Black strings in Hořava-Lifshitz gravity

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We examine a class of cylindrically symmetric solutions in Hořava-Lifshitz gravity. For the relativistic value of the coupling constant, $\lambda = 1$, we find the hedgehog-type static black string solution with the nonvanishing radial shift in the Arnowitt-Deser-Misner-type decomposition of the spacetime metric. With zero radial shift, this solution corresponds to the usual Banados-Teitelboim-Zanelli (BTZ) black string in general relativity. However, unlike the general relativity case, the BTZ-type black strings do naturally exist in Hořava-Lifshitz gravity, without the need for any specific source term. We also find a rotating BTZ-type black string solution which requires the nonvanishing radial shift for its very existence. We calculate the mass and the angular momentum of this solution, using the canonical Hamiltonian approach. Next, we discuss the Lemos-type black string, which is inherent in general relativity with a negative cosmological constant, and present the static metric for any value of $\lambda > 1/3$. Finally, we show that while, for $\lambda = 1$, the entropy of the Lemos-type black string is given by *one quarter* of the horizon area, the entropy of the static BTZ-type black string is *one half* of its horizon area.

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I. INTRODUCTION

Recently, Hořava put forward the idea of gravity endowed with Lifshitz-type anisotropic scaling [1,2]. This is an intriguing attempt to formulate a consistent quantum field theory of gravity in $3 + 1$ dimensions by invoking the anisotropy between space and time, first introduced in condensed matter systems [3]. The degree of the anisotropy given by a number z, the "dynamical critical exponent," plays the role of an important observable in the theory, determining its behavior at short scales. The Hořava-Lifshitz (HL) theory of gravity exhibits an anisotropic scaling with $z = 3$ fixed point at short distances, thereby becomes a power-counting renormalizable in the ultraviolet (UV) regime. Thus, in this approach the classical theory of gravity acquires UV completion, being driven to a quantum field theory of nonrelativistic gravitons in $3 + 1$ dimensions. Meanwhile, at long distances the scaling becomes isotropic, flowing to $z = 1$, and the theory restores its relativistic invariance in the infrared (IR) regime where it resembles, through some relevant deformations, many familiar features of general relativity.

Because of its fundamentally nonrelativistic nature, HL gravity admits a natural description in terms of the Arnowitt-Deser-Misner (ADM)-type variables, appearing in the $3 + 1$ foliation of the spacetime metric in general relativity. These variables form triplet which consists of the spatial metric as a dynamical field, the lapse function and the shift vector. However, unlike in general relativity, the privileged role of time in HL gravity leads to a "preferred foliation" of spacetime by slices of constant time. Consequently, the full spacetime symmetries of the theory reduce to time reparametrization symmetry (space-independent) and spatial diffeomorphisms (time-dependent), which preserve the spacetime foliation. Clearly, the lapse function and the shift vector can be viewed as two gauge fields of the foliation-preserving diffeomorphisms. This fact is also encoded in the physical spectrum of the theory around flat spacetime where an extra scalar polarization of the graviton appears. With the foliation-preserving diffeomorphisms one can naturally assume that the lapse is a function of time alone, while the shift is a spacetime field, thereby fitting the "projectable" theory of foliation [2]. Altogether, these properties form a minimal basis for the realization of anisotropic scaling in gravity. The minimal realization also involves the concept of the "detailed balance" condition. This implies that the potential term in the action is effectively a square of a prepotential, appearing in a one dimension fewer Euclidean theory. In further developments, to improve the physical content of the theory, both the projectability condition and the detailed balance condition were relaxed in a number of cases (see a review [4], for details). Moreover, it was shown that an extension of the foliation-preserving diffeomorphisms by an Abelian gauge symmetry, eliminates the scalar polarization of the graviton that appeared in the minimal realization of the idea of anisotropic scaling [5].

Among possible applications of HL gravity, its phenomenological consequences in our universe are of great importance. It is interesting that the theory results in a new mechanism for scale-invariant cosmological perturbations, even without inflation [6,7]. The early history of the universe is also significantly changed with HL gravity which admits regular cyclic and bouncing solutions [6,8,9]. However, it should be emphasized that HL gravity suffers from a number of inconsistency problems as well. For instance, the scalar mode becomes unstable in the UV regime [8] when keeping the detailed balance condition, but abandoning the projectability condition. There also exist scalar instabilities in the IR regime [10], which may result in strong coupling problems [11–13]. Furthermore, scale-invariant perturbations [14] are generated provided that the detailed balance condition is broken in the UV regime [15]. Another issue is the existence of black hole solutions. In [16], it was shown that the theory admits a static and spherically symmetric AdS-type black hole solution. The asymptotic behavior of this solution is essentially different from that of the Schwarzschild-AdS black hole in general relativity. Meanwhile, the counterpart of the usual asymptotically flat Schwarzschild solution was found in [17] by a relevant deformation of the HL action. This solution turned out to be very useful to figure out the observational consequences of HL gravity in both weak and strong gravity regimes [18,19]. Further, these type of solutions, as well as their certain extension in the framework of the most general spherically symmetric ansatz, were studied in [20–26]. As for the rotating counterparts of these solutions, they still remain unknown. In a recent work [27], some progress in this direction was achieved in the limit of slow rotation (see also Ref. [28]).

In this paper, we examine a class of cylindrically symmetric solutions in HL gravity, which can be thought of as counterparts of black strings in general relativity. In Sec. II, we begin by describing the physical content of HL gravity using the ADM-type decomposition of the spacetime metric and present the equations of motion underlying the theory. In Sec. III, we discuss the general stationary and cylindrically symmetric ansatz for spacetime metric. Focusing on the spacetimes, for which the Cotton tensor in the HL action vanishes, we delineate two intriguing examples of the cylindrically symmetric spacetimes which are the counterparts of those for the Banados-Teitelboim-Zanelli (BTZ) and Lemos types black strings in general relativity [29,30]. The BTZ black strings in general relativity are obtained by adding an extra spacelike flat dimension to the metric of the three-dimensional BTZ black hole [31]. Next, for $\lambda = 1$, we discuss the static BTZ-type black string solutions with zero and nonzero radial shift. In the latter case, we call it the *hedgehog* type solution. In this section, we also present the stationary and cylindrically symmetric solution that describes the BTZ-type rotating black string in HL gravity. This solution is of a hedgehog type as well, since the radial "hair" is inevitable to support the rotational dynamics. We calculate the mass and the angular momentum of this solution, employing the canonical Hamiltonian approach. We further discuss the Lemos-type black string and present the corresponding static solution for any value of the coupling constant $\lambda > 1/3$. In Sec. IV, we examine the thermodynamical properties of the static black string configurations in HL gravity using the Euclidean path integral approach.

