Maxwell theory, the so-called dilaton gravity being the low-energy limit of the heterotic string theory. In dilaton gravity one has to work with the nontrivial coupling of the dilaton field with the $U(1)$-gauge field. Our considerations will be valid for an arbitrary coupling constant. In our research we shall consider a static spherically charged black hole solution in dilaton gravity which has quite a different topological structure than the one studied in [32]. Contrary to the research conducted in the aforementioned reference we shall not only pay attention to Dirac zero fermion modes but we shall elaborate the nonzero fermion modes in the underlying spacetime. To our knowledge, the problem of nonzero Dirac fermion modes in the spacetime of the black hole cosmic string system has not been studied before. We will also pay attention to the near-horizon behavior of fermionic fields causing superconductivity in the case of an extremal charged dilaton black hole. As was mentioned before, in the spacetime of an extremal dilaton black hole one can observe the analog of the Meissner effect.

The layout of our paper will be as follows. In Sec. II we start with a discussion of the vortex itself in the background of the Euclidean dilaton black hole being the static solution of dilaton gravity equations with an arbitrary α-coupling constant. Section III will be devoted to the superconducting cosmic string piercing the black hole in question. We shall elaborate the behavior of zero Dirac fermion modes both on a nonextremal and extremal Euclidean dilaton black hole. In Sec. IV we take into account fermion modes for the case when $k > 0$ on the same kinds of black holes. We find that Dirac fermion modes may be regarded as hair on the considered black holes. In the next section we conclude our studies.

II. EUCLIDEAN DILATON BLACK HOLE/ABELIAN HIGGS VORTEX SYSTEM

In the following section we shall consider a Euclidean charged dilaton black hole/string vortex configuration. One assumes the complete separation between the degrees of freedom of each of the objects in question. We shall treat a static charged dilaton black hole line element as the background solution and numerically justify the existence of the vortex solution, for an arbitrary coupling constant in the considered theory.

The system under consideration will be described by the action of the form as

$$S = S_1 + S_{\text{bos}}, \tag{1}$$

where S_1 is the dilaton gravity action being the low-energy limit of the string action with an arbitrary coupling constant. It is provided by

$$S_1 = \int \sqrt{-g} d^4x [R - 2(\nabla\phi)^2 - e^{-2\alpha\phi} F_{\mu\nu}F^{\mu\nu}], \tag{2}$$

where $F_{\alpha\beta} = 2\nabla_{[\alpha}A_{\beta]}$, ϕ is the dilaton field, and α is a coupling constant which determines the interaction between dilaton and Abelian gauge fields. In action S_1, the corresponding Abelian gauge field can be thought of as the everyday Maxwell one.

The other gauge field is hidden in the action S_{bos} and it is subject to the spontaneous symmetry breaking. Its action implies

$$S_{\text{bos}} = \int \sqrt{-g} d^4x \bigg[-(d_\mu \Phi)^\dagger d^\mu \Phi - \frac{1}{4} B_{\mu\nu}B^{\mu\nu} - \frac{\lambda}{4}(\Phi^\dagger \Phi - \eta^2)^2 \bigg], \tag{3}$$

where $B_{\mu\nu} = 2\nabla_{[\mu}B_{\nu]}$ is the field strength associated with the B_μ-gauge field, η is the energy scale of symmetry breaking, and λ is the Higgs coupling. The covariant derivative has the form $d_\mu = \nabla_\mu + ie_R B_\mu$, where e_R is the gauge coupling constant.

The line element of the general static spherically symmetric Euclidean dilaton black hole yields

$$ds^2 = A^2 d\tau^2 + B^2 dr^2 + C^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4}$$

where in order to Euclideanize the metric we set the Euclidean time as $t \to i\tau$. For the case when $B(r)^2 = 1/A(r)^2$, the explicit forms of the metric coefficients are as follows:

$$A^2 = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{(1-\alpha^2)/(1+\alpha^2)}, \tag{5}$$

$$C^2 = r^2\left(1 - \frac{r_-}{r}\right)^{2\alpha^2/(1+\alpha^2)}, \tag{6}$$

where r_+, r_- are related to the mass and charge Q of the black hole due to the relations

$$2M = r_+ + \frac{1 - \alpha^2}{1 + \alpha^2} r_-, \tag{7}$$

$$Q^2 = \frac{r_+ r_-}{1 + \alpha^2}. \tag{8}$$

On the other hand, the dilaton field is given as

$$e^{2\alpha\phi} = \left(1 - \frac{r_-}{r}\right)^{2\alpha^2/(1+\alpha^2)}. \tag{9}$$

The location of the event horizon is $r = r_+$. The $r = r_-$ is another singularity, but one can ignore it for $r_- < r_+$. Having in mind the above charged dilaton black hole solution one can see that the structure of the black hole in question is drastically changed due to the presence of the dilaton field. Moreover, the arbitrary α-coupling constant is the other nontrivial element in the studies. Recently, the numerical studies of the dynamical collapse of the complex charged scalar field [34] reveal that due to the coupling between the dilaton and $U(1)$-gauge field the collapse leads to the Schwarzschild black hole rather than the collapse of the charged field in Einstein-Maxwell gravity—though, when one puts the coupling constant to zero we obtain the behavior leading to a black hole with a Cauchy horizon.

Treating a nonlinear system coupled to gravity is a very difficult problem but it is worth mentioning that it was found in Refs. [20–24] that the self-gravitating Nielsen-Olesen vortex can act as a long hair for various kinds of black holes. In what follows, we refine our attention to the vortex itself and elaborate its behavior in the background of a Euclidean dilaton black hole in the theory with the arbitrary coupling constant α. To begin with we choose X and P fields provided by the expressions

$$\Phi(x^i) = \eta X(r) e^{iN\tau\kappa}, \tag{10}$$

$$B_\mu(x^i) = \frac{\kappa}{e_R}[P_\mu(r) - N\nabla_\mu \tau], \tag{11}$$

where $\kappa = \frac{1}{2}[\partial_r A^2(r)]_{r=r_+}$ is the surface gravity of the Euclidean dilaton black hole. Further, we assume that B_τ is the only nonvanishing coefficient of the gauge field which is subject to the spontaneous symmetry breaking.

