Diagrammatic analysis of charmless three-body B decays

Nicolas Rey-Le Lorier,[1,*] Maxime Imbeault,[2,†] and David London[1,‡]

[1]*Physique des Particules, Université de Montréal, Case postale 6128, succursale centre-ville, Montréal, Quebec, Canada H3C 3J7*
[2]*Département de Physique, Cégep de Baie-Comeau, 537 boulevard Blanche, Baie-Comeau, Quebec, Canada G5C 2B2*

We express the amplitudes for charmless three-body B decays in terms of diagrams. In addition, we show how to use Dalitz-plot analyses to obtain decay amplitudes which are symmetric or antisymmetric under the exchange of two of the final-state particles. When annihilation/exchange-type diagrams are neglected, as in two-body decays, many of the exact, purely isospin-based results are modified, leading to new tests of the standard model. Some of the tests can be performed now, and we find that present data agree with the predictions of the standard model. Furthermore, contrary to what was thought previously, it is possible to cleanly extract weak-phase information from three-body decays, and we discuss methods for $B \to K\pi\pi$, $KK\bar{K}$, $K\bar{K}\pi$, and $\pi\pi\pi$.

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I. INTRODUCTION

The B factories *BABAR* and Belle ran for over ten years, and made an enormous number of measurements of observables in B decays. For the most part, these decays were of the form $B \to M_1 M_2$ (M_i is a meson), as these are most accessible experimentally. Nevertheless, there have still been some probes of three-body $B \to M_1 M_2 M_3$ decays. To be specific, experiments have obtained Dalitz plots for many of the decay modes in $B \to K\pi\pi$, $KK\bar{K}$, $K\bar{K}\pi$, $\pi\pi\pi$, and made measurements of (or obtained upper limits on) the branching ratios and indirect (mixing-induced) CP asymmetries of a number of these decays [1].

Things are similar on the theory side. The vast majority of theoretical analyses involve two-body B decays. This is in part due to the relative angular momentum of the final-state particles. For example, consider $B_d^0 \to \pi^+ \pi^-$. Because there are two particles in the final state, it has a fixed value of l (in this case $l = 0$), and so $\pi^+ \pi^-$ is a CP eigenstate. On the other hand, in the decay $B_d^0 \to K_S \pi^+ \pi^-$, the $\pi^+ \pi^-$ can have even or odd relative angular momentum, so that $K_S \pi^+ \pi^-$ is not a CP eigenstate. This makes it much more difficult to find clean predictions of the standard model (SM) to compare with experimental measurements. This is a general property of three-body decays.

Still, there have been some theoretical analyses of CP-conserving observables in three-body $B \to K\pi\pi$, $KK\bar{K}$ decays [2–5]. In general, these studies examined the isospin decomposition of the decay amplitudes, and the symmetry relations among them. The analyses were carried out using isospin amplitudes.

In this paper, we examine the amplitudes of the three-body charmless decays $B \to K\pi\pi$, $KK\bar{K}$, $K\bar{K}\pi$, and $\pi\pi\pi$

using diagrams. In addition, using Dalitz-plot analyses of such decays, we show how to separate the amplitudes into pieces which are symmetric or antisymmetric under the exchange of two of the final-state particles. This is useful for any decay that contains particles that are identical under isospin. Now, as has been shown in Ref. [6], the amplitudes for two-body B decays can be expressed in terms of 9 diagrams. However, 3 of these—the annihilation/exchange-type diagrams—are expected to be quite a bit smaller than the dominant diagrams, and can be neglected, to a good approximation. This same procedure can be applied to three-body decays.

The point of this is as follows. When one neglects annihilation/exchange-type diagrams, new features appear. A given set of three-body decays (e.g. $B \to K\pi\pi$) contains a number of different transitions (e.g. $B^+ \to K^+ \pi^+ \pi^-$, $B_d^0 \to K^+ \pi^0 \pi^-$, etc.). There are exact relations among the symmetric or antisymmetric amplitudes for these specific decays. However, when one neglects certain diagrams, these relations can be modified, and this can lead to new effects. For example, some linear combinations of the isospin amplitudes vanish for certain decays. Also, there are additional tests of the SM. In some cases, it is even possible to obtain clean information about the CP-violating phases.

In Sec. II, we present the diagrams describing $B \to M_1 M_2 M_3$ processes. We review Dalitz-plot analyses of three-body decays in Sec. III, and show how to obtain amplitudes which are symmetric or antisymmetric under the exchange of two of the final-state particles. The decays $B \to K\pi\pi$, $B \to KK\bar{K}$, $B \to K\bar{K}\pi$, and $B \to \pi\pi\pi$ are discussed in Secs. IV, V, VI, and VII, respectively. In all cases, we give the expressions for the decay amplitudes in terms of diagrams, and examine the prospects for the clean extraction of weak-phase information. Other subjects related to the particular decays are also discussed: resonances and penguin dominance in $B \to K\pi\pi$ (Sec. IV), penguin dominance and isospin amplitudes in $B \to KK\bar{K}$

*nicolas.rey-le.lorier@umontreal.ca
†imbeault.maxime@gmail.com
‡london@lps.umontreal.ca

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(Sec. V), T dominance in $B \to K\bar{K}\pi$ (Sec. VI), and Dalitz plots in $B \to \pi\pi\pi$ (Sec. VII). We conclude in Sec. VIII.

II. DIAGRAMS

It has been shown in Ref. [6] that the amplitudes for two-body B decays can be expressed in terms of 9 diagrams: the color-favored and color-suppressed tree amplitudes T and C, the gluonic-penguin amplitudes P_{tc} and P_{uc}, the color-favored and color-suppressed electroweak-penguin (EWP) amplitudes P_{EW} and P_{EW}^C, the annihilation amplitude A, the exchange amplitude E, and the penguin-annihilation amplitude PA. These last three all involve the interaction of the spectator quark and are expected to be much smaller than the other diagrams. It is standard to neglect them.