II. BASICS OF HORíAVA-LIFSHITZ GRAVITY

The privileged role of time in HL gravity with Lifshitz-type anisotropic scaling makes it fundamentally nonrelativistic and results in a preferred foliation of spacetime by slices of constant time. As a consequences of this, the full spacetime symmetries of the system reduce to the foliation-preserving diffeomorphisms which are generated by

$$t \to \tilde{t}(t), \qquad x^i \to \tilde{x}^i(t, x^i). \tag{1}$$

With this in mind, it natural to employ the ADM-type 3 + 1 decomposition of the spacetime metric. We have

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{2}$$

where the three-dimensional spatial metric g_{ij} is a dynamical field, the lapse function N and the shift vector N^i play the role of gauge fields of diffeomorphisms (1) and therefore one can suppose that they respect the same functional dependence. That is, the lapse is only the function of time, $N = N(t)$, while the shift is a spacetime function, $N^i = N^i(t, x^i)$. We recall that such a decomposition of the spacetime metric corresponds to the projectable version of the HL gravity.

With the metric decomposition in (2), the usual Einstein-Hilbert action decomposes as

$$I_{EH} = \frac{1}{16\pi G}\int dt d^3x \sqrt{g} N(K_{ij}K^{ij} - K^2 + R - 2\Lambda), \tag{3}$$

where G is the gravitational constant, K_{ij} is the extrinsic curvature, $R = g^{ij}R_{ij}$ is the Ricci scalar, Λ is the cosmological constant and

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - D_i N_j - D_j N_i), \qquad K = g^{ij}K_{ij}, \qquad N_i = g_{ij}N^j. \tag{4}$$

Here the dot denotes the derivative with respect to time and D is the derivative operator with respect to the spatial metric g_{ij}.

The action governing the dynamics of HL gravity with the detailed balance condition is given by (see Ref. [2])

$$I = \int dt d^3x \sqrt{g} N\{g_0(K_{ij}K^{ij} - \lambda K^2) + g_1(R - 3\Lambda_W) + g_2 R^2 + g_3 Z_{ij}Z^{ij}\}, \tag{5}$$

where, for further convenience, we have used the notations $Z_{ij} = C_{ij} + g_4 R_{ij}$,

$$g_0 = \frac{2}{\kappa^2}, \qquad g_1 = \frac{\kappa^2\mu^2\Lambda_W}{8(1-3\lambda)}, \qquad g_2 = \frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}, \qquad g_3 = -\frac{\kappa^2}{2\omega^4}, \qquad g_4 = -\frac{\mu\omega^2}{2}. \tag{6}$$

We note that κ, λ, μ and ω are coupling constants of the theory, Λ_W is a three-dimensional cosmological constant. The Cotton tensor C^{ij} is symmetric, traceless and covariantly constant and it is given by

$$C^{ij} = \frac{\epsilon^{ikl}}{\sqrt{g}} D_k\left(R^j{}_l - \frac{1}{4}\delta^j{}_l R\right), \tag{7}$$

where ϵ^{ikl} is the usual Levi-Civita symbol. From action (5) it follows that in HL gravity the speed of light, the Newtonian constant and the cosmological constant appear as emergent quantities. Indeed, taking the IR limit of this action, where the quadratic in curvature terms are omitted, and rescaling the time coordinate as $t \to ct$, we compare the result with the Einstein-Hilbert action in (3). This yields the emergent relations

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda_W}{1-3\lambda}}, \qquad G = \frac{\kappa^2 c^2}{32\pi}, \qquad \Lambda = \frac{3}{2}\Lambda_W. \tag{8}$$

In what follows, we shall focus only on the case of a negative cosmological constant. Then, from the emergent relation for the speed of light, it follows that the dynamical coupling constant of HL gravity λ must obey the inequality $\lambda > 1/3$. We shall also take $c = 1$, without loss of generality.

The equations of motion that follow from action (5) were obtained in [6,16]. Variation of the action with respect to the lapse N yields the Hamiltonian constraint

$$-g_0(K_{ij}K^{ij} - \lambda K^2) + g_1(R - 3\Lambda_W) + g_2R^2 + g_3Z_{ij}Z^{ij} = 0, \tag{9}$$

and its variation with respect to the shift N^i gives us the momentum constraint

$$D_j(K^{ij} - \lambda g^{ij}K) = 0. \tag{10}$$

Meanwhile, variation of the action with respect to the dynamical variable g^{ij} yields the equation of motion given by

$$E_{ij} \equiv g_0(E^{(1)}_{ij} - \lambda E^{(2)}_{ij}) + g_1E^{(3)}_{ij} + g_2E^{(4)}_{ij} + g_3(g_4E^{(5)}_{ij} + E^{(6)}_{ij}) = 0, \tag{11}$$

where

$$\begin{aligned}
E^{(1)}_{ij} &= 2N_{(i}D_{|k|}K^k{}_{j)} - 2K^k{}_{(i}D_{j)}N_k - N^kD_kK_{ij} - 2NK_{ik}K^k{}_j - \frac{1}{2}g_{ij}NK_{kl}K^{kl} + NKK_{ij} + \dot{K}_{ij},\\
E^{(2)}_{ij} &= \left(\frac{1}{2}NK^2 - N^k\partial_kK + \dot{K}\right)g_{ij} + 2N_{(i}\partial_{j)}K, \qquad E^{(3)}_{ij} = \left[R_{ij} - \frac{1}{2}g_{ij}(R - 3\Lambda_W) - D_iD_j + g_{ij}D^2\right]N,\\
E^{(4)}_{ij} &= 2\left(R_{ij} - \frac{1}{4}g_{ij}R - D_iD_j + g_{ij}D^2\right)NR, \qquad E^{(5)}_{ij} = -2D_kD_{(i}[Z_{j)}{}^kN] + D^2(NZ_{ij}) + g_{ij}D_kD_l(NZ^{kl}),\\
E^{(6)}_{ij} &= \left(-\frac{1}{2}g_{ij}Z_{kl}Z^{kl} + 2Z_{ik}Z^k{}_j - 2Z_{k(i}C_{j)}{}^k + g_{ij}Z_{kl}C^{kl}\right)N - D_k[T^{kl}{}_{(i}R_{j)l}] + R^n{}_lD_n[T^{kl}{}_{(i}g_{j)k}] - D^n[T^{kl}{}_ng_{k(i}R_{j)l}]\\
&\quad - D^2D_k[T^{kl}{}_{(i}g_{j)l}] + D^n[g_{l(i}D_{j)}]D_kT^{kl}{}_n + D_lD_{(i}D_{|k|}T^{kl}{}_{j)} + g_{ij}D^nD_kD_lT^{kl}{}_n.
\end{aligned} \tag{12}$$

We note that in these expressions $D^2 = D_iD^i$, $T^{ij}{}_k = N(\sqrt{g})^{-1}\epsilon^{ijl}Z_{lk}$ and round parentheses over indices denote a symmetrization procedure. Despite the fact that these equations look rather complicated, the authors of works [6,16] were the first to find the simple exact solutions to them, using a standard spherically symmetric ansatz for the spacetime metric. In further developments, these type of solutions were also studied in the framework of the most general spherically symmetric metric ansatz (see, for instance, Refs. [23,24]).

