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Kaluza-Klein brane cosmology with a bulk scalar field

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The brane-world cosmological model in higher-dimensional spacetime with a bulk scalar field is investigated. We derive the $(4 + n)$-dimensional gravitational field equations for the scalar field on the $(3 + n)$-brane in a $(5 + n)$-dimensional bulk with Einstein gravity plus a self-interacting scalar field. The $(4 + n)$-dimensional gravitational field equations can be formulated to standard form with the extra component. Using this formalism we study the Kaluza-Klein brane cosmology. We derive the Friedmann equation and a possible energy leak out of the brane into the bulk. We present some exact solutions corresponding to vacuum brane and matter on the brane.

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I. INTRODUCTION

The idea that our universe has dimensions more than four has been around since the first attempts to unify fundamental forces. The Kaluza-Klein theories were one of the first attempts towards this direction. According to the Kaluza-Klein picture, extra dimensions are compactified to a very small length scale (naturally the Planck scale), and as a result spacetime appears to be effectively four dimensions, insofar as low energies are concerned. On the other hand, inspired by the discovery of D-branes within string theories, the brane-world scenario is now one of the most important ideas in high energy physics. According to the brane-world scenario, our physical Universe is envisioned as a four-dimensional hypersurface in a five-dimensional bulk spacetime. The standard model matter is confined to the brane but gravity, by its universal character, can propagate in the extra dimensions. Much efforts to reveal cosmology on the brane have been done in the context of five-dimensional spacetime, especially after the stimulating proposals by Randall and Sundrum [1,2]. In this model, a five-dimensional realization of the Horava-Witten solution [3], the hierarchy problem can be solved by introducing an appropriated exponential warp factor in the metric. The various properties and characteristics of the Randall-Sundrum model have been extensively analyzed: the cosmology framework [4–8], the low-energy effective theory [9–20], the black hole physics [21–26], and the Lorentz violation [27–37].

The brane-world models with scalar field in the bulk have been discussed by various authors; see [38] and references therein. It is believed that in the unified theory approach, a dilatonic gravitational scalar field term is required in the Einstein-Hilbert action [38]. One of the first motivations to introduce a bulk scalar field is to stabilize the distance between the two branes [39] in the context of the first model introduced by Randall and Sundrum. A second motivation for studying scalar fields in the bulk is due to the possibility that such a setup could provide some clue to solve the famous cosmological constant problem. Models with inflation driven by the bulk scalar field have been studied, and it is shown that inflation is possible without inflaton on the brane [40]. Later, the quantum fluctuations of brane inflation and reheating issues are also addressed in the brane-world scenario with a bulk scalar field [41]. The creation of a brane world with a bulk scalar field using an instanton solution in a five-dimensional Euclidean Einstein equations is also considered [42].

In this paper, our main purpose is to construct brane cosmological models in higher-dimensional spacetime with a bulk scalar field. We generalize the case of four-dimensional brane models to $(4 + n)$-dimensional brane models where n represents internal dimensions of the brane. In order to obtain a four-dimensional description of our Universe, we combine two representative and contrastive approaches, that is, the Randall-Sundrum and the Kaluza-Klein scenario. In the former case, the compactification is done by the localization of the configuration along the extra dimension, while in the latter one the reduction is achieved by the compactification of the internal space of the brane. Such a way of construction is called Kaluza-Klein brane world, and it has been investigated by various authors [43–51].

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This paper is organized as follows. In Sec. II, we study a higher brane-world model in a $(5+n)$-dimensional spacetime with a bulk scalar field. We derive the $(4+n)$-dimensional Einstein equations using the geometrical approach. We transform the nonconventional kinetic term in the Einstein equations into the standard form. In Sec. III, the Kaluza-Klein brane worlds cosmology is presented by using the extra component formalism. Then we discuss the vacuum brane solutions with an initial singularity and a possible energy leak out of the brane into the bulk. For completeness, we study the static internal dimension with matter on the brane in Sec. IV. We derive the Friedmann equation and a possible energy leak out of the brane into the bulk. In Sec. V, we study dimensional reduction of the higher-dimensional theory and then performing a conformal transformation in order to recover ordinary four-dimensional general relativity. Section VI is devoted to the conclusions.

II. EFFECTIVE GRAVITATIONAL EQUATIONS

A. Action and field equations

We consider the higher-dimensional dilatonic brane world, i.e. the higher-dimensional brane world where the bulk is allowed to contain a single nongravitational degree of freedom, the scalar field ϕ. A $(3+n)$ brane with the $(4+n)$-dimensional spacetime embedded in the $(5+n)$-dimensional spacetime is located at $y=0$, where the y direction is compactified on the orbifold. This higher-dimensional dilatonic brane-world model is described by the action

$$S = \int d^{5+n}x\sqrt{-g}\left[\frac{1}{2\kappa^2}\mathcal{R} - \frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right] + \int d^{4+n}x\sqrt{-h}[-\sigma(\phi)+\mathcal{L}_m], \quad (1)$$

where $\mathcal{R}$ is the Ricci scalar of the $(5+n)$-dimensional metric g_{ab}. σ is the brane tension that is allowed to the function of ϕ, and $\mathcal{L}_m$ is the Lagrangian for matter localized on the brane. A metric $h_{\mu\nu}$ is the induced metric on the brane.

We write the coordinate system for the bulk spacetime in the form

$$ds^2 = g_{ab}dx^a dx^b = dy^2 + g_{\mu\nu}(y,x)dx^\mu dx^\nu, \quad (2)$$

and we may assume that the position of the brane is $y=0$ in this coordinate system so that the induced metric on the brane is $h_{\mu\nu}(x) = g_{\mu\nu}(y=0,x)$ and the extrinsic curvature is defined as

$$K_{\mu\nu} = -\frac{1}{2}\frac{\partial}{\partial y}g_{\mu\nu} \equiv -\frac{1}{2}g_{\mu\nu,y}. \quad (3)$$

The Einstein equations can be obtained by varying the action (1) with respect to the gravitational field g_{ab},

$$\mathcal{G}_{ab} = \kappa^2\mathcal{T}_{ab} + \kappa^2\delta^\mu_a\delta^\nu_b(-\sigma g_{\mu\nu} + t_{\mu\nu})\delta(y), \quad (4)$$

where $\mathcal{G}_{ab} = \mathcal{R}_{ab} - g_{ab}\mathcal{R}/2$ is the $(5+n)$-dimensional Einstein tensor, and $\mathcal{T}_{ab}$ is the energy-momentum tensor for the scalar field,

$$\mathcal{T}_{ab} = \nabla_a\phi\nabla_b\phi - g_{ab}\left(\frac{1}{2}g^{cd}\nabla_c\phi\nabla_d\phi + V(\phi)\right), \quad (5)$$

while the total energy-momentum tensor on the brane $t_{\mu\nu}$ is defined as

$$t_{\mu\nu} \equiv -2\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_m. \quad (6)$$

The equation of motion for the scalar field reads

$$\nabla^a\nabla_a\phi - V'(\phi) - \sigma'(\phi)\delta(y) = 0, \quad (7)$$

where a prime denotes a derivative with respect to ϕ.

