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Higgs multiplets of the quark-lepton family group

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It is shown that realistic models can be constructed in which the standard model Higgs field is in a nontrivial multiplet of a non-Abelian family group of the quarks and leptons. It is shown that the observed quark and lepton masses and mixing angles can be fit, while the coefficients of flavor-changing four-fermion operators mediated by the extra Higgs doublets are determined in terms of only a few unknown parameters.

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I. INTRODUCTION

A peculiar feature of the standard model is that there are many multiplets of fermions, but only one multiplet of spin-0 bosons, the Higgs doublet. Supersymmetrizing the standard model would produce a balance between spin-0 and spin-$1/2$, but still would not explain why there are so many matter multiplets (i.e. quarks and leptons) and so few Higgs multiplets.

In this paper we pursue a different idea than supersymmetry. We suppose that there is a non-Abelian family group [1] under which *both* the Higgs fields and the matter fields transform as nontrivial multiplets. The particular model we shall describe as an example has an $SO(4)_F$ family group (or equivalently $SU(2) \times SU(2)$), under which four-quark and lepton families transform as a 4-plet, a mirror family transforms as a singlet, and nine Higgs doublets transform as a 9-plet, i.e. as a rank-2, symmetric, traceless tensor. [Under the $SU(2) \times SU(2)$ to which $SO(4)_F$ is isomorphic, the four families transform as (2, 2) and the 9 Higgs doublets as (3, 3).] We shall call all nine of the scalar doublets "Higgs" doublets, even though only the lightest of them—the standard model Higgs doublet—actually gets a nonzero vacuum expectation value (VEV).

Such a rich Higgs sector would yield new physics beyond the standard model. Most obviously, it would imply the existence of flavor-changing couplings of the "extra" Higgs doublets. The non-Abelian family group, besides explaining to some extent why there are families of quarks and leptons, and giving a rich Higgs sector, would also greatly constrain the form of the quark and lepton mass matrices and the couplings of the extra Higgs doublets. There is therefore the potential of great predictivity. For example, in the illustrative $SO(4)_F$ model discussed in this paper we shall show that there are sufficiently many model parameters to give a good fit to the quark and lepton masses and mixings, but still few enough parameters that the coefficients of all the flavor-changing four-fermion operators are almost completely determined.

One might worry that these flavor-changing effects would be too large. However, in the kind of model we are describing there is a mass hierarchy within the family multiplet of Higgs fields that mirrors the mass hierarchy among the families of quarks and leptons. Therefore, most of the extra Higgs doublets (particularly those that couple most strongly to the first family of quarks and leptons) are much heavier than the standard model Higgs doublet, and excessive flavor-changing effects can be avoided. Nevertheless, as will be seen, there typically is a "lightest extra Higgs doublet" (LED) that can give flavor-changing near the current limits.

This raises another question: given that there is no low-energy supersymmetry to protect them, should not all the extra Higgs doublets "naturally" be superheavy? In other words, would not a multiplicity of Higgs doublets make the "gauge hierarchy problem" much worse, since there are now many such fields whose masses have to be tuned? The answer is that family symmetry protects the masses of the extra Higgs doublets and there is no extra tuning. We assume that the mass-squared of the standard model Higgs field (the lightest Higgs field in the $SO(4)_F$ 9-plet) is set "anthropically". Under reasonable assumptions this means that it must be negative and have magnitude of order $(100\ \mathrm{GeV})^2$ [2,3]. Since all the Higgs doublets are in one irreducible multiplet (the 9-plet) of the $SO(4)_F$ family group, the masses of the extra Higgs fields are tied to that of the standard model Higgs field by that symmetry. (One can also use the language of $SU(2) \times SU(2)$, to which $SO(4)$ is isomorphic: under $SU(2) \times SU(2)$, the 9-plet of Higgs transforms as (3, 3), which is clearly irreducible.) Since all the Higgs doublets are in one irreducible multiplet of $SO(4)$ (or $SU(2) \times SU(2)$), they would have equal masses in the limit of exact flavor symmetry. But given that the flavor symmetry is broken, the mass splittings among the Higgs doublets are controlled by this breaking. In particular, since tuning makes the mass-squared of the lightest Higgs doublet (the standard model Higgs doublet) quite small compared to the $SO(4)_F$-breaking splittings within the 9-plet, the masses of the extra eight Higgs doublets are of the order of magnitude of some $SO(4)_F$-breaking vacuum expectation value. This breaking is assumed to be dynamical, and therefore can occur without fine-tuning at a low enough scale to produce observable effects.

In a previous paper [4], one of us proposed a much more ambitious version of this model, in which unification of the standard model gauge couplings was achieved through the group $SU(3) \times SU(3) \times SU(3) \times Z_3$. This led to a much more involved model. Here, by staying with the standard model gauge group $G_{\rm SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$ we have a model that is considerably simpler and easier to analyze.

II. THE MODEL

The model has the gauge group $G_{\rm SM} \times SO(4)_F \times SU(N)_{\rm DSB}$, where $SU(N)_{\rm DSB}$ is a confining group that plays the role of dynamically breaking the family group $SO(4)_F$. The field content is shown in Table I:

In Table I and throughout the paper, the $SO(4)_F$ indices are denoted by Latin letters i, j, k and range from 1 to 4. The fact that the Higgs fields are in a rank-2 symmetric tensor multiplet of $SO(4)$ allows them to couple directly by a renormalizable Yukawa term to the quarks and leptons, schematically as $Y(\psi^i \psi^j)\Phi^{(ij)}$. Note that $SO(4)_F$ symmetry and the pattern of its breaking controls the form of $\langle \Phi^{(ij)} \rangle$ and thus the form of the "textures" of the quark and lepton mass matrices. So we now consider how $SO(4)_F$ is broken and how this breaking is communicated to the standard model fields.

The dynamical symmetry breaking is done by a $\langle \bar{\chi}_a \chi^i \rangle$ condensate, where as shown in Table I the χ^i are N's of $SU(N)_{\rm DSB}$ in a 4 of $SO(4)_F$ and the $\bar{\chi}_a$ are four $\bar{N}$'s of $SU(N)_{\rm DSB}$ that are singlets of $SO(4)_F$ with the subscript a being just a label that distinguishes them. Since renormalizable couplings of the χ, $\bar{\chi}$ fields to the standard model fields are forbidden by the gauge symmetries of the model, as is easily seen, the standard model fields can only learn of the breaking of the family group $SO(4)_F$ through "messenger fields", which are the η_I^i shown in Table I. These are real scalars that are vectors under $SO(4)_F$ and singlets under the other groups. There are several such messenger multiplets, which are distinguished by a capital Latin subscript.