III. BLACK STRING SOLUTIONS

In this section, we discuss a class of exact cylindrically symmetric solutions to HL gravity. We begin with the general stationary and cylindrically symmetric metric ansatz in the form

$$ds^2 = (-\tilde{N}^2f + N_rN^r + N_\phi N^\phi)dt^2 + 2(N_rdr + N_\phi d\phi)dt + f^{-1}dr^2 + r^2d\phi^2 + gdz^2, \tag{13}$$

where all the metric functions are assumed to depend on the radial coordinate r alone and we have redefined the lapse function as $N = \tilde{N}\sqrt{f}$ for further convenience. The shift vector $N_i = g_{ij}N^j = \{N_r, N_\phi, 0\}$ and the three-dimensional spatial metric possesses cylindrical symmetry, involving the functions $f = f(r)$ and $g = g(r)$. We note that, just like in the spherically symmetric case [24], the presence of the radial shift in the metric (13) is inherent in HL gravity as the foliation-preserving invariance of the theory is not enough for eliminating it from the metric. That is, in contrast to general relativity, in HL gravity cylindrically symmetric metrics with $N_r = 0$ and $N_r \neq 0$ are not physically equivalent.

In order to simplify the consideration, we focus on the solutions for which the Cotton tensor (7) vanishes. It is straightforward to show that with metric ansatz (13), the only nonvanishing component of this tensor is given by

$$C_{\phi z} = \frac{\sqrt{f}}{8rg^{5/2}}\{rg^2f''(rg' - 2g) + gf'(2g^2 - 2r^2g'^2 + 3r^2gg'') + 2f[r^2g'^3 - r^2(g^2)'g'' - g^2(g' - rg'' - r^2g''')]\}. \tag{14}$$

Here and in what follows, the prime denotes differentiation with respect to r. For $g = \text{const}$, this expression takes the most simple form

$$C_{\phi z} = \frac{\sqrt{fg}}{4r}(f' - rf'') \tag{15}$$

and the equation $C_{\phi z} = 0$ is immediately solved by

$$f = \eta r^2 - m, \tag{16}$$

where η and m are constants of integration. Thus, for constant g and with f given in (16) the Cotton tensor $C_{ij} = 0$. For these solutions, as seen from metric (13), one can set $g = 1$ by rescaling of the z-coordinate. Furthermore, we see that the resulting metric, with $N_r = 0$, matches the form of the stationary BTZ black string spacetime in general relativity [29] (see also Ref. [32] for the BTZ string in Cherns-Simon gravity). We recall that the BTZ black string configurations are obtained by adding an extra flat dimension to the metric of the three-dimensional BTZ black hole [31] and they require for their very existence a specific source term in the corresponding field equations. Using this analogy, we will call the class of solutions with $g = 1$ and $N_r \neq 0$, the ‘‘BTZ-type black string’’ solutions. Below, we will see that the BTZ-type black string solutions do naturally exist in HL gravity (without the need for any specific source term).

Meanwhile, it is not difficult to see that for $g = \alpha^2 r^2$, where α is a constant parameter, expression (14) vanishes identically, irrespective of the form of the function $f(r)$. That is, we again have $C_{ij} = 0$. In this case, the metric ansatz in (13), with $N_r = 0$, matches the form which describes the stationary black string solution of general relativity, found by Lemos [30]. This type of string solution is inherent in general relativity with a negative cosmological constant. Below, we will also discuss the Lemos-type static string solution in HL gravity for any value of the coupling constant $\lambda > 1/3$.

A. BTZ-type solutions

We now need to substitute metric (13), with $g = 1$, into the field equations of HL gravity. In doing this, we find that the Hamiltonian constraint (9) takes the form

$$\begin{aligned} &2\eta^2 r^6 N_r^2 f^{-1}(\lambda-1) + 2r^2 f[(\lambda-1)(N_r^2 + r^2 N_r'^2) \\ &\quad + 2\lambda r N_r N_r'] + \frac{\kappa^4\mu^2 r^4 \tilde{N}^2}{8(1-3\lambda)}[(2\lambda-1)\eta^2 - 2\eta\Lambda_W - 3\Lambda_W^2] \\ &\quad - (2N_\phi - rN_\phi')^2 + 4\eta r^4 N_r[\lambda N_r + (\lambda-1)rN_r'] = 0, \end{aligned} \tag{17}$$

and the momentum constraint (10) reduces to the following two equations

$$\begin{aligned} &\eta r^2 f^{-2}(\lambda-1)\{\eta r^2 \tilde{N} N_r + f[rN_r\tilde{N}' - 2\tilde{N}(N_r + rN_r')]\} \\ &\quad + r\tilde{N}'[\lambda N_r + (\lambda-1)rN_r'] \\ &\quad + (\lambda-1)\tilde{N}[N_r - r(N_r' + rN_r'')] = 0, \end{aligned} \tag{18}$$

$$\tilde{N}'(2N_\phi - rN_\phi') - \tilde{N}(N_\phi' - rN_\phi'') = 0. \tag{19}$$

Meanwhile, calculations show that the nontrivial components of Eq. (11) are given by

$$\begin{aligned} &\frac{\kappa^4\mu^2 r^3 \tilde{N}^2}{1-3\lambda}\{[(2\lambda-1)\eta^2 - 2\eta\Lambda_W - 3\Lambda_W^2]r\tilde{N} + 2f[(2\lambda-1)\eta - \Lambda_W]\tilde{N}'\} + 16\eta r^5 f^{-1} N_r^2(\lambda-1)(3\eta r\tilde{N} + 2f\tilde{N}') \\ &\quad + 16r^2 f\{2rN_r\tilde{N}'[\lambda N_r + (\lambda-1)rN_r'] + (\lambda-1)(3N_r^2 + r^2N_r'^2)\tilde{N} + 2r\tilde{N}N_r[N_r' - (\lambda-1)rN_r'']\} \\ &\quad - 8\tilde{N}\{(2N_\phi - rN_\phi')^2 + 4\eta r^4 N_r[(\lambda-2)N_r + (\lambda-1)rN_r']\} = 0, \qquad (E_{rr} = 0), \end{aligned} \tag{20}$$

$$\begin{aligned} &-\frac{\kappa^4\mu^2 r^4 \tilde{N}^2}{1-3\lambda}\{[(2\lambda-1)\eta^2 - 2\eta\Lambda_W - 3\Lambda_W^2]\tilde{N} + 2[(2\lambda-1)\eta - \Lambda_W](3\eta r\tilde{N}' + f\tilde{N}'')\} + 16\eta^2 r^6 f^{-1}\tilde{N}N_r^2(\lambda-1) \\ &\quad - 32r\tilde{N}'[N_\phi(2N_\phi - rN_\phi') - \lambda\eta r^4 N_r^2] - 8\tilde{N}\{3(2N_\phi - rN_\phi')^2 + 4\eta r^4 N_r[(3\lambda-2)N_r + (1+3\lambda)rN_r'] - 4rN_\phi(N_\phi' - rN_\phi'')\} \\ &\quad + 16r^2 f\{2rN_r\tilde{N}'[(\lambda-1)N_r + \lambda rN_r'] + \tilde{N}[(\lambda-1)N_r^2 - (\lambda+1)r^2N_r'^2] - 2rN_r\tilde{N}[2(\lambda-1)N_r' + \lambda rN_r'']\} = 0, \\ &(E_{\phi\phi} = 0), \end{aligned} \tag{21}$$