At this point it is useful to revisit the expected size of these diagrams. In Ref. [6] it was noted that, when the spectator quark is involved, there is an additional suppression factor f_B/m_B, where $f_B \simeq 200$ MeV is the B-meson decay constant. This is $O(\bar{\lambda}^2)$, where $\bar{\lambda} \simeq 0.2$. There is an additional factor of $\bar{\lambda}$ due to color suppression, so that $|A/T|$, $|E/T|$, and $|PA/P|$ are all estimated to be $O(\bar{\lambda}^3)$. However, a few years later it was shown in Ref. [7] that rescattering effects can be important. One decay examined in detail was $B_d^0 \to D_s^- K^+$, which proceeds through the W-exchange diagram. The contributing Cabibbo-Kobayashi-Maskawa (CKM) matrix elements are $V_{cb}^* V_{ud}$. Now, $B_d^0 \to D_s^- K^+$ also receives a contribution due to the rescattering of the T decay $B_d^0 \to D^- \pi^+$, which is nonsuppressed and governed by the same CKM matrix elements. It was shown that this contribution is roughly $1/\bar{\lambda}$ larger than that due to E, so that rescattering is indeed important.

Now, the key point here is that *all* E transitions potentially have associated with them a T rescattering contribution. It is therefore reasonable to combine them into an effective E diagram:

$$E_{\text{eff}} = E_{\text{quark}} + E_{\text{rescatt}}, \qquad (1)$$

in which $|E_{\text{quark}}/T| = O(\bar{\lambda}^3)$, but $|E_{\text{rescatt}}/T|$ can be larger. The A and PA transitions can be redefined similarly (though the rescattering diagram for PA is P). We therefore see that the annihilation/exchange-type decays can be larger than previously thought. This is qualitatively consistent with the QCD-based theoretical approaches (QCD factorization [8], perturbative QCD [9], soft collinear effective theory [10]), in which annihilation/exchange-type transitions can be important.

The branching ratio for $B_d^0 \to D_s^- K^+$ has been measured to be 1.99×10^{-5} [11], to be compared with the branching ratio for $B_d^0 \to D^- \pi^+$: 2.7×10^{-3} [12]. From these, we deduce that $|E_{\text{rescatt}}^{B_d^0 \to D_s K}/T|$, and hence $|E_{\text{eff}}^{B_d^0 \to D_s K}/T|$, is $\simeq 0.09$. The decay $B_s^0 \to \pi^+ \pi^-$, which was recently observed by the CDF Collaboration [13] with a branching ratio of 0.6×10^{-6}, has a similar behavior. Here the rescattering comes from $B_s^0 \to K^+ K^-$, which is dominated by P (so that $B_s^0 \to \pi^+ \pi^-$ is mainly PA), and whose branching ratio is 26.5×10^{-6}. This implies that $|PA_{\text{rescatt}}^{B_s^0 \to \pi\pi}/P|$, and hence $|PA_{\text{eff}}^{B_s^0 \to \pi\pi}/P|$, is $\simeq 0.15$. Given that $|PA_{\text{quark}}/P| = O(\bar{\lambda}^3)$, this is a significant enhancement.

However, it is not clear that all $E/A/PA$ diagrams are similarly increased in size. The above decays are both pure E or PA transitions and the rescattering inelastic. But what about a decay like $B_s^0 \to D_s^- K^+$? There is an $E_{\text{quark}}^{B_s^0 \to D_s K}$ contribution: $|E_{\text{quark}}^{B_s^0 \to D_s K}/T| = O(\bar{\lambda}^3)$. But now the T decays directly to the same final state, so that the rescattering contribution to E is elastic. This has not been calculated, so it is not certain that the enhancement is as large as in the inelastic case. Most of the decays discussed in this paper are of this form, so the actual size of the annihilation/exchange-type contributions is uncertain. It might be as large as above, e.g. $|E_{\text{eff}}/T| \simeq 0.1$, but it might also be smaller.

For the three-body decays considered in this paper, we adopt a similar procedure to that used in two-body decays. That is, we neglect all annihilation/exchange-type diagrams (whose size is estimated to be $\lesssim 0.1|T|$), and express all amplitudes in terms of tree, penguin, and EWP diagrams. We assume isospin invariance, but not flavor SU(3) symmetry. [It is straightforward to modify our analysis by imposing SU(3).] The diagrams are shown in Fig. 1. A few words of explanation are in order. These diagrams are for the decay $B \to \pi\pi\pi$. There are changes of notation for the other decays:

(i) For $\bar{b} \to \bar{d}$ transitions ($B \to K\bar{K}\pi$, $\pi\pi\pi$), the diagrams are written without primes; for $\bar{b} \to \bar{s}$ transitions ($B \to K\pi\pi$, $KK\bar{K}$), they are written with primes.

(ii) In all diagrams, it is necessary to "pop" a quark pair from the vacuum. It is assumed that this pair is $u\bar{u}$ or $d\bar{d}$ ($\equiv q\bar{q}$); if the popped pair is $s\bar{s}$, the diagram is written with an additional subscript s. Thus, for $B \to K\bar{K}\pi$, $KK\bar{K}$, in the penguin or EWP diagrams with a popped $q\bar{q}$ pair, the virtual particle decays to $s\bar{s}$; if the popped quark pair is $s\bar{s}$ (so that the diagram is written with an additional subscript s), the virtual particle decays to $q\bar{q}$.

(iii) The subscript 1 indicates that the popped quark pair is between two (nonspectator) final-state quarks; the subscript 2 indicates that the popped quark pair is between two final-state quarks including the spectator.