In the coordinate system (2) and by the application of the Gauss-Codazzi equations, one has the $(5+n)$-dimensional field equations as follows:

$$\mathcal{G}^y{}_y = -\tfrac{1}{2}R + \tfrac{1}{2}K^2 - \tfrac{1}{2}K^{\alpha\beta}K_{\alpha\beta} = \kappa^2\mathcal{T}^y{}_y, \quad (8)$$

$$\mathcal{G}^y{}_\mu = -\nabla_\alpha K_\mu{}^\alpha + \nabla_\mu K = \kappa^2\mathcal{T}^y{}_\mu, \quad (9)$$

$$\mathcal{G}^\mu{}_\nu = G^\mu{}_\nu + (K^\mu{}_\nu - \delta^\mu{}_\nu K)_{,y} - KK^\mu{}_\nu + \tfrac{1}{2}\delta^\mu{}_\nu(K^2 + K^{\alpha\beta}K_{\alpha\beta}) = \kappa^2\mathcal{T}^\mu{}_\nu + \kappa^2(-\sigma\delta^\mu{}_\nu + t^\mu{}_\nu)\delta(y), \quad (10)$$

where $G^\mu{}_\nu = R^\mu{}_\nu - \delta^\mu{}_\nu R/2$ is the $(4+n)$-dimensional Einstein tensor and the covariant derivative ∇_μ is calculated with respect to the metric $g_{\mu\nu}$. Combining Eqs. (8) with (10), and using the $(5+n)$-dimensional Weyl tensor, we obtain the following $(4+n)$-dimensional Einstein equations

$$G^\mu{}_\nu = -K^{\mu\alpha}K_{\alpha\nu} + KK^\mu{}_\nu + \frac{1}{2}\delta^\mu{}_\nu(K_{\alpha\beta}K^{\alpha\beta} - K^2) + \frac{2+n}{3+n}\kappa^2\left[\mathcal{T}^\mu{}_\nu - \frac{1}{4+n}\delta^\mu{}_\nu\mathcal{T}^\alpha{}_\alpha + \frac{3+n}{4+n}\delta^\mu{}_\nu\mathcal{T}^y{}_y\right] - E^\mu{}_\nu. \quad (11)$$

Here, we have defined that the term $\mathcal{T}^\alpha{}_\alpha$ as the trace defined with respect to the $(4+n)$-dimensional metric $g_{\mu\nu}$, and the projected Weyl tensor is defined as $E_{\mu\nu} = C_{y\mu y\nu}|_{y=0}$. To eliminate the extrinsic curvature, we need the junction conditions. It can be obtained by collecting together the terms in field equations that contain a δ function. By assuming Z_2 symmetry we find

$$[K^\mu{}_\nu - \delta^\mu{}_\nu K]|_{y=0} = \frac{\kappa^2}{2}(-\sigma\delta^\mu{}_\nu + t^\mu{}_\nu), \quad (12)$$

$$[\partial_y\phi]|_{y=0} = \tfrac{1}{2}\sigma'(\phi). \quad (13)$$

Using these junction conditions, we find the $(4+n)$-dimensional generalization of the Shiromizu-Maeda-Sasaki equations [5],

$$G^{\mu}{}_{\nu} = \frac{(2+n)\kappa^4}{4(3+n)}\sigma t^{\mu}{}_{\nu} + \frac{(2+n)\kappa^2}{(3+n)} \times \left[\nabla^{\mu}\phi\nabla_{\nu}\phi - \frac{(5+n)}{2(4+n)}\delta^{\mu}{}_{\nu}\nabla^{\alpha}\phi\nabla_{\alpha}\phi\right] - \Lambda_b\delta^{\mu}{}_{\nu} + \kappa^4\pi^{\mu}{}_{\nu} - E^{\mu}{}_{\nu}, \quad (14)$$

where the induced cosmological constant on the brane is given by

$$\Lambda_b(\phi) = \frac{(2+n)\kappa^2}{(4+n)}\left[V + \frac{(4+n)\kappa^2}{8(3+n)}\sigma^2 - \frac{1}{8}\sigma'^2\right], \quad (15)$$

and the local quadratic energy-momentum tensor on the brane is

$$\pi^{\mu}{}_{\nu} = -\frac{1}{4}t^{\mu}{}_{\alpha}t^{\alpha}{}_{\nu} + \frac{1}{4(3+n)}tt^{\mu}{}_{\nu} + \frac{1}{8}\delta^{\mu}{}_{\nu}\left(t_{\alpha\beta}t^{\alpha\beta} - \frac{1}{3+n}t^2\right). \quad (16)$$

We note that Eq. (15) may not be constant in general, as is clear from its expression. The first term comes from the scalar field potential. The second term is a contribution from the brane tension, which yields extrinsic curvature of the brane and its quadratic. The third term is the first derivative of the brane tension, which leads to a discontinuity in the scalar field gradient normal to the brane.