$$\begin{aligned} &\frac{\kappa^4\mu^2 r^3 \tilde{N}^2 f}{1-3\lambda}\{[(1+2\lambda)\eta^2 + 6\eta\Lambda_W + 3\Lambda_W^2]r\tilde{N} + 2(\lambda\eta + \Lambda_W)[(3\eta r^2 + f)\tilde{N}' + rf\tilde{N}'']\} \\ &\quad + 32\lambda r^3 f N_r\tilde{N}'[(\eta r^2 + f)N_r + rfN_r'] - 8\tilde{N}\{4fN_\phi(N_\phi - rN_\phi') + 2[(1-\lambda)m^2 + 2\eta r^2 f(1+2\lambda)]r^2N_r^2 \\ &\quad + r^2 f[N_\phi'^2 + 2(\lambda+1)r^2 fN_r'^2] + 4r^3 fN_r[(\eta r^2 + 5\lambda\eta r^2 - 2\lambda m)N_r' + \lambda rfN_r'']\} = 0, \qquad (E_{zz} = 0), \end{aligned} \tag{22}$$

$$N_\phi\{\eta r^2 f^{-2}(\lambda-1)(\eta r^2\tilde{N}N_r + f[rN_r\tilde{N}' - 2\tilde{N}(N_r + rN_r')]) + r\tilde{N}'[\lambda N_r + (\lambda-1)rN_r'] + (\lambda-1) \times \tilde{N}[N_r - r(N_r' + rN_r'')]\} = 0, \qquad (E_{r\phi}=0), \tag{23}$$

$$(8\eta^2 r^4 - 4\eta m r^2 - m^2)\tilde{N}' + rf[(7\eta r^2 - m)\tilde{N}'' + rf\tilde{N}'''] = 0, \qquad (E_{z\phi}=0). \tag{24}$$

We recall that the function f, appearing in these equations is given in (16). For the general value of λ these equations look somewhat complicated, but they are drastically simplified for the relativistic value $\lambda = 1$. To make further consideration more illustrative, it is fitting to begin with the special cases and then go up to the general case.

(i) *The static solution* ($N_r = 0$ and $N_\phi = 0$). In this case, from the Hamiltonian constraint (17), we find that

$$\eta = \frac{1 \pm \sqrt{6\lambda - 2}}{2\lambda - 1}\Lambda_W. \tag{25}$$

whereas, the momentum constraints in Eqs. (18) and (19) are trivially satisfied. Taking this value of η into account in Eq. (20), we immediately fix the function $\tilde{N}$ as $\tilde{N} = \tilde{N}_0$, where $\tilde{N}_0$ is a constant of integration. With these quantities in mind, it is easy to check the remaining equations of motion. As a consequence, we find that Eq. (22) takes the simple form given by

$$\frac{(1+2\lambda)(3\lambda-1) \pm (4\lambda-1)\sqrt{6\lambda-2}}{(1-2\lambda)^2} = 0. \tag{26}$$

All other equations are satisfied automatically. Solving Eq. (26) for $\lambda > 1/3$, which is the case in our consideration, we arrive at the relativistic value $\lambda = 1$. This value corresponds to the lower sign in Eqs. (25) and (26). In other words, starting with the general value of $\lambda > 1/3$, we are driven, by the equations of motion, to the value $\lambda = 1$.

Thus, the static and cylindrically symmetric solution for $\lambda = 1$ is given by

$$ds^2 = -f dt^2 + f^{-1}dr^2 + r^2 d\phi^2 + dz^2, \tag{27}$$

where the metric function $f = -\Lambda_W r^2 - m$. Furthermore, we have set $\tilde{N}_0 = 1$ by adjusting the time coordinate, and used Eqs. (16) and (25). It is easy to see that this metric can be interpreted as describing the spacetime of a static BTZ string in HL gravity. It possesses an event horizon located at the radius

$$r_+ = \sqrt{\frac{m}{-\Lambda_W}}, \tag{28}$$

where the quantity m plays the role of a mass parameter and $m > 0$.

As we have mentioned above, the BTZ black string configurations do not exist in general relativity without introducing a specific source term into the Einstein field equations. It is remarkable that a particular higher derivative structure of HL gravity provides a natural place for the BTZ-type black strings in this theory. We note that this solution was also discussed in [33].

(ii) *The static hedgehog solution*. This is the general static and cylindrically symmetric spacetime with the nonvanishing radial shift, $N_r \neq 0$. We present now this solution for $\lambda = 1$. From the momentum constraint Eq. (18) we see that the quantity $\tilde{N}$ again remains constant, i.e. $\tilde{N} = \tilde{N}_0$. Since $N_\phi = 0$ as well, Eqs. (19), (23), and (24) become trivial. Meanwhile, the Hamiltonian constraint (17) gives

$$N_r = \pm f^{-1/2}\sqrt{\xi + \frac{\kappa^4\mu^2}{64}\tilde{N}_0^2 r^2(\eta - 3\Lambda_W)(\eta + \Lambda_W)}, \tag{29}$$

where ξ is a constant of integration. It is not difficult to verify that solution (29) is also subject to Eqs. (20) and (21). On the other hand, substituting this solution in Eq. (22), focusing on the case when the associated spacetime metric for $\xi = 0$ goes over into that given in (27), we find that

$$\eta = -\Lambda_W. \tag{30}$$

Finally, we arrive at the spacetime metric in the form

$$ds^2 = -(\tilde{N}_0^2 - N_r^2)f dt^2 + 2N_r dr dt + f^{-1}dr^2 + r^2 d\phi^2 + dz^2, \tag{31}$$

where the metric functions are given by

$$f = -\Lambda_W r^2 - m, \qquad N_r = \pm\sqrt{\xi/f}. \tag{32}$$

For $\xi \neq 0$, the solution describes the BTZ-type static black string with the radial hair, i.e. the black string with a hedgehog behavior. Taking $\tilde{N}_0 = 1$, we find that the horizon radius, at which $g^{rr} = 0$, is given by

$$r_+ = \sqrt{\frac{m+\xi}{-\Lambda_W}}. \tag{33}$$

We see that the radial hair contributes to the mass parameter. Clearly, the quantity $m + \xi$ must be positive.