Let us introduce quantities defined by

$$\sqrt{\lambda}\eta(M, Q_{\rm BH}, r, r_+, r_-, \kappa) \equiv (\bar{M}, \bar{Q}_{\rm BH}, \bar{r}, \bar{r}_+, \bar{r}_-, \bar{\kappa}). \tag{12}$$

Taking the background solution as the spacetime of the Euclidean charged dilaton black hole, one reaches the following equations of motion for X and P fields:

$$\frac{1}{C^2}(C^2 A^2 X_{,\bar{r}})_{,\bar{r}} - \bar{\kappa}^2 \frac{P^2 X}{A^2} - \frac{1}{2}X(X^2 - 1) = 0, \tag{13}$$

$$\frac{1}{C^2}(C^2 P_{,\bar{r}})_{,\bar{r}} - \frac{1}{\nu}\frac{X^2 P}{A^2} = 0, \tag{14}$$

where we denoted $\nu = \frac{\lambda}{2e_R^2}$. The above equations can be rearranged in the forms which imply

$$\left(1 - \frac{\bar{r}_+}{\bar{r}}\right)\left(1 - \frac{\bar{r}_-}{\bar{r}}\right)^{(1-\alpha^2)/(1+\alpha^2)} \frac{d^2}{d\bar{r}^2}X + \left\{\frac{2}{\bar{r}} + \frac{2\alpha^2}{1+\alpha^2}\frac{\bar{r}_-}{\bar{r}} \times \frac{1}{\bar{r} - \bar{r}_-} + \frac{1}{\bar{r}^2}\left(1 - \frac{\bar{r}_-}{\bar{r}}\right)^{(1-\alpha^2)/(1+\alpha^2)} \times \left[\bar{r}_+ + \bar{r}_- \frac{1-\alpha^2}{1+\alpha^2}\frac{\bar{r} - \bar{r}_+}{\bar{r} - \bar{r}_-}\right]\right\}\frac{d}{d\bar{r}}X - \bar{\kappa}^2\left(\frac{\bar{r}}{\bar{r} - \bar{r}_+}\right)\left(\frac{\bar{r}}{\bar{r} - \bar{r}_+}\right)^{(1-\alpha^2)/(1+\alpha^2)} P^2 X - \frac{1}{2}X(X^2 - 1) = 0, \tag{15}$$

$$\frac{d^2}{d\bar{r}^2}P + \left\{\frac{2}{\bar{r}} + \frac{2\alpha^2}{1+\alpha^2}\frac{\bar{r}_-}{\bar{r}}\frac{1}{\bar{r} - \bar{r}_-}\right\}\frac{d}{d\bar{r}}P - \left(\frac{\bar{r}}{\bar{r} - \bar{r}_+}\right)\left(\frac{\bar{r}}{\bar{r} - \bar{r}_-}\right)^{(1-\alpha^2)/(1+\alpha^2)} \frac{PX^2}{\nu} = 0. \tag{16}$$

We solve this set of equations numerically using the relaxation technique [35]. As in Ref. [32] we shall work in the so-called supersymmetric limit when $\nu = 1$. The behavior of the fields in question is depicted in Figs. 1 and 2, respectively. For the completeness of the studies we also plotted in Fig. 3 the α dependence of the surface gravity. It can be seen that the bounded solution at the event horizon for the P field integrated out the exponentially decaying at infinity while the X field tends to the constant value equal to 1 at infinity. We take the coupling constant equal to 0.0, 0.5, 1.0, 1.5, respectively. On the other hand, the charge of the considered black hole was taken as $Q_{\rm BH} = 0.96 Q_{\rm BH\ max}$, where $Q_{\rm BH\ max} = \sqrt{1+\alpha^2}M$. In our numerical analysis we set r/r_+ as the radial coordinate for the Euclidean black hole in question. For the flat spacetime we chose the ordinary r coordinate. We obtained a perfect agreement with the previous numerical studies for the

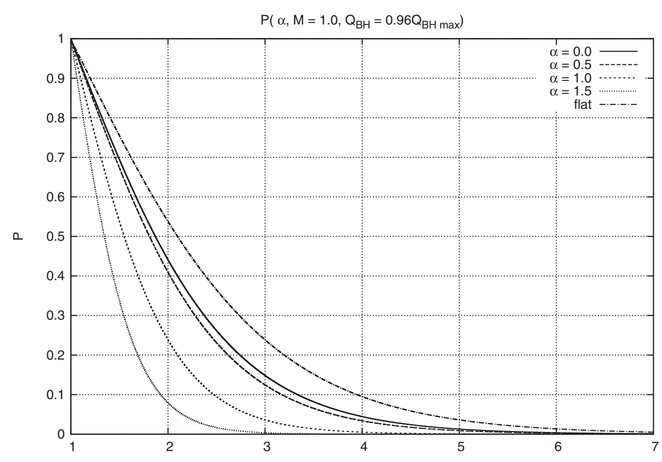

FIG. 1. Plot of the gauge field P for the different values of the coupling constant α. $Q_{\rm BH\ max}$ is the charge of the extreme black hole, given by $Q_{\rm BH\ max} = \sqrt{1+\alpha^2}M$. The cosmic string winding number is equal to 1.0.

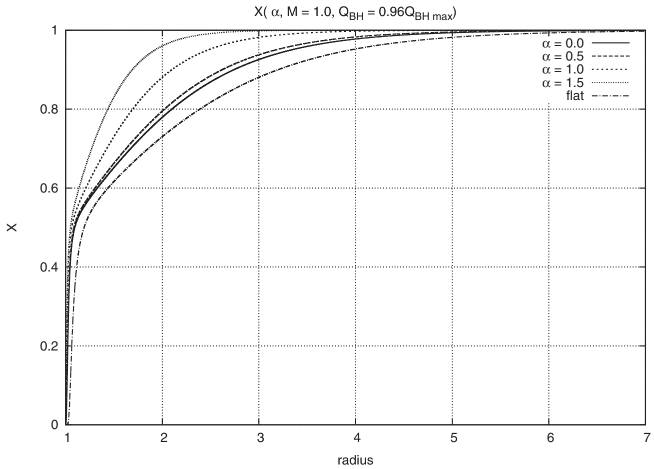

FIG. 2. Plot of the gauge X for the different values of the coupling constant α and for the winding number $N = 1.0$.

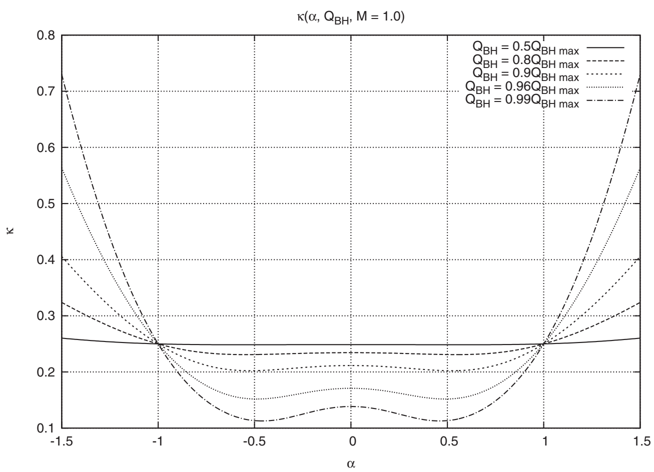

FIG. 3. Behavior of the surface gravity κ for the different values of the coupling constant α.

case $\alpha = 0$, Ref. [30], and when $\alpha = 1$, Ref. [31]. Our numerical investigations reveal that the larger coupling constant α is, the quicker field P tends to zero. For the other vortex field X the conclusion is similar; i.e., the larger the value of α one considers, the quicker field X tends to the constant value equal to 1.