The scalar field equation of motion is given by

$$\nabla^{\alpha}\nabla_{\alpha}\phi - \frac{1}{\kappa^2}\Lambda_b'(\phi) = J_n, \quad (17)$$

where a possible energy leak out of the brane into the bulk, J_n, is defined by

$$J_n = \frac{(2+n)}{4(4+n)}\left(\sigma'\sigma'' - \frac{\kappa^2}{3+n}\sigma t\right) + \frac{2}{4+n}\nabla^{\alpha}\nabla_{\alpha}\phi - \frac{2+n}{4+n}[\partial_y^2\phi]_{y=0}. \quad (18)$$

B. Extra component formalism

The induced Einstein equations on the brane, Eq. (14), contain the nonconventional kinetic term. However, we can transform the nonconventional kinetic term into the standard form, then Eq. (14) can be rewritten as

$$G^{\mu}{}_{\nu} = \kappa^2\left(\nabla^{\mu}\phi\nabla_{\nu}\phi - \frac{1}{2}\delta^{\mu}{}_{\nu}\nabla^{\alpha}\phi\nabla_{\alpha}\phi\right) - \kappa^2 V_{\rm eff}\delta^{\mu}{}_{\nu} + X^{\mu}{}_{\nu}, \quad (19)$$

where the extra component $X^{\mu}{}_{\nu}$ is defined as

$$X^{\mu}{}_{\nu} \equiv Y^{\mu}{}_{\nu} - \mathcal{Z}^{\mu}{}_{\nu}, \quad (20)$$

with

$$Y^{\mu}{}_{\nu} \equiv \frac{(2+n)\kappa^4}{4(3+n)}\sigma t^{\mu}{}_{\nu} + \kappa^4\pi^{\mu}_{\nu}, \quad (21)$$

and

$$\mathcal{Z}^{\mu}{}_{\nu} \equiv E^{\mu}{}_{\nu} + \frac{1}{(3+n)}\left(\nabla^{\mu}\phi\nabla_{\nu}\phi - \frac{1}{4+n}\delta^{\mu}{}_{\nu}\nabla^{\alpha}\phi\nabla_{\alpha}\phi\right). \quad (22)$$

Following Eq. (17) we have defined the effective potential $V_{\rm eff} = \Lambda_b/\kappa^2$, where the induced cosmological constant on the brane Λ_b is given by Eq. (15). Then the scalar field equation of motion becomes

$$\nabla^{\alpha}\nabla_{\alpha}\phi - V'_{\rm eff} = J_n. \quad (23)$$

From Eq. (19), the Bianchi identity implies

$$\nabla_{\mu}X^{\mu}{}_{\nu} = -\kappa^2 J_n\nabla_{\nu}\phi. \quad (24)$$

Note that the energy-momentum tensor of the extra component is not conserved due to the existence of the bulk scalar field. In the absence of matter on the brane, $Y^{\mu}{}_{\nu} = 0$, then we have $X^{\mu}{}_{\mu} = -\mathcal{Z}^{\mu}{}_{\mu} = 0$. In this case, we can interpret the extra component as the energy-momentum tensor for the dark radiation.

In the following section, we attempt to study analytically the cosmological consequences of a higher-dimensional brane world. We used the extra component formalism to discuss cosmology in the brane Universe in the context of the Kaluza-Klein brane-world scheme, i.e., to consider Kaluza-Klein compactifications on the brane.

III. DILATONIC KALUZA-KLEIN BRANE-WORLD COSMOLOGY

For the cosmological applications, we are interested in homogeneous and isotropic geometries on the brane, hence the metric on the brane is taken as the Friedmann-Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j + b^2(t)\delta_{\alpha\beta}dz^{\alpha}dz^{\beta}, \quad (25)$$

where δ_{ij} represents the metric of three-dimensional ordinary spaces with the spatial coordinates x^i $(i = 1, 2, 3)$, while $\delta_{\alpha\beta}$ represents the metric of n-dimensional compact spaces with the coordinates z^{α} $(\alpha = 4, \ldots, 3+n)$. We assume that the internal space is given by n-dimensional torus. The scale factor b denotes the size of the internal dimensions, while the scale factor a is the usual scale factor for the external space.

In the background metric (25) and assuming that ϕ only depends on time, we obtain the following equations:

$$G^{0}{}_{0} = -\kappa^2\left(\tfrac{1}{2}\dot{\phi}^2 + V_{\rm eff}\right) + X^{0}{}_{0}, \quad (26)$$

$$G^{i}{}_{j} = \kappa^2\left(\tfrac{1}{2}\dot{\phi}^2 - V_{\rm eff}\right)\delta^{i}{}_{j} + X^{i}{}_{j}, \quad (27)$$

$$G^{\alpha}{}_{\beta} = \kappa^2(\tfrac{1}{2}\dot{\phi}^2 - V_{\rm eff})\delta^{\alpha}{}_{\beta} + X^{\alpha}{}_{\beta}, \tag{28}$$

$$\ddot{\phi} + 3H_a\dot{\phi} + nH_b\dot{\phi} + V'_{\rm eff} = -J_n, \tag{29}$$

$$\dot{t}^0{}_0 + 3H_a(t^0{}_0 - t^1{}_1) + nH_b(t^0{}_0 - t^4{}_4) = 0, \tag{30}$$

where $H_a = \dot{a}/a$ and $H_b = \dot{b}/b$, and the components of the $(4+n)$-dimensional Einstein tensor are

$$G^0{}_0 = -3\left(H_a^2 + \frac{k_a}{a^2}\right) - n\left[\frac{(n-1)}{2}H_b^2 + 3H_aH_b + \frac{(n-1)}{2}\frac{k_b}{b^2}\right], \tag{31}$$

$$G^i{}_j = -\left(2\dot{H}_a + 3H_a^2 + \frac{k_a}{a^2}\right)\delta^i{}_j - n\left[\dot{H}_b + \frac{(n+1)}{2}H_b^2 + 2H_aH_b + \frac{(n-1)}{2}\frac{k_b}{b^2}\right]\delta^i{}_j, \tag{32}$$

$$G^{\alpha}{}_{\beta} = -3\left(\dot{H}_a + 2H_a^2 + \frac{k_a}{a^2}\right)\delta^{\alpha}{}_{\beta} - (n-1)\left[\dot{H}_b + \frac{n}{2}H_b^2 + 3H_aH_b + \frac{(n-2)}{2}\frac{k_b}{b^2}\right]\delta^{\alpha}{}_{\beta}. \tag{33}$$

The values of k_a and k_b are related to the curvatures of the external space and the internal space, respectively. In this paper we assume $k_b = 0$ for simplicity. Equations (26)–(30) are our basic equations to study cosmology in the Kaluza-Klein brane world.

