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Measuring γ in $B \to K\pi\pi$ decays

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We reexamine the question of measuring the weak phase γ in $B \to K\pi\pi$ decays. To this end, we express all $B \to K\pi\pi$ amplitudes in terms of diagrams. We show that, as in $B \to K\pi$, relations exist between certain tree and electroweak-penguin diagrams. The imposition of these relations allows the extraction of γ from measurements of the $B \to K\pi\pi$ observables. We estimate the theoretical error in this method to be $O(5\%)$.

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I. INTRODUCTION

In the standard model, CP violation is due to a phase in the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix. The CKM phase information is conventionally parametrized in terms of the unitarity triangle, in which the interior (CP-violating) angles are known as α, β, and γ [1]. In this paper, we discuss a method for measuring γ in $B \to K\pi\pi$ decays. In order to put this discussion into context, we begin with a review of weak phases in $B \to K\pi$.

At the end of the 1980s, it was thought that $B \to K\pi$ receives contributions only from tree-type diagrams (proportional to $e^{i\gamma}$) and penguin diagrams (no weak phase). The appearance of two contributions with different weak phases meant that it was not possible to obtain clean weak-phase information from the measurement of the indirect CP asymmetry. In 1991, Nir and Quinn (NQ) [2] showed that one can use an isospin analysis to eliminate the "penguin pollution," so that one could indeed obtain γ from $B \to K\pi$ decays. However, several years later it was noted that, in fact, these decays receive significant electroweak-penguin (EWP) contributions [3], and that their appearance makes the NQ analysis fail. Several years after that, it was shown that, under flavor SU(3) symmetry, the EWP diagrams are proportional to the tree diagrams (apart from their weak phases) [4,5]. Finally, in 2004, all this information was put together, and it was found that it is possible to modify the NQ analysis using the EWP-tree relations, and cleanly extract γ from $B \to K\pi$ [6].

Weak phases in $B \to K\pi\pi$ follow a similar story (up to a point). (Note: assuming isospin symmetry, the wave function in $B \to K\pi\pi$ decays must be symmetrized with respect to the exchange of the final-state pions. Depending on their relative angular momentum, the $\pi\pi$ isospin state must be symmetric or antisymmetric.) In 1991, Lipkin, Nir, Quinn, and Snyder (LNQS) performed an isospin analysis of $K\pi\pi$, and obtained the relations among the amplitudes for the various $B \to K\pi\pi$ decays, for both the symmetric and antisymmetric cases [7]. Assuming the experimental separation of these cases, they noted that the relations permit one to extract clean weak-phase information from $B \to K\pi\pi$ decays. However, their analysis was based in part on that of Nir and Quinn, i.e. EWP contributions were neglected. Once these are included, the LNQS method fails. In 2003, Deshpande, Sinha, and Sinha (DSS) attempted to revive the LNQS analysis for the case of symmetric $\pi\pi$ isospin states [8]. They included EWPs in a schematic way, and assumed that these can be related to the tree diagrams, as in Refs. [4,5]. Within their assumptions, they argued that it is possible to extract γ from $B \to K\pi\pi$. However, it was subsequently noted that the assumed EWP-tree relation in $K\pi\pi$ does not hold [9], so that we are back to the situation of being unable to obtain weak-phase information from $B \to K\pi\pi$. This is how things stand presently.

In light of this, in this paper we reexamine the question of whether it is possible to measure γ in $B \to K\pi\pi$ decays. To this end, we express the $B \to K\pi\pi$ amplitudes in terms of diagrams and note that the number of unknown theoretical parameters does indeed exceed the number of observables. Thus, one cannot extract weak phases without additional information.

This input comes from EWP-tree relations. It is true that the relation assumed by DSS does not hold. However, we show that there are other relations between certain EWP and tree diagrams. If these are taken into account, this reduces the number of unknown theoretical parameters, so that the extraction of γ is possible. Experimentally, it is not easy, but it is fairly clean theoretically.

In Sec. II, we introduce the diagrams and show how to express the $B \to K\pi\pi$ amplitudes in terms of these. EWP-tree relations are discussed in Sec. III. The contractions formalism is used to derive such relations for $B \to K\pi\pi$ decays. In Sec. IV, we show how the EWP-tree relations permit the measurement of γ in $B \to K\pi\pi$ decays. We conclude in Sec. V.

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II. $B \to K\pi\pi$ AMPLITUDES

There are six processes in $B \to K\pi\pi$ decays: $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^+\pi^0\pi^0$, $B^+ \to K^0\pi^+\pi^0$, $B_d^0 \to K^+\pi^-\pi^0$, $B_d^0 \to K^0\pi^+\pi^-$, and $B_d^0 \to K^0\pi^0\pi^0$. For the moment, we assume only isospin symmetry, as in Refs. [7,8]. In all of these decays, the overall wave function of the final $\pi\pi$ pair must be symmetrized with respect to the exchange of these two particles. If the relative $\pi\pi$ angular momentum is even (odd), the isospin state must be symmetric (antisymmetric). We refer to these two cases as $I_{\pi\pi}^{\rm sym}$ and $I_{\pi\pi}^{\rm anti}$.

In Ref. [10] it was shown that $I_{\pi\pi}^{\rm sym}$ and $I_{\pi\pi}^{\rm anti}$ can be determined experimentally. We briefly summarize the argument. Consider, for example, $B_d^0 \to K^0\pi^+\pi^-$ (other decays are treated similarly). The events in the Dalitz plot can be described by the following two variables:

$$s_+ = m_{K^0\pi^+}^2 = (p_{K^0} + p_{\pi^+})^2,$$
$$s_- = m_{K^0\pi^-}^2 = (p_{K^0} + p_{\pi^-})^2. \qquad (1)$$

Now, a Dalitz-plot analysis permits the extraction of the decay amplitude, $\mathcal{M}(s_+, s_-)$, including both resonant and nonresonant contributions. The key point is that, under the exchange of the two pions, we have $p_{\pi^+} \leftrightarrow p_{\pi^-}$, i.e. $s_+ \leftrightarrow s_-$. Thus, the symmetric and antisymmetric amplitudes are simply $\frac{1}{\sqrt{2}}[\mathcal{M}(s_+, s_-) \pm \mathcal{M}(s_-, s_+)]$.

In fact, the full amplitude cannot be obtained—its global phase is undetermined. Thus, it is really $|\mathcal{M}|$ which is extracted. Similarly, one can obtain $|\bar{\mathcal{M}}|$ from the CP-conjugate decay. Therefore, for each decay one measures the momentum-dependent branching ratio ($\propto |\mathcal{M}|^2 + |\bar{\mathcal{M}}|^2$) and the momentum-dependent direct CP asymmetry ($\propto |\mathcal{M}|^2 - |\bar{\mathcal{M}}|^2$). In addition, for $K^0\pi^+\pi^-$ (where the K^0 is seen as K_S), the momentum-dependent indirect CP asymmetry[1] can be measured and gives $\mathcal{M}^*\bar{\mathcal{M}}$ for this decay.

In the $I_{\pi\pi}^{\rm sym}$ scenario, there are several relations among the amplitudes, including $A(B^+ \to K^0\pi^+\pi^0)_{\rm sym} = -A(B_d^0 \to K^+\pi^-\pi^0)_{\rm sym}$ [7]. This implies that there are only five independent decays. For $I_{\pi\pi}^{\rm anti}$, there are only four processes: $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^0\pi^+\pi^0$, $B_d^0 \to K^+\pi^-\pi^0$, and $B_d^0 \to K^0\pi^+\pi^-$ (one cannot antisymmetrize a $\pi^0\pi^0$ state).

Now, the goal here is to extract the weak phase γ from measurements of $B \to K\pi\pi$ decays. This can be done if the number of unknown theoretical parameters in the amplitudes is less than or equal to the number of observables. In the $I_{\pi\pi}^{\rm sym}$ case, there are 11 observables: the momentum-dependent branching ratios and direct CP asymmetries of $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^+\pi^0\pi^0$, $B_d^0 \to K^+\pi^-\pi^0$, $B_d^0 \to K^0\pi^+\pi^-$, and $B_d^0 \to K^0\pi^0\pi^0$, and the momentum-dependent indirect CP asymmetry of $B_d^0 \to K^0\pi^+\pi^-$ (the indirect CP asymmetry of $B_d^0 \to K^0\pi^0\pi^0$ will essentially be impossible to measure). For $I_{\pi\pi}^{\rm anti}$, there are 9 observables: the momentum-dependent branching ratios and direct CP asymmetries of $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^0\pi^+\pi^0$, $B_d^0 \to K^+\pi^-\pi^0$, and $B_d^0 \to K^0\pi^+\pi^-$, and the momentum-dependent indirect CP asymmetry of $B_d^0 \to K^0\pi^+\pi^-$. We therefore conclude that the $I_{\pi\pi}^{\rm sym}$ scenario is the more promising for extracting γ, and we concentrate on it exclusively below.