In principle, one can also include the gluonic-penguin diagrams in which the popped quark pair is between the pair of quarks produced by the gluon. This corresponds to the case where the virtual spin-1 gluon decays to two spin-0 mesons (with relative angular momentum $l = 1$). In order to account for the color imbalance, additional gluons
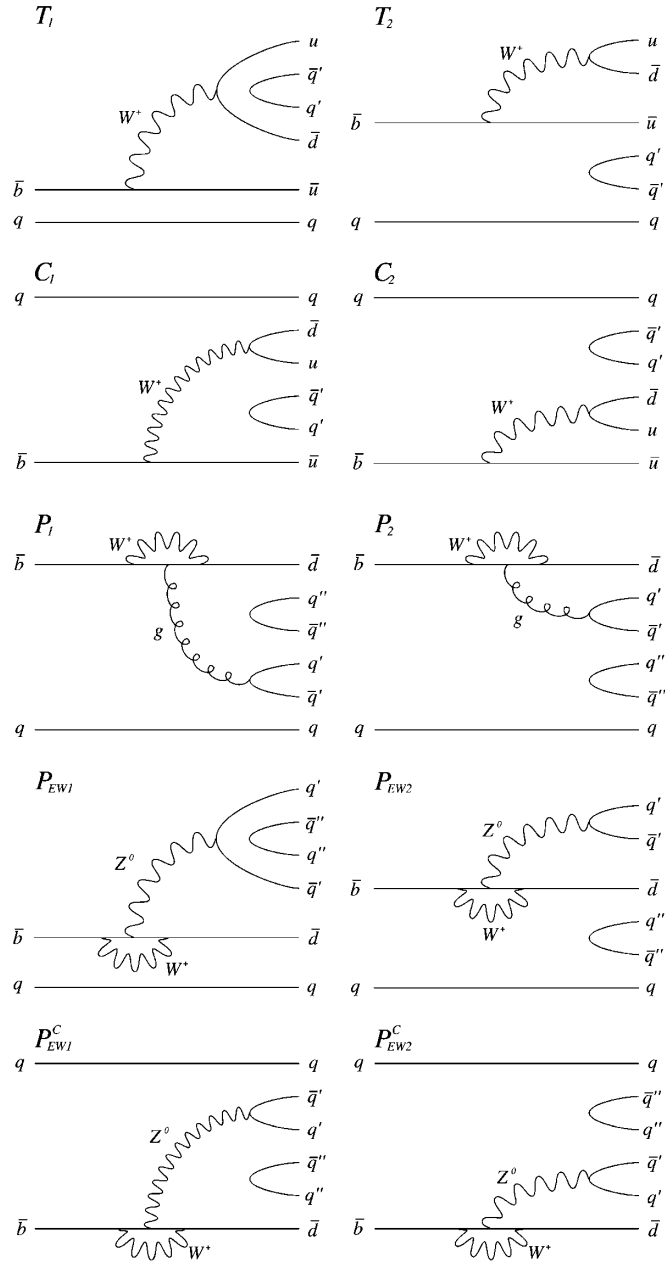
FIG. 1.   Diagrams contributing to $B \to \pi\pi\pi$.

must be exchanged. Although this can take place at low energy, it will still suppress these diagrams somewhat, and so we do not include them here. (Note: their inclusion does not change any of our conclusions.)

One important difference compared to two-body B-decay diagrams is momentum dependence. In two-body decays, in the rest frame of the B, the three-momenta of the final-state particles are equal and opposite. One does not have the same type of behavior in three-body decays. Although the sum of the three-momenta of the final particles is zero, there is no constraint on any individual particle. As such, the three-body diagrams are momentum dependent, and this must be taken into account whenever the diagrams are used.

III. DALITZ PLOTS

In this section, we review certain aspects of the Dalitz-plot analysis. To illustrate these, we focus on the decay $B^+ \to K^+ \pi^- \pi^+$ [14]. However, a similar type of analysis can be applied to any three-body B decay.

$B^+ \to K^+ \pi^- \pi^+$ can take place via intermediate resonances, as well as nonresonant decays. The events in the Dalitz plot are therefore described by the following two variables:

$$
\begin{aligned}
x &= m^2_{K^+\pi^-} = (p_{K^+} + p_{\pi^-})^2, \\
y &= m^2_{\pi^+\pi^-} = (p_{\pi^+} + p_{\pi^-})^2.
\end{aligned}
\tag{2}
$$

Now, one of the great advantages of a Dalitz-plot analysis is that it allows one to extract the full amplitude of the decay. To this end, we write

$$
\mathcal{M}(B^+ \to K^+ \pi^- \pi^+) = \sum_j c_j e^{i\theta_j} F_j(x, y),
\tag{3}
$$

where the sum is over all decay modes (resonant and nonresonant). c_j and θ_j are the magnitude and phase of the j contribution, respectively, measured relative to one of the contributing channels. The distributions F_j, which depend on x and y, describe the dynamics of the individual decay amplitudes. In the experimental analyses, these take different (known) forms for the various contributions. The key point is that a maximum likelihood fit over the entire Dalitz plot gives the best values of the c_j and θ_j. Thus, the decay amplitude can be obtained.

In this paper, the following issue is of central importance. In $B^+ \to K^+ \pi^- \pi^+$, since the π's are identical particles under isospin, the overall $\pi^- \pi^+$ wave function must be symmetric. If the $\pi\pi$ pair is in a state of even (odd) isospin, the wave function (or, equivalently, the $B^+ \to K^+ \pi^- \pi^+$ decay amplitude) must be symmetric (antisymmetric) under the exchange $p_{\pi^+} \leftrightarrow p_{\pi^-}$. Unfortunately, the amplitude of Eq. (3) does not possess such a symmetry.