The constraint equation for the extra component is given by

$$\dot{X}^0{}_0 + 3H_a(X^0{}_0 - X^1{}_1) + nH_b(X^0{}_0 - X^4{}_4) = -\kappa^2 J_n\dot{\phi}. \tag{34}$$

By combining Eqs. (26)–(28) one can write

$$X^0{}_0 + 3X^1{}_1 - 2X^4{}_4 = G^0{}_0 + 3G^1{}_1 - 2G^4{}_4 + 2\kappa^2 V_{\rm eff}. \tag{35}$$

A. Solutions with $\Lambda_b = 0$ and $J_n = 0$

Let us first consider that the bulk and brane potentials obey the generalized Randall-Sundrum condition $\Lambda_b = 0$, so that $V_{\rm eff} = 0$, and the energy conservation for scalar field on the brane is satisfied, $J_n = 0$. The solution of Eq. (29) is given by

$$\dot{\phi} = \frac{c_\phi}{a^3 b^n}, \tag{36}$$

where c_ϕ is an integration constant. Note that the scalar field depends on both the external scale factor and the internal scale factor, $\phi(t) = \phi(a(t), b(t))$.

In the following we consider a special case in which the brane world evolves with two scale factors. We take a simple relation between the scale factors on the brane of the form $b(t) = a^\gamma(t)$, where γ is a constant. For the internal scale factor $b(t)$ to be small compared to the external scale factor $a(t)$, the constant γ should be negative. In this choice we have $H_b = \gamma H_a \equiv \gamma H$, and then Eq. (34) becomes

$$\dot{X}^0{}_0 + (4+n\gamma)HX^0{}_0 = -n\gamma(1-\gamma)[\dot{H} + (3+n\gamma)H^2]H. \tag{37}$$

The Friedmann equation is given by

$$3(1+\alpha_0)H^2 = -3\frac{k_a}{a^2} + \frac{\kappa^2}{2}\frac{c_\phi^2}{a^{2(3+n\gamma)}} - X^0{}_0, \tag{38}$$

where

$$\alpha_0 = \frac{n\gamma(6 - \gamma + n\gamma)}{6}. \tag{39}$$

The extra component $X^0{}_0$ is determined by the solution of Eq. (37) that can be rewritten as

$$\dot{X}^0{}_0 + (4+\beta_0)X^0{}_0 H = \beta_1\frac{k_a}{a^2}H, \tag{40}$$

where

$$\beta_0 = \frac{n\gamma(2+\gamma+n\gamma)}{3+n\gamma}, \qquad \beta_1 = \frac{6n\gamma(1-\gamma)}{3+n\gamma}, \qquad \beta_2 = \frac{3n(1-\gamma)(n-1)(4+\gamma+3n\gamma)}{4(3+n\gamma)}. \tag{41}$$

Note that the effect of internal dimension is given by definitions (39) and (41). Equation (40) can be integrated to give

$$X^0{}_0 = \frac{\beta_1}{2+\beta_0}\frac{k_a}{a^2} - \frac{\varepsilon_0}{a^{4+\beta_0}}, \tag{42}$$

where ε_0 is an integration constant. Inserting Eq. (42) into Eq. (38), we find the modified Friedmann equation

$$3(1+\alpha_0)H^2 = -3\left[1 + \frac{\beta_1}{3(2+\beta_0)}\right]\frac{k_a}{a^2} + \frac{\kappa^2}{2}\frac{c_\phi^2}{a^{2(3+n\gamma)}} + \frac{\varepsilon_0}{a^{4+\beta_0}}. \tag{43}$$

In what follows we study the solutions with an initial singularity ($a = 0$). Using the conformal time variable η defined by the differential relation $dt = ad\eta$, we can rewrite Eq. (43) as

$$3(1+\alpha_0)\left(\frac{da}{d\eta}\right)^2 = -3\left[1 + \frac{\beta_1}{3(2+\beta_0)}\right]k_a a^2 + \frac{\kappa^2}{2}\frac{c_\phi^2}{a^{2(1+n\gamma)}} + \frac{\varepsilon_0}{a^{\beta_0}}, \tag{44}$$

and Eq. (36) as

$$\frac{d\phi}{d\eta} = \frac{c_\phi}{a^{(2+n\gamma)}}. \tag{45}$$

In the following we consider three cases: $\gamma = 0, \pm 1$.

In the case $\gamma = -1$, the internal scale factor $b(t)$ is proportional to $b(t) = 1/a(t)$. We have $a \to 0$, and the infinitely large internal space, $b \to \infty$.

In the case $\gamma = 1$, the internal scale factor $b(t)$ is related to $a(t)$ as $b(t) = a(t)$. The external scale factor evolves as

$$a^{2+n}(\tau) = \frac{\varepsilon_0 \tau(\tau + \tau_c)}{3(1 + k_a \tau^2)}, \tag{46}$$

where the new variable time is given by

$$\tau(\eta) = \begin{cases} \sqrt{\frac{3(2+n)}{2(3+n)}}|\eta|, & \text{for } k_a = 0 \\ \tan\sqrt{\frac{3(2+n)}{2(3+n)}}|\eta|, & \text{for } k_a = +1, \\ \tanh\sqrt{\frac{3(2+n)}{2(3+n)}}|\eta|, & \text{for } k_a = -1 \end{cases} \tag{47}$$

and

$$\tau_c = \frac{\sqrt{6}\kappa|c_\phi|}{\varepsilon_0}. \tag{48}$$

The scalar field evolves as

$$\phi - \phi_0 = \pm\sqrt{\frac{3+n}{(2+n)\kappa^2}} \ln\left|\frac{\tau}{\varepsilon_0(\tau + \tau_c)}\right|, \tag{49}$$

for $\varepsilon_0 \neq 0$, or

$$\phi - \phi_0 = \pm\sqrt{\frac{3+n}{(2+n)\kappa^2}} \ln|\tau|, \tag{50}$$

for $\varepsilon_0 = 0$, where $\pm$ corresponds to the sign of c_ϕ. Note that the initial singularity appears at $\tau = 0$. If we convert to proper time the evolution of the scale factor with $c_\phi \neq 0$ is given by

$$a(t) = \left(\frac{2}{3}\right)^{1/2(2+n)} \left(\frac{3+n}{2+n}\right)^{5+n/2(2+n)(3+n)} (\kappa c_\phi)^{1/3+n} t^{1/3+n}. \tag{51}$$

Then, the Universe starts expanding away from the initial singularity.