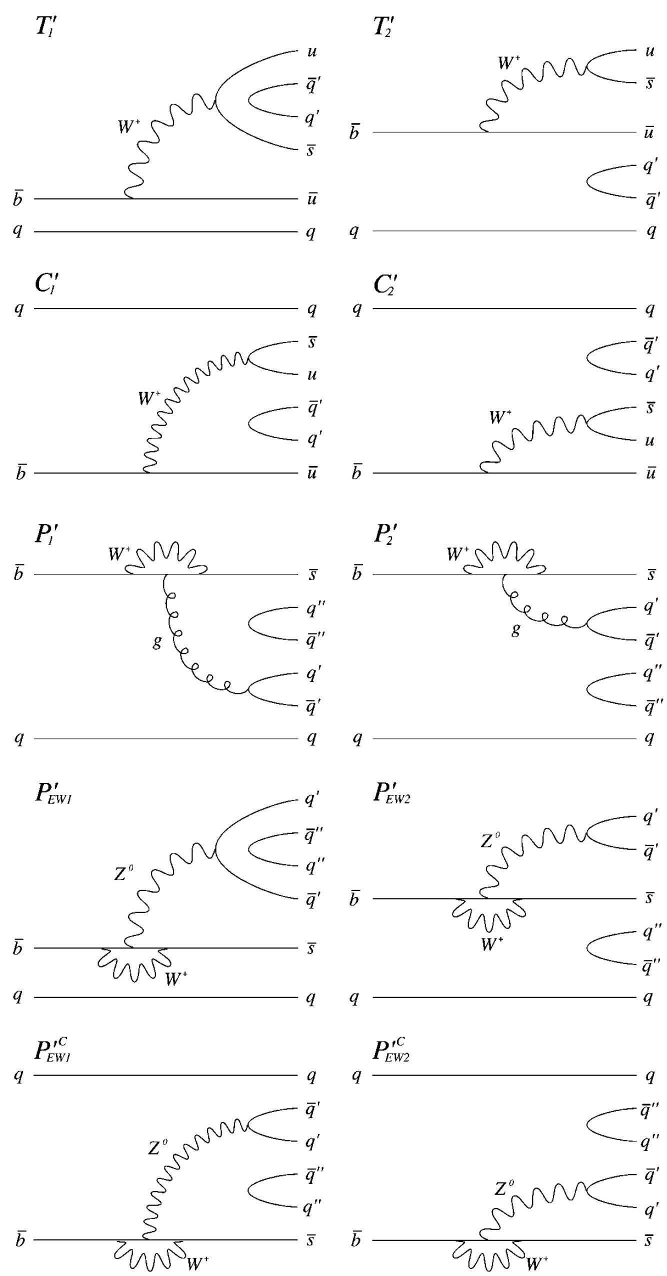

FIG. 1. Diagrams contributing to $B \to K\pi\pi$.

[1] The indirect CP asymmetry depends on the CP of the final state, and *a priori* $K^0\pi^+\pi^-$ is a mixture of $CP+$ and $CP-$. However, the separation of symmetric and antisymmetric $\pi\pi$ states also fixes the final state CP: $K^0(\pi\pi)_{\rm sym}$ and $K^0(\pi\pi)_{\rm anti}$ have $CP+$ and $-$, respectively.

It was shown in Ref. [10] that the amplitudes for three-body B decays can be expressed in terms of diagrams. The diagrams are shown in Fig. 1 (all annihilation- and exchange-type diagrams have been neglected). Note the following:

(i) In all diagrams, it is necessary to "pop" a quark pair from the vacuum. This pair is $u\bar{u}$ or $d\bar{d}$.

(ii) The subscript 1 indicates that the popped quark pair is between two (nonspectator) final-state quarks; the subscript 2 indicates that the popped quark pair is between two final-state quarks including the spectator.

One difference compared to two-body B decays is that here, because the final state contains three particles, the diagrams are momentum dependent. However, this does not pose a problem. The diagrams (magnitudes and relative strong phases) are determined via a fit to the data. But since the experimental observables are themselves momentum dependent, the fit will yield the momentum dependence of each diagram.

In terms of diagrams, the $B \to K\pi\pi$ amplitudes are given by

$$\begin{aligned}
\sqrt{2}A(B^+ \to K^0\pi^+\pi^0)_{\rm sym} &= -T_1' e^{i\gamma} - C_2' e^{i\gamma} + P'_{\rm EW2} + P'^C_{\rm EW1},\\
A(B^0_d \to K^0\pi^+\pi^-)_{\rm sym} &= -T_1' e^{i\gamma} - C_1' e^{i\gamma} - \tilde{P}'_{uc} e^{i\gamma} + \tilde{P}'_{tc} + \tfrac{1}{3}P'_{\rm EW1} + \tfrac{2}{3}P'^C_{\rm EW1} - \tfrac{1}{3}P'^C_{\rm EW2},\\
\sqrt{2}A(B^0_d \to K^0\pi^0\pi^0)_{\rm sym} &= C_1' e^{i\gamma} - C_2' e^{i\gamma} + \tilde{P}'_{uc} e^{i\gamma} - \tilde{P}'_{tc} - \tfrac{1}{3}P'_{\rm EW1} + P'_{\rm EW2} + \tfrac{1}{3}P'^C_{\rm EW1} + \tfrac{1}{3}P'^C_{\rm EW2},\\
A(B^+ \to K^+\pi^+\pi^-)_{\rm sym} &= -T_2' e^{i\gamma} - C_1' e^{i\gamma} - \tilde{P}'_{uc} e^{i\gamma} + \tilde{P}'_{tc} + \tfrac{1}{3}P'_{\rm EW1} - \tfrac{1}{3}P'^C_{\rm EW1} + \tfrac{2}{3}P'^C_{\rm EW2},\\
\sqrt{2}A(B^+ \to K^+\pi^0\pi^0)_{\rm sym} &= T_1' e^{i\gamma} + T_2' e^{i\gamma} + C_1' e^{i\gamma} + C_2' e^{i\gamma} + \tilde{P}'_{uc} e^{i\gamma} - \tilde{P}'_{tc} - \tfrac{1}{3}P'_{\rm EW1} - P'_{\rm EW2} - \tfrac{2}{3}P'^C_{\rm EW1} - \tfrac{2}{3}P'^C_{\rm EW2},\\
\sqrt{2}A(B^0_d \to K^+\pi^0\pi^-)_{\rm sym} &= T_1' e^{i\gamma} + C_2' e^{i\gamma} - P'_{\rm EW2} - P'^C_{\rm EW1},
\end{aligned} \tag{2}$$

where $\tilde{P}' \equiv P_1' + P_2'$, and all amplitudes have been multiplied by $\sqrt{2}$. Above we have explicitly written the weak-phase dependence [this includes γ and the minus sign from $V^*_{tb}V_{ts}$ ($\tilde{P}'_{tc}$ and EWPs)], while the diagrams contain strong phases.