It is the use of the parameters x and y which is problematic. A better choice of variables would be s_+ and s_-, where

$$
\begin{aligned}
s_+ &= m^2_{K^+\pi^+} = (p_{K^+} + p_{\pi^+})^2, \\
x = s_- &= m^2_{K^+\pi^-} = (p_{K^+} + p_{\pi^-})^2.
\end{aligned}
\tag{4}
$$

Now, under the exchange $p_{\pi^+} \leftrightarrow p_{\pi^-}$, we simply have $s_+ \leftrightarrow s_-$. Thus, if we had started with the amplitude $\mathcal{M}(B^+ \to K^+ \pi^- \pi^+) = g(s_+, s_-)$, the symmetric combination would be $\frac{1}{\sqrt{2}}[g(s_+, s_-) + g(s_-, s_+)]$, i.e. it would correspond to the production of the $\pi^- \pi^+$ pair with a symmetric wave function; $\frac{1}{\sqrt{2}}[g(s_+, s_-) - g(s_-, s_+)]$ would be antisymmetric.

The problem is that the wave function of Eq. (3) is not given in terms of s_+ and s_-. Fortunately, there is a

resolution to this problem: the independent Mandelstam variables y, s_+, and s_- satisfy

$$y = m_B^2 + 2m_\pi^2 + m_{K^+}^2 - s_+ - s_-. \quad (5)$$

This implies that $f(x, y) = f(s_-, y) = f(s_-, m_B^2 + 2m_\pi^2 + m_{K^+}^2 - s_+ - s_-) \equiv g(s_+, s_-)$. Given the decay amplitude $\mathcal{M}(x, y)$ of Eq. (3), one can therefore easily construct the amplitude which is symmetric/antisymmetric in $p_{\pi^+} \leftrightarrow p_{\pi^-}$. The same method applies to other $B \to K\pi\pi$ decays, and indeed to all three-body decays. Thus, if there are identical particles in the final state, the B-decay Dalitz plot allows us to construct the amplitude for the production of these particles in a symmetric/antisymmetric state.

Above, we argued that the Dalitz-plot analysis allows one to obtain the amplitude $\mathcal{M}$ of any three-body B decay. Actually, this is not quite accurate—the global phase of the amplitude is undetermined. Thus, it is really $|\mathcal{M}|$ that should be compared with theory. Similarly, one can obtain $|\bar{\mathcal{M}}|$ of the CP-conjugate decay. In the rest of the paper, we refer to the momentum-dependent branching ratio and direct CP asymmetry of a particular decay. These are proportional to $|\mathcal{M}|^2 + |\bar{\mathcal{M}}|^2$ and $|\mathcal{M}|^2 - |\bar{\mathcal{M}}|^2$, respectively. Finally, for a self-conjugate final state such as $K^0\pi^+\pi^-$ (where the K^0 is seen as K_S), the momentum-dependent indirect CP asymmetry[1] can be measured, and gives $\mathcal{M}^*\bar{\mathcal{M}}$ for this decay.

IV. $B \to K\pi\pi$ DECAYS

We begin with $B \to K\pi\pi$ decays, a $\bar{b} \to \bar{s}$ transition. There are six processes: $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^+\pi^0\pi^0$,

$B^+ \to K^0\pi^+\pi^0$, $B_d^0 \to K^+\pi^0\pi^-$, $B_d^0 \to K^0\pi^+\pi^-$, and $B_d^0 \to K^0\pi^0\pi^0$. In all of these, the overall wave function of the final $\pi\pi$ pair must be symmetrized with respect to the exchange of these two particles. There are two possibilities. If the relative angular momentum is even (odd), the isospin state must be symmetric (antisymmetric). We refer to these two cases as $I_{\pi\pi}^{\text{sym}}$ and $I_{\pi\pi}^{\text{anti}}$. As shown in Sec. III, they can be determined experimentally. We discuss them in turn.

We first consider $I_{\pi\pi}^{\text{sym}}$, i.e. $I = (0, 2)$. The final state has $I = \frac{1}{2}$, $\frac{3}{2}$, or $\frac{5}{2}$. The B meson has $I = \frac{1}{2}$ and the weak Hamiltonian has $\Delta I = 0$ or 1. The final state with $I = \frac{5}{2}$ cannot be reached. So there are three different ways of getting to the final state. Given that there are six decays, this means that there should be three relations among their amplitudes. This conclusion is an exact result; the relations can be found by applying the Wigner-Eckart theorem:

$$A(B^+ \to K^0\pi^+\pi^0)_{\text{sym}} = -A(B_d^0 \to K^+\pi^0\pi^-)_{\text{sym}},$$

$$\sqrt{2}A(B^+ \to K^0\pi^+\pi^0)_{\text{sym}} = A(B_d^0 \to K^0\pi^+\pi^-)_{\text{sym}}$$
$$+ \sqrt{2}A(B_d^0 \to K^0\pi^0\pi^0)_{\text{sym}},$$

$$\sqrt{2}A(B_d^0 \to K^+\pi^0\pi^-)_{\text{sym}} = A(B^+ \to K^+\pi^+\pi^-)_{\text{sym}}$$
$$+ \sqrt{2}A(B^+ \to K^+\pi^0\pi^0)_{\text{sym}}. \quad (6)$$

These relations were first given (implicitly) in Ref. [5]. The subscript "sym" indicates that the $\pi\pi$ isospin state is symmetrized.