A more interesting case is in the static internal dimension, $\gamma = 0$; we obtain

$$a^2(\tau) = \frac{\varepsilon_0 \tau(\tau + \tau_c)}{3(1 + k_a \tau^2)}, \tag{52}$$

where

$$\tau(\eta) = \begin{cases} |\eta|, & \text{for } k_a = 0 \\ \tan|\eta|, & \text{for } k_a = +1, \\ \tanh|\eta|, & \text{for } k_a = -1 \end{cases} \tag{53}$$

and

$$\phi - \phi_0 = \pm\sqrt{\frac{3}{2\kappa^2}} \ln\left|\frac{\tau}{\varepsilon_0(\tau + \tau_c)}\right|, \tag{54}$$

for $\varepsilon_0 \neq 0$, or

$$\phi - \phi_0 = \pm\sqrt{\frac{3}{2\kappa^2}} \ln|\tau|, \tag{55}$$

for $\varepsilon_0 = 0$. In a spatially flat Universe the evolution of the scale factor is the same as the standard cosmological solution with stiff matter,

$$a(t) = \left(\frac{3}{2} c_\phi^2 \kappa^2\right)^{1/6} t^{1/3}. \tag{56}$$

We also find that the Universe starts expanding away from the initial singularity.

B. Solutions with exponential potential

We may now consider the case when the bulk scalar field potential and the brane tension are given as exponents, with some constant parameters V_0, σ_0, and λ,

$$V(\phi) = V_0 \exp\left(-\frac{2}{3+n}\lambda\kappa\phi\right), \tag{57}$$

$$\sigma(\phi) = \sigma_0 \exp\left(-\frac{1}{3+n}\lambda\kappa\phi\right). \tag{58}$$

These are string-inspired values for the potentials [52–54]. Then the effective potential is given by

$$V_{\text{eff}} = V_{\text{eff},0} \exp\left(-\frac{2}{3+n}\lambda\kappa\phi\right) = \frac{\Lambda_b}{\kappa^2}, \tag{59}$$

where

$$V_{\text{eff},0} = \frac{2+n}{4+n} V_0 + \frac{(2+n)\kappa^2\sigma_0^2}{8(3+n)}\left[1 - \frac{\lambda^2}{3(4+n)}\right]. \tag{60}$$

We also assume the proportionality relation between the scalar field and the logarithm of the scale factors,

$$\phi = \frac{\lambda}{\kappa} \ln(ab) = \frac{\lambda}{\kappa} \ln(a^{1+\gamma}). \tag{61}$$

From the above model, if $\lambda = 0$ we have $\phi = 0$ and then $V = V_0$, $\sigma = \sigma_0$, and $V_{\text{eff}} = V_{\text{eff},0}$. The induced cosmological constant on the brane, Λ_b, is exactly constant. If we set $\Lambda_b = 0$, the generalized Randall-Sundrum condition is given by

$$\sigma_0 = \sqrt{\frac{8(3+n)}{(4+n)\kappa}(-V_0)}. \tag{62}$$

For the positive brane tension, V_0 must be negative. Thus V_0 can be interpreted as a bulk cosmological constant and the Randall-Sundrum brane is a slice in $(5+n)$-dimensional anti–de Sitter spacetime. For

vanishing the extra component $X_{\mu\nu}$ or the Weyl tensor $E_{\mu\nu}$, we have the Minkowski spacetime on the brane.

Applying the above model, Eq. (34) becomes

$$\dot{X}^0{}_0 + (4+\beta_{0\lambda})X^0{}_0 H = \beta_{1\lambda}\frac{k_a}{a^2}H + \beta_3\kappa^2 V_{\rm eff}H, \quad (63)$$

where

$$\beta_{0\lambda} = \beta_0 + \frac{(1+\gamma)^2}{3+n\gamma}\lambda^2, \qquad \beta_{1\lambda} = \beta_1 - \frac{6(1+\gamma)^2}{3+n\gamma}\lambda^2,$$
$$\beta_{2\lambda} = \beta_2 - \frac{(n-1)(3+5n)(1+\gamma)^2}{4(3+n\gamma)}\lambda^2,$$
$$\beta_3 = \frac{6n(1-\gamma)^2}{(2+n)(3+n\gamma)} + \frac{2(1+\gamma)(3+3n+n^2+4n\gamma+9\gamma)}{(2+n)(3+n)(3+n\gamma)}\lambda^2. \quad (64)$$

The solution of Eq. (63) is given by

$$X^0{}_0 = \frac{\beta_{1\lambda}}{2+\beta_{0\lambda}}\frac{k_a}{a^2} + \frac{(3+n)\beta_3}{(3+n)(4+\beta_{0\lambda})-2\lambda^2}\kappa^2 V_{\rm eff} - \frac{\varepsilon_{0\lambda}}{a^{(4+\beta_{0\lambda})}}, \quad (65)$$

where $\varepsilon_{0\lambda}$ is an integration constant. The Friedmann equation is given by

$$3(1+\alpha_{0\lambda})H^2 = -3\left[1+\frac{\beta_{1\lambda}}{3(2+\beta_{0\lambda})}\right]\frac{k_a}{a^2} + \left[1-\frac{(3+n)\beta_3}{(3+n)(4+\beta_{0\lambda})-2\lambda^2}\right]\frac{\kappa^2 V_{\rm eff,0}}{a^{2\lambda^2/(3+n)}} + \frac{\varepsilon_{0\lambda}}{a^{(4+\beta_{0\lambda})}}, \quad (66)$$

where

$$\alpha_{0\lambda} = \alpha_0 - \tfrac{1}{6}(1+\gamma)^2\lambda^2. \quad (67)$$

We also find that a possible energy leak out of the brane into the bulk is in the form

$$J_n = -\frac{3(1+\alpha_{0\lambda})}{3+n\gamma}H\dot{\phi} - \frac{(1+\gamma)\lambda}{(3+n\gamma)\kappa} \times\left[\frac{n(1+n)+\gamma(12+3n-n^2)}{(2+n)(3+n)(1+\gamma)}\frac{\kappa^2 V_{\rm eff,0}}{a^{2\lambda^2/(3+n)}} - 3\frac{k_a}{a^2}\right]. \quad (68)$$

Note that Eqs. (66) and (68) include a five-dimensional case, corresponding to $n = 0$ [54]. By contrast, in higher dimensions, J_n depends on the effective potential or the induced cosmological constant on the brane. For example, in the static internal dimension, $\gamma = 0$, and assuming $k_a = 0$ for simplicity, we find

$$J_n = -\left(1-\frac{\lambda^2}{6}\right)H_a\dot{\phi} - \frac{n(1+n)\lambda}{3(2+n)(3+n)\kappa}\Lambda_b, \quad (69)$$

where we have used $\kappa^2 V_{\rm eff} = \Lambda_b$. If we impose the generalized Randall-Sundrum condition, $\Lambda_b = 0$, the energy is flowing onto the brane when $\lambda^2 < 6$ and flowing out to the brane when $\lambda^2 > 6$. For $\Lambda_b > 0$, the energy is flowing onto the brane when $0 < \lambda < \sqrt{6}$ and flowing out to the brane when $\lambda < -\sqrt{6}$. While for $\Lambda_b < 0$, the energy is flowing onto the brane when $-\sqrt{6} < \lambda < 0$ and flowing out to the brane when $\lambda > \sqrt{6}$.