III. FERMIONS IN THE EUCLIDEAN DILATON BLACK HOLE BACKGROUND

In Ref. [33] it was shown that cosmic strings can behave like superconductors and are able to carry electric currents. In principle there are two varieties of superconducting strings, i.e., fermionic or bosonic. In the fermionic case superconductivity takes place due to the appearance of charged Jackiw-Rossi [36] zero modes which effectively can be regarded as Nambu-Goldstone bosons in $1 + 1$ dimensions. They give a longitudinal component to the photon field on the cosmic string, and may be trapped in the string as massless zero modes. On the other hand, bosonic superconductivity occurs when a charged Higgs field acquires an expectation value in the core of the cosmic string. In this case the current in the aforementioned object is carried by bosonic modes.

In this section we shall consider a fermionic superconducting cosmic string piercing a Euclidean charged dilaton black hole. As was mentioned above, it turned out that it is possible for the currents to be carried by fermionic degrees of freedom confined to the cosmic string core [33]. If one takes into account the electromagnetic one, then the cosmic string will behave as superconducting. It can be performed by extending the $U(1) \times U(1)$ Lagrangian by adding the following fermionic sector:

$$S_{\mathrm{FE}} = \int \sqrt{-g} d^4x [\bar{\psi}\gamma^\mu D_\mu \psi + \bar{\chi}\gamma^\mu D_\mu \chi + i\tilde{\alpha}(\Phi\psi^T C\chi - \Phi^* \bar{\psi} C \bar{\chi}^T)], \tag{17}$$

where $\tilde{\alpha}$ is a coupling constant while the Dirac operator satisfies the relation of the form as

$$D_\mu = \nabla_\mu + iRe_R B_\mu + iQe_q A_\mu. \tag{18}$$

We take that $A_\mu = A_\phi = -q_M \cos\theta$. The covariant derivative for spinor fields is given by the standard relation $\nabla_\mu = \partial_\mu + \frac{1}{2}\omega_\mu^{ab}\gamma_a\gamma_b$. Dirac gamma matrices forming the chiral basis for the problem in question yield

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^a = \begin{pmatrix} 0 & i\sigma^a \\ -i\sigma^a & 0 \end{pmatrix}, \tag{19}$$

where the Pauli matrices are provided by

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{20}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{21}$$

Moreover, the charge conjugation matrix implies

$$C = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \tag{22}$$

$$C^\dagger = C^T = -C. \tag{23}$$

In what follows we consider the line element of the Euclidean spherically symmetric static black hole with a vortex passing through it. Its metric is given by (4), with the line element on the S^2 sphere defected by the presence of the vortex. Hence, it yields

$$d\Omega^2 = C^2(r)(d\theta^2 + \tilde{b}^2 \sin\theta^2 d\phi^2), \tag{24}$$

where $\tilde{b} = 1 - 4\mu$ is a cosmic string parameter.

For the above metric the curved spacetime gamma matrices are related to those given by Eq. (19) by the relations

$$\gamma^\tau = A^{-1}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^r = B^{-1}\begin{pmatrix} 0 & i\sigma^1 \\ -i\sigma^1 & 0 \end{pmatrix}, \tag{25}$$

$$\begin{aligned} \gamma^\theta &= C^{-1}\begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \\ \gamma^\phi &= \frac{1}{C\tilde{b}\sin\theta}\begin{pmatrix} 0 & i\sigma^3 \\ -i\sigma^3 & 0 \end{pmatrix}. \end{aligned} \tag{26}$$

In the spacetime under consideration spinors ψ and χ and their complex conjugations must be regarded as independent fields. Following this idea, variation of the fermion action S_{FE} with respect to ψ and χ fields implies the following equations of motion,

$$\begin{aligned} \gamma^\mu D_\mu \psi - i\tilde{\alpha}\Phi^* C\bar{\chi}^T = 0, \\ \gamma^\mu D_\mu \bar{\chi}^\dagger - i\tilde{\alpha}\Phi^* C\psi^* = 0, \end{aligned} \tag{27}$$

plus the analogous relations achieved by conjugations of the adequate relations in question.

With the above definitions one finds the exact form of the Dirac operator. Accordingly, it yields the result that

$$\not{D} = \gamma^\mu D_\mu = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}, \tag{28}$$

where we have denoted by D^+ and D^- the following parts of the Dirac operator defined above:

$$D^+ = \sigma^\tau D_\tau + i\sigma^k D_k, \tag{29}$$

$$D^- = \sigma^\tau D_\tau - i\sigma^j D_j. \tag{30}$$

Consequently, with the remark that in Euclidean spacetime ψ and $\bar{\psi}$ must be treated as independent fields, we implicitly choose ψ and χ as left-handed, while $\bar{\psi}$ and $\bar{\chi}$ are right-handed. Namely, they can be brought to the forms

$$\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \qquad \chi = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}, \tag{31}$$

$$\bar{\psi} = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}, \qquad \bar{\chi} = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix}. \tag{32}$$

Returning to the equations of motion, they can be rewritten as

$$\begin{aligned} D^+\psi_R + i\tilde{\alpha}\Phi^*(-i\sigma^2\chi_L^*) = 0, \\ D^-\chi_L + i\tilde{\alpha}\Phi^*(i\sigma^2\psi_R^*) = 0. \end{aligned} \tag{33}$$

As in Ref. [32] we choose the following forms of ψ and χ spinors:

$$\psi_L = f_L\xi_-, \qquad \chi_R = g_R\xi_+, \tag{34}$$

$$\psi_R = f_R\xi_-, \qquad \chi_L = g_L\xi_+, \tag{35}$$

where $\xi_\pm$ we choose as the right- and left-handed ones. They obey the relation of the form as $i\sigma^2\sigma^3\xi_\pm = \pm\xi_\pm$, which ensures that fermions propagate along the cosmic string. On the other hand, taking into account the normalization conditions $\langle\xi_\pm|\xi_\pm\rangle = 1$, one arrives at the explicit form of $\xi_\pm$. Namely, they may be written in the form as

$$\xi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \xi_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{36}$$