The $SO(4)_F$-breaking condensate $\langle \bar{\chi}_a \chi^i \rangle$ generates vacuum expectation values for the messenger fields through the terms

$$f_{aI} \langle \bar{\chi}_a \chi^i \rangle \eta_I^i + \tfrac{1}{2} M_{IJ}^2 \eta_I^i \eta_J^i, \tag{1}$$

where here and throughout we always sum over repeated indices of any type. These terms give $\langle \eta_I^i \rangle = -M_{IK}^{-2} f_{aK} \langle \bar{\chi}_a \chi^i \rangle$. If the scale of the $\langle \bar{\chi}\chi \rangle$ condensate is called Λ^3, and the mass of the messenger fields η is assumed to be superheavy (near the Planck scale), then the messenger VEVs are typically of order $\Lambda^3/M_{P\ell}^2$. Since the scale Λ is set by dynamical symmetry breaking, it can naturally be of any magnitude, depending on the $SU(N)_{\rm DSB}$ gauge coupling. Thus the VEVs of the messenger fields can be quite near the weak scale in a "technically natural" way. If we suppose that the VEVs of the messenger fields are in the 10 to 1000 TeV range, as will be assumed later, then Λ is of order 10^{14} GeV. This is the scale at which the local $SO(4)_F$ symmetry is broken, and thus the mass scale of the $SO(4)_F$ gauge bosons, which are consequently far too heavy to affect low-energy physics. And since the messenger fields are superheavy, their exchange is also irrelevant to low-energy physics. The VEVs of the messenger fields, by contrast, can be small enough to produce significant effects at low energy, and, in particular, to split the 9-plet of Higgs fields and determine the pattern of quark and lepton masses. Note that since the matrices M_{IJ}^2 and f_{aI} in Eq. (1) are arbitrary parameters, they can have a nontrivial and perhaps hierarchical form, and therefore so can the VEVs of the messenger fields.

There are two types of renormalizable couplings of the messenger fields to the standard model fields. They couple directly to the fermions through terms that are schematically of the form $y_I(\psi^i \bar{\psi})\eta_I^i$. Such terms, which will be discussed in more detail later, have the effect of "mating" the mirror family with one of the four families to give them a large mass, leaving three light families.

The messenger fields also couple directly to the Higgs doublets through a renormalizable term of the form

$$\mathcal{L}_{\Delta M_\Phi^2} = \tfrac{1}{2} \lambda_{KI} \Phi^{(ij)\dagger} \Phi^{(jk)} \eta_K^k \eta_I^i. \tag{2}$$

Defining what we shall call the "master matrix" m^2 by

$$(m^2)^{ij} \equiv \lambda_{IJ} \langle \eta_I^i \rangle \langle \eta_J^j \rangle, \tag{3}$$

TABLE I. The field content of the model. F stands for the $G_{\rm SM}$ family representation $(3, 2, \frac{1}{6}) + (\bar{3}, 1, -\frac{2}{3}) + (\bar{3}, 1, \frac{1}{3}) + (1, 2, -\frac{1}{2}) + (1, 1, 1)$; and H for the $G_{\rm SM}$ Higgs representation $(1, 2, -\frac{1}{2})$. The subscript DSB stands for "dynamical symmetry breaking."

Field	$G_{\rm SM} \times SO(4)_F \times SU(N)_{\rm DSB}$	Symbol
4 families	$(F, 4, 1)$	$\psi^i = Q^i, (u^c)^i, (d^c)^i, L^i, (\ell^c)^i$
Mirror family	$(\bar{F}, 1, 1)$	$\bar{\psi} = \bar{Q}, \bar{u}^c, \bar{d}^c, \bar{L}, \bar{\ell}^c$
Higgs doublets	$(H, 9, 1)$	$\Phi^{(ij)}$
Messenger scalar fields	$(1, 4, 1)$	η_I^i
DSB fermions	$(1, 4, N)$	χ^i
DSB fermions	$4 \times (1, 1, \bar{N})$	$\bar{\chi}_a$, $a = 1, \ldots, 4$

we can write the mass terms of the nine Higgs doublets as

$$\begin{aligned}\mathcal{L}_{M^2_\Phi} &= -\tfrac{1}{2}M^2\Phi^{(ij)\dagger}\Phi^{(ji)} - (m^2)^{ki}\Phi^{(ij)\dagger}\Phi^{(jk)} \\ &= -\tfrac{1}{2}M^2\,\mathrm{Tr}[\Phi^\dagger\Phi] - \mathrm{Tr}[m^2\Phi^\dagger\Phi]. \qquad (4)\end{aligned}$$

The parameter M^2 in Eq. (4) is the overall $SO(4)_F$-invariant mass of the Higgs 9-plet. The matrix m^2 in Eq. (4) gives the splittings within the 9-plet. As a result of these splittings, one linear combination of the $\Phi^{(ij)}$ is lighter than the rest. It is assumed that anthropic tuning of the parameter M^2 causes the mass-squared of this lightest doublet to be negative and of order $(100\ \mathrm{GeV})^2$, meaning that it is the standard model Higgs field. (In other words, M^2 varies among domains or subuniverses of the Universe, so that there exist domains in which the mass-squared of the lightest doublet has the value required for life to be possible.) Let the standard model Higgs doublet be the following linear combination: $\Phi_{\mathrm{SM}} = \frac{1}{2}\Sigma_{ij}a_{ij}\Phi^{(ij)}$, with $\Sigma_{ij}|a_{ij}|^2 = 2$, where a_{ij} (like $\Phi^{(ij)}$) is a symmetric traceless matrix. It then follows that $\langle\Phi^{(ij)}\rangle = a_{ij}\langle\Phi_{\mathrm{SM}}\rangle = a_{ij}v/\sqrt{2}$. This directly gives a nontrivial "texture" for the mass matrices of the four families of quarks and leptons, through the Yukawa terms of the form $Y(\psi^i\psi^j)\Phi^{(ij)}$. One sees immediately, however, that it gives a texture of exactly the same form ($\propto a_{ij}$) for the mass matrices of the up quarks, down quarks, and charged leptons of the four families. However, there are also the mass terms of the form $y_I(\psi^i\bar{\psi})\eta^i_I$ that couple the four families to the mirror family. Since, as we shall now see, these terms can be different for the up quarks, down quarks and charged leptons, a realistic spectrum for the three light families of quarks and leptons can result.