Although there are a large number of diagrams in these amplitudes, they can be combined into a smaller number of effective diagrams:

$$\begin{aligned}
\sqrt{2}A(B^+ \to K^0\pi^+\pi^0)_{\rm sym} &= -T_a' e^{i\gamma} - T_b' e^{i\gamma} + P'_{{\rm EW},a} + P'_{{\rm EW},b},\\
A(B^0_d \to K^0\pi^+\pi^-)_{\rm sym} &= -T_a' e^{i\gamma} - P_a' e^{i\gamma} + P_b',\\
\sqrt{2}A(B^0_d \to K^0\pi^0\pi^0)_{\rm sym} &= -T_b' e^{i\gamma} + P_a' e^{i\gamma} - P_b' + P'_{{\rm EW},a} + P'_{{\rm EW},b},\\
A(B^+ \to K^+\pi^+\pi^-)_{\rm sym} &= -P_a' e^{i\gamma} + P_b' - P'_{{\rm EW},a},\\
\sqrt{2}A(B^+ \to K^+\pi^0\pi^0)_{\rm sym} &= T_a' e^{i\gamma} + T_b' e^{i\gamma} + P_a' e^{i\gamma} - P_b' - P'_{{\rm EW},b},\\
\sqrt{2}A(B^0_d \to K^+\pi^0\pi^-)_{\rm sym} &= T_a' e^{i\gamma} + T_b' e^{i\gamma} - P'_{{\rm EW},a} - P'_{{\rm EW},b},
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
&T_a' \equiv T_1' - T_2', \quad T_b' \equiv C_2' + T_2', \quad P_a' \equiv \tilde{P}'_{uc} + T_2' + C_1',\\
&P_b' \equiv \tilde{P}'_{tc} + \tfrac{1}{3}P'_{\rm EW1} + \tfrac{2}{3}P'^C_{\rm EW1} - \tfrac{1}{3}P'^C_{\rm EW2},\\
&P'_{{\rm EW},a} \equiv P'^C_{\rm EW1} - P'^C_{\rm EW2}, \quad P'_{{\rm EW},b} \equiv P'_{\rm EW2} + P'^C_{\rm EW2}.
\end{aligned} \tag{4}$$

The amplitudes can therefore be written in terms of 6 effective diagrams. This corresponds to 12 theoretical parameters: 6 magnitudes of diagrams, 5 relative (strong) phases, and γ. However, as noted above, there are only 11 experimental observables. Therefore, in order to extract γ, one requires additional input.

One obvious idea is the following. In two-body $\bar{b} \to \bar{s}$ B decays, the diagrams are expected to obey the approximate hierarchy [11]

$$1{:}\ P'_{tc}, \qquad \bar{\lambda}{:}\ T', P'_{\rm EW}, \qquad \bar{\lambda}^2{:}\ C', P'_{uc}, P'^C_{\rm EW}, \tag{5}$$

where $\bar{\lambda} \simeq 0.2$. If the three-body decay diagrams obey a similar hierarchy, one can neglect C_1', C_2', $\tilde{P}'_{uc}$, $P'^C_{\rm EW1}$, and $P'^C_{\rm EW2}$, with only a $\sim$5% theoretical error. But if these diagrams are neglected, then two of the effective diagrams vanish: $P'_{{\rm EW},a} \to 0$ and $T_b' - P_a' \to 0$ [Eq. (4)]. In this case, the amplitudes can be written in terms of 4 effective diagrams, corresponding to 8 theoretical parameters: 4 magnitudes of diagrams, 3 relative (strong) phases, and γ. Given that there are 11 experimental observables, the weak phase γ can be extracted.

The problem here is that it is difficult to test the assumption that C_1', C_2', $\tilde{P}'_{uc}$, $P'^C_{\rm EW1}$, and $P'^C_{\rm EW2}$ are negligible, so that the theoretical error is really unknown. Given this, it is perhaps better to look for another method, in which the theoretical error is better under control.

As mentioned in the Introduction, in Ref. [8], DSS proposed a new method for measuring γ in $B \to K\pi\pi$ decays. Although the details are different, at its heart the method is similar to that outlined above. While DSS do not write the amplitudes in terms of diagrams, they do note that each decay amplitude receives two contributions, one proportional to $e^{i\gamma}$, the other with no weak phase. The key point is that there is no gluonic-penguin contribution to $B^+ \to K^0\pi^+\pi^0$—its amplitude has only tree and EWP pieces. DSS's assumption, which provides the additional input and allows γ to be extracted, is that the EWP and tree contributions in $B^+ \to K^0\pi^+\pi^0$ are related to one another as in Refs. [4,5]. Unfortunately, it was then shown that this relation does not hold [9], so that γ cannot be obtained using DSS's method.

Now, in terms of diagrams, the DSS assumption is that $T'_1 + C'_2$ is related to $P'_{\rm EW2} + P'^C_{\rm EW1}$ [Eq. (2)]. Although this is not true, it does not preclude other EWP-tree relations. Indeed, as we will see in the next section, such relations do exist, and their imposition does allow γ to be extracted from $B \to K\pi\pi$ decays.

Finally, we return to the issue of the underlying symmetry. The above discussion is for the case where only isospin symmetry is considered. However, below we will see that it may be necessary to assume full flavor SU(3) symmetry. In this case, the final state involves three identical particles, so that the six permutations of these particles (the group S_3) must be taken into account. Correspondingly, there are six possible wave functions, in which the three particles are in a totally symmetric state, a totally antisymmetric state, or one of four mixed states. These six states can be chosen such that the $\pi\pi$ wave function is either symmetric or antisymmetric. A symmetric $\pi\pi$ state is then a linear combination of the totally symmetric S_3 state and one mixed state. Consequently, the parametrization of Eq. (3) holds even under full SU(3) symmetry, as long as the state is symmetric under $\pi\pi$ exchange.

III. EWP-TREE RELATIONS

EWP-tree relations are well known in the context of $B \to PP$ decays (P is a pseudoscalar meson), particularly $B \to K\pi$. They have been very useful for reducing the number of free theoretical parameters. The starting point is the electroweak effective Hamiltonian for quark-level $\bar{b}$ decays [12]:

$$H_{\rm eff} = \frac{G_F}{\sqrt{2}} \sum_{q=d,s} \left(\sum_{p=u,c} \lambda_p^{(q)} (c_1(\mu) O_1^p(\mu) + c_2(\mu) O_2^p(\mu)) - \lambda_t^{(q)} \sum_{i=3}^{10} c_i(\mu) O_i(\mu) \right), \tag{6}$$

where $\lambda_p^{(q)} = V^*_{pb} V_{pq}$. μ is the renormalization point, typically taken to be $O(m_b)$. All physical quantities must be independent of μ. The Wilson coefficients c_i include gluons (QCD corrections) whose energy is above μ (short distance), while the operators O_i include QCD corrections of energy less than μ (long distance). Note: factors of $G_F/\sqrt{2}$ are omitted for the remainder of this paper.

The operators take the following form:

$$\begin{aligned} O_1^p &= (\bar{b}_\alpha p_\alpha)_{V-A} (\bar{p}_\beta q_\beta)_{V-A}, \\ O_2^p &= (\bar{b}_\alpha p_\beta)_{V-A} (\bar{p}_\beta q_\alpha)_{V-A}, \end{aligned} \tag{7}$$

summed over color indices α and β. These are the usual (tree-level) current-current operators induced by W-boson exchange.

$$\begin{aligned} O_3 &= (\bar{b}_\alpha q_\alpha)_{V-A} \sum_{q'} (\bar{q}'_\beta q'_\beta)_{V-A}, \\ O_4 &= (\bar{b}_\alpha q_\beta)_{V-A} \sum_{q'} (\bar{q}'_\beta q'_\alpha)_{V-A}, \\ O_5 &= (\bar{b}_\alpha q_\alpha)_{V-A} \sum_{q'} (\bar{q}'_\beta q'_\beta)_{V+A}, \\ O_6 &= (\bar{b}_\alpha q_\beta)_{V-A} \sum_{q'} (\bar{q}'_\beta q'_\alpha)_{V+A}, \end{aligned} \tag{8}$$

summed over the light flavors $q' = u, d, s$, and c. These are referred to as QCD (gluonic) penguin operators.

$$\begin{aligned} O_7 &= \frac{3}{2} (\bar{b}_\alpha q_\alpha)_{V-A} \sum_{q'} e_{q'} (\bar{q}'_\beta q'_\beta)_{V+A}, \\ O_8 &= \frac{3}{2} (\bar{b}_\alpha q_\beta)_{V-A} \sum_{q'} e_{q'} (\bar{q}'_\beta q'_\alpha)_{V+A}, \\ O_9 &= \frac{3}{2} (\bar{b}_\alpha q_\alpha)_{V-A} \sum_{q'} e_{q'} (\bar{q}'_\beta q'_\beta)_{V-A}, \\ O_{10} &= \frac{3}{2} (\bar{b}_\alpha q_\beta)_{V-A} \sum_{q'} e_{q'} (\bar{q}'_\beta q'_\alpha)_{V-A}, \end{aligned} \tag{9}$$

with $e_{q'}$ denoting the electric charges of the quarks. These are the electroweak-penguin operators. The quark current $(\bar{q}_1 q_2)_{V\pm A}$ denotes $\bar{q}_1 \gamma^\mu (1 \pm \gamma_5) q_2$. The key observation is that the Wilson coefficients $c_{7,8}$ are small compared to $c_{9,10}$. Neglecting them, the tree and EWP operators then have exactly the same structure, up to a Fierz transformation of the fermions, and can be related.