In terms of diagrams, the amplitudes are given by

$$\sqrt{2}A(B^+ \to K^0\pi^+\pi^0)_{\text{sym}} = -T_1'e^{i\gamma} - C_2'e^{i\gamma} + P_{\text{EW2}}' + P_{\text{EW1}}'^C,$$

$$A(B_d^0 \to K^0\pi^+\pi^-)_{\text{sym}} = -T_1'e^{i\gamma} - C_1'e^{i\gamma} - \tilde{P}_{uc}'e^{i\gamma} + \tilde{P}_{tc}' + \frac{1}{3}P_{\text{EW1}}' + \frac{2}{3}P_{\text{EW1}}'^C - \frac{1}{3}P_{\text{EW2}}'^C,$$

$$\sqrt{2}A(B_d^0 \to K^0\pi^0\pi^0)_{\text{sym}} = C_1'e^{i\gamma} - C_2'e^{i\gamma} + \tilde{P}_{uc}'e^{i\gamma} - \tilde{P}_{tc}' - \frac{1}{3}P_{\text{EW1}}' + P_{\text{EW2}}' + \frac{1}{3}P_{\text{EW1}}'^C + \frac{1}{3}P_{\text{EW2}}'^C,$$

$$A(B^+ \to K^+\pi^+\pi^-)_{\text{sym}} = -T_2'e^{i\gamma} - C_1'e^{i\gamma} - \tilde{P}_{uc}'e^{i\gamma} + \tilde{P}_{tc}' + \frac{1}{3}P_{\text{EW1}}' - \frac{1}{3}P_{\text{EW1}}'^C + \frac{2}{3}P_{\text{EW2}}'^C, \quad (7)$$

$$\sqrt{2}A(B^+ \to K^+\pi^0\pi^0)_{\text{sym}} = T_1'e^{i\gamma} + T_2'e^{i\gamma} + C_1'e^{i\gamma} + C_2'e^{i\gamma} + \tilde{P}_{uc}'e^{i\gamma} - \tilde{P}_{tc}' - \frac{1}{3}P_{\text{EW1}}' - P_{\text{EW2}}' - \frac{2}{3}P_{\text{EW1}}'^C - \frac{2}{3}P_{\text{EW2}}'^C,$$

$$\sqrt{2}A(B_d^0 \to K^+\pi^0\pi^-)_{\text{sym}} = T_1'e^{i\gamma} + C_2'e^{i\gamma} - P_{\text{EW2}}' - P_{\text{EW1}}'^C,$$

where $\tilde{P}' \equiv P_1' + P_2'$. (Note: all amplitudes have been multiplied by $\sqrt{2}$.) Above we have explicitly written the weak-phase dependence [including the minus sign from $V_{tb}^*V_{ts}$ ($\tilde{P}_{tc}'$ and EWP's)], while the diagrams contain strong phases. [The phase information in the Cabibbo-Kobayashi-Maskawa quark mixing matrix is conventionally

parametrized in terms of the unitarity triangle, in which the interior (CP-violating) angles are known as α, β, and γ [12].] It is straightforward to verify that the three relations of Eq. (6) are reproduced. Thus, in this case, there is no difference between the exact and diagrammatic amplitude relations.

We now turn to $I_{\pi\pi}^{\text{anti}}$, i.e. $I = 1$. Here there are four processes: $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^0\pi^+\pi^0$, $B_d^0 \to K^+\pi^0\pi^-$, and $B_d^0 \to K^0\pi^+\pi^-$ (one cannot antisymmetrize a $\pi^0\pi^0$ state). The final state has $I = \frac{1}{2}$ or $\frac{3}{2}$, so there are still three different paths to get to the final state. We

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[1]The indirect CP asymmetry depends on the CP of the final state, and *a priori* $K^0\pi^+\pi^-$ is a mixture of $CP+$ and $CP-$. However, the separation of symmetric and antisymmetric $\pi\pi$ states also fixes the final-state CP: $K^0(\pi\pi)_{\text{sym}}$ and $K^0(\pi\pi)_{\text{anti}}$ have $CP+$ and $-$, respectively.

therefore expect one relation among the four amplitudes. Reference [5] notes that it is similar to that in $B \to \pi K$:

$$\sqrt{2}A(B^+ \to K^+\pi^+\pi^-)_{\text{anti}} + A(B^+ \to K^0\pi^+\pi^0)_{\text{anti}}$$
$$= \sqrt{2}A(B^0_d \to K^0\pi^+\pi^-)_{\text{anti}} + A(B^0_d \to K^+\pi^0\pi^-)_{\text{anti}},$$
$$(8)$$

where the subscript "anti" indicates that the $\pi\pi$ isospin state is antisymmetrized.

Writing the amplitudes in terms of diagrams is a bit more complicated because antisymmetrization is involved. Depending on the order of the pions, there might be an extra minus sign. To account for this, we use the following prescription:

(i) All diagrams with the pions in order of decreasing charge from top to bottom are unmodified; all diagrams with the pions in order of increasing charge from top to bottom get an additional factor of -1.

This requires that diagrams always be drawn the same way. For example, the spectator quark for all tree diagrams should always appear in the same place (e.g. at the bottom of the diagram), and the decay products of the neutral bosons in penguin and EWP diagrams should always appear in the same order (e.g. quark on top, antiquark on bottom).

With this rule, the amplitudes take the form[2]

$$\sqrt{2}A(B^+ \to K^0\pi^+\pi^0)_{\text{anti}} = -T'_1 e^{i\gamma} - C'_2 e^{i\gamma} - 2\tilde{P}'_{uc} e^{i\gamma} + 2\tilde{P}'_{tc} - P'_{\text{EW2}} - \tfrac{1}{3}P'^C_{\text{EW1}} + \tfrac{2}{3}P'^C_{\text{EW2}},$$
$$A(B^0_d \to K^0\pi^+\pi^-)_{\text{anti}} = -T'_1 e^{i\gamma} - C'_1 e^{i\gamma} - \tilde{P}'_{uc} e^{i\gamma} + \tilde{P}'_{tc} + P'_{\text{EW1}} - \tfrac{2}{3}P'^C_{\text{EW1}} + \tfrac{1}{3}P'^C_{\text{EW2}},$$
$$A(B^+ \to K^+\pi^+\pi^-)_{\text{anti}} = T'_2 e^{i\gamma} - C'_1 e^{i\gamma} + \tilde{P}'_{uc} e^{i\gamma} - \tilde{P}'_{tc} + P'_{\text{EW1}} - \tfrac{1}{3}P'^C_{\text{EW1}} + \tfrac{2}{3}P'^C_{\text{EW2}},$$
$$\sqrt{2}A(B^0_d \to K^+\pi^0\pi^-)_{\text{anti}} = T'_1 e^{i\gamma} + 2T'_2 e^{i\gamma} - C'_2 e^{i\gamma} + 2\tilde{P}'_{uc} e^{i\gamma} - 2\tilde{P}'_{tc} - P'_{\text{EW2}} + \tfrac{1}{3}P'^C_{\text{EW1}} + \tfrac{4}{3}P'^C_{\text{EW2}}.$$
$$(9)$$

(As above, all amplitudes have been multiplied by $\sqrt{2}$.) The relation of Eq. (8) is reproduced. Therefore, there is no difference between the exact and diagrammatic amplitude relations in the antisymmetric case.