IV. STATIC INTERNAL DIMENSION SOLUTIONS WITH MATTER ON THE BRANE

In the previous section we focused upon dilaton-vacuum solutions on the brane. For completeness we now study the presence of matter on the brane. In this case we do not have $X^\mu{}_\mu = 0$. Indeed, we have $\mathcal{Z}^\mu{}_\mu = 0$ that implies

$$\mathcal{Z}^0{}_0 + 3\mathcal{Z}^1{}_1 + n\mathcal{Z}^4{}_4 = 0. \quad (70)$$

Using Eq. (20) and by combining Eqs. (35) and (70) we find

$$-n\mathcal{Z}^4{}_4 = \mathcal{Z}^0{}_0 + 3\mathcal{Z}^1{}_1 = -\frac{n}{2+n}(G^0{}_0 + 3G^1{}_1 - 2G^4{}_4) - \frac{2n}{2+n}\kappa^2 V_{\rm eff} + \frac{n}{2+n}(Y^0{}_0 + 3Y^1{}_1 - 2Y^4{}_4), \quad (71)$$

where $Y^\mu{}_\nu$ is given by Eq. (21) with the components of $\pi^\mu{}_\nu$ are

$$\pi^0{}_0 = -\frac{1}{8(3+n)}\{2(t^0{}_0)^2 + n[(t^0{}_0)^2 - 3(t^1{}_1 - t^4{}_4)^2]\}, \quad (72)$$

$$\pi^i{}_j = \frac{1}{8(3+n)}\{2(t^0{}_0)^2 - 4t^0{}_0 t^1{}_1 + n[(t^0{}_0)^2 - 2t^0{}_0 t^4{}_4 + (t^1{}_1 - t^4{}_4)(t^1{}_1 - 3t^4{}_4)]\}\delta^i{}_j, \quad (73)$$

$$\pi^\alpha{}_\beta = \frac{1}{8(3+n)}\{2(t^0{}_0)^2 - 6t^0{}_0 t^1{}_1 + 2t^4{}_4(t^0{}_0 + 3t^1{}_1 - 3t^4{}_4) + n[(t^0{}_0)^2 - 2t^0{}_0 t^4{}_4 + 3(t^1{}_1 - t^4{}_4)^2]\}\delta^\alpha{}_\beta. \quad (74)$$

Here $-t^0{}_0$ is the total energy density, $t^1{}_1 = t^2{}_2 = t^3{}_3$ the total external pressure, and $t^4{}_4 = t^5{}_5 = \ldots = t^{3+n}{}_{3+n}$ the total internal pressure. Inserting Eq. (20) into Eq. (34) and then eliminating $\mathcal{Z}^1{}_1$ and $\mathcal{Z}^4{}_4$ by using Eq. (71), we find

$$\dot{\mathcal{Z}}^0{}_0 + (4H_a + nH_b)\mathcal{Z}^0{}_0 = \kappa^2 J_n \dot{\phi} - \frac{n}{2+n}(G^0{}_0 + 3G^1{}_1 - 2G^4{}_4 + 2\kappa^2 V_{\rm eff})(H_a - H_b) + \dot{Y}^0{}_0 + \frac{2}{2+n}[(3+2n)Y^0{}_0 - 3Y^1{}_1 - nY^4{}_4]H_a + \frac{n}{2+n}[(1+n)Y^0{}_0 - 3Y^1{}_1 - nY^4{}_4]H_b. \quad (75)$$

Note that in the absence of matter on the brane, $\mathcal{Z}^\mu{}_\nu = -X^\mu{}_\nu$, Eq. (75) reduced to Eq. (34).

In the following we study the static internal dimension case for simplicity. In this case we have

$$H_b = 0, \quad \text{and} \quad G^0{}_0 + 3G^1{}_1 - 2G^4{}_4 = 0. \quad (76)$$

Then, Eq. (75) becomes

$$\dot{\mathcal{Z}}^0{}_0 + 4\mathcal{Z}^0{}_0 H_a = \kappa^2 J_n \dot{\phi} - \frac{2n}{2+n}\kappa^2 V_{\rm eff} H_a + \dot{Y}^0{}_0 + \frac{2}{2+n}[(3+2n)Y^0{}_0 - 3Y^1{}_1 - nY^4{}_4]H_a. \quad (77)$$

The conservation of the energy-momentum tensor for the matter field on the brane and the equation of motion of the scalar field, respectively, are given by

$$\dot{t}^0{}_0 = -3H_a(t^0{}_0 - t^1{}_1), \quad (78)$$

$$\ddot{\phi} + 3H_a\dot{\phi} + V'_{\rm eff} = -J_n. \quad (79)$$

Eliminating $Y^\mu{}_\nu$ in Eq. (77), we obtain

$$\dot{\mathcal{Z}}^0{}_0 + 4\mathcal{Z}^0{}_0 H_a = \kappa^2 J_n \dot{\phi} - \frac{2n}{2+n}\kappa^2 V_{\rm eff} H_a + \frac{(2+n)\kappa^4}{4(3+n)}\sigma'\dot{\phi}t^0{}_0 + \frac{n\kappa^4}{4(3+n)}\sigma[t^0{}_0 + 3t^1{}_1 - 2t^4{}_4]H_a + \kappa^4\dot{\pi}^0{}_0 + \frac{2}{2+n} \times [(3+2n)\pi^0{}_0 - 3\pi^1{}_1 - n\pi^4{}_4]H_a. \quad (80)$$