Various approaches have been used to exploit this fact for $B \to K\pi$ decays. Neubert and Rosner (NR) showed that a basic SU(3) EWP-tree relation can be obtained by manipulating the effective Hamiltonian itself at the level of quark operators [4]. Later, Gronau, Pirjol, and Yan (GPY) used a more general technique based on group theory to find additional SU(3) EWP-tree relations [5]. Recently, it was shown that these relations can be obtained by studying Wick contractions of the effective Hamiltonian [13].

In this section, we will apply the contractions approach to $B \to K\pi\pi$ decays. As we will see, the correct SU(3) EWP-tree relations in $B \to K\pi\pi$ are between specific diagrams. For example, $P'_{\rm EW1}$ is related to T'_1 and C'_1, and not to T'_2 and C'_2. Since different diagrams such as T'_1 and T'_2 cannot be distinguished at the level of operators or group theory, the NR and GPY approaches may not be applicable [14]. In the following section, we give a brief review of the contractions formalism.

A. Contractions

The formalism of contractions gives a bridge between the effective Hamiltonian and the language of diagrams. Contractions include all the short-distance information of Wilson coefficients, and also exploit the fact that trees and EWPs arise from long-distance operators with almost identical structures. In Ref. [13], contractions are discussed at length for $B \to PP$ decays (see also Ref. [15]). Here only isospin symmetry is assumed initially. It is shown that all diagrams can be expressed in terms of contractions, and the EWP-tree relations of Refs. [4,5] are reproduced. However, these relations hold only if SU(3) symmetry is imposed. For this reason, in our review below, we assume SU(3) from the beginning. Also, for definitiveness, and to make the comparison with $B \to K\pi\pi$ clearer, we focus on the decay $B \to K\pi$.

The idea is as follows: (i) one symmetrizes or antisymmetrizes the final state, (ii) one takes the operators of effective Hamiltonian, (iii) one adds initial and final states, and (iv) one computes the sum of all possible Wick contractions, applying the basic rules of quantum field theory. This gives the decomposition of the decay amplitude in terms of contractions. This can be compared with the decomposition in terms of diagrams and therefore gives us the structure of each diagram in terms of contractions. It is this comparison which allows us to match diagrams and contractions and thus yields the EWP-tree relations.

Since the spinless B meson decays into a pair of pseudoscalar mesons K and π, these are necessarily in an S wave. Under SU(3), K and π mesons are identical particles, and so one must symmetrize the final state $|f\rangle$:

$$|f\rangle = \frac{1}{\sqrt{2}}(|K(p_1)\pi(p_2)\rangle + |\pi(p_1)K(p_2)\rangle). \tag{10}$$

When calculating the amplitude for a particular $B \to K\pi$ decay, one must "sandwich" all operators of the effective Hamiltonian between initial and final states. All such terms have the form

$$\langle \bar{q}_1 q_2 \bar{q}_3 q_4 | \bar{b} q_5 \bar{q}_6 q_7 | \bar{b} q_8 \rangle. \tag{11}$$

(Dirac and color structures are omitted for notational convenience.) $\bar{b}q_8$ is the B meson. The final-state mesons contain the quarks $\bar{q}_1$, q_2, $\bar{q}_3$, and q_4. The two choices are $K = \bar{q}_1 q_2$ and $\pi = \bar{q}_3 q_4$, or $\pi = \bar{q}_1 q_2$ and $K = \bar{q}_3 q_4$, and these correspond to the two states in Eq. (10).

For a given B decay, there are $4! = 24$ possible contractions. However, not all are independent. For example, consider the two contractions[2]

$$EM'(1) = \langle \bar{q}_1 q_2 \bar{q}_3 q_4 | \bar{b} q_5 \bar{q}_6 q_7 | \bar{b} q_8 \rangle, \quad EM'(2) = \langle \bar{q}_1 q_2 \bar{q}_3 q_4 | \bar{b} q_5 \bar{q}_6 q_7 | \bar{b} q_8 \rangle. \tag{12}$$

(The prime indicates a $\bar{b} \to \bar{s}$ transition.) Here the labels (1) and (2) correspond, respectively, to the momentum assignments $K(p_1)\pi(p_2)$ and $\pi(p_1)K(p_2)$. It is clear that the above contractions are not independent since one can be obtained from the other with an exchange of mesons, so that $EM'(1) = EM'(2)$.

Now, if one performs the contractions with the operators O_1^u and O_2^u of Eq. (6), one finds that the T' diagram is related to the EM'-type contractions [13]:

$$\begin{aligned} T' &= \frac{1}{\sqrt{2}}|\lambda_u^{(s)}|(c_1 EM_1'(1) + c_1 EM_1'(2) \\ &\quad + c_2 EM_2'(1) + c_2 EM_2'(2)) \\ &= \frac{1}{\sqrt{2}}|\lambda_u^{(s)}|c_1\bigg(EM_1'(1) + EM_1'(2) \\ &\quad + \frac{c_2}{c_1}EM_2'(1) + \frac{c_2}{c_1}EM_2'(2)\bigg), \end{aligned} \tag{13}$$

where EM_i' is an EM'-type contraction of the operator O_i. Similarly, the $P'_{\rm EW}$ diagram is related to the EM_i' contraction of the operators O_9 and O_{10}:

$$\begin{aligned} P'_{\rm EW} &= -\frac{1}{\sqrt{2}}\frac{3}{2}|\lambda_t^{(s)}|(c_9 EM_9'(1) + c_9 EM_9'(2) \\ &\quad + c_{10} EM_{10}'(1) + c_{10} EM_{10}'(2)) \\ &= -\frac{1}{\sqrt{2}}\frac{3}{2}|\lambda_t^{(s)}|c_9\bigg(EM_9'(1) + EM_9'(2) \\ &\quad + \frac{c_2}{c_1}EM_{10}'(1) + \frac{c_2}{c_1}EM_{10}'(2)\bigg). \end{aligned} \tag{14}$$

Here, we have used the fact that the Wilson coefficients obey $c_1/c_2 = c_9/c_{10}$ to about 5%. (In the rest of the paper, we assume this equality.)

Now, the T' diagram contains EM'-type contractions of $O_{1,2}^u$, while the $P'_{\rm EW}$ diagram contains EM'-type contractions of $O_{9,10}$. However, since s-quark contractions are equal to u- or d-quark contractions in the SU(3) limit, $O_9^q \sim (\bar{b}_\alpha s_\alpha)_{V-A}(\bar{q}_\beta q_\beta)_{V-A} = (\bar{b}_\alpha u_\alpha)_{V-A}(\bar{u}_\beta s_\beta)_{V-A} \sim O_1^u$. That is, O_9^q and O_1^u have the same form under SU(3). Things are similar for O_{10}^q and O_2^u. We therefore see that $P'_{\rm EW}$ is proportional to T':

$$P'_{\rm EW} = -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\frac{c_9 + c_{10}}{c_1 + c_2}T'. \tag{15}$$

[2]Here EM stands for "emission." See Ref. [13] for details.