By virtue of the above, one can readily verify that they satisfy the following:

$$i\sigma^1\xi_\pm = \mp i\xi_\pm, \qquad -i\sigma^2\xi_\pm = \pm\xi_\mp, \tag{37}$$

$$\sigma^3\xi_\pm = \xi_\mp, \qquad \sigma^0\xi_\pm = \xi_\pm. \tag{38}$$

IV. *S* WAVES

First of all, we shall elaborate the s-wave case. The behavior of the Dirac operator acting on the S^2 sphere with a magnetic monopole and pierced by an Abelian vortex will be crucial in this case. Namely, on evaluating the action of the Dirac operator D_{S^2} on the ξ_+ spinor, we find that it is proportional to ξ_-. Having in mind the orthogonality condition for $\xi_\pm$ one can draw the conclusion that the only admissible eigenvalue of k is $k = 0$. Taking into account the explicit form of the Dirac operators and Eqs. (34)–(37), after a little algebra, equations of motion (33) reduce to

$$\begin{aligned} A^{-1}(i\partial_\tau + R\kappa(P-N))f_R^* + \Big(B^{-1}\partial_r - \frac{1}{2}A^{-1}B^{-1}\partial_r A \\ + B^{-1}C^{-1}\partial_r C\Big)f_R^* + \tilde{\alpha}\Phi g_L = 0, \end{aligned} \tag{39}$$

and

$$\begin{aligned} A^{-1}(-i\partial_\tau + R\kappa(P-N))g_L + \Big(B^{-1}\partial_r + \frac{1}{2}A^{-1}B^{-1}\partial_r A \\ + B^{-1}C^{-1}\partial_r C\Big)g_L + \tilde{\alpha}\Phi^* f_R^* = 0. \end{aligned} \tag{40}$$

In what follows we assume that the time dependence of g_L and f_R will be of the form $g_L(\tau, r) = e^{-i\omega_g\tau}g_L(r)$ and $f_R(\tau, r) = e^{-i\omega_f\tau}f_R(r)$, where $\omega_{f/g}\epsilon\mathbb{C}$. The explicit form of the field Φ given by Eq. (10), as well as Eqs. (39) and (40), enables us to deduce that $e^{-i\omega_g\tau} = e^{-iN\kappa\tau}e^{i\omega_f^*\tau}$. Consequently, it leads to the condition

$$\omega_g = N\kappa - \omega_f^*. \tag{41}$$

As was argued in Ref. [32] the action of the operators R and Q appearing in the definition of the covariant derivative given by Eq. (18) can be defined as $Rf = \tilde{r}f$ and $Rg = -(\tilde{r}+1)g$, while $Qf = qf$ and $Qg = -qg$. On this account it is customary to rewrite equations of motion as follows:

$$A^{-1}(-\omega_f^* + \tilde{r}\kappa(P-N))f_R^* + \left(B^{-1}\partial_r - \frac{1}{2}A^{-1}B^{-1}\frac{d}{dr}A + B^{-1}C^{-1}\frac{d}{dr}C\right)f_R^* + m_{\rm fer}Xg_L = 0, \quad (42)$$

$$A^{-1}(-N\kappa + \omega_f^* - (\tilde{r}+1)\kappa(P-N))g_L + \left(B^{-1}\partial_r + \frac{1}{2}A^{-1}B^{-1}\frac{d}{dr}A + B^{-1}C^{-1}\frac{d}{dr}C\right)g_L + m_{\rm fer}Xf_R^* = 0, \quad (43)$$

where for brevity we have denoted $m_{\rm fer} = \tilde{\alpha}\eta$.

A. Nonextremal Euclidean dilaton black hole

First we elaborate the behavior of Dirac fermions in the vicinity of the black hole event horizon. We shall begin with considering the nonextremal case of the dilaton Euclidean black hole. Condition $A(r_+) = 0$ determines the outer black hole event horizon, while $C^2(r_+) = \frac{\mathcal{A}}{4\pi}$, $\mathcal{A}$ is the area of the event horizon. Having in mind the form of the line element (4), we assume the regularity of the solution at r_+. On this account, we can choose locally cylindrical coordinates in which the aforementioned metric is regular. Namely, one has that $\hat{r} = \beta_E/2\pi A(r)$, where β_E is the period of the Euclidean time. Moreover, regularity yields that $(A^2)'_{|r_+} = \frac{4\pi}{\beta_E}$. To proceed further, let us suppose that f_R^* and g_L are provided by the following:

$$f_R^* = x_+(\hat{r})\exp\left(\int\left(\frac{1}{2}\frac{(P-N)\kappa}{A} - m_{\rm fer}X\right)d\hat{r} + i\omega_f^*\tau\right), \quad (44)$$

$$g_L = x_-(\hat{r})\exp\left(\int\left(\frac{1}{2}\frac{(P-N)\kappa}{A} - m_{\rm fer}X\right)d\hat{r} - i\omega_g^*\tau\right). \quad (45)$$

It helps us to rewrite the equations of motion (42) and (43) in the form

$$\frac{d}{d\hat{r}}x_\pm \pm \left(\tilde{r} + \frac{1}{2}\right)\frac{(P-N)}{\hat{r}}x_\pm - \frac{1}{\hat{r}}\left(\omega_\pm \pm \frac{1}{2}\right) \pm m_{\rm fer}X(x_- - x_+) = 0, \quad (46)$$

where we set $\omega_+ = \omega_f^*/\kappa$ and $\omega_- = \omega_g/\kappa$.

One can draw a conclusion that f_R^* and g_L are of the order of unity as we reach the event horizon, i.e., $\hat{r} \to 0$. They can be regarded as hair on the Euclidean dilaton black hole in the sense of the nontrivial field configurations maintained by the black hole event horizon. The same observation was revealed in the case of the Euclidean RN black hole solution pierced by superconducting string [32]. Returning to the relations (46), we observe that they resemble equations of motion obtained for cosmic string with fermion modes in flat Minkowski spacetime [32].

An alternative way of treating the problem is to expand the metric coefficients of the considered line element in the vicinity of the black hole event horizon. They will be given by the following:

$$A^2(r) \simeq a(r_+)(r-r_+), \qquad B^2(r) \simeq b(r_+)(r-r_+)^{-1}, \qquad C^2(r) = C^2(r_+). \quad (47)$$

Next, changing the variables described by the relations

$$\rho^2 = 4b(r_+)(r-r_+), \qquad T = \frac{1}{2}\sqrt{\frac{a(r_+)}{b(r_+)}}\tau, \quad (48)$$

it can be shown that the line element of the nonextremal Euclidean black hole yields

$$ds^2 = \rho^2 dT^2 + d\rho^2 + C^2(r_+)d\Omega^2. \quad (49)$$

On the other hand, the asymptotic behavior of the background solutions subject to the equations of motion for Abelian vortex fields are provided by [20,32]

$$X \sim (r-r_+)^{|N|/2} = \rho^{|N|}, \qquad P \sim N - \mathcal{O}(r-r_+) = N + \mathcal{O}(\rho^2). \quad (50)$$

Returning to the equations of motion for the Dirac fermion, one can easily verify by the above relations that they reduce to the forms

$$\frac{d}{d\rho}f_R^* - \left(\frac{\omega_f^* + \frac{1}{2}}{\rho}\right)f_R^* + m_{\rm fer}\rho^{|N|}g_L = 0, \quad (51)$$

$$\frac{d}{d\rho}g_L + \left(\frac{-N\kappa + \omega_f^* + \frac{1}{2}}{\rho}\right)g_L + m_{\rm fer}\rho^{|N|}f_R^* = 0. \quad (52)$$

It will be interesting to consider the influence of the winding number N on the behavior of the fermion modes in question. We shall elaborate two limiting cases of the aforementioned problem, i.e., the case when $|N| \gg 1$ and the case for which the winding number tends to 1.