The quark and lepton Yukawa terms given schematically above have the actual forms

$$\begin{aligned}\mathcal{L}_{\mathrm{Yuk}} &= \mathcal{L}_{4\times4} + \mathcal{L}_{4\times1} \\ \mathcal{L}_{4\times4} &= Y_u\Phi^{(ij)*}(u^iu^{cj}) + Y_d\Phi^{(ij)}(d^id^{cj}) + Y_\ell\Phi^{(ij)}(\ell^i\ell^{cj}) \\ \mathcal{L}_{4\times1} &= y^I_Q\eta^i_I(u^i\bar{u} + d^i\bar{d}) + y^I_u\eta^i_I(u^{ci}\bar{u}^c) + y^I_d\eta^i_I(d^{ci}\bar{d}^c) \\ &\quad + y^I_L\eta^i_I(\ell^i\bar{\ell}) + y^I_\ell\eta^i_I(\ell^{ci}\bar{\ell}^c). \qquad (5)\end{aligned}$$

$\mathcal{L}_{4\times4}$ contains the Yukawa couplings of the four families to each other, and $\mathcal{L}_{4\times1}$ contains the Yukawa couplings of the four families to the mirror family. In order to express the mass terms coming from $\mathcal{L}_{4\times1}$ more compactly, it is convenient to define the following vectors in the $SO(4)_F$ family space:

$$X^i_f \equiv \sum_I y^I_f\langle\eta^i_I\rangle/m, \qquad f = Q, u, d, L, \ell. \qquad (6)$$

where $m \equiv Y_uv/\sqrt{2}$. Then the fermion mass matrices have the forms

$$\begin{aligned}\mathcal{L}_{M,\mathrm{up}} &= Y_u\frac{v}{\sqrt{2}}(u^1, u^2, u^3, u^4, \bar{u}^c)\begin{pmatrix} & & a_{ij} & & X^1_Q \\ & & & & X^2_Q \\ & & & & X^3_Q \\ & & & & X^4_Q \\ X^1_u & X^2_u & X^3_u & X^4_u & 0\end{pmatrix}\begin{pmatrix}u^{c1}\\u^{c2}\\u^{c3}\\u^{c4}\\\bar{u}\end{pmatrix}, \\ \mathcal{L}_{M,\mathrm{down}} &= Y_u\frac{v}{\sqrt{2}}(d^1, d^2, d^3, d^4, \bar{d}^c)\begin{pmatrix} & & ra_{ij} & & X^1_Q \\ & & & & X^2_Q \\ & & & & X^3_Q \\ & & & & X^4_Q \\ X^1_d & X^2_d & X^3_d & X^4_d & 0\end{pmatrix}\begin{pmatrix}d^{c1}\\d^{c2}\\d^{c3}\\d^{c4}\\\bar{d}\end{pmatrix}, \qquad (7) \\ \mathcal{L}_{M,\mathrm{lepton}} &= Y_u\frac{v}{\sqrt{2}}(\ell^1, \ell^2, \ell^3, \ell^4, \bar{\ell}^c)\begin{pmatrix} & & sa_{ij} & & X^1_L \\ & & & & X^2_L \\ & & & & X^3_L \\ & & & & X^4_L \\ X^1_\ell & X^2_\ell & X^3_\ell & X^4_\ell & 0\end{pmatrix}\begin{pmatrix}\ell^{c1}\\\ell^{c2}\\\ell^{c3}\\\ell^{c4}\\\bar{\ell}\end{pmatrix},\end{aligned}$$

where $r \equiv Y_d/Y_u$, $s \equiv Y_\ell/Y_u$. Note that the elements in the 1×4 and 4×1 blocks of these matrices are very large ($O(\langle\eta^i_I\rangle)$) compared to the elements in the 4×4 blocks, which are $O(\Phi^{(ij)})$, i.e. the weak scale or smaller. All these matrices can be brought by change of bases to the general form

$$\begin{pmatrix} & & & & 0 \\ & A_{ij} & & & 0 \\ & & & & 0 \\ & & & & B \\ 0 & 0 & 0 & C & 0 \end{pmatrix}. \tag{8}$$

When this is done, one sees that the fermions of the fourth family (in this basis) obtain very large Dirac masses with the fermions of the mirror family, while the first three families remain light. The effective 3×3 mass matrix of the light families is then just given by the first three rows and columns of what we call A_{ij} in Eq. (8) (with corrections that are $O(v/\langle\eta\rangle)$) or smaller and thus utterly negligible). One sees from this that the *magnitudes* of the "vectors" X_f^i, in the 1×4 and 4×1 blocks of the mass matrices in Eq. (7) do not affect the spectrum of the light three families, only their *directions* do.

The three mass matrices in Eq. (7) depend on several groups of parameters. (a) r, s, which are just ratios of Yukawa couplings ($r \equiv Y_d/Y_u$, $s \equiv Y_\ell/Y_u$). (b) a_{ij}, which is just the direction of the VEV of $\Phi^{(ij)}$ in $SO(4)_F$ space, and is determined by the mass matrix of the $\Phi^{(ij)}$, which in turn is controlled by the "master matrix" m^2 defined in Eq. (3). And (c) the vectors defined in Eq. (6). Most of the parameters are in this last category. These five vectors could be independent of each other, in which case the number of parameters would be too large to have a predictive model.

There are a number of ways in which the five vectors could be related to each other, thus reducing the number of free parameters. One is through unification of the standard model gauge group in a larger group. This was the approach discussed in [4], where $G_{\rm SM}$ was embedded in the "trinification group" $SU(3) \times SU(3) \times SU(3)$. Such unification symmetries relate quarks to leptons and thus relate some of these vectors to each other. As can be seen from [4], however, there are significant costs to such unification. It makes models considerably more involved.