The argument is much the same for C' and $P'^{C}_{\rm EW}$. Two other contractions are

$$EM'_C(1) = \langle \bar{q}_1 q_2 \bar{q}_3 q_4 | \bar{b} q_5 \bar{q}_6 q_7 | \bar{b} q_8 \rangle \ , \quad EM'_C(2) = \langle \bar{q}_1 q_2 \bar{q}_3 q_4 | \bar{b} q_5 \bar{q}_6 q_7 | \bar{b} q_8 \rangle \ . \tag{16}$$

The diagrams C' and $P'^{C}_{\rm EW}$ are related to the EM'_C-type contractions:

$$\begin{aligned} C' &= \frac{1}{\sqrt{2}} |\lambda_u^{(s)}| (c_1 EM'_{C1}(1) + c_1 EM'_{C1}(2) + c_2 EM'_{C2}(1) + c_2 EM'_{C2}(2)) \\ &= \frac{1}{\sqrt{2}} |\lambda_u^{(s)}| c_1 \left(EM'_{C1}(1) + EM'_{C1}(2) + \frac{c_2}{c_1} EM'_{C2}(1) + \frac{c_2}{c_1} EM'_{C2}(2) \right), \\ P'^{C}_{\rm EW} &= -\frac{1}{\sqrt{2}} \frac{3}{2} |\lambda_t^{(s)}| (c_9 EM'_{C9}(1) + c_9 EM'_{C9}(2) + c_{10} EM'_{C10}(1) + c_{10} EM'_{C10}(2)) \\ &= -\frac{1}{\sqrt{2}} \frac{3}{2} |\lambda_t^{(s)}| c_9 \left(EM'_{C9}(1) + EM'_{C9}(2) + \frac{c_2}{c_1} EM'_{C10}(1) + \frac{c_2}{c_1} EM'_{C10}(2) \right). \end{aligned} \tag{17}$$

In the SU(3) limit, $EM'_{C9}(n) = EM'_{C1}(n)$ and $EM'_{C10}(n) = EM'_{C2}(n)$ ($n = 1, 2$), so that $P'^{C}_{\rm EW}$ is proportional to C':

$$P'^{C}_{\rm EW} = -\frac{3}{2} \frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|} \frac{c_9 + c_{10}}{c_1 + c_2} C'. \tag{18}$$

Above, we have described the formalism of contractions in the context of two-body decays. Our aim now is to apply this to the problem of $B \to K\pi\pi$ decays and derive EWP-tree relations. As we saw above, different contractions can be made equal through the imposition of SU(3). However, this can lead to some subtleties in the case of three-body decays.

Under SU(3), π and K mesons are treated as identical particles, and the total wave function of the final state must be symmetric under the exchange of these particles. For $B \to K\pi$ decays, since the final state has to be in an S wave, it is automatically symmetric under the exchange of the final-state mesons. However, for $B \to K\pi\pi$, higher states of angular momentum are possible, and the final state is then not necessarily symmetric under permutations of the mesons. As mentioned earlier, the group of permutations is S_3, and there are six possible states: the three particles can be in a totally symmetric state, a totally antisymmetric state, or one of four mixed states. To be completely explicit, we define

$$\begin{aligned} |1\rangle &\equiv |K(p_1)\pi_1(p_2)\pi_2(p_3)\rangle, \\ |2\rangle &\equiv |K(p_1)\pi_2(p_2)\pi_1(p_3)\rangle, \\ |3\rangle &\equiv |\pi_2(p_1)K(p_2)\pi_1(p_3)\rangle, \\ |4\rangle &\equiv |\pi_2(p_1)\pi_1(p_2)K(p_3)\rangle, \\ |5\rangle &\equiv |\pi_1(p_1)\pi_2(p_2)K(p_3)\rangle, \\ |6\rangle &\equiv |\pi_1(p_1)K(p_2)\pi_2(p_3)\rangle, \end{aligned} \tag{19}$$

where the p_i are the momenta of the final-state mesons. The six states of S_3 can then be defined as

$$\begin{aligned} |S\rangle &\equiv \frac{1}{\sqrt{6}}(|1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle), \\ |M_1\rangle &\equiv \frac{1}{\sqrt{12}}(2|1\rangle + 2|2\rangle - |3\rangle - |4\rangle - |5\rangle - |6\rangle), \\ |M_2\rangle &\equiv \frac{1}{\sqrt{4}}(|3\rangle - |4\rangle - |5\rangle + |6\rangle), \\ |M_3\rangle &\equiv \frac{1}{\sqrt{4}}(-|3\rangle - |4\rangle + |5\rangle + |6\rangle), \\ |M_4\rangle &\equiv \frac{1}{\sqrt{12}}(2|1\rangle - 2|2\rangle - |3\rangle + |4\rangle - |5\rangle + |6\rangle), \\ |A\rangle &\equiv \frac{1}{\sqrt{6}}(|1\rangle - |2\rangle + |3\rangle - |4\rangle + |5\rangle - |6\rangle). \end{aligned} \tag{20}$$

Note that $|S\rangle$, $|M_1\rangle$, and $|M_2\rangle$ are all symmetric under the exchange of the two pions, while $|M_3\rangle$, $|M_4\rangle$, and $|A\rangle$ are all antisymmetric.

Below, we present two cases which illustrate the features of all six S_3 states. First, we examine the totally symmetric SU(3) state $|S\rangle$. This can be determined experimentally as follows. Consider again the decay $B_d^0 \to K^0\pi^+\pi^-$. In Sec. II it was noted that the Dalitz-plot events can be described by s_+ and s_- [Eq. (1)], and that the decay amplitude, $\mathcal{M}(s_+, s_-)$, can be extracted. We introduce the third Mandelstam variable, $s_0 = (p_{\pi^+} + p_{\pi^-})^2$. It is related to s_+ and s_- as follows:

$$s_0 = m_B^2 + 2m_\pi^2 + m_{K^0}^2 - s_+ - s_-. \tag{21}$$

The totally symmetric SU(3) decay amplitude is then given by

$$\frac{1}{\sqrt{6}}[\mathcal{M}(s_+, s_-) + \mathcal{M}(s_-, s_+) + \mathcal{M}(s_+, s_0) + \mathcal{M}(s_0, s_+) + \mathcal{M}(s_0, s_-) + \mathcal{M}(s_-, s_0)]. \tag{22}$$

Other decays can be treated similarly.

Second, we examine the state which is symmetric only under the exchange of the two pions (we denote this state as $|S_{\pi\pi}\rangle$, and refer to it as $\pi\pi$ symmetric). Previous analyses of $B \to K\pi\pi$ concentrated on the $\pi\pi$-symmetric case with isospin symmetry [7,8,10,16]. It is written as

$$|S_{\pi\pi}\rangle = \frac{1}{\sqrt{2}}(|K(p_1)\pi_1(p_2)\pi_2(p_3)\rangle + |K(p_1)\pi_2(p_2)\pi_1(p_3)\rangle) = \sqrt{\frac{1}{3}}|S\rangle + \sqrt{\frac{2}{3}}|M_1\rangle. \quad (23)$$

Thus, the $\pi\pi$-symmetric state is a mixture of the totally symmetric state and a mixed state of S_3.

B. Totally symmetric case

We begin with the totally symmetric state $|S\rangle$. The amplitude is obtained by summing over all possible contractions:

$$\mathcal{A}(B \to K\pi_1\pi_2)_{\text{tot-sym}} = \sum_{\text{contractions}} \langle S|\mathcal{H}_{\text{eff}}|B\rangle. \quad (24)$$

Here there are $5! = 120$ possible contractions. Even after removing those which are not independent, there are a large number of contractions involved.

Below we concentrate on the tree and EWP contractions/diagrams. We use the same notation as for $B \to K\pi$. To be specific, $X_i(n)$ ($n = 1$–6) is an X-type contraction of the operator O_i of $\mathcal{H}_{\text{eff}}$ arising from the momentum assignments of the states $|n\rangle = |1\rangle, |2\rangle, \ldots, |6\rangle$. For example, $T'_{1,2}(2)$ denotes a contraction of the tree operator O_2 related to the T'_1 diagram with momentum assignments $K(p_1)$, $\pi_1(p_3)$, and $\pi_2(p_2)$. The explicit forms of contractions for the trees and EWPs that interest us are the following:

$$C'_1(1) = \langle \bar q_1 q_2 \bar q_3 q_4 \bar q_5 q_6|\bar b q_7 \bar q_8 q_9|\bar b q_{10}\rangle, \quad T'_1(1) = \langle \bar q_1 q_2 \bar q_3 q_4 \bar q_5 q_6|\bar b q_7 \bar q_8 q_9|\bar b q_{10}\rangle,$$
$$C'_2(1) = \langle \bar q_1 q_2 \bar q_3 q_4 \bar q_5 q_6|\bar b q_7 \bar q_8 q_9|\bar b q_{10}\rangle, \quad T'_2(1) = \langle \bar q_1 q_2 \bar q_3 q_4 \bar q_5 q_6|\bar b q_7 \bar q_8 q_9|\bar b q_{10}\rangle,$$
$$P'^C_{EW1}(1) = \langle \bar q_1 q_2 \bar q_3 q_4 \bar q_5 q_6|\bar b q_7 \bar q_8 q_9|\bar b q_{10}\rangle, \quad P'_{EW1}(1) = \langle \bar q_1 q_2 \bar q_3 q_4 \bar q_5 q_6|\bar b q_7 \bar q_8 q_9|\bar b q_{10}\rangle,$$
$$P'^C_{EW2}(1) = \langle \bar q_1 q_2 \bar q_3 q_4 \bar q_5 q_6|\bar b q_7 \bar q_8 q_9|\bar b q_{10}\rangle, \quad P'_{EW2}(1) = \langle \bar q_1 q_2 \bar q_3 q_4 \bar q_5 q_6|\bar b q_7 \bar q_8 q_9|\bar b q_{10}\rangle. \quad (25)$$

These are easy to verify from Fig. 1.