(iii) *The stationary hedgehog solutions*. Using equations of motion given in (17)–(24), it is straightforward to show that with the vanishing radial shift, $N_r = 0$, HL gravity does not support the stationary and cylindrically symmetric solution, i.e., the rotating BTZ-type black string. However, such a solution does exist in the general stationary and cylindrically symmetric case (with $N_r \neq 0$), whereby it inevitably behaves as a hedgehog type solution. Turning now to this solution for $\lambda = 1$, we first note that the quantity N_ϕ does not enter in Eq. (18) at all. Therefore, as in the static case, this equation gives us $\tilde{N} = \tilde{N}_0$. With this in mind, from Eq. (19) we find that

$$N_\phi = \sigma r^2 + \gamma, \tag{34}$$

where σ and γ are constants of integration. Substituting now this expression in Eq. (17) and solving it, we find that

$$N_r = \pm f^{-1/2}\sqrt{\xi + \frac{\kappa^4\mu^2}{64}\tilde{N}_0^2 r^2(\eta - 3\Lambda_W)(\eta + \Lambda_W) - \frac{\gamma^2}{r^2}}. \tag{35}$$

Straightforward calculations show that with $\tilde{N} = \tilde{N}_0$, the expressions in (34) and (35) solve all the remaining field equations, provided that the relation in (30) holds. Altogether, these expressions enable us to write down the spacetime metric in the form

$$ds^2 = -(\tilde{N}_0^2 - N_r^2 - r^{-2}f^{-1}N_\phi^2)fdt^2 + 2(N_r dr + 2N_\phi d\phi)dt + f^{-1}dr^2 + r^2 d\phi^2 + dz^2, \tag{36}$$

where

$$N_r = \pm f^{-1/2}\sqrt{\xi - \frac{\gamma^2}{r^2}}, \tag{37}$$

and the functions f and N_ϕ are given in Eqs. (32) and (34), respectively. Here the constant parameter γ can be thought of as a rotation parameter and, as seen from Eq. (37), it necessarily requires the nonvanishing radial hair, $\xi \neq 0$.

The horizon structure of this solution is determined by the equation $g^{rr} = 0$ and we have

$$-\Lambda_W r^2 - m - \xi + \gamma^2/r^2 = 0. \tag{38}$$

The two roots of this equation, r_+ and r_- are given by

$$r_\pm^2 = \frac{m+\xi}{-2\Lambda_W}\left[1 \pm \sqrt{1 - \frac{4\gamma^2(-\Lambda_W)}{(m+\xi)^2}}\right], \tag{39}$$

and provided that

$$m + \xi > 0, \qquad |\gamma| \leq \frac{m+\xi}{2\sqrt{-\Lambda_W}}, \tag{40}$$

they give the radii of outer and inner horizons, respectively. In the extreme limit of rotation, where the equality in the second expression in (40) holds, the outer and inner horizons coincide and we find that

$$r_+^2 = r_-^2 = \frac{m+\xi}{-2\Lambda_W}. \tag{41}$$

The Hawking temperature can be calculated using the standard formulae

$$T = \frac{\kappa}{2\pi} = \sqrt{-\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)}, \tag{42}$$

where κ is the surface gravity and the Killing vector $\chi = \partial_t + \Omega_H \partial_\phi$ describes the isometry of the horizon that rotates with the angular velocity $\Omega_H = -\gamma/r_+^2$. Performing explicit calculations, we find that

$$T = -\frac{r_+^2 - r_-^2}{2\pi r_+}\Lambda_W. \tag{43}$$

We see that in the limit of extreme rotation the Hawking temperature vanishes, just as for the extreme BTZ black hole in three-dimensional general relativity [31].

Next, we calculate the physical mass and angular momentum of solution (36), using the canonical Hamiltonian formalism [31] in HL gravity [20,21,27]. It is straightforward to show that in this approach the action in (5) takes the form

$$I = \int dt d^3x(\pi^{ij}\dot{g}_{ij} - N\mathcal{H} - N^i\mathcal{H}_i) + B, \tag{44}$$

where

$$\begin{aligned}\pi^{ij} &= g_0\sqrt{g}(K^{ij} - \lambda K g^{ij}),\\ \mathcal{H} &= \sqrt{g}\{g_0(K_{ij}K^{ij} - \lambda K^2) - g_1(R - 3\Lambda_W)\\ &\quad - g_2R^2 - g_3 Z_{ij}Z^{ij}\},\\ \mathcal{H}_i &= -2D_j\pi_i{}^j,\end{aligned} \tag{45}$$

and B denotes a boundary term. Evaluating this action for the metric in (36) and taking the result per unit length of the string, we find that

$$\mathcal{I} = -2\pi(t_2 - t_1)\int dr(\tilde{N}\sqrt{f}\mathcal{H} + N^r\mathcal{H}_r + N^\phi\mathcal{H}_\phi) + \mathcal{B}, \tag{46}$$

where

$$\mathcal{H} = \frac{1}{\sqrt{f}}\left\{g_1(3\Lambda_W r + f') - \frac{2g_2 + g_3g_4^2}{2r}f'^2 - \frac{g_3}{8r^3}f(f' - rf'')^2 + \frac{g_0 r^3}{2}\left(\frac{N^{\phi\prime}}{\tilde{N}}\right)^2 - \frac{g_0}{\tilde{N}^2}\left(\frac{N^{r2}}{f}\right)'\right\}, \tag{47}$$

$$\mathcal{H}_r = 2g_0 f^{-1}\frac{N^r\tilde{N}'}{\tilde{N}^2}, \tag{48}$$

$$\mathcal{H}_\phi = g_0\left(\frac{r^3N^{\phi\prime}}{\tilde{N}}\right)'. \tag{49}$$

We note that, with the z-coordinate being noncompact, physically meaningful quantities are those taken per unit length of the string. Varying this action with respect to the associated fields and omitting the terms which vanish when the equation of motion hold, we arrive at the expression

$$\delta I = -2\pi(t_2 - t_1)\Big\{\tilde{N}\Big[g_1 - \frac{2g_2 + g_3 g_4^2}{r} f' + \frac{g_3}{4r^3} f(f' - rf'')\Big]\delta f - \frac{g_3}{4r^3}[\tilde{N} f(f' - rf'')]'\delta f + \frac{g_3}{4r^2}\tilde{N} f(f' - rf'')\delta f' - g_0 \tilde{N}\delta\Big(\frac{N^{r2}}{f\tilde{N}^2}\Big) + g_0 N^\phi \delta\Big(\frac{r^3 N^{\phi\prime}}{\tilde{N}}\Big)\Big\} + \delta\mathcal{B}. \tag{50}$$

Clearly, this quantity must vanish under extremizing of the action with appropriate boundary conditions. This implies adjusting the boundary term $\mathcal{B}$ in such a way that to cancel all the preceding terms in (50). With this in mind and demanding that the fields at infinity are determined by the solution in (36), we find that the boundary term is given by

$$\mathcal{B} = (t_2 - t_1)\Big\{-\tilde{N}_\infty\Big(2\pi m[g_1 + 2(2g_2 + g_3 g_4^2)\Lambda_W] + \frac{2\pi g_0}{\tilde{N}_\infty^2}\xi\Big) + N_\infty^\phi\Big(-\frac{4\pi g_0}{\tilde{N}_\infty}\gamma\Big)\Big\} + \mathcal{B}_0, \tag{51}$$

where B_0 is an arbitrary constant and we have renamed the constants of integration $\tilde{N}_0$ and σ in solution (36) as asymptotic displacements $\tilde{N}(\infty)$ and $N^\phi(\infty)$, respectively. From this expression, we see that the mass $\mathcal{M}$ and the angular momentum $\mathcal{J}$ appear as conjugates to these asymptotic displacements, as it must be in the Hamiltonian approach under consideration. Therefore, we have

$$\mathcal{M} = 2\pi m[g_1 + 2(2g_2 + g_3 g_4^2)\Lambda_W] + \frac{2\pi g_0}{\tilde{N}_\infty^2}\xi + C,$$
$$\mathcal{J} = -\frac{4\pi g_0}{\tilde{N}_\infty}\gamma, \tag{52}$$

where the appearance of an arbitrary constant C in the expression for the mass is induced by the constant B_0 present in (51). We can now set $\tilde{N}(\infty) = 1$ and $N^\phi(\infty) = 0$, without loss of generality.