We shall begin with the case $|N| \gg 1$. The close inspection of formulas (51) and (52) reveals that the mass term proportional to $\rho^{|N|}$ can be neglected because of the fact that $\rho \to 0$ near the black hole event horizon. On this account, one has

$$\frac{d}{d\rho}f_R^* - \left(\frac{\omega_f^* + \frac{1}{2}}{\rho}\right)f_R^* = 0, \qquad \frac{d}{d\rho}g_L + \left(\frac{-N\kappa + \omega_f^* + \frac{1}{2}}{\rho}\right)g_L = 0. \quad (53)$$

It can be easily checked that the solutions of the above set of differential equations imply

$$f_R^* = c_1\rho^{\omega_f^* + (1/2)}, \qquad g_L = c_2\rho^{N\kappa - \omega_f^* - (1/2)}, \quad (54)$$

where c_1 and c_2 are constants.

Consistent with the requirement of the finiteness of the solutions in question on the black hole event horizon we arrive at the condition

$$N\kappa - \frac{1}{2} \geq \mathrm{Re}(\omega_f) \geq -\frac{1}{2}. \tag{55}$$

It could be also verified, by the direct calculations, that g_L and f_R belong to the square integrable class of functions, i.e., $\int_0^\rho \sqrt{g} d\rho |g_L|^2 < \infty$ and $\int_0^\rho \sqrt{g} d\rho |f_R|^2 < \infty$.

In the case when $|N| \sim 1$, equations of motion for the Dirac fermion fields are as follows:

$$\begin{aligned} &\frac{d}{d\rho} f_R^* - \left(\frac{\omega_f^* + \frac{1}{2}}{\rho}\right) f_R^* + m_{\mathrm{fer}} \rho^{|N|} g_L = 0, \\ &\frac{d}{d\rho} g_L + \left(\frac{-\omega_g + \frac{1}{2}}{\rho}\right) g_L + m_{\mathrm{fer}} \rho^{|N|} f_R^* = 0, \end{aligned} \tag{56}$$

where we have used relation (41) to eliminate N dependence in the second term on the left-hand side of Eq. (56). In order to solve Eq. (56), we assume the following *Ansatz* for the spinors f_R^* and g_L:

$$f_R^* = \rho^{\omega_f^* + (1/2)} \tilde{f}, \qquad g_L = \rho^{N\kappa - \omega_f^* - (1/2)} \tilde{g}. \tag{57}$$

Defining $\beta = |N| + N\kappa - 2\omega_f^* - 1$ and after a little of algebra we get

$$\frac{d}{d\rho} \tilde{f} + m_{\mathrm{fer}} \rho^{\beta} \tilde{g} = 0, \qquad \frac{d}{d\rho} \tilde{g} + m_{\mathrm{fer}} \rho^{2|N| - \beta} \tilde{f} = 0. \tag{58}$$

From the first equation we find that $\tilde{g} = -1/m_{\mathrm{fer}} \rho^{-\beta} \frac{d}{d\rho} \tilde{f}$. Substituting the expression for $\frac{d}{d\rho} \tilde{f}$ into the second equation of the underlying system, we reach the second order differential equation for $\tilde{f}$; i.e., one gets

$$\frac{d}{d\rho}\left(\rho^{-\beta} \frac{d}{d\rho} \tilde{f}\right) - m_{\mathrm{fer}}^2 \rho^{2|N| - \beta} \tilde{f} = 0. \tag{59}$$

Having in mind the explicit form of f_R^* and g_L given by (57), and the condition for β, one finds that

$$N\kappa - \frac{1}{2} \geq \mathrm{Re}(\omega_f) \geq -\frac{1}{2}. \tag{60}$$

After some further calculations in which we make use of the so-called Lommel's transformation for Bessel functions, i.e., we look for the solution of Eq. (59) in the form of

$$\tilde{f} = \rho^p G_\nu(\lambda \rho^a), \tag{61}$$

where G_ν is the Bessel function while p, λ, a are constants, it can be shown that the solution of (59) can be written in the form of

$$\begin{aligned} \tilde{f} = {} & C_1 \rho^{(1+\beta)/2} I_\nu\left(\frac{im}{N+1} \rho^{N+1}\right) \\ & + C_2 \rho^{(1+\beta)/2} K_\nu\left(\frac{im}{N+1} \rho^{N+1}\right), \end{aligned} \tag{62}$$

where C_1 and C_2 are constants while $\nu = 1 + \beta/2(N+1)$. I_ν stands for the modified Bessel function of the first kind while K_ν is the Macdonald's function. Assuming that $C_2 = 0$ and having in mind the behavior of I_ν when $\rho \to 0$, one can conclude that the spinor function near the event horizon tends to the constant value described by I_0. This condition leads us to $\beta = -1$ and to the conclusion that for $\omega_f^* = N(1+\kappa)/2$ one gets the greatest value of hair near the black hole event horizon.

B. Extremal dilaton black hole

In the following section we shall discuss zero fermion modes in the background of the extremal Euclidean dilaton black hole with a vortex passing through it. In the extreme black hole case the outer event horizon coincides with the inner one, i.e., $r_- = r_+$. In order to describe the system we use coordinates given by relation (47)with the only modification in the $C^2(r)$ coefficient. Because of the condition for the black hole is an extremal one, we have

$$C^2(r) = c(r_+)(r - r_+). \tag{63}$$

By virtue of this relation the line element describing the near-horizon geometry of the extremal Euclidean dilaton black hole yields

$$ds^2 = \rho^2 dT^2 + d\rho^2 + \frac{c(r_+)}{4b(r_+)} \rho^2 d\Omega^2. \tag{64}$$

Thus, in this picture, equations of motion are provided by

$$\begin{aligned} &\frac{d}{d\rho} f_R^* + \left(-\rho^{-1}\omega_f^* - \frac{1}{2}\rho^{-1} + \rho^{-1}\right) f_R^* + m_{\mathrm{fer}} \rho^N g_L = 0, \\ &\frac{d}{d\rho} g_L + \left(\rho^{-1}(-N\kappa + \omega_f^*) + \frac{1}{2}\rho^{-1} + \rho^{-1}\right) g_L \\ &\quad + m_{\mathrm{fer}} \rho^N f_R^* = 0. \end{aligned} \tag{65}$$