We now consider the exponential potential as discussed in the previous section with static internal dimension. By defining the equations of state $t^1{}_1 = -w_i t^0{}_0$, $t^4{}_4 = -w_\alpha t^0{}_0$, and inserting Eq. (79) into Eq. (80) for J_n, one can integrate the following equation:

$$\dot{\mathcal{Z}}^0{}_0 + \left(4 + \frac{\lambda^2}{3}\right)\mathcal{Z}^0{}_0 H_a = -\frac{6n(3+n) + 2(3+3n+n^2)b^2}{3(2+n)(3+n)}\kappa^2 V_{\rm eff} H_a + 2\lambda^2\frac{k_a}{a^2}H_a + \frac{3n(3+n)(1-3w_i+2w_\alpha) + [3+3n+n^2-(3+n)(3w_i+nw_\alpha)]\lambda^2}{12(3+n)^2}\kappa^4\sigma t^0{}_0 H_a + \frac{3n(1-3w_i+3w_\alpha)(w_\alpha+3(1+w_i)(w_i-w_\alpha)) + [3w_i+nw_\alpha+3n(w_i-w_\alpha)^2]\lambda^2}{12(3+n)}\kappa^4(t^0{}_0)^2 H_a, \quad (81)$$

and yields

$$\mathcal{Z}^0{}_0 = -\frac{6n(3+n) + 2(3+3n+n^2)\lambda^2}{(2+n)[12(3+n) + (n-3)\lambda^2]}\kappa^2 V_{\rm eff} - \frac{3n(3+n)(1-3w_i+2w_\alpha) + [3+3n+n^2-(3+n)(3w_i+nw_\alpha)]\lambda^2}{4(3+n)[3(3+n)(1-3w_i) + n\lambda^2]}\kappa^4\sigma t^0{}_0 + \frac{3n(1-3w_i+3w_\alpha)[w_\alpha+3(1+w_i)(w_i-w_\alpha)] + [3w_i+nw_\alpha+3n(w_i-w_\alpha)^2]\lambda^2}{4(3+n)[6(1+3w_i) - \lambda^2]}\kappa^4(t^0{}_0)^2. \quad (82)$$

Inserting Eq. (82) into the (00) component of the Einstein equations, we find

$$H_a^2 + \frac{\bar{k}_a}{a^2} = \beta_4 \kappa^2 V_{\rm eff} + 8\pi G_{\rm eff}\rho + \beta_5 \rho^2 + \frac{\bar{C}}{a^{4+\lambda^2/3}}, \tag{83}$$

$$J_n = -\left(1 - \frac{\lambda^2}{6}\right) H_a \dot{\phi} + \frac{\lambda}{\kappa}\frac{k_a}{a^2} - \frac{n(1+n)\lambda\kappa}{3(2+n)(3+n)} V_{\rm eff} - \frac{(1 - 3w_i - nw_\alpha)\lambda\kappa^3}{12(3+n)}\sigma\rho + \frac{[4(1+3w_i) + 2n(2+2w_i+3w_\alpha+3(w_i-w_\alpha)^2) + n^2(1+2w_i+3(w_i-w_\alpha)^2)]\lambda\kappa^3}{24(2+n)(3+n)}\rho^2, \tag{84}$$

where $\rho = -t^0{}_0$, and

$$\bar{k}_a = \left(1 + \frac{\lambda^2}{6}\right)^{-1} k_a, \tag{85}$$

$$\beta_4 = \frac{2(3+n)(4+n)}{(2+n)[12(3+n) + (n-3)\lambda^2]}, \tag{86}$$

$$8\pi G_{\rm eff} = \frac{(1 - 3w_i - nw_\alpha)}{2(3+n)(1-3w_i) + 2n\lambda^2}\kappa^4\sigma, \tag{87}$$

$$\beta_5 = \frac{2(1+3w_i) + n(1+2w_\alpha) + 3(w_i - w_\alpha)^2}{4(3+n)[6(1+3w_i) - \lambda^2]}\kappa^4, \tag{88}$$

$$\bar{C} = \frac{1}{3}\left(1 - \frac{\lambda^2}{6}\right)^{-1} C. \tag{89}$$

The first two terms on the right-hand side of (83) are what we would expect for standard four-dimensional cosmology. The third term is quadratic in the brane energy-momentum tensor, and the fourth term is a generalized dark radiation energy component. Note that in contrast with the five-dimensional case, the effective gravitational constant depends on time through the brane tension and the equation of state. One can obtain the variation rate of the effective gravitational constant as

$$\frac{\dot{G}_{\rm eff}}{G_{\rm eff}} = -\frac{\lambda^2}{3+n} H_a. \tag{90}$$

We see that the variation rate of the effective gravitational constant is smaller than the five-dimensional case ($n = 0$) with the exponential potential [54], and it is relatively small if the number of the internal dimension n is large. Observational bounds on $\dot{G}_{\rm eff}/G_{\rm eff}$ then constrain the parameters of the theory. If we assume $|\dot{G}_{\rm eff}/(G_{\rm eff}H_{a,0})| \lesssim 10^{-2}$, where $H_{a,0}$ is the value of the Hubble parameter today, we have $\lambda^2 \lesssim (3+n)/100$.

V. COSMOLOGICAL EVOLUTION OF THE SOURCES IN THE EINSTEIN FRAME

In the previous sections, we discussed the cosmological evolutions in the induced metric frame, which is given by Eq. (14). In this section, we study the four-dimensional cosmological evolution in the Einstein frame and discuss the properties of the resulting four-dimensional cosmological evolution in this frame. A complementary way of studying higher-dimensional theories is to dimensionally reduce the action by integrating out the internal dimensions and then performimg a conformal transformation.

From Eq. (14) we define the Newton gravitational constant in $(4+n)$-dimensions as

$$8\pi G_{4+n} = \frac{(2+n)\kappa^4}{4(3+n)}\sigma. \tag{91}$$

If the brane tension is not depending on the bulk scalar field (minimally coupled), the $(4+n)$-dimensional Newton gravitational constant is truly constant. In this section we assume $\sigma = $ const. Note that recovering the four-dimensional effective Newton gravitational constant requires an additional time-dependent factor,

$$8\pi G_4 = \frac{8\pi G_{4+n}}{V_n}, \tag{92}$$

where V_n denotes the volume of the n-dimensional internal spaces, while the four-dimensional cosmological constant is given by

$$\Lambda_4 = V_n \Lambda_b, \tag{93}$$

where Λ_b is given by Eq. (15). Note that both G_4 and Λ_4 are time-dependent. However, it must be remembered that we are not in the Einstein frame. Similarly, the four-dimensional energy density is given by $\rho_4 = V_n\rho$.