Another possibility is that a small number of messenger fields give the dominant contributions to the vectors of Eq. (6). To take an extreme example, if only one messenger field, say η_1^i, contributed, then the sums $\Sigma_I y_f^I \langle\eta_I^i\rangle$ in Eq. (6) would collapse to single terms proportional to $\langle\eta_1^i\rangle$, and all the vectors would be parallel. This is too extreme, however, because it would mean that the effective 3×3 mass matrices of the up quarks, down, quarks, and charged leptons of the three light families would all be of the same form, which is unrealistic.

An interesting possibility, which we will discuss briefly later, is that all the vectors in Eq. (6) get their dominant contribution from *two* of the messenger fields. Then the five vectors defined in Eq. (6) would all lie in a two dimensional subspace. The number of parameters would thereby be reduced so much that the model would be very predictive—as predictive as the version of the model we discuss below.

In this paper we follow a somewhat different path. We assume that certain of the vectors (but not all of them) are dominated by a single messenger field VEV and therefore parallel. We will consider two cases for illustration, which we will call "Case A" and "Case B". In Case A, the vectors X_u and X_Q are assumed parallel. In Case B, the vectors X_d and X_Q are assumed parallel. We will only explicitly work out the quark sector couplings (the charged lepton sector is quite similar, as will be seen), so we make no assumption about the vectors X_L and X_ℓ here.

III. FITTING THE QUARK SPECTRUM IN CASE A

We make the further assumption (to be justified later when we discuss the spectrum of Higgs doublet masses) that the matrix a_{ij} is real. The forms of the mass matrices given in Eq. (7) can then be simplified by a choice of $SO(4)_F$ basis. One can do an $SO(4)_F$ transformation that makes the vectors X_Q^i and X_u^i, which are parallel in Case A, point in the 4 direction, i.e. have the forms $(0, 0, 0, X_Q)$ and $(0, 0, 0, X_u)$. (This can be done with a real orthogonal transformation, because these vectors are assumed proportional to a single messenger field VEV, and each messenger field is a *real* $SO(4)_F$ vector field. Moreover, because a_{ij} is real, an $SO(4)_F$ transformation preserves its character as a *traceless* symmetric matrix.) One can follow this by an $SO(3)$ transformation involving only the indices $i = 1, 2, 3$ (which thus preserves the special forms of X_Q^i and X_u) that diagonalizes the upper-left 3×3 block of the mass matrices. In the resulting basis the mass matrices have the forms

$$M_{\rm up} = \begin{pmatrix} c & 0 & 0 & d & 0 \\ 0 & b & 0 & e & 0 \\ 0 & 0 & a & f & 0 \\ d & e & f & -\Sigma & X_Q \\ 0 & 0 & 0 & X_u & 0 \end{pmatrix} m,$$

$$M_{\rm down} = r \begin{pmatrix} c & 0 & 0 & d & 0 \\ 0 & b & 0 & e & 0 \\ 0 & 0 & a & f & 0 \\ d & e & f & -\Sigma & \frac{1}{r} X_Q \\ \frac{1}{r} X_d^1 & \frac{1}{r} X_d^2 & \frac{1}{r} X_d^3 & \frac{1}{r} X_d^4 & 0 \end{pmatrix} m, \tag{9}$$

$$M_{\rm lep} = s \begin{pmatrix} c & 0 & 0 & d & \frac{1}{s} X_L^1 \\ 0 & b & 0 & e & \frac{1}{s} X_L^2 \\ 0 & 0 & a & f & \frac{1}{s} X_L^3 \\ d & e & f & -\Sigma & \frac{1}{s} X_L^4 \\ \frac{1}{s} X_\ell^1 & \frac{1}{s} X_\ell^2 & \frac{1}{s} X_\ell^3 & \frac{1}{s} X_\ell^4 & 0 \end{pmatrix} m,$$

where $\Sigma \equiv a + b + c$. Note that because we have obtained this form by a real orthogonal transformation, and because we are assuming that a_{ij} is real, the parameters a, b, c, d, e, f in these matrices are real.

Since X_Q and X_u (which are of order $\langle \eta_l^i \rangle$) are several orders of magnitude larger than the elements a, b, c, d, e, f, it is easily seen that the three light families of up quarks (namely u, c, t) correspond almost exactly to the first three rows and columns of $M_{\rm up}$ in Eq. (9). Thus, the effective mass matrix for the three observed families of up quarks is given in this basis simply by

$$\tilde{M}_{\rm up} = \begin{pmatrix} c & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix} m. \tag{10}$$

Therefore, $c/b = m_u/m_c \ll 1$ and $b/a = m_c/m_t \ll 1$. Since it will turn out that b, c, d, e, f are all small compared to 1, and a_{ij} is normalized so that $\Sigma_{ij}|a_{ij}|^2 = 2$, one has $a^2 \cong 1$. Without loss of generality we can take $a \cong +1$, and $m \equiv Y_u v/\sqrt{2} \cong m_t$.

To find the effective mass matrix for the three light families of down quarks, we must do a further change of basis of the d^{ci} to bring the complex vector $(X_d^1, X_d^2, X_d^3, X_d^4)$ to the form $(0, 0, 0, X_d)$. This is done by multiplying $M_{\rm down}$ from the right by a unitary transformation of the form

$$U = \begin{pmatrix} c_\alpha & s_\alpha^* & 0 & 0 \\ -s_\alpha & c_\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\beta & s_\beta^* & 0 \\ 0 & -s_\beta & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_\gamma & s_\gamma \\ 0 & 0 & -s_\gamma & c_\gamma^* \end{pmatrix} = \begin{pmatrix} c_\alpha & c_\beta s_\alpha^* & c_\gamma s_\beta^* s_\alpha^* & s_\gamma s_\beta^* s_\alpha^* \\ -s_\alpha & c_\beta c_\alpha & c_\gamma s_\beta^* c_\alpha & s_\gamma s_\beta^* c_\alpha \\ 0 & -s_\beta & c_\gamma c_\beta & s_\gamma c_\beta \\ 0 & 0 & -s_\gamma & c_\gamma^* \end{pmatrix} \tag{11}$$