Recall that the momentum assignment (1) corresponds to $K(p_1)$, $\pi_1(p_2)$, and $\pi_2(p_3)$, while (2) corresponds to $K(p_1)$, $\pi_2(p_2)$, and $\pi_1(p_3)$. Contractions of type (2) can be obtained by acting with P_{23}, where P_{ij} is the permutation operator which exchanges the ith and jth mesons of the final state. For example, the contraction $C'_1(2)$ is

$$C'_1(2) = P_{23}C'_1(1) = \langle \bar q_1 q_2 \bar q_5 q_6 \bar q_3 q_4|\bar b q_7 \bar q_8 q_9|\bar b q_{10}\rangle. \quad (26)$$

In the same vein, contractions of type (n) can be obtained by acting on contractions of type (1) with the appropriate permutation operator (exchanges, cyclic or anticyclic permutations).

With these, it is straightforward to express the tree and EWP diagrams in terms of contractions. We have

$$T'_j = \frac{|\lambda_u^{(s)}|}{\sqrt{6}} c_i(T'_{j,i}(1) + \cdots + T'_{j,i}(6)), \qquad C'_j = \frac{|\lambda_u^{(s)}|}{\sqrt{6}} c_i(C'_{j,i}(1) + \cdots + C'_{j,i}(6)),$$
$$P'_{\text{EW}j} = -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{\sqrt{6}} c_i(P'_{\text{EW}j,i}(1) + \cdots + P'_{\text{EW}j,i}(6)), \qquad P'^C_{\text{EW}j} = -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{\sqrt{6}} c_i(P'^C_{\text{EW}j,i}(1) + \cdots + P'^C_{\text{EW}j,i}(6)), \quad (27)$$

where the sum is over $i = 1, 2$ for trees and $i = 9, 10$ for EWPs.

The point is that, with a totally symmetric state, the contractions $T'_{j,i}(m)$ and $P'_{\text{EW}j,i}(n)$ are simply different ways of writing the same thing. Applying the permutation operator P_{13} (for example), it is easy to show that

$$P_{13}T'_{j,i}(1) = P'_{\mathrm{EW}j,i}(1), \qquad P_{13}T'_{j,i}(2) = P'_{\mathrm{EW}j,i}(6),$$
$$P_{13}T'_{j,i}(3) = P'_{\mathrm{EW}j,i}(5), \qquad P_{13}T'_{j,i}(4) = P'_{\mathrm{EW}j,i}(4),$$
$$P_{13}T'_{j,i}(5) = P'_{\mathrm{EW}j,i}(3), \qquad P_{13}T'_{j,i}(6) = P'_{\mathrm{EW}j,i}(2). \quad (28)$$

The situation here is very similar to what we found in the $B \to K\pi$ decay. In that case, both T' and P'_{EW} diagrams were written in terms of EM'-type contractions. Here, we use a slightly different notation, but the above equation proves that T'_i and $P'_{\mathrm{EW}i}$ diagrams ($i = 1, 2$) actually contain the same type of contraction.

Similar relations exist between the $C'_{j,i}(m)$ and $P'^C_{\mathrm{EW}j,i}(n)$ contractions. Thus, assuming the Wilson coefficients respect the approximate equality $c_1/c_2 \approx c_9/c_{10}$, it is straightforward to find the following SU(3) relations from Eq. (27):

$$P'_{\mathrm{EW1}} = -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\frac{c_9 + c_{10}}{c_1 + c_2}T'_1,$$
$$P'_{\mathrm{EW2}} = -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\frac{c_9 + c_{10}}{c_1 + c_2}T'_2,$$
$$P'^C_{\mathrm{EW1}} = -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\frac{c_9 + c_{10}}{c_1 + c_2}C'_1,$$
$$P'^C_{\mathrm{EW2}} = -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\frac{c_9 + c_{10}}{c_1 + c_2}C'_2. \quad (29)$$

Now, these relations assume only SU(3) symmetry and the approximate ratio of Wilson coefficients. The expected error due to SU(3)-breaking effects is $O(30\%)$. However, when all contributions to $B \to K\pi\pi$ are taken into account, the net error is much smaller, $O(5\%)$, since EWPs and trees are subleading effects. This is consistent with the error estimates for EWP-tree relations in $B \to K\pi$ given in Ref. [4].

Finally, we note that the assumption $c_1/c_2 = c_9/c_{10}$ is not necessary. It is actually possible to prove EWP-tree relations which are exact under SU(3). They are

$$P'_{\mathrm{EW}i} = -\frac{3}{4}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\left[\frac{c_9 + c_{10}}{c_1 + c_2}(T'_i + C'_i) + \frac{c_9 - c_{10}}{c_1 - c_2}(T'_i - C'_i)\right],$$
$$P'^C_{\mathrm{EW}i} = -\frac{3}{4}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\left[\frac{c_9 + c_{10}}{c_1 + c_2}(T'_i + C'_i) - \frac{c_9 - c_{10}}{c_1 - c_2}(T'_i - C'_i)\right], \quad (30)$$

for $i = 1, 2$. These are similar to the exact SU(3) EWP-tree relations for $B \to K\pi$ given in Ref. [5]. [When we assume that $c_1/c_2 = c_9/c_{10}$, we recover the relations of Eq. (29).]

C. $\pi\pi$-symmetric case

We now consider the $\pi\pi$-symmetric state. Applying the formalism of contractions to $|S_{\pi\pi}\rangle$ with the effective Hamiltonian $\mathcal{H}_{\mathrm{eff}}$, we obtain the amplitude from

$$\mathcal{A}(B \to K\pi_1\pi_2)_{\pi\pi\text{-sym}} = \sum_{\text{contractions}} \langle S_{\pi\pi}|\mathcal{H}_{\mathrm{eff}}|B\rangle. \quad (31)$$

Again, there are many contractions involved.

We use the same notation as in the previous section, but now the number in parentheses only goes from 1 to 2. Thus, $X_i(1)$ [$X_i(2)$] denotes an X-type contraction of operator O_i of $\mathcal{H}_{\mathrm{eff}}$ arising from the first (second) term of the first relation in Eq. (23). The expressions for the trees and EWPs in terms of contractions are the same as for the totally symmetric state $|S\rangle$ [Eq. (27)], but with only two permutation terms:

$$T'_j = \frac{|\lambda_u^{(s)}|}{\sqrt{2}}c_i(T'_{j,i}(1) + T'_{j,i}(2)),$$
$$C'_j = \frac{|\lambda_u^{(s)}|}{\sqrt{2}}c_i(C'_{j,i}(1) + C'_{j,i}(2)),$$
$$P'_{\mathrm{EW}j} = -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{\sqrt{2}}c_i(P'_{\mathrm{EW}j,i}(1) + P'_{\mathrm{EW}j,i}(2)),$$
$$P'^C_{\mathrm{EW}j} = -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{\sqrt{2}}c_i(P'^C_{\mathrm{EW}j,i}(1) + P'^C_{\mathrm{EW}j,i}(2)), \quad (32)$$

where, as usual, the sum is over $i = 1, 2$ for trees and $i = 9, 10$ for EWPs.

Based on the EWP-tree relations in $B \to K\pi$, from the previous equation we would expect to find a relation between T'_j and $P'_{\mathrm{EW}j}$ (or between C'_j and $P'^C_{\mathrm{EW}j}$) under SU(3) symmetry. And indeed, there is such a relation: for example, $P'_{\mathrm{EW1}}(1)$ can be obtained from $T'_1(1)$ by applying the permutation operator P_{13}:

$$P'_{\mathrm{EW1}}(1) = P_{13}T'_1(1). \quad (33)$$

The above equality can be verified easily from Eq. (25). Other pairs of contractions are related similarly. The problem is that P_{13} corresponds to the exchange of the K meson and one of the π's. But a $K \leftrightarrow \pi$ exchange is not a valid operation here since the initial state is not defined as being symmetric under such an exchange. More generally, this conclusion applies to all four states of mixed symmetry. Thus, there are no exact SU(3) EWP-tree relations for the mixed states in $B \to K\pi\pi$ decays. This means that, for these states, we need different additional input in order to reduce the number of effective diagrams.