From Eqs. (87), (91), and (92), we get

$$G_4 = \frac{(2+n)[2(3+n)(1-3w_i) + 2n\lambda^2]}{4(3+n)(1-3w_i-nw_\alpha)}\frac{G_{\rm eff}}{V_n}. \tag{94}$$

Here, V_n is a constant, corresponding to the static internal dimensions. As we expected

$$\frac{\dot{G}_4}{G_4} = \frac{\dot{G}_{\rm eff}}{G_{\rm eff}}. \tag{95}$$

However, for the static internal dimensions case with a constant brane tension, the effective Newton gravitational constant is exactly constant. It depends on the equations of state, the potential parameter λ, and the number of internal dimensions.

In order to recover a four-dimensional cosmological evolution in the Einstein frame, we proceed through three

steps: (1) a dimensional reduction of the $(4+n)$-action by integrating out the internal dimensions; (2) a conformal transformation of the four-dimensional induced metric into the Einstein frame; (3) a redefinition of the scale factor b into a new scalar field to give its kinetic term a canonical form. From the gravitational sector, the resulting four-dimensional Einstein frame action is given by

$$S_* = \int d^4x\sqrt{-h_*} \times \left[\frac{1}{16\pi G_{4*}}R_* - \frac{1}{2}h^{*\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V_*(\Phi)\right], \quad (96)$$

where we mark the quantities in the Einstein frame with the subscript * and a new scalar field is defined by

$$\Phi = \sqrt{\frac{n(2+n)}{16\pi G_{4*}}}\ln b. \quad (97)$$

We also have the relations between the variables in the four-dimensional induced metric frame and those in the Einstein frame as

$$dt_* = b^{n/2}dt, \qquad a_* = b^{n/2}a, \quad (98)$$

where the scale factor b is appropriately expressed in Φ. Hence we have the relation

$$H_{a*} = b^{-n/2}\left(H_a + \frac{n}{2}H_b\right), \quad (99)$$

where H_{a*} is the Hubble parameter in the Einstein frame,

$$H_{a*} = \frac{1}{a_*}\frac{da_*}{dt_*}. \quad (100)$$

In the static internal dimensions case, $b = b_0 = 1$, we have $a_* = a$ so that $H_{a*} = H_a$.

From the matter sector, we have

$$\rho_* = b^{-2n}\rho_4. \quad (101)$$

Assuming $V_n \sim b^n$, the evolution of the four-dimensional energy density in the Einstein frame is given by

$$\rho_* \propto b^{-(n/2)(1-3w_i+2w_\alpha)}a_*^{-3(1+w_i)}. \quad (102)$$

Note that we recover standard four-dimensional energy density for $n = 0$ or if the equations of state satisfy the following constraint:

$$1 - 3w_i + 2w_\alpha = 0. \quad (103)$$

For instance, in the radiation-dominated era $w_i = 1/3$, we find $w_\alpha = 0$, the internal pressure drops to zero. Moreover, in order to understand the physical interpretations of the constraint (103), let us consider Eq. (76). Because the components of the Einstein tensor are corresponding to the matter field: ${G^0}_0 \sim {t^0}_0$, ${G^1}_1 \sim {t^1}_1$, and ${G^4}_4 \sim {t^4}_4$, and using the relations ${t^1}_1 = -w_i{t^0}_0$, ${t^4}_4 = -w_\alpha{t^0}_0$, the constraint (103) can be interpreted as a necessary condition for static internal dimensions.

We have shown that we can obtain the standard cosmology in the Einstein frame by assuming a constant brane tension. In the case of the nonminimally coupled to bulk scalar field, in which the brane tension is a function of the bulk scalar field, a conformal transformation is determined by the form of brane tension. For instance, if the brane tension is given by Eq. (58), the Einstein frame is obtained by a conformal transformation of the four-dimensional metric: $h_{*\mu\nu} = b^n e^{2\varphi}h_{\mu\nu}$, where $2\varphi = \lambda\kappa\phi/(3+n)$. Then the Hubble parameter in the Einstein frame is related to the Hubble parameter in the induced metric frame as

$$H_{a*} = b^{-n/2}e^{-\varphi}\left(H_a + \frac{n}{2}H_b + \dot{\varphi}\right), \quad (104)$$

and the evolution of the four-dimensional energy density in the Einstein frame is given by $\rho_* = b^{-n}e^{-4\varphi}\rho$, with ρ being the higher-dimensional energy density.

VI. CONCLUSIONS

In this paper we studied the brane-world cosmological model in higher-dimensional spacetime with a bulk scalar field. The $(4+n)$-dimensional gravitational field equations can be formulated to standard form with the extra component. In the absence of the matter on the brane, we can interpret the extra component as the energy-momentum tensor for the dark radiation. We used this formalism to discuss cosmology in the brane Universe in the context of the Kaluza-Klein brane-world scheme, i.e., to consider Kaluza-Klein compactifications on the brane. By assuming the cosmological symmetry on the Kaluza-Klein brane, we derived the Friedmann equation and a possible energy leak out of the brane into the bulk.

If the bulk and brane potentials obey the generalized Randall-Sundrum condition and the energy conservation for scalar field on the brane is satisfied, we find that the brane Universe starts expanding away from the initial singularity both for the internal scale factor that is proportional to the external scale factor and the static internal scale factor. A significant result in the present study is that of the possibility of energy flow out of the brane into the bulk, which depends on the effective potential or the effective cosmological constant on the brane. It is different with the five-dimensional dilatonic brane-world cosmologies. By constraining the potential parameter λ and the effective cosmological constant, there exist two possibilities of the energy flows: on to and out of the brane. Finally, we derive the Friedmann equation with matter on the brane. We find that the effective gravitational constant depends on time through the brane tension and the equation of state. However, after appropriate conformal transformation, the

four-dimensional Newton gravitational constant in the Einstein frame is a true constant. A new result in Sec. V is that we recover standard four-dimensional energy density in the Einstein frame if the equations of state satisfy the constraint Eq. (103).

There could be various extensions of our considerations. By introducing a bulk scalar field, it is interesting to stabilize the distance between the two Kaluza-Klein branes as is done in the context of the first model introduced by Randall and Sundrum. It also important to understand inflation without inflaton and the quantum fluctuations of brane-inflation in the Kaluza-Klein brane scenario with a bulk scalar field. It is intriguing to consider the low energy description of this brane-world model [51]. We leave these issues for future studies.

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