Substituting into these expressions the quantities given in (6), with the emergent relations (8) in mind, and choosing the constant C so as to obtain zero mass for the disappearing event horizon (see Eqs. (33) and (39)), we find that the mass and the angular momentum, per unit length of the rotating black string (36), are given by

$$\mathcal{M} = \frac{m + \xi}{4G}, \qquad \mathcal{J} = -\frac{\gamma}{4G}. \tag{53}$$

As it was mentioned above, from these expressions it follows that the quantities $m + \xi$ and γ can be thought of as the mass and the rotation parameters, respectively.

B. Lemos-type solutions

We turn now to the solutions for which the function $g(r)$ in metric (13) is given as $g = \alpha^2 r^2$. Unfortunately, for the nonvanishing shift vector we were unable to solve the field equations even in the relativistic limit $\lambda = 1$. Therefore, we restrict ourselves to the static case with zero shifts, but with any value of $\lambda > 1/3$. With these in mind, we substitute the metric ansatz (13) into the equations of motion. As a consequence, we find that the momentum constraint (10) is trivially fulfilled, while the Hamiltonian constraint (9) gives the equation

$$(2\lambda - 1)\frac{f^2}{r^2} - 2\lambda\frac{ff'}{r} + \frac{\lambda - 1}{2}f'^2 - 2\Lambda_W(rf)' - 3\Lambda_W^2 r^2 = 0. \tag{54}$$

It is also straightforward to show that for the components of the tensor E_{ij} in Eq. (11), the relation $E_{zz} = \alpha^2 E_{\phi\phi}$ holds. Therefore, we have only two independent components of Eqs. (11). These are given by

$$(\ln\tilde{N})'\Big[(\lambda - 1)f' - 2\lambda\frac{f}{r} - 2\Lambda_W r\Big] + (\lambda - 1)\Big(f'' - 2\frac{f}{r^2}\Big) = 0, \qquad (E_{rr} = 0), \tag{55}$$

$$\tilde{N}^{-1}\tilde{N}''\Big[(\lambda - 1)f' - 2\lambda\frac{f}{r} - 2\Lambda_W r\Big] - \frac{(\ln\tilde{N})'}{2r^2 f}\{3r^2 f'[(1-\lambda)f' + 2\Lambda_W r] + 2f[5\lambda rf' - 2f + 2(1-\lambda)r^2 f'' + 2\Lambda_W r^2]\} - \frac{1}{r^3 f}\{(3 - 2\lambda)f^2 + \Lambda_W r^4(f'' + 3\Lambda_W)[+r^2 f'[\lambda f' + 2(1-\lambda)rf'' + 2\Lambda_W r] + rf[2(\lambda - 2)f' + \lambda rf'' + (1-\lambda)r^2 f''']]\} = 0, \qquad (E_{\phi\phi} = 0). \tag{56}$$

We note that in obtaining Eq. (55) we have used Eq. (54). Next, introducing a new radial function $\mathcal{F}(r)$ through the relation

$$f = -\Lambda_W r^2 - \mathcal{F}(r), \tag{57}$$

we put Eqs. (54) and (55) in the form

$$\frac{\lambda - 1}{2}\mathcal{F}'^2 - 2\lambda\frac{\mathcal{F}\mathcal{F}'}{r} + (2\lambda - 1)\frac{\mathcal{F}^2}{r^2} = 0, \tag{58}$$

$$(\ln\tilde{N})'\Big[(\lambda - 1)\mathcal{F}' - 2\lambda\frac{\mathcal{F}}{r}\Big] + (\lambda - 1)\Big(\mathcal{F}'' - 2\frac{\mathcal{F}}{r^2}\Big) = 0. \tag{59}$$

These equations as well as Eq. (56) admit the trivial solution $\mathcal{F} = 0$, leaving unconstrained the function $\tilde{N}$. Furthermore, we have two other solutions given by

$$f = -\Lambda_W r^2 - M r^p, \qquad \tilde{N} = \tilde{N}_0 r^{1-2p}, \tag{60}$$

where M and $\tilde{N}_0$ are constants of integration and

$$p = \frac{2\lambda \pm \sqrt{6\lambda - 2}}{\lambda - 1}. \tag{61}$$

The associated spacetime metric is given by

$$ds^2 = -\tilde{N}^2 f dt^2 + f^{-1} dr^2 + r^2 d\phi^2 + \alpha^2 r^2 dz^2. \tag{62}$$

We are interested in the solution that has a clear physical meaning in the relativistic limit $\lambda = 1$. This corresponds to the lower sign in (61) with $\lambda \in (1/3, \infty)$ or $p \in (-1, 2)$. Evaluating the scalar curvature for this solution, we find the expression

$$R = 2\Lambda_W(11 - 12p + 4p^2) + 3M r^{p-2}(2 - 2p + p^2), \tag{63}$$

which clearly shows that at $r = 0$ there exists a curvature singularity. It is also easy to see that for this solution the radius of the event horizon is given by

$$r_+ = \left[\frac{M}{-\Lambda_W}\right]^{1/(2-p)}, \tag{64}$$

where the parameter M is supposed to be positive and it is related to the mass per unit length of the string (see Eq. (82)). Meanwhile, for the Hawking temperature evaluated by means of formulae (42) we find

$$T = -\frac{\tilde{N}_0(2-p)}{4\pi} r_+^{2(1-p)} \Lambda_W. \tag{65}$$

On the other hand, for $\lambda = 1$ (or $p = 1/2$) we have solution (62) with

$$f = -\Lambda_W r^2 - M\sqrt{r}, \qquad \tilde{N} = \tilde{N}_0, \tag{66}$$

where $\tilde{N}_0$ can be set equal to one. We note that these results are in agreement with those obtained in [20] for topological black holes.

IV. THERMODYNAMICS

One of the most striking properties of black holes in general relativity is that they obey the laws of thermodynamics and have an entropy which is always given by one quarter of the horizon area. However, the simple area law breaks down for black holes in higher derivative gravity theories [34]. Recently, this question was also raised in the context of HL gravity [20,21]. In particular, it was shown that the entropy of spherically symmetric black holes (as well as the topological ones) in HL gravity involves a logarithmic term, in addition to the leading "one quarter of area" term. The logarithmic term disappears only for black holes for which the scalar curvature of two-dimensional Einstein space vanishes. This fact motivates us to study the area law for the black string configurations as well. In this section, we calculate the thermodynamical quantities and study the area law for both the static BTZ- and Lemos-type black string solutions, given in (27) and (62), respectively.