A. Resonances

It is possible that the B decays to an intermediate on-shell M_1M_2 state, which then subsequently decays to $K\pi\pi$. Examples of such resonances are $M_1M_2 = K\rho$, $K^*\pi$, and $Kf_0(980)$. The question now is how does the diagrammatic analysis presented above jibe with resonant decays? To answer this, we examine the resonances in turn.

Consider first $M_1M_2 = K\rho$. The four decays are $B^+ \to K^+\rho^0$, $B^+ \to K^0\rho^+$, $B^0_d \to K^0\rho^0$, and $B^0_d \to K^+\rho^-$, whose amplitudes take the form

$$\sqrt{2}A(B^+ \to K^+\rho^0) = -T'_V e^{i\gamma} - C'_P e^{i\gamma} - P'_{uc,V} e^{i\gamma} + P'_{tc,V} + P'_{\text{EW},P} + \tfrac{2}{3}P'^C_{\text{EW},V},$$
$$A(B^+ \to K^0\rho^+) = P'_{uc,V} e^{i\gamma} - P'_{tc,V} + \tfrac{1}{3}P'^C_{\text{EW},V},$$
$$\sqrt{2}A(B^0_d \to K^0\rho^0) = -C'_P e^{i\gamma} + P'_{uc,V} e^{i\gamma} - P'_{tc,V} + P'_{\text{EW},P} + \tfrac{1}{3}P'^C_{\text{EW},V},$$
$$A(B^0_d \to K^+\rho^-) = -T'_V e^{i\gamma} - P'_{uc,V} e^{i\gamma} + P'_{tc,V} + \tfrac{2}{3}P'^C_{\text{EW},V},$$
$$(10)$$

where the subscripts P or V indicate which final-state meson [pseudoscalar (K) or vector (ρ)] contains the spectator quark of the B meson [15]. [Note that the diagrams that describe resonant decays are a subset of those used for $B \to K\pi\pi$ (Fig. 1). Above, the diagram D_V (D_P) is the same as D_2 (D_1).] The relation among the amplitudes is

$$\sqrt{2}A(B^+ \to K^+\rho^0) + A(B^+ \to K^0\rho^+)$$
$$= \sqrt{2}A(B^0_d \to K^0\rho^0) + A(B^0_d \to K^+\rho^-). \quad (11)$$

Given that $\rho^0 \to \pi^+\pi^-$, $\rho^+ \to \pi^+\pi^0$, and $\rho^- \to \pi^0\pi^-$, this reproduces Eq. (8), which is the relation for the antisymmetric $\pi\pi$ isospin state. This makes sense, since the ρ decays to $(\pi\pi)_{\text{anti}}$.

Consider now $M_1M_2 = Kf_0(980)$. There are two decays: $B^+ \to K^+f_0(980)$ and $B^0_d \to K^0f_0(980)$. It is straightforward to show that there is no relation between the two amplitudes. However, the $f_0(980)$ decays to a pion pair in a symmetric isospin state, with $A(f_0 \to (\pi^+\pi^-)_{\text{sym}}) = -\sqrt{2}A(f_0 \to \pi^0\pi^0)$. This leads to

$$A(B^0_d \to K^0\pi^+\pi^-) + \sqrt{2}A(B^0_d \to K^0\pi^0\pi^0) = 0,$$
$$A(B^+ \to K^+\pi^+\pi^-) + \sqrt{2}A(B^+ \to K^+\pi^0\pi^0) = 0.$$
$$(12)$$

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[2]Note: even though the diagrams of Eq. (9) have the same names as those of Eq. (7), they are not the same diagrams. That is, in general, they take different values.

Given that the $Kf_0(980)$ resonance does not contribute to $A(B^+ \to K^0\pi^+\pi^0)$, $A(B_d^0 \to K^+\pi^0\pi^-)$, or $A(B^+ \to K^0\pi^+\pi^0)$, the decays $B \to Kf_0(980) \to K\pi\pi$ satisfy Eq. (6), which are the relations for the symmetric $\pi\pi$ isospin state.

Finally, consider $M_1M_2 = K^*\pi$. The four decays are $B^+ \to K^{*0}\pi^+$, $B^+ \to K^{*+}\pi^0$, $B_d^0 \to K^{*+}\pi^-$, and $B_d^0 \to K^{*0}\pi^0$. The amplitudes are [15]

$$A(B^+ \to K^{*0}\pi^+) = P'_{uc,P}e^{i\gamma} - P'_{tc,P} + \tfrac{1}{3}P'^C_{EW,P},$$

$$\sqrt{2}A(B^+ \to K^{*+}\pi^0)$$
$$= -T'_P e^{i\gamma} - C'_V e^{i\gamma} - P'_{uc,P}e^{i\gamma} + P'_{tc,P}$$
$$+ P'_{EW,V} + \tfrac{2}{3}P'^C_{EW,P},$$

$$A(B_d^0 \to K^{*+}\pi^-) = -T'_P e^{i\gamma} - P'_{uc,P}e^{i\gamma} + P'_{tc,P} + \tfrac{2}{3}P'^C_{EW,P},$$

$$\sqrt{2}A(B_d^0 \to K^{*0}\pi^0)$$
$$= -C'_V e^{i\gamma} + P'_{uc,P}e^{i\gamma} - P'_{tc,P}$$
$$+ P'_{EW,V} + \tfrac{1}{3}P'^C_{EW,P}. \tag{13}$$

The relation among the amplitudes is

$$A(B^+ \to K^{*0}\pi^+) + \sqrt{2}A(B^+ \to K^{*+}\pi^0)$$
$$= A(B_d^0 \to K^{*+}\pi^-) + \sqrt{2}A(B_d^0 \to K^{*0}\pi^0). \tag{14}$$

What does this relation imply for $K\pi\pi$ decays?