Fortunately, there is a possible piece of additional information. The EWP-tree relations of Eqs. (15) and (18) hold for $B \to K\pi$ to all orders in α_s. However, it was shown in Ref. [13] that one can also work order by order in α_s, i.e. perform the contractions analysis for processes with 0, 1, 2, etc. internal gluons. At leading order (LO), different EWP-tree relations appear. As we see below, a similar behavior holds for $B \to K\pi\pi$.

Contractions are related to Fierz tranformations [$q_7 \leftrightarrow q_9$ in Eq. (25)] in the following way:

$$C'_{1,1} \overset{\text{Fierz}}{=} P'_{\text{EW1},10}, \qquad C'_{1,2} \overset{\text{Fierz}}{=} P'_{\text{EW1},9}, \qquad C'_{2,1} \overset{\text{Fierz}}{=} P'_{\text{EW2},10}, \qquad C'_{2,2} \overset{\text{Fierz}}{=} P'_{\text{EW2},9},$$
$$T'_{1,1} \overset{\text{Fierz}}{=} P'^{C}_{\text{EW1},10}, \qquad T'_{1,2} \overset{\text{Fierz}}{=} P'^{C}_{\text{EW1},9}, \qquad T'_{2,1} \overset{\text{Fierz}}{=} P'^{C}_{\text{EW2},10}, \qquad T'_{2,2} \overset{\text{Fierz}}{=} P'^{C}_{\text{EW2},9}. \tag{34}$$

That is, since Fierz relations hold at the level of operators, contractions of operators $O_{1,2}$ are related to those of operators $O_{10,9}$, respectively. Applying this to diagram P'_{EW1} of Eq. (32) for example, we have

$$\begin{aligned} P'_{\text{EW1}} &= -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{\sqrt{2}} c_i (P'_{\text{EW1},i}(1) + P'_{\text{EW1},i}(2)) \\ &= -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{\sqrt{2}} (c_9 P'_{\text{EW1},9}(1) + c_{10} P'_{\text{EW1},10}(1) + c_9 P'_{\text{EW1},9}(2) + c_{10} P'_{\text{EW1},10}(2)) \\ &\overset{\text{Fierz}}{=} -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{\sqrt{2}} (c_9 C'_{1,2}(1) + c_{10} C'_{1,1}(1) + c_9 C'_{1,2}(2) + c_{10} C'_{1,1}(2)). \end{aligned} \tag{35}$$

We also have

$$\begin{aligned} C'_1 &= \frac{|\lambda_u^{(s)}|}{\sqrt{2}} c_i (C'_{1,i}(1) + C'_{1,i}(2)) \\ &= \frac{|\lambda_u^{(s)}|}{\sqrt{2}} (c_1 C'_{1,1}(1) + c_2 C'_{1,2}(1) + c_1 C'_{1,1}(2) + c_2 C'_{1,2}(2)). \end{aligned} \tag{36}$$

We therefore see that P'_{EW1} is not proportional to the C'_1. It would be if the Wilson coefficients respected the equality $c_1/c_2 = c_{10}/c_9$, but this obviously does not hold (what is true is that $c_1/c_2 \approx c_9/c_{10}$).

This can be ameliorated by working only to LO in α_s. In this case, color effects can be extracted out [13], so that the above equations become

$$\begin{aligned} P'_{\text{EW1}} &= -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{\sqrt{2}} (c_9 N_c^2 \bar{C}'_1(1) + c_{10} N_c \bar{C}'_1(1) \\ &\quad + c_9 N_c^2 \bar{C}'_1(2) + c_{10} N_c \bar{C}'_1(2)) \\ &= -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{\sqrt{2}} (c_9 N_c^2 + c_{10} N_c)(\bar{C}'_1(1) + \bar{C}'_1(2)) + O(\alpha_s), \\ C'_1 &= \frac{|\lambda_u^{(s)}|}{\sqrt{2}} (c_1 N_c \bar{C}'_1(1) + c_2 N_c^2 \bar{C}'_1(1) \\ &\quad + c_1 N_c \bar{C}'_1(2) + c_2 N_c^2 \bar{C}'_1(2)) \\ &= \frac{|\lambda_u^{(s)}|}{\sqrt{2}} (c_1 N_c + c_2 N_c^2)(\bar{C}'_1(1) + \bar{C}'_1(2)) + O(\alpha_s), \end{aligned} \tag{37}$$

in which $N_c = 3$ is the number of colors in QCD and the overline notation indicates color-extracted contractions. We therefore obtain the relation

$$P'_{\text{EW1}} \approx -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\frac{c_9 + c_{10}/N_c}{c_1/N_c + c_2} C'_1, \tag{38}$$

which is valid at LO and under isospin symmetry [SU(3) was not used above]. The same procedure can be applied to other diagrams, with the result that

$$\begin{aligned} P'_{\text{EW2}} &\approx -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\frac{c_9 + c_{10}/N_c}{c_1/N_c + c_2} C'_2, \\ P'^{C}_{\text{EW1}} &\approx -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\frac{c_9/N_c + c_{10}}{c_1 + c_2/N_c} T'_1, \\ P'^{C}_{\text{EW2}} &\approx -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\frac{c_9/N_c + c_{10}}{c_1 + c_2/N_c} T'_2. \end{aligned} \tag{39}$$

We refer to these as "crossed" EWP-tree relations.

As noted above, the crossed relations hold only at LO—these are not reproduced by the higher-order diagrams. The error is therefore $O(\alpha_s)$. The size of this error then depends crucially on what the value of α_s is for this calculation. For example, if soft gluons are important, then α_s is large, and the use of these relations is not a good approximation. To address this question, we rely on theoretical input. There are basically three approaches used in calculations of hadronic B decays: QCD factorization (QCDf) [17], perturbative QCD (pQCD) [18], and soft collinear effective theory (SCET) [19]. All three methods perform their studies of two-body decays by taking the $m_b \to \infty$ limit and separating the nonperturbative low-energy (soft) effects from those at high energies (hard effects).[3] All gluons (soft and hard) between two quarks in the same meson are absorbed into the parameters describing hadronization (decay constants). Other gluons between quarks of two different mesons are absorbed into the form factors. For the remaining gluons, in all three approaches it was found that soft gluons are suppressed, so that $\alpha_s = \alpha_s(m_b) \sim 20\%$ [13]. This permits an expansion in α_s, and this was done in QCDf, pQCD, and SCET. Here we assume that this also holds in the case of three-body decays, so that the use of crossed EWP-tree relations is, in fact, a reasonable

[3]In fact, there are three energy scales for gluons: Λ_{QCD} (soft), m_b (hard), and $\sqrt{\Lambda_{\text{QCD}} m_b}$ (hard collinear). The presence of these three scales affects calculations within a specific model, but does not change our conclusions regarding the value of α_s in the expansion.

approximation. (There have been some studies of such decays, and they support this assumption [20].)

The relations suffer from an additional error due to the fact that the ratio of Wilson coefficients is strongly dependent on the choice of renormalization scale. This effect can be taken into account by considering a large range of values for this ratio. On the whole, we estimate that the total error is roughly comparable to that when SU(3) is assumed, $O(30\%)$. However, since EWPs and trees are themselves subleading effects in $B \to K\pi\pi$ decays, the net effect in concrete applications is much smaller, $O(5\%)$.

IV. MEASURING γ

The EWP-tree relations found above do indeed permit the extraction of γ from $B \to K\pi\pi$ decays. However, the precise procedure used depends on what the $K\pi\pi$ state is. One can use only events corresponding to the totally symmetric final state $|S\rangle$. Or one can combine the two S_3 states, both symmetric under the exchange of the two pions, whose sum forms $|S_{\pi\pi}\rangle$ [Eq. (23)].

If the $K\pi\pi$ state is $|S\rangle$, the exact SU(3) EWP-tree relations [Eq. (29)] hold. For the effective diagrams, this implies that

$$P'_{\mathrm{EW},b} = -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\frac{c_9 + c_{10}}{c_1 + c_2}T'_b. \tag{40}$$

Thus, there are only five effective diagrams in the six $B \to K\pi\pi$ decay amplitudes. This corresponds to 10 theoretical parameters: 5 magnitudes of diagrams, 4 relative (strong) phases, and γ. Since there are 11 experimental observables, γ can be extracted by doing a fit.