As in the previous case of the nonextremal Euclidean dilaton black hole, we first we consider the $|N| \gg 1$ case. The mass term proportional to $\rho^{|N|} \sim 0$ will tend to zero. Then, the solutions of (65) may be written in the following form:

$$f_R^* = d_1 \rho^{\omega_f^* - (1/2)}, \qquad g_L = d_2 \rho^{N\kappa - \omega_f^* - (3/2)}, \tag{66}$$

where d_1 and d_1 are constants. The finiteness and the regularity conditions on the event horizon imply that the following conditions are satisfied:

$$N\kappa - \frac{3}{2} \geq \mathrm{Re}(\omega_f) \geq \frac{1}{2}. \tag{67}$$

On the other hand, in the case when $|N| \sim 1$, equations of motion may be rewritten in the form of

$$\frac{d}{d\rho} f_R^* + \left(\frac{1 - \omega_f^* - \frac{1}{2}}{\rho}\right) f_R^* + m_{\rm fer}\rho^N g_L = 0, \qquad \frac{d}{d\rho} g_L + \left(\frac{\frac{3}{2} - N\kappa + \omega_f^*}{\rho}\right) g_L + m_{\rm fer}\rho^N f_R^* = 0. \tag{68}$$

Introducing the *Ansatz* for fermion zero modes given by

$$f_R^* = \rho^{\omega_f^* - (1/2)} \tilde{f}, \qquad g_L = \rho^{N\kappa - \omega_f^* - (3/2)} \tilde{g}, \tag{69}$$

we obtain the same set of equations as described by relations (58). Having in mind formula (69) and the requirement of the finiteness of $\tilde{f}$ and $\tilde{g}$ on the black hole event horizon, we reach the condition for $\mathrm{Re}(\omega_f)$. Thus, the result is provided by

$$N\kappa - \frac{3}{2} \geq \mathrm{Re}(\omega_f) \geq \frac{1}{2}. \tag{70}$$

For the completeness of our research we remark that by the same procedure as we followed in the case of the nonextremal Euclidean black hole we can cast the set of the first order ordinary differential equations into the second order one for $\tilde{f}$, which has a solution in terms of generalized Bessel functions. The range of the parameters and conclusions about hair on the extremal Euclidean dilaton black hole is the same as in the nonextremal case.

V. $k > 0$ DIRAC FERMION MODES

Now, we turn our attention to the case when $k > 0$. Our main aim will be to solve Eq. (33) and to discuss the behavior of Dirac fermions in the case in question. Spinors ψ and χ are singled out in such a way to correspond with the case $k = 0$. Namely, one has that $i\gamma^2\gamma^3\tilde{\xi}_\pm = \pm\tilde{\xi}_\pm$, where $\tilde{\xi}$ is a linear combination of ξ. It can be done by preferring the basis in the form

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & i\sigma^3 \\ -i\sigma^3 & \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & i\sigma^1 \\ -i\sigma^1 & \end{pmatrix}. \tag{71}$$

Thus, the zero mode condition can be cast into $i\sigma^2\sigma^1\tilde{\xi}_\pm = \pm\tilde{\xi}_\pm$, which provides the following relation:

$$\tilde{\xi}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \tilde{\xi}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{72}$$

In the case under consideration, the exact form of the Dirac operators will be given by the expressions

$$D^+ = A^{-1}(\partial_\tau + iR(P-N)\kappa)\sigma^0 + \left(B^{-1}\partial_r - \frac{1}{2}B^{-1}A^{-1}\partial_r A + B^{-1}C^{-1}\partial_r C\right)i\sigma^3 + \frac{D_{S^2}}{C}, \tag{73}$$

$$D^- = A^{-1}(\partial_\tau + iR(P-N)\kappa)\sigma^0 + \left(-B^{-1}\partial_r - \frac{1}{2}B^{-1}A^{-1}\partial_r A - B^{-1}C^{-1}\partial_r C\right)i\sigma^3 + \frac{D_{S^2}}{C}, \tag{74}$$

where the Dirac operator on an S^2 sphere with a magnetic monopole pierced by an Abelian vortex yields

$$D_{S^2} = i\sigma^2\left(\partial_\theta + \frac{1}{2}\cot\theta\right) + \sigma^1\left(\frac{i\partial_\phi}{\tilde{b}\sin\theta} + Qe_M\tilde{b}^{-1}\cot\theta\right). \tag{75}$$

Moreover, the action of the D_{S^2} operator on ψ and χ spinors implies

$$iD_{S^2}\psi = k\psi, \qquad iD_{S^2}\chi = k\chi, \tag{76}$$

where k are eigenvalues of the operator in question. As far as spinors ψ and χ are concerned, we set them to be linear combinations of $\tilde{\xi}_\pm$:

$$\psi_R = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}, \qquad \chi_L = \begin{pmatrix} g_+ \\ g_- \end{pmatrix}. \tag{77}$$

By virtue of the above, equations of motion are provided by the following relations:

$$A^{-1}(\partial_\tau + iR(P-N)\kappa)f_+ + \left(B^{-1}\partial_r - \frac{1}{2}A^{-1}B^{-1}\partial_r A + B^{-1}C^{-1}\partial_r C\right)if_+ - \frac{ik}{C}f_+ - i\alpha\Phi^* g_-^* = 0, \tag{78}$$

$$A^{-1}(\partial_\tau + iR(P-N)\kappa)f_- - \left(B^{-1}\partial_r - \frac{1}{2}A^{-1}B^{-1}\partial_r A + B^{-1}C^{-1}\partial_r C\right)if_- - \frac{ik}{C}f_- + i\alpha\Phi^* g_+^* = 0, \tag{79}$$

$$A^{-1}(\partial_\tau + iR(P-N)\kappa)g_+ + \left(-B^{-1}\partial_r - \frac{1}{2}A^{-1}B^{-1}\partial_r A - B^{-1}C^{-1}\partial_r C\right)ig_+ + \frac{ik}{C}g_+ + i\alpha\Phi^* f_-^* = 0, \tag{80}$$

$$A^{-1}(\partial_\tau + iR(P-N)\kappa)g_- - \left(-B^{-1}\partial_r - \frac{1}{2}A^{-1}B^{-1}\partial_r A - B^{-1}C^{-1}\partial_r C\right)ig_- + \frac{ik}{C}g_- - i\alpha\Phi^* f_+^* = 0. \tag{81}$$

In order to get the regular solutions on the Euclidean black hole event horizon, we take only f_+ and g_- components to be nonzero. Consequently, this choice reduces the number of equations to the following set of equations:

$$A^{-1}(i\partial_\tau + \tilde{r}(P-N)\kappa)f_+^* + \Big(B^{-1}\partial_r - \frac{1}{2}A^{-1}B^{-1}\partial_r A + B^{-1}C^{-1}\partial_r C - \frac{k}{C}\Big)f_+^* - \alpha\Phi g_- = 0, \tag{82}$$

$$A^{-1}(i\partial_\tau + (\tilde{r}+1)(P-N)\kappa)g_- + \Big(-B^{-1}\partial_r - \frac{1}{2}A^{-1}B^{-1}\partial_r A - B^{-1}C^{-1}\partial_r C - \frac{k}{C}\Big)g_- + \alpha\Phi^* f_+^* = 0. \tag{83}$$