Where the angles s_α, s_β and s_γ are in general complex. It turns out that to get a realistic fit to the quark masses, one needs to assume that c, $d \ll b$, $e \ll f < a \cong 1$, and that $|s_\alpha|$, $|s_\beta|$, and $|c_\gamma|$ are small compared to 1. This allows us to write $M_{\rm down}$ in the new basis as

$$M_{\rm down} \cong r \begin{pmatrix} 0 & 0 & -d & c_\gamma^* & 0 \\ -s_\alpha b & b & -e & s_\beta^* + c_\gamma^* e & 0 \\ 0 & -s_\beta & c_\gamma - f & 1 + c_\gamma^* f & 0 \\ d - s_\gamma e & e - s_\beta f & c_\gamma f + 1 & f - c_\gamma^* & \frac{1}{r} X_Q \\ 0 & 0 & 0 & \frac{1}{r} X_d & 0 \end{pmatrix} m \tag{12}$$

From this one can read off that the effective 3×3 mass matrix of the three light families of down quarks is simply

$$\tilde{M}_{\rm down} \cong r \begin{pmatrix} 0 & 0 & -d \\ -s_\alpha b & b & -e \\ 0 & -s_\beta & c_\gamma - f \end{pmatrix} m \tag{13}$$

The parameter s_α can be made real by redefining the phase of d^{c1} in this basis. The parameter s_β can be made real by redefining the phase of u^3 in this basis. (These phase redefinitions do not affect the fitting of known quantities, but do affect the phases of the Yukawa couplings of the extra scalar doublets, which are therefore undetermined by just fitting the known quark masses and mixing angles.) Calling $c_\gamma - f \equiv Fe^{i\phi}$ and remembering that we have normalized a to be 1, the quark mass matrices can be written

$$\tilde{M}_{\rm up} = \begin{pmatrix} c & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} m, \quad \tilde{M}_{\rm down} \cong r \begin{pmatrix} 0 & 0 & -d \\ -s_\alpha b & b & -e \\ 0 & -s_\beta & Fe^{i\phi} \end{pmatrix} m. \tag{14}$$

These depend on nine real parameters (r, m, b, c, d, e, F, s_α, s_β) and one phase ($e^{i\phi}$). This is just the right number of parameters to fit the six quark masses, three Cabibbo-Kobayashi-Maskawa (CKM) angles and the CKM phase. The results of the fit are given in Table II.

Note that the parameter f is not determined, and the parameter c_γ is given by $c_\gamma = Fe^{i\phi} + f$. These numbers determine (except for the parameter f) the 4×4 mass matrix of the four families in the basis of Eq. (9):

$$Y_u \langle \Phi^{(ij)} \rangle = Y_u \left(\frac{v}{\sqrt{2}} \right) a_{ij} = Y_u \frac{v}{\sqrt{2}} \begin{pmatrix} c & 0 & 0 & d \\ 0 & b & 0 & e \\ 0 & 0 & a & f \\ d & e & f & -\Sigma \end{pmatrix}. \tag{15}$$

TABLE II. Parameter values in Case A of the model that reproduce the known quark masses and CKM mixing matrix.

Parameter	Value
a	1.0
b	3.6×10^{-3}
c	7.4×10^{-6}
d	4.7×10^{-4}
e	2.2×10^{-3}
F	5.7×10^{-2}
ϕ	0.98
$\sin\alpha$	0.105
$\sin\beta$	0.076
r	0.177

In the next section, we will use this information to determine the spectrum of the scalars $\Phi^{(ij)}$. This is possible because the matrix a_{ij} is enough to determine the master matrix $(m^2)^{ij}$ (if that is assumed real).

IV. THE SCALAR SPECTRUM IN CASE A

The masses of the Higgs doublets $\Phi^{(ij)}$ are controlled by the "master matrix" m^2 defined in Eq. (3). (There are also contributions to the mass-squared of the extra scalar doublets that come from the quartic self-couplings of $\Phi^{(ij)}$ once $\Phi_{\rm SM}$ gets a VEV, but these are negligible if, as will turn out to be the case, the masses of the extra doublets are much larger than the mass of the standard model Higgs.) From Eq. (3), one easily sees that m^2 is Hermitian. (In that equation the coupling matrix λ_{IJ} is in general complex and Hermitian, whereas the VEVs $\langle \eta_I^i \rangle$ are real.) To obtain a realistic hierarchy among the quark and lepton masses, it turns out that m^2 must be very hierarchical, as will be seen. In simple cases where m^2 is hierarchical, it also tends to be approximately real. (To take an extreme case, suppose, that one of the η_I^i, say η_1^i, gave the largest contribution to m^2. Then $(m^2)^{ij} \cong \lambda_{11}\langle \eta_1^i \rangle \langle \eta_1^j \rangle$, which is rank 1, and thus hierarchical, and also manifestly real.) We therefore make the approximation that m^2 is real, since this greatly simplifies the analysis of the model. (It is also possible to imagine that the master matrix arises primarily from the VEVs of other messenger fields $\eta^{(ij)}$ that are real 9-plets of $SO(4)_F$. Those contributions would be *exactly* real.)

If m^2 is taken to be real, then it is also symmetric, and it can be diagonalized by an $SO(4)_F$ rotation, i.e. by a choice of $SO(4)_F$ basis, which we will call the "scalar-mass basis". Since the terms in Eq. (4) can be written $\mathrm{Tr}[(\frac{1}{2}M^2 I + m^2)\Phi^\dagger \Phi]$, it is clear that without loss of generality one can make one of the diagonal elements of m^2 vanish by shifting the parameter M^2. Thus m^2 can be taken (in the "scalar-mass basis") to be of the form

$$m^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} m_0^2. \qquad (16)$$

We assume that $\delta \ll \epsilon \ll 1$, which will lead directly to a hierarchy in the quark and lepton mass matrices, as will be seen. Writing the 9-plet of Higgs fields as

$$\Phi^{(ij)} = \begin{pmatrix} \frac{3\bar{\Phi}_{11}}{\sqrt{6}} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\ \Phi_{12} & \frac{2\sqrt{2}\bar{\Phi}_{22} - \bar{\Phi}_{11}}{\sqrt{6}} & \Phi_{23} & \Phi_{24} \\ \Phi_{13} & \Phi_{23} & \frac{\sqrt{6}\bar{\Phi}_{33} - \sqrt{2}\bar{\Phi}_{22} - \bar{\Phi}_{11}}{\sqrt{6}} & \Phi_{34} \\ \Phi_{14} & \Phi_{24} & \Phi_{34} & \frac{-\sqrt{6}\bar{\Phi}_{33} - \sqrt{2}\bar{\Phi}_{22} - \bar{\Phi}_{11}}{\sqrt{6}} \end{pmatrix} \qquad (17)$$

and substituting Eqs. (16) and (17) into Eq. (4), one finds the spectrum given in Table III:

Note that the Higgs fields in the first row/column ($\Phi^{(1i)}$) get contributions of order m_0^2 from the master matrix; those in the second (but not first) row/column ($\Phi^{(2i)}$, $i \neq 1$) get contributions of order ϵm_0^2, and the remaining ones get contributions of order δm_0^2, as an inspection of Eqs. (4) and (15) would suggest. The fields denoted $\bar{\Phi}^{(ii)\prime}$ are linear combinations of the fields denoted $\bar{\Phi}^{(ii)}$ in Eq. (17). The lightest of the Higgs doublets, which is the standard model Higgs doublet, turns out to be the linear combination

$$\Phi_{\rm SM} = \bar{\Phi}^{(33)\prime} \cong \bar{\Phi}^{(33)} + \frac{\sqrt{3}}{4}\frac{\delta}{\epsilon}\bar{\Phi}^{(22)} + \frac{5}{6\sqrt{6}}\delta\bar{\Phi}^{(11)}. \quad (18)$$

One sees, then, that the standard model Higgs doublet has diagonal Yukawa couplings in the "scalar-mass basis". The mass-squared of the standard model Higgs doublet is fine-tuned (presumably anthropically) to be $-\mu^2$, where $\mu \sim 100$ GeV. This gives $M^2 \cong -(\delta - \frac{\delta^2}{4\epsilon})m_0^2 - \mu^2$. Substituting this into the mass-squared of the other Higgs fields in the 9-plet gives the results in the last column of Table III.

The next lightest Higgs doublet is $\Phi^{(34)}$. We will call this $\Phi_{\rm LED}$, where LED stands for "lightest extra doublet". From Table III, one sees that the mass of $\Phi_{\rm LED}$ is $\frac{\delta}{2\epsilon}$ times that of the next lightest Higgs doublets $\Phi^{(23)}$ and $\Phi^{(24)}$. Shortly, we will see that this is 3.6×10^{-3}. Thus, it turns out that flavor-violating effects are dominated by the exchange of $\Phi_{\rm LED}$. In the scalar-mass basis, $\Phi_{\rm LED} = \Phi^{(34)}$ couples very simply to the quarks and leptons: it only

TABLE III. The mass spectrum of the 9-plet of Higgs doublets.

Field	$(\text{mass})^2$	After tuning SM Higgs
$\bar{\Phi}^{(11)\prime}$	$\cong M^2 + \frac{3}{2}m_0^2$	$\cong \frac{3}{2}m_0^2$
$\Phi^{(12)}$	$M^2 + (1+\epsilon)m_0^2$	$\cong (1+\epsilon)m_0^2$
$\Phi^{(13)}$	$M^2 + (1+\delta)m_0^2$	$\cong m_0^2$
$\Phi^{(14)}$	$M^2 + m_0^2$	$\cong (1-\delta)m_0^2$
$\bar{\Phi}^{(22)\prime}$	$\cong M^2 + \frac{1}{3}\epsilon m_0^2$	$\cong \frac{1}{3}\epsilon m_0^2$
$\Phi^{(23)}$	$M^2 + (\epsilon+\delta)m_0^2$	$\cong \epsilon m_0^2$
$\Phi^{(24)}$	$M^2 + \epsilon m_0^2$	$\cong (\epsilon-\delta)m_0^2$
$\Phi^{(34)}$	$M^2 + \delta m_0^2$	$\cong \frac{\delta^2}{4\epsilon}m_0^2$
$\bar{\Phi}^{(33)\prime}(\equiv \Phi_{\rm SM})$	$\cong M^2 + (\delta - \frac{\delta^2}{4\epsilon})m_0^2$	$\equiv -\mu^2$

couples the third to the fourth family, with strength 1 for the up quarks, r for the down quarks, and s for the charged leptons.

From Eqs. (18) one sees that $\langle \bar{\Phi}^{(11)} \rangle = (\frac{5\delta}{6\sqrt{6}}) v/\sqrt{2}$, $\langle \bar{\Phi}^{(22)} \rangle = (\frac{\sqrt{3}\delta}{4\epsilon}) v/\sqrt{2}$, and $\langle \bar{\Phi}^{(33)} \rangle = v/\sqrt{2}$. Substituting this into Eq. (17), one finds that the matrix a_{ij} that appears in the mass matrices given in Eq. (7) is just given in the scalar-mass basis by

$$\langle \Phi^{(ij)} \rangle = a_{ij} v/\sqrt{2} \cong \begin{pmatrix} \frac{5}{12}\delta & 0 & 0 & 0 \\ 0 & \frac{1}{2}(\delta/\epsilon) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} v/\sqrt{2}. \quad (19)$$

This is related to the form of a_{ij} in Eqs. (9) and (15) by a change of basis of the fermions. Indeed, since the parameters that appear in Eq. (15) are given in Table II (except for f), one simply diagonalizes the form in Eq. (15) to determine the parameters ϵ and δ in Eq. (19). In this way, one finds that $\epsilon \cong 2.5 \times 10^{-3}$, $\delta \cong 1.8 \times 10^{-5}$, and $\delta/\epsilon \cong 7.2 \times 10^{-3}$

Since the transformation between these two bases is known (in terms of one unknown, namely f), one can determine the Yukawa couplings of $\Phi_{\rm LED}$, and indeed all the other extra Higgs doublets, in the basis of Eqs. (9) and (15). The basis of Eqs. (9) and (15) is in fact the physical basis of the up quarks u, c and t, as explained before Eq. (10). Thus we know how all nine of the Higgs doublets couple to u, c, and t. To get to the physical basis of the down quarks d, s, and b, one must do two further changes of basis of the down quarks: first, that shown in Eq. (11), the parameters of which are given in Table II (except for the phases of s_α and s_β); and second, the change of basis needed to diagonalize the matrix in Eq. (13), which are completely determined from Table II.

In other words, one is in a position to compute the couplings of the *all* nine of the $\Phi^{(ij)}$ to *all* of the known quarks in terms of the unknown parameter f and the unknown phases of s_α and s_β.

The results are given in Table IV, for $f = 0.05$ and the phases of s_α and s_β equal to zero. One gets similar results for other values of these parameters.