We consider decays of the general form $B \to K(p_1)\pi(p_2)\pi(p_3)$, and take into account both $K^* \to K(p_1)\pi(p_2)$ and $K^* \to K(p_1)\pi(p_3)$. We define the reduced matrix elements:

$$D_2 \equiv A(K^*(-p_2) \to K(p_1)\pi(p_3)),$$
$$D_3 \equiv A(K^*(-p_3) \to K(p_1)\pi(p_2)). \tag{15}$$

In addition, a K^* can decay to two different $K\pi$ states. The relative contributions are given by the Clebsch-Gordan coefficients:

$$K^{*+} \to \sqrt{1/3}K^+\pi^0 - \sqrt{2/3}K^0\pi^+,$$
$$K^{*0} \to \sqrt{2/3}K^+\pi^- - \sqrt{1/3}K^0\pi^0. \tag{16}$$

We begin with $B \to K^*\pi \to K(\pi\pi)_{\text{sym}}$. There are three B^+ decays: $B^+ \to K^+(\pi^+\pi^-)_{\text{sym}}$, $B^+ \to K^0(\pi^+\pi^0)_{\text{sym}}$, and $B^+ \to K^+\pi^0\pi^0$, and three B^0 decays: $B^0 \to K^+(\pi^0\pi^-)_{\text{sym}}$, $B^0 \to K^0(\pi^+\pi^-)_{\text{sym}}$, and $B^0 \to K^0\pi^0\pi^0$. The expressions for the decay amplitudes are given in terms of the above reduced matrix elements. For example, the amplitude for the resonant contribution to $K^0(\pi^+\pi^0)_{\text{sym}}$ is

$$-\frac{D_2}{\sqrt{2}} \cdot A(B^+ \to K^{*0}(-p_2)\pi^+(p_2))$$

$$- D_3 \cdot A(B^+ \to K^{*+}(-p_3)\pi^0(p_3))$$

$$- D_2 \cdot A(B^+ \to K^{*+}(-p_2)\pi^0(p_2))$$

$$- \frac{D_3}{\sqrt{2}} \cdot A(B^+ \to K^{*0}(-p_3)\pi^+(p_3)). \tag{17}$$

The other B decay amplitudes can be found similarly.

Now, the general linear combination of the amplitudes for B^+ decays can be written in terms of the arbitrary coefficients X, Y, and Z; for B^0 decays, the coefficients are T, U, and V. The expressions are

$$X \cdot A(B^+ \to K^+(\pi^+\pi^-)_{\text{sym}})_{\text{res}}$$
$$+ Y \cdot A(B^+ \to K^0(\pi^+\pi^0)_{\text{sym}})_{\text{res}}$$
$$+ Z \cdot A(B^+ \to K^+\pi^0\pi^0)_{\text{res}}$$
$$= D_2[(X - Y/\sqrt{2})A(B^+ \to K^{*0}(-p_2)\pi^+(p_2))$$
$$+ (-Y/\sqrt{2} + Z/\sqrt{2})\sqrt{2}A(B^+ \to K^{*+}(-p_2)\pi^0(p_2))]$$
$$+ (2 \leftrightarrow 3), \tag{18}$$

$$T \cdot A(B^0 \to K^+(\pi^0\pi^-)_{\text{sym}})_{\text{res}}$$
$$+ U \cdot A(B^0 \to K^0(\pi^+\pi^-)_{\text{sym}})_{\text{res}}$$
$$+ V \cdot A(B^0 \to K^0\pi^0\pi^0)_{\text{res}}$$
$$= D_2[(T/\sqrt{2} - U)A(B^0 \to K^{*+}(-p_2)\pi^-(p_2))$$
$$+ (T/\sqrt{2} - V/\sqrt{2})\sqrt{2}A(B^0 \to K^{*0}(-p_2)\pi^0(p_2))]$$
$$+ (2 \leftrightarrow 3). \tag{19}$$

The requirement that Eqs. (18) and (19) be consistent with Eq. (14) puts constraints on the parameters. Specifically, we have $Z = \sqrt{2}X$ and $V = \sqrt{2}U$ from the B^+ and B^0 pieces of Eq. (14). And setting Eqs. (18) and (19) equal requires

$$(X - Y/\sqrt{2}) = (T/\sqrt{2} - U) \Rightarrow U = T/\sqrt{2} + Y/\sqrt{2} - X. \tag{20}$$

Thus, the $K^*\pi$ relation of Eq. (14) leads to the following for $K\pi\pi$ decays:

$$X \cdot A(B^+ \to K^+(\pi^+\pi^-)_{\text{sym}})_{\text{res}}$$
$$+ Y \cdot A(B^+ \to K^0(\pi^+\pi^0)_{\text{sym}})_{\text{res}}$$
$$+ \sqrt{2}X \cdot A(B^+ \to K^+\pi^0\pi^0)_{\text{res}}$$
$$= T \cdot A(B^0 \to K^+(\pi^0\pi^-)_{\text{sym}})_{\text{res}}$$
$$+ (T/\sqrt{2} + Y/\sqrt{2} - X) \cdot A(B^0 \to K^0(\pi^+\pi^-)_{\text{sym}})_{\text{res}}$$
$$+ (T + Y - \sqrt{2}X) \cdot A(B^0 \to K^0\pi^0\pi^0)_{\text{res}}. \tag{21}$$