The dependence on the Euclidean time will be given by Eq. (41). First, we shall proceed to study equations of motion for the nonextremal Euclidean dilaton black hole. As in the preceding section when $k=0$, we study the behavior of Dirac fermions near the event horizon of the Euclidean dilaton black hole. Suppose that f_+^* and g_- yield

$$f_+^* = x_+(\hat{r})\exp\Big(\int\Big(\frac{1}{2}\frac{(P-N)\kappa}{A} - m_{\rm fer}X\Big)d\hat{r} + i\omega_f^*\tau\Big),$$
$$g_- = x_-(\hat{r})\exp\Big(\int\Big(\frac{1}{2}\frac{(P-N)\kappa}{A} - m_{\rm fer}X\Big)d\hat{r} - i\omega_g\tau\Big), \tag{84}$$

where $\hat{r}$ is defined in the same manner as in the s-wave case. In terms of the above relations the considered set of equations implies

$$\frac{d}{d\hat{r}}x_\pm \pm \Big(\tilde{r}+\frac{1}{2}\Big)\frac{(P-N)}{\hat{r}}x_\pm - \frac{1}{\hat{r}}\Big(\omega_\pm \pm \frac{1}{2}\Big)x_\pm \mp \sqrt{\frac{4\pi}{\mathcal{A}}}kx_\pm - m_{\rm fer}X(x_+ + x_-) = 0, \tag{85}$$

where we set $\omega_+ = \frac{\omega_f^*}{\kappa}$ and $\omega_- = \frac{\omega_g}{\kappa}$. As we take into account that near the black hole event horizon $A(r) \simeq 2\pi\hat{r}/\beta_E$, one concludes that the situation is similar to the zero mode case and the spinors in question are of order of unity. Thus, they constitute hair on the considered Euclidean black hole.

As in Sec. IVA, to proceed further, we expand the line element in the vicinity of the black hole event horizon. The corresponding relations for spinors g_- and f_+^* may be written as

$$\frac{d}{d\rho}g_- + \Big(\frac{\omega_f^* - N\kappa}{\rho} + \frac{1}{2\rho} + \frac{k}{C(r_+)}\Big)g_- - m_{\rm fer}\rho^{|N|}f_+^* = 0,$$
$$\frac{d}{d\rho}f_+^* - \Big(\frac{\omega_f^*}{\rho} + \frac{1}{2\rho} + \frac{k}{C(r_+)}\Big)f_+^* - m_{\rm fer}\rho^{|N|}g_- = 0. \tag{86}$$

One elaborates two cases of the cosmic string winding number. We shall begin with $|N| \gg 1$. One can neglect the mass term in relations (86) and the adequate solutions can be expressed in terms of

$$f_+^* = C_1\rho^{\omega_f^*+(1/2)}e^{(k/C(r_+))\rho}, \tag{87}$$

$$g_- = C_2\rho^{-\omega_f^*+N\kappa-(1/2)}e^{-(k/C(r_+))\rho}, \tag{88}$$

where we denote C_1 and C_2 as constants. On the other hand, the requirement of regularity on the black hole event horizon implies the following:

$$N\kappa - \frac{1}{2} \geq {\rm Re}(\omega_f) \geq -\frac{1}{2}. \tag{89}$$

One can remark that it has the same form as in the $k=0$ case.

Proceeding to the case of $|N| \sim 1$ it turned out that equations of motion can be simplified by setting the following *Ansätze* for the Dirac spinors in question:

$$g_- = \rho^{-\omega_f^*+N\kappa-(1/2)}e^{-(k/C(r_+))\rho}\tilde{g},$$
$$f_+^* = \rho^{\omega_f^*+(1/2)}e^{(k/C(r_+))\rho}\tilde{f}. \tag{90}$$

Consequently, one can readily verify that we arrive at the expressions

$$\frac{d}{d\rho}\tilde{g} - m_{\rm fer}\rho^{2|N|-\beta}e^{\int b d\rho}\tilde{f} = 0, \tag{91}$$

$$\frac{d}{d\rho}\tilde{f} - m_{\rm fer}\rho^{\beta}e^{-\int b d\rho}\tilde{g} = 0, \tag{92}$$

where we have denoted $b = k/C(r_+)$.

As in the previous considerations, extracting from Eq. (92) $\tilde{g}$ and setting in the remaining expression enables one to obtain the second order differential equation for $\tilde{f}$. Namely, it has the form

$$\frac{d}{d\rho}\Big(e^{\int b d\rho}\rho^{-\beta}\frac{d}{d\rho}\tilde{f}\Big) - m_{\rm fer}^2\rho^{2|N|-\beta}e^{\int b d\rho}\tilde{f} = 0. \tag{93}$$

Consequently, from the relations in (90) we have the following:

$$N\kappa - \frac{1}{2} \geq {\rm Re}(\omega_f) \geq -\frac{1}{2}. \tag{94}$$

Summing our results for the nonzero Dirac fermion modes in the spacetime of the nonextremal Euclidean dilaton black hole with a superconducting fermion vortex, we draw a conclusion that for both $N \gg 1$ and $N \sim 1$, $k > 0$ does not modify the intervals of admissible values of ω_f. We attain the same conditions on ${\rm Re}(\omega_f)$ as in the case when $k=0$. Namely, relations (55) and (89), as well as Eqs. (60) and (94), have the same forms. However, despite the fact that the form of Eq. (93) is similar to that studied before, it seems that it has no solution in terms of the known special functions.

Extremal dilaton black hole and nonzero fermion modes

In what follows, we shall establish some main features of the behavior of nonzero Dirac fermion modes in the vicinity of the event horizon of an extremal Euclidean dilaton black hole pierced by a vortex. Using the coordinate transformation (47) and (63) we arrive at the following set of the first order differential equations for g_- and f_+^*:

$$\frac{d}{d\rho} g_- + \left(\frac{\omega_f^* - N\kappa + \frac{3}{2} + \tilde{k}}{\rho}\right) g_- - m_{\rm fer}\rho^{|N|} f_+^* = 0, \tag{95}$$

$$\frac{d}{d\rho} f_+^* + \left(\frac{\omega_f^* + \frac{1}{2} - \tilde{k}}{\rho}\right) f_+^* - m_{\rm fer}\rho^{|N|} g_- = 0, \tag{96}$$

where we have denoted $\tilde{k} = \sqrt{\frac{4b(r_+)}{c(r_+)}}k$.