The analysis of the charged leptons is very similar to that of the down quarks. There are a number of assumptions

TABLE IV. Values for the Yukawa couplings of the lightest extra Higgs doublet (LED) to the quarks, in Case A of the model, with $f = 0.05$, and $s_\alpha = s_\beta = 0$.

Yukawa of LED	Value
$Y^u_{12} = Y^u_{21}$	2.1×10^{-7}
$Y^u_{13} = Y^u_{31}$	-4.7×10^{-4}
$Y^u_{23} = Y^u_{32}$	2.2×10^{-3}
Y^u_{11}	-1.1×10^{-8}
Y^u_{22}	-9.7×10^{-7}
Y^u_{33}	0.12
Y^d_{12}	$(-0.37 - i1.2) \times 10^{-3}$
Y^d_{21}	$(6.2 + i8.5) \times 10^{-4}$
Y^d_{13}	$(2.8 + i9.4) \times 10^{-4}$
Y^d_{31}	$(1.5 - i2.3) \times 10^{-2}$
Y^d_{23}	$(2.4 + i0.34) \times 10^{-3}$
Y^d_{32}	$(-0.77 + i1.2) \times 10^{-1}$
Y^d_{11}	$(0.7 + i2.3) \times 10^{-4}$
Y^d_{22}	$(3.2 + i4.4) \times 10^{-3}$
Y^d_{33}	$(6.0 - i8.9) \times 10^{-2}$

that could be made about the vectors X^i_L and X^i_ℓ in Eq. (9). Suppose, for example, one assumed that X^i_ℓ is parallel to X^i_Q and X^i_u. Then the diagonalization of $M_{\rm lep}$ proceeds in the same way as the diagonalization of $M_{\rm down}$ above, except that $M_{\rm lep}$ in Eq. (9) is multiplied on the *left* by a unitary matrix $U'^\dagger$, where U' has the same form as U in Eq. (11) but with different angles α', β', and γ'. The phases of these parameters turn out not to affect the fitting of the charged lepton masses significantly. So there are four additional parameters in the lepton sector (s, α', β', and γ') available to fit the three masses m_e, m_μ, and m_τ. Consequently, the Yukawa coupling matrices of all 9 Higgs doublets to the charged leptons are determined in terms of only a small number of additional unknown parameters. Here we will only discuss the quark sector for purposes of illustration.

The Yukawa couplings in Table IV allow us to write down the coefficients flavor-changing four-fermion operators. The most interesting involving the down-type quarks are given in Table V.

In Table V, the limits on $M_{\rm LED}$ are obtained from the limits on the coefficients of flavor-changing operators given in [5]. One sees from Table V that the contribution to ϵ_K from the CP-violating part of the $(\bar{s}_R d_L)(\bar{s}_L d_R)$

TABLE V. The predicted coefficients of the most important flavor-changing four-quark operators and the resulting lower limits on the mass of the LED [5].

Operator	Coefficient	Limit on $M_{\rm LED}$
$c_{sd}(\bar{s}_R d_L)(\bar{s}_L d_R)$	$\lvert c_{sd}\rvert = 1.33 \times 10^{-6}/M^2_{\rm LED}$	≥ 14 TeV
	${\rm Im}(c_{sd}) = 1.33 \times 10^{-6} \frac{\arg(c_{sd})}{M^2_{\rm LED}}$	≥ 230 TeV$[\arg(c_{sd})]^{1/2}$
$c_{bs}(\bar{b}_R s_L)(\bar{b}_L s_R)$	$\lvert c_{bs}\rvert = 3.45 \times 10^{-4}/M^2_{\rm LED}$	≥ 5.1 TeV
$c_{bd}(\bar{b}_R d_L)(\bar{b}_L d_R)$	$\lvert c_{bd}\rvert = 2.7 \times 10^{-5}/M^2_{\rm LED}$	≥ 6.9 TeV

operator gives an extremely severe constraint on the mass of the lightest extra doublet in this model if the phase of c_{sd} is order one. If that phase happens to be very small, then δm_K still constrains the LED mass to be greater than 14 TeV. (It should be pointed out that these numbers turn out to be fairly insensitive to the value of the unknown parameter f.)

These bounds are considerably tighter than one might have expected for a flavor-changing Higgs if its Yukawa couplings were similar to those of the standard model Higgs. These bounds are very sensitive to the details of the model. We will now look at another version of the model (Case B), since the comparison is instructive.

V. RESULTS FOR CASE B

The analysis of Case B is quite similar to that of Case A. In Case B the mass matrices in the same basis as Eq. (9) take the form

$$M_{\rm up} = \begin{pmatrix} c & 0 & 0 & d & 0 \\ 0 & b & 0 & e & 0 \\ 0 & 0 & a & f & 0 \\ d & e & f & -\Sigma & X_Q \\ X_u^1 & X_u^2 & X_u^3 & X_u^4 & 0 \end{pmatrix} m,$$
$$M_{\rm down} = r\begin{pmatrix} c & 0 & 0 & d & 0 \\ 0 & b & 0 & e & 0 \\ 0 & 0 & a & f & 0 \\ d & e & f & -\Sigma & \frac{1}{r}X_Q \\ 0 & 0 & 0 & \frac{1}{r}X_d & 0 \end{pmatrix} m. \tag{20}$$

In this case, it is apparent that (neglecting terms of order $v/\langle\eta\rangle$) the mass matrix of the observed down quarks, d, s, b is just given by the upper-left 3×3 block of $M_{\rm down}$, i.e.

$$\tilde{M}_{\rm down} = r\begin{pmatrix} c & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix} m, \tag{21}$$

so that this is already in the physical basis of these quarks. Thus, in Case B, $b/a = m_s/m_b$ and $c/a = m_d/m_b$.

For the up-type quarks, however, one must make a further change of basis for the u^{ci} in order to bring the complex vector $(X_u^1, X_u^2, X_u^3, X_u^4)$ to the form $(0, 0, 0, X_u)$. This involves rotating the matrix $M_{\rm up}$ in Eq. (20) from the right by a matrix of the same form shown in Eq. (11) This gives for $M_{\rm up}$ the same form as $M_{\rm down}$ has in Case A, shown in Eq. (12) (with $r = 1$). However, it turns out that the fit to the quark masses and mixing angles implies that here $a < 1$ and $f \cong 1$, and, unlike Case A, the parameter c is not negligible. Moreover, the angle β is large enough here that it is not a good approximation to set $\cos\beta = 1$, but it is a good approximation here, as in Case A, to set $\cos\alpha = 1$ and $\sin\gamma = 1$. With these approximations, one has for the effective 3×3 mass matrix of the u, c, and t quarks

TABLE VI. The parameter values in Case B of the model that reproduce the masses of the quarks and the CKM mixing matrix.