The fit itself is somewhat unusual. All experimental observables are momentum dependent, as are the diagrams. In obtaining the best-fit ‘‘values’’ of the diagrams, one will determine the momentum dependence of their magnitudes and relative strong phases. However, γ is independent of the particles’ momenta. Thus, the fit must yield a momentum-independent value for γ. The error on γ must take into account any momentum dependence.

Now, the extraction of the decay amplitudes from the Dalitz plots is rather difficult and has a certain amount of input—distributions of resonant effects (e.g. Breit-Wigner), treatment of nonresonant contributions, etc. It is possible the input chosen is imprecise and can lead to a momentum-dependent value for γ. In this sense, the requirement that γ be momentum independent may provide some hints regarding the form of the decay amplitudes.

In the $\pi\pi$-symmetric case, we have shown above that there are no exact SU(3) EWP-tree relations for $|S_{\pi\pi}\rangle$. However, the crossed EWP-tree relations [Eq. (39)] do hold. By applying these to the effective diagrams of Eq. (4) we have

$$P'_{\mathrm{EW},a} \approx -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\frac{c_9/N_c + c_{10}}{c_1 + c_2/N_c}T'_a. \tag{41}$$

Once again, the number of effective diagrams is reduced to five, which makes the extraction of γ possible. One can even use both the exact and crossed EWP-tree relations, in which case fewer observables are needed to obtain γ.

In both cases, the theoretical error is $O(5\%)$. The advantage of using $|S_{\pi\pi}\rangle$ rather than $|S\rangle$ is that the number of events is somewhat larger.

V. CONCLUSIONS

It has been known for some time that there are relations between the EWP and the tree contributions to $B \to K\pi$ decays. In particular, apart from the weak phases, the diagrams P'_{EW} and P'^{C}_{EW} are proportional to T' and C', respectively, to a good approximation. In 2003, Deshpande, Sinha, and Sinha attempted to use these EWP-tree relations to extract γ from $B \to K\pi\pi$ decays. Working with the $\pi\pi$-symmetric $K\pi\pi$ states ($|S_{\pi\pi}\rangle$), they noted that $B^+ \to K^0\pi^+\pi^0$ receives only tree and EWP contributions. DSS’s assumption was that these are related as in $B \to K\pi$, and this additional information allowed γ to be obtained. Unfortunately, it was subsequently shown that the EWP-tree relation does not hold, so that γ cannot be extracted using DSS’s method.

In this paper, we revisit the question of measuring γ in $B \to K\pi\pi$ decays, and we show that, in fact, it is possible. We first define the diagrams contributing to $B \to K\pi\pi$. Because there are three particles in the final state, there are two types of each diagram. We call them T'_1, T'_2, $P'_{\mathrm{EW}1}$, $P'_{\mathrm{EW}2}$, etc. We then express each $B \to K\pi\pi$ amplitude in terms of these diagrams. DSS’s assumption is that $P'_{\mathrm{EW}2} + P'^{C}_{\mathrm{EW}1}$ is proportional to $T'_1 + C'_2$. Using the contractions formalism, we are able to express all diagrams in terms of contractions and thereby show that there are, in fact, EWP-tree relations. To be specific, we find that $P'_{\mathrm{EW}i} \propto T'_i$ $(i = 1,\ 2)$ and $P'^{C}_{\mathrm{EW}i} \propto C'_i$ $(i = 1,\ 2)$. From this, we see immediately that the DSS assumption is indeed false.

Now, when one writes the amplitudes in terms of diagrams, one sees that there are more unknown theoretical parameters than there are observables, so that weak-phase information cannot be obtained without additional input. The EWP-tree relations provide this input and allow γ to be measured in $B \to K\pi\pi$ decays. But there is a complication. EWP-tree relations require flavor SU(3) symmetry. Since K and π are equivalent under this symmetry, one has to deal with three identical particles in the $K\pi\pi$ final states. The permutation group of three objects is S_3, which has as eigenstates a totally symmetric state of the three objects, a totally antisymmetric state, and four mixed states. However, since the relative angular momentum between the particles is not fixed (due to the fact that we have a three-particle final state), the state is not necessarily symmetric under permutations of the mesons. On the other hand, it turns out that the EWP-tree relations apply only to the totally symmetric state ($|S\rangle$). Thus, if one wants to apply these relations, one must isolate those events which correspond to the state $|S\rangle$.

The state $|S_{\pi\pi}\rangle$ is a combination of $|S\rangle$ and one of the S_3 mixed states. As such, the above EWP-tree relations do not apply to it. Fortunately, there is an alternative. If one works to LO in α_s, we find crossed EWP-tree relations: $P'^{C}_{\mathrm{EW}i} \propto T'_i$ ($i = 1, 2$) and $P'_{\mathrm{EW}i} \propto C'_i$ ($i = 1, 2$). We expect these to hold approximately, since $\alpha_s(m_b) \sim 0.2$. The crossed EWP-tree relations can be used with $|S_{\pi\pi}\rangle$. They are valid under isospin symmetry—SU(3) is not used.

In both cases, we estimate the theoretical error to be $O(5\%)$. Experimentally, one can choose to use either state. The advantage of $|S\rangle$ over $|S_{\pi\pi}\rangle$ is that the EWP-tree relations are exact, as opposed to LO. On the other hand, the advantage of $|S_{\pi\pi}\rangle$ over $|S\rangle$ is that the number of events is somewhat larger.

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[1] C. Amsler *et al.* (Particle Data Group), Phys. Lett. B **667**, 1 (2008).
[2] Y. Nir and H.R. Quinn, Phys. Rev. Lett. **67**, 541 (1991).
[3] N.G. Deshpande and X.G. He, Phys. Rev. Lett. **74**, 26 (1995); **74**, 4099(E) (1995).
[4] M. Neubert and J.L. Rosner, Phys. Lett. B **441**, 403 (1998); **441**, 403 (1998).
[5] M. Gronau, D. Pirjol, and T.M. Yan, Phys. Rev. D **60**, 034021 (1999); **69**, 119901(E) (2004).
[6] M. Imbeault, A.L. Lemerle, V. Page, and D. London, Phys. Rev. Lett. **92**, 081801 (2004).
[7] H.J. Lipkin, Y. Nir, H.R. Quinn, and A. Snyder, Phys. Rev. D **44**, 1454 (1991).
[8] N.G. Deshpande, N. Sinha, and R. Sinha, Phys. Rev. Lett. **90**, 061802 (2003).
[9] M. Gronau, Phys. Rev. Lett. **91**, 139101 (2003).
[10] N. Rey-Le Lorier, M. Imbeault, and D. London, preceding paper, Phys. Rev. D **84**, 034040 (2011).
[11] M. Gronau, O.F. Hernandez, D. London, and J.L. Rosner, Phys. Rev. D **50**, 4529 (1994); **52**, 6374 (1995).
[12] G. Buchalla, A.J. Buras, and M.E. Lautenbacher, Rev. Mod. Phys. **68**, 1125 (1996).
[13] M. Imbeault, A. Datta, and D. London, Int. J. Mod. Phys. A **22**, 2057 (2007).
[14] However, for a discussion of $B \to K\pi\pi$ decays with SU(3) symmetry, see M. Ciuchini, M. Pierini, and L. Silvestrini, Phys. Rev. D **74**, 051301 (2006); Phys. Lett. B **645**, 201 (2007); M. Gronau, D. Pirjol, A. Soni, and J. Zupan, Phys. Rev. D **75**, 014002 (2007).
[15] A.J. Buras and L. Silvestrini, Nucl. Phys. **B569**, 3 (2000).
[16] M. Gronau and J.L. Rosner, Phys. Rev. D **72**, 094031 (2005).
[17] M. Beneke, G. Buchalla, M. Neubert, and C.T. Sachrajda, Phys. Rev. Lett. **83**, 1914 (1999); Nucl. Phys. **B591**, 313 (2000); **B606**, 245 (2001).
[18] Y.Y. Keum, H.n. Li, and A.I. Sanda, Phys. Lett. B **504**, 6 (2001); Phys. Rev. D **63**, 054008 (2001).
[19] C.W. Bauer, S. Fleming, D. Pirjol, and I.W. Stewart, Phys. Rev. D **63**, 114020 (2001).
[20] C.H. Chen and H.n. Li, Phys. Lett. B **561**, 258 (2003); C.H. Chen, Phys. Lett. B **560**, 178 (2003).