As in the previous sections we start with the case of $|N| \gg 1$ and neglect the mass term. Next, we take into account the *Ansatz* for Dirac fermions

$$g_- = D_1 \rho^{-\omega_f^* + N\kappa - (3/2) - \tilde{k}}, \tag{97}$$

$$f_+^* = D_2 \rho^{\omega_f^* - (1/2) + \tilde{k}}, \tag{98}$$

where D_1 and D_2 are constants. Inspection of the resultant equations of motion leads us to the conclusion that the admissible range of $\mathrm{Re}(\omega_f)$ is given by

$$N\kappa - \frac{3}{2} - \tilde{k} \geq \mathrm{Re}(\omega_f) \geq \frac{1}{2} - \tilde{k}. \tag{99}$$

Next, our attempts will be to study the case when $|N| \sim 1$. Without loss of generality, we assume the following *Ansatz*:

$$g_- = \rho^{-\omega_f^* + N\kappa - (3/2) - \tilde{k}} \tilde{g}, \tag{100}$$

$$f_+^* = \rho^{\omega_f^* - (1/2) + \tilde{k}} \tilde{f}. \tag{101}$$

Hence, the underlying equations of motion simplify to the form

$$\frac{d}{d\rho}\tilde{g} - m_{\rm fer}\rho^{2|N| - \tilde{\beta}} \tilde{f} = 0, \tag{102}$$

$$\frac{d}{d\rho}\tilde{f} - m_{\rm fer}\rho^{\tilde{\beta}} \tilde{g} = 0, \tag{103}$$

where now we set

$$\tilde{\beta} = |N| + N\kappa - 1 - 2\tilde{k} - 2\omega_f^*. \tag{104}$$

The same reasoning as in the previous section leads us to the second order differential equation for $\tilde{f}$, which yields

$$\frac{d}{d\rho}\left(\rho^{-\tilde{\beta}} \frac{d}{d\rho}\tilde{f}\right) - m_{\rm fer}^2 \rho^{2|N| - \tilde{\beta}} \tilde{f} = 0. \tag{105}$$

It can be established from Eqs. (100) and (101) that the following is satisfied:

$$N\kappa - \frac{3}{2} - \tilde{k} \geq \mathrm{Re}(\omega_f) \geq \frac{1}{2} - \tilde{k}. \tag{106}$$

It is worth pointing out that in both cases $N \gg 1$ and $N \sim 1$ one has diminishing of the admissible interval of $\mathrm{Re}(\omega_f)$. On the other hand, the form of Eq. (105) is the same as relation (59) so the arguments in the preceding section can be repeated, leading to the conclusion that the solution of (105) may be written as a combination of modified Bessel function of the first kind plus Macdonald's function. We also conclude that the condition for the largest hair in the near-horizon limit will be diminished by the value of $\tilde{k}$. It will be provided by $\omega_f^* = (N(1+\kappa) - 2\tilde{k})/2$.

VI. CONCLUSIONS

In our paper we have analyzed the problem of an Abelian Higgs vortex on the Euclidean dilaton black hole in the presence of Dirac fermion modes. Fermions were coupled to the fields in question as in Witten's model of superconducting string [33]. Assuming the complete separation of the degrees of freedom of the fields in our considerations we examined behavior of Dirac fermion modes in the vicinity of the Euclidean dilaton black hole event horizon. We studied both the case of zero and $k > 0$ Dirac fermion modes. We took into account dilaton theory with arbitrary coupling constant α determining the interaction between the dilaton and $U(1)$-gauge field. Moreover, we elaborated a nonextremal and extremal Euclidean dilaton black hole pierced by a vortex as a background of our considerations.

For zero Dirac fermion modes we obtain a different interval of the $\mathrm{Re}(\omega_f)$ parameter for different kinds of the considered Euclidean black holes. It happened that for a nonextremal one, we get $N\kappa - 1/2 \geq \mathrm{Re}(\omega_f) \geq -1/2$ for the winding number $N \gg 1$ and for $N \sim 1$. On the other hand, for an extremal Euclidean dilaton black hole one has that $N\kappa - 3/2 \geq \mathrm{Re}(\omega_f) \geq 1/2$ for both aforementioned cases of choosing N. Having these in mind, one can draw a conclusion that for the nonextremal black hole the admissible interval is bigger compared to the interval for the extremal black hole. For both kinds of the considered black holes when $N \gg 1$ one reaches effectively massless fermions in the near-horizon region. On the other hand, for the winding number $N \sim 1$ the fermion mass term is not a crucial ingredient of the underlying equations of motion.

For the case when $k > 0$, we did not observe any modification of the range of $\mathrm{Re}(\omega_f)$ for nonextremal black holes. On the contrary, in the case of the extremal Euclidean dilaton black hole, the nonzero value of k diminishes the admissible intervals of $\mathrm{Re}(\omega_f)$.

By virtue of the above, one can readily see that localization of the Dirac fermions strongly depends on the string winding number as well as the value of the black hole surface gravity κ. It just leads to the conclusion that in some situations (the adequate value of κ which is bounded with the black hole mass) the presence of a black hole can destroy superconductivity, in the sense of not satisfying the inequalities for $\mathrm{Re}(\omega_f)$. It turned out that superconductivity was achieved when one had to do this with a small black hole.

To conclude one remarks that spinor fields $\bar{\psi}$ and χ can be regarded as hair on the Euclidean dilaton black hole in the dilaton gravity theory with the arbitrary coupling constant α, for both extremal and nonextremal black holes. Hair on the considered black holes can be understood in the sense that there is a nontrivial spinor field configuration supported by the black hole event horizon.

Moreover, citing the same arguments as presented in Ref. [32] reveals that the background of a magnetically charged Euclidean dilaton black hole pierced by a vortex is a lack of the physical effect of the discrete charge. Fermions' condensate appears outside the event horizon.

Namely, it was stated in [32] that in the case of a Euclidean RN black hole superconducting cosmic string system one has to do this with another $U(1)$-type global symmetry which is orthogonal to the symmetries $U(1)_R$ and $U(1)_Q$ (orthogonal in the sense that the trace of the product of the operators connected with those symmetries is equal to zero). But this global symmetry is anomalous because of the fact that the aforementioned trace of the operators' products is not equal to zero. It leads to the conclusion that this symmetry is spontaneously broken in the background of a black hole and any alleged discrete charge can be absorbed by making a $U(1)$-type transformation. Summing it all up, in the background of magnetically charged black holes one has the lack of the physical effect of the discrete charge. On the contrary, the vicinity of the black hole event horizon is furnished with a fermion condensate violating global anomalous symmetry.

The key point of this phenomenon is that for zero modes there is no contribution to the partition function as well as to the other correlation functions which do not involve the essential number of fermions. There is no contribution to the temperature of a black hole by a discrete electric charge and the screened electric charge does not acquire an exponentially small probability outside the event horizon of the considered black hole. However, this is not the case for $k \neq 0$.

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