The thermodynamical properties of the black string configurations in HL gravity can be discussed in a similar way to those of black holes in general relativity, using the Euclidean path integral approach [35]. Within this approach, the free energy F of a thermodynamical ensemble divided by the temperature T is identified with the Euclidean action evaluated on the Euclidean continuation of the black hole solutions. Thus, keeping in mind that in our case all the related quantities are taken per unit length of the black string, we have

$$\mathcal{I}_E = \frac{F}{T} = \frac{\mathcal{M}}{T} - \mathcal{S}, \tag{67}$$

where $\mathcal{S}$ denotes the entropy of the system and the Euclidean action is related to that given in (44) as $\mathcal{I}_E = -i\mathcal{I}$. We recall that we are interested in the static case with $N_r = 0$. Therefore, passing to the imaginary time $\tau = it$ and using Eq. (46), we obtain that

$$\mathcal{I}_E = 2\pi\beta \int_{r_+}^{\infty} dr \tilde{N} \sqrt{f}\, \mathcal{H} + \mathcal{B}_E, \tag{68}$$

where $\beta = \tau_2 - \tau_1$ is the period of the Euclidean "time" that in turn determines the Hawking temperature

$$T = \beta^{-1} = \frac{\tilde{N} f'}{4\pi}\bigg|_{r_+}, \tag{69}$$

implying the absence of singularities at the black string horizon. An equivalent definition is given in (42) as well. Since for the static black string solutions under consideration we have $\mathcal{H} = 0$, the boundary term $\mathcal{B}_E$ plays a crucial role in the variation of the action. This term must be adjusted so as to provide a true extremum of the action on these solutions. When performing the variation, as in the case of the mass and angular momentum calculations described in the previous section, one must allow changes in the corresponding field variables which contribute to the boundary term, while keeping fixed their conjugates. In our case, the conjugate is the temperature which we keep fixed under the variation.

We first begin with the static BTZ-type metric (27) for which the Euclidean action (68), as follows from (46), can be written in the form

$$\mathcal{I}_E = 2\pi\beta \int_{r_+}^{\infty} dr \tilde{N}\bigg\{ g_1(3\Lambda_W r + f') - \frac{2g_2 + g_3 g_4^2}{2r} f'^2 - \frac{g_3}{8r^3} f(f' - r f'')^2 \bigg\} + \mathcal{B}_E. \tag{70}$$

The extremum of this action on metric (27) enables us to fix the variation of the boundary term as

$$\delta\mathcal{B}_E = -2\pi\beta\Big\{\tilde{N}\Big[g_1 - \tfrac{2g_2+g_3g_4^2}{r}f' + \tfrac{g_3}{4r^3}f(f'-rf'')\Big]\delta f - \frac{g_3}{4r^3}[\tilde{N}f(f'-rf'')]'\delta f + \frac{g_3}{4r^2}\tilde{N}f(f'-rf'')\delta f'\Big\}_{r_+}^{\infty}. \tag{71}$$

With the BTZ-type solution in (27), evaluating the boundary term at infinity, we find that

$$\mathcal{B}_E(\infty) = 2\pi\beta m[g_1 + 2(2g_2 + g_3g_4^2)\Lambda_W] + \mathcal{B}_1. \tag{72}$$

Meanwhile, for the boundary term at the horizon, similar calculations yield

$$\mathcal{B}_E(r_+) = 8\pi^2 r_+[g_1 + 2(2g_2 + g_3g_4^2)\Lambda_W] + \mathcal{B}_2. \tag{73}$$

Here $\mathcal{B}_1$ and $\mathcal{B}_2$ are constants of integration, and in obtaining (73) we have used Eq. (69) along with the fact that

$$(\delta f)_{r_+} = -(f)'|_{r_+}\delta r_+. \tag{74}$$

Since the on-shell value of the Euclidean action is determined by the boundary term alone, $I_E = \mathcal{B}_E(\infty) - \mathcal{B}_E(r_+)$, then comparing this result with Eqs. (67) and (69), it is not difficult to see that the mass and the entropy (per unit length) of the static BTZ-type black string are given by

$$\begin{aligned}\mathcal{M} &= 2\pi m[g_1 + 2(2g_2 + g_3g_4^2)\Lambda_W],\\ \mathcal{S} &= 8\pi^2 r_+[g_1 + 2(2g_2 + g_3g_4^2)\Lambda_W],\end{aligned} \tag{75}$$

where we have omitted an arbitrary constant of integration, requiring that $I_E = 0$ for $r_+ \to 0$. We note that the expression for the mass is precisely the same as that given in (52) for $\xi = 0$ and $C = 0$. Remarkably, these expressions clearly delineate the contribution from higher-derivative terms in the action through the combination of constants g_2, g_3 and g_4. Moreover, for $\lambda = 1$ using the value of these constants given in (6), it is easy to show that $g_1 = 2(2g_2 + g_3g_4^2)$. That is, the higher-derivative contributions to the mass and entropy result in the doubling of the ordinary Einstein-Hilbert contribution. As a consequence, we find that

$$\mathcal{S} = \frac{\mathcal{A}}{2G}, \tag{76}$$

where $\mathcal{A} = 2\pi r_+$ is the area of the horizon per unit length and we have also used the emergent relations in (8). Thus, the entropy of the static BTZ-type black string in HL gravity is one half of its horizon area. It is easy to check that the entropy, with the mass and with the temperature given in (75) and (69), respectively, satisfies the first law of thermodynamics

$$d\mathcal{M} = Td\mathcal{S}. \tag{77}$$

We note that this entropy-area relation was first obtained in [36] for a particular class of Calabi-Yau black holes. See also [37], for further developments.

Next, we turn to the Lemos-type black string solution (62) with the metric functions given in (60). For this solution the Euclidean action is given by

$$I_E = 2\pi\beta\alpha\int_{r_+}^{\infty} dr\tilde{N}\Big\{g_1[2(rf)' + 3\Lambda_W r^2] - \frac{4g_2}{r^2}(rf)'^2 - \frac{g_3g_4^2}{2r^2}[4f(rf)' + 3r^2f'^2]\Big\} + \mathcal{B}_E. \tag{78}$$

Again, the variation of the boundary term can be found from the extremum of this action on solution (62). We have

$$\delta\mathcal{B}_E = -2\pi\beta\alpha\Big\{\tilde{N}\Big[2g_1 r - (8g_2 + 3g_3g_4^2)f' - \frac{2(4g_2 + g_3g_4^2)}{r}f\Big]\delta f\Big\}_{r_+}^{\infty}. \tag{79}$$

Performing similar calculations, as in the case of BTZ-type black strings, and comparing the on-shell value of the Euclidean action $I_E = \mathcal{B}_E(\infty) - \mathcal{B}_E(r_+)$ with that given in (67), we find that the mass and the entropy of the Lemos-type black strings are given by

$$\mathcal{M} = \pi\alpha\tilde{N}_0 M^2[2(4g_2 + g_3g_4^2) + p(8g_2 + 3g_3g_4^2)], \tag{80}$$

$$\mathcal{S} = 4\pi^2 r_+^2\alpha[2g_1 + (2-p)(8g_2 + 3g_3g_4^2)\Lambda_W]. \tag{81}$$