Parameter	Value
a	0.137
b	2.54×10^{-3}
c	1.27×10^{-4}
d	7.5×10^{-3}
e	-0.04
f	1.0
$\sin\alpha$	$0.24e^{i0.42}$
$\sin\beta$	$0.08e^{i1.05}$
r	0.132

$$\tilde{M}_{\rm up} \cong \begin{pmatrix} c & c_\beta s_\alpha c & -d \\ -s_\alpha^* b & c_\beta b & -e \\ 0 & -s_\beta^* a & -f \end{pmatrix} m \tag{22}$$

Fitting the quark masses and mixing angles leads to the parameter values given in Table VI.

It turns out that $\cos\gamma$ is almost unconstrained, since the quark masses and mixing angles are only very weakly dependent on it. With the parameters given in Table VI, the Yukawa couplings of all the Higgs doublets to all the quarks can be straightforwardly computed in terms of $\cos\gamma$.

What distinguishes Cases A and B is that in Case A the strongest flavor-changing effects are in the down-quark sector, whereas for Case B the strongest flavor-changing effects are in the up-quark sector. What matters most in Case B, therefore, are the couplings of u to c, which give the operator $(\bar{c}_L u_R)(\bar{c}_R u_L)$. The coefficient of this operator c_{uc} is somewhat insensitive to the value of $\cos\gamma$. With $\cos\gamma = 0.1$, one finds $c_{uc} = 7.5\times 10^{-5}/M_{\rm LED}^2$. The current limit from $D-\bar{D}$ mixing gives $M_{\rm LED} \gtrsim 36$ TeV. The limits from the B_s and B_d system turn out to be much weaker: they only constrain $M_{\rm LED}$ to be larger than about 1.4 TeV. The limit from the ϵ_K parameter is that $M_{\rm LED} > 7$ TeV, for CP phases of order 1.

One sees, then, that in the two special cases of the model that we have analyzed the lightest extra Higgs doublet has to be too heavy to be seen at accelerators or to give significant flavor-changing effects in rare processes. In Case A this because of the $K-\bar{K}$ mixing limits and in Case B it is because of the $D-\bar{D}$ mixing limits. However, Cases A and B do not exhaust the possibilities of this model. For example, as noted near the end of Sec. II, the assumption that all the vectors in Eq. (6) arise from just two messenger fields reduces the number of parameters almost as much as in the two cases we have studied here. It may be that this assumption or other assumptions or limits of the model can allow the lightest extra Higgs doublet to be lighter than in Cases A and B. Moreover, what has been studied here is only one particular model that realizes the basic idea of putting multiple Higgs doublets into a representation of a non-Abelian flavor group.

VI. CONCLUSIONS

The repetition of quark and lepton families has long suggested the possibility of a non-Abelian family symmetry [1]. It is quite natural, therefore, to consider the possibility that the Higgs field of the standard model belongs to a multiplet of the same family group. In the model we have presented as an example of this idea, the family symmetry tightly constrains the forms of the quark and lepton mass matrices. Nevertheless, it has been shown that the observed fermion masses and mixing angles can be reproduced. The family symmetry also severely constrains the Yukawa couplings of all the extra Higgs doublets; and it has been seen that after fitting the known quark and lepton masses, the coefficients of all the flavor-changing four-fermion operators that come from the exchange of extra Higgs doublets are predicted in terms of only a few parameters. It turns out that in the specific model we have studied, the constraints from limits on flavor-changing in the $K - \bar{K}$ and $D - \bar{D}$ systems require the lightest extra Higgs doublet to have a mass of tens of TeV, which is too heavy to lead to testable phenomenology in the near future. This may, however, be a feature of the specific model we have studied rather than an inevitable consequence of the general approach we are proposing.

One of the interesting features of the approach being described in this paper is that the spectrum of the Higgs fields is closely connected to the spectrum of the quarks and leptons. The pattern of couplings of the standard model Higgs to the quark families—i.e. the so-called textures of the Yukawa matrices—is determined by which component within the ‘‘family’’ of Higgs fields is the lightest, i.e. is the standard model Higgs field. This is determined, in turn, by the pattern of family-symmetry-breaking within the Higgs family multiplet. Thus, both the spectrum of fermion masses and the spectrum of Higgs boson masses is largely determined by what we have called a ‘‘master matrix’’.

A key feature of the present approach is that the breaking of the family symmetry takes place dynamically in a sector of fields that are standard model singlets and is communicated to the standard model degrees of freedom by ‘‘messenger’’ fields. If the messenger sector is simple, then the pattern of masses of the standard model fields, including the extra Higgs doublets, is highly constrained.

It would be interesting to see if other non-Abelian family groups and particular choices of family representations for the quarks, leptons, and Higgs fields could lead to realistic models that predict flavor-changing effects at observable levels.

[1] T. Maehara and T. Yanagida, Prog. Theor. Phys. **60**, 822 (1978); **61**, 1434 (1979); F. Wilczek and A. Zee, Phys. Rev. Lett. **42**, 421 (1979); R. N. Cahn and H. Harari, Nucl. Phys. **B176**, 135 (1980); O. Shanker, Phys. Rev. D **23**, 1555 (1981); D. R. T. Jones and G. L. Kane, Nucl. Phys. **B198**, 45 (1982); D. B. Reiss, Phys. Lett. B **115**, 217 (1982).

[2] V. Agrawal, S. M. Barr, J. F. Donoghue, and D. Seckel, Phys. Rev. D **57**, 5480 (1998); Phys. Rev. Lett. **80**, 1822 (1998).

[3] S. M. Barr and Almas Khan, Phys. Rev. D **76**, 045002 (2007).

[4] S. M. Barr, Phys. Rev. D **82**, 055010 (2010).

[5] G. Isidori, Y. Nir, and G. Perez, Annu. Rev. Nucl. Part. Sci. **60**, 355 (2010).