We note that the mass is determined only by the contribution from higher-derivative terms. On the other hand, it is curious that in both expressions the second term in the square bracket disappears in the relativistic limit $\lambda = 1$. Indeed, using Eq. (6), it is easy to show that the combination $4g_2 + g_3g_4^2$ vanishes identically. That is, for $\lambda = 1$, the entropy must obey the usual area law just as for black holes in general relativity. Indeed, using relations (6) and (8) in Eqs. (80) and (81), we find that

$$\mathcal{M} = -\frac{\alpha\tilde{N}_0 M^2\sqrt{6\lambda - 2}}{16G\Lambda_W}, \qquad \mathcal{S} = \frac{\sqrt{6\lambda - 2}}{2}\frac{\mathcal{A}}{4G}, \tag{82}$$

where $\mathcal{A} = 2\pi\alpha r_+^2$ is the horizon area per unit length. We see that, for $\lambda = 1$, the entropy of the Lemos-type black string in HL gravity is given by one quarter of the horizon area. It is straightforward to verify that with the temperature given in (65), the quantities in (82) fulfil the first law of thermodynamics (see Eq. (77)).

V. CONCLUSION

In this paper, we have shown that HL gravity admits a class of cylindrically symmetric solutions which can be interpreted as counterparts of black strings in general relativity. Using the general stationary and cylindrically symmetric ansatz for the spacetime metric and focusing on the cases when the Cotton tensor in the HL action vanishes, we have distinguished two examples of the cylindrically symmetric spacetimes. In the first example, the metric ansatz matches, for the vanishing radial shift in the

ADM-type decomposition, the form of the stationary BTZ black string metric in general relativity. On this ground, one can think of the resulting solutions as describing the BTZ-type black strings in HL gravity. In the second example, the metric ansatz corresponds to the Lemos-type black string configuration which does exist in general relativity with a negative cosmological constant.

For the relativistic value of the coupling constant, $\lambda = 1$, we have given the static BTZ-type black string solutions with both zero and nonzero radial shift. The solution with the radial shift, the hedgehog type solution, is inherent in HL gravity alone, as the foliation-preserving invariance of the theory is not enough to eliminate the shift from the metric. Moreover, unlike general relativity, HL gravity provides a natural place for the BTZ-type black string configurations, due to its particular higher derivative structure. As is known, in general relativity such configurations require a specific source term for the Einstein field equations. We have also found the stationary and cylindrically symmetric solution with the radial shift, which corresponds to a rotating BTZ-type black string. It is important to note that this solution requires the presence of the radial hair for its very existence. In other words, the radial hair is necessary for rotation.

With the Lemos-type black string, restricting ourselves to the static case with zero shifts, we have presented the exact solution for any value of $\lambda > 1/3$. Further, exploring the thermodynamical properties of the black strings in the framework of the Euclidean path integral approach, we have shown that for $\lambda = 1$ the entropy (per unit length) of the Lemos-type static black string is one quarter of the horizon area. Meanwhile, the corresponding entropy of the static BTZ-type black string is equal to one half of its horizon area.

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[1] P. Hořava, Phys. Rev. D **79**, 084008 (2009).
[2] P. Hořava, J. High Energy Phys. 03 (2009) 020.
[3] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **11**, 255 (1941); **11** 269 1941.
[4] S. Mukohyama, Classical Quantum Gravity **27**, 223101 (2010).
[5] P. Hořava and C. M. Melby-Thompson, Phys. Rev. D **82**, 064027 (2010).
[6] E. Kiritsis and G. Kofinas, Nucl. Phys. B **821**, 467 (2009).
[7] S. Mukohyama, J. Cosmol. Astropart. Phys. 06 (2009) 001.
[8] G. Calcagni, J. High Energy Phys. 09 (2009) 112.
[9] A. Wang and Y. Wu, J. Cosmol. Astropart. Phys. 07 (2009) 012.
[10] C. Bogdanos and E. N. Saridakis, Classical Quantum Gravity **27**, 075005 (2010).
[11] D. Blas, O. Pujolas, and S. Sibiryakov, Phys. Rev. Lett. **104**, 181302 (2010).
[12] A. Wang and Q. Wu, Phys. Rev. D **83**, 044025 (2011).
[13] M. Li and Y. Pang, J. High Energy Phys. 08 (2009) 015; M. Henneaux, A. Kleinschmidt, and G. L. Gomez, Phys. Rev. D **81**, 064002 (2010).
[14] X. Gao, Y. Wang, R. Brandenberger, and A. Riotto, Phys. Rev. D **81**, 083508 (2010).
[15] A. Wang, D. Wands, and R. Maartens, J. Cosmol. Astropart. Phys. 03 (2010) 013.
[16] H. Lü, J. Mei, and C. N. Pope, Phys. Rev. Lett. **103**, 091301 (2009).
[17] A. Kehagias and K. Sfetsos, Phys. Lett. B **678**, 123 (2009).
[18] R. A. Konoplya, Phys. Lett. B **679**, 499 (2009).
[19] T. Harko, Z. Kovács, and F. S. N. Lobo, Phys. Rev. D **80**, 044021 (2009).
[20] R. G. Cai, L. M. Cao, and N. Ohta, Phys. Lett. B **679**, 504 (2009).
[21] R. G. Cai, L. M. Cao, and N. Ohta, Phys. Rev. D **80**, 024003 (2009).
[22] H. W. Lee, Y. W. Kim, and Y. S. Myung, Eur. Phys. J. C **68**, 255 (2010).
[23] J. Greenwald, A. Papazoglou, and A. Wang, Phys. Rev. D **81**, 084046 (2010).
[24] D. Capasso and A. Polychronakos, Phys. Rev. D **81**, 084009 (2010).
[25] N. Varghese and V. C. Kuriakose, arXiv:1011.6608.
[26] E. Gruss, Classical Quantum Gravity **28**, 085007 (2011).
[27] A. N. Aliev and Ç. Şentürk, Phys. Rev. D **82**, 104016 (2010).
[28] H. W. Lee, Y. W. Kim, and Y. S. Myung, Eur. Phys. J. C **70**, 367 (2010).
[29] J. P. S. Lemos and V. T. Zanchin, Phys. Rev. D **53**, 4684 (1996).
[30] J. P. S. Lemos, Phys. Lett. B **353**, 46 (1995).
[31] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).
[32] H. Ahmedov and A. N. Aliev, Phys. Rev. D **82**, 024043 (2010).
[33] I. Cho and G. Kang, J. High Energy Phys. 07 (2010) 034.
[34] M. Lu and M. B. Wise, Phys. Rev. D **47**, R3095 (1993); T. Jacobson and R. C. Myers, Phys. Rev. Lett. **70**, 3684 (1993).
[35] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
[36] A. Dabholkar, R. Kallosh, and A. Maloney, J. High Energy Phys. 12 (2004) 059.
[37] R. G. Cai, C. M. Chen, K. I. Maeda, N. Ohta, and D. W. Pang, Phys. Rev. D **77**, 064030 (2008).