This decomposes into three independent relations, one for each arbitrary coefficient:

$$A(B^+ \to K^+(\pi^+\pi^-)_{\text{sym}})_{\text{res}} + \sqrt{2}A(B^+ \to K^+\pi^0\pi^0)_{\text{res}}$$
$$= -A(B^0 \to K^0(\pi^+\pi^-)_{\text{sym}})_{\text{res}} - \sqrt{2}A(B^0 \to K^0\pi^0\pi^0)_{\text{res}},$$
$$A(B^+ \to K^0(\pi^+\pi^0)_{\text{sym}})_{\text{res}}$$
$$= \frac{1}{\sqrt{2}}A(B^0 \to K^0(\pi^+\pi^-)_{\text{sym}})_{\text{res}} + A(B^0 \to K^0\pi^0\pi^0)_{\text{res}},$$
$$0 = A(B^0 \to K^+(\pi^0\pi^-)_{\text{sym}})_{\text{res}}$$
$$+ \frac{1}{\sqrt{2}}A(B^0 \to K^0(\pi^+\pi^-)_{\text{sym}})_{\text{res}} + A(B^0 \to K^0\pi^0\pi^0)_{\text{res}}. \tag{22}$$

It is straightforward to show that these are equivalent to the symmetric $K\pi\pi$ relations in Eq. (6).

We now turn to the antisymmetric case. We have

$$K^+(\pi^+\pi^-)_{\text{anti}} = \frac{1}{\sqrt{2}}(K^+(p_1)\pi^+(p_2)\pi^-(p_3)$$
$$- K^+(p_1)\pi^+(p_3)\pi^-(p_2)),$$
$$K^0(\pi^+\pi^0)_{\text{anti}} = \frac{1}{\sqrt{2}}(K^0(p_1)\pi^+(p_2)\pi^0(p_3)$$
$$- K^0(p_1)\pi^+(p_3)\pi^0(p_2)),$$
$$K^+(\pi^0\pi^-)_{\text{anti}} = \frac{1}{\sqrt{2}}(K^+(p_1)\pi^0(p_2)\pi^-(p_3)$$
$$- K^+(p_1)\pi^0(p_3)\pi^-(p_2)),$$
$$K^0(\pi^+\pi^-)_{\text{anti}} = \frac{1}{\sqrt{2}}(K^0(p_1)\pi^+(p_2)\pi^-(p_3)$$
$$- K^0(p_1)\pi^+(p_3)\pi^-(p_2)). \tag{23}$$

There are no antisymmetric $K^+\pi^0\pi^0$ or $K^0\pi^0\pi^0$ states. Using the above, we can easily write out the amplitudes for the resonant contributions to $K(\pi\pi)_{\text{anti}}$.

Here, the general linear combination of the amplitudes for B^+ decays is written as a function of X and Y; for B^0 decays, the coefficients are T and U. The expressions are

$$X \cdot A(B^+ \to K^+(\pi^+\pi^-)_{\text{anti}})_{\text{res}} + Y \cdot A(B^+ \to K^0(\pi^+\pi^0)_{\text{anti}})_{\text{res}}$$
$$= D_2[(X - Y/\sqrt{2})A(B^+ \to K^{*0}(-p_2)\pi^+(p_2))$$
$$+ (Y/\sqrt{2})\sqrt{2}A(B^+ \to K^{*+}(-p_2)\pi^0(p_2))] - (2 \leftrightarrow 3), \tag{24}$$

$$T \cdot A(B^0 \to K^+(\pi^0\pi^-)_{\text{anti}})_{\text{res}} + U \cdot A(B^0 \to K^0(\pi^+\pi^-)_{\text{anti}})_{\text{res}}$$
$$= D_2[(U - T/\sqrt{2})A(B^0 \to K^{*+}(-p_2)\pi^-(p_2))$$
$$+ (T/\sqrt{2})\sqrt{2}A(B^0 \to K^{*0}(-p_2)\pi^0(p_2))] - (2 \leftrightarrow 3). \tag{25}$$

Comparing Eqs. (24) and (25) with Eq. (14), we require

$$Y = X/\sqrt{2}, \qquad T = U/\sqrt{2}, \qquad X = U. \tag{26}$$

Only one arbitrary coefficient remains, leading to the antisymmetric relation

$$\sqrt{2}A(B^+ \to K^+(\pi^+\pi^-)_{\text{anti}})_{\text{res}} + A(B^+ \to K^0(\pi^+\pi^0)_{\text{anti}})_{\text{res}}$$
$$= A(B^0 \to K^+(\pi^0\pi^-)_{\text{anti}})_{\text{res}}$$
$$+ \sqrt{2}A(B^0 \to K^0(\pi^+\pi^-)_{\text{anti}})_{\text{res}}. \tag{27}$$

This is the same as the antisymmetric $K\pi\pi$ relation of Eq. (8).

We therefore see that resonances reproduce the symmetric/antisymmetric $K\pi\pi$ relations.

B. Penguin dominance

In general, the dominant contribution to $\bar{b} \to \bar{s}$ transitions comes from the penguin amplitude. In Ref. [3], Gronau and Rosner explore the consequences for $B \to K\pi\pi$ decays of assuming penguin dominance and neglecting all other contributions. They note that, in this limit, the amplitudes must respect isospin reflection (i.e. $u \leftrightarrow d$), which implies that

$$A(B^+ \to K^+\pi^+\pi^-) = A(B^0_d \to K^0\pi^+\pi^-),$$
$$A(B^+ \to K^0\pi^+\pi^0) = A(B^0_d \to K^+\pi^0\pi^-), \tag{28}$$
$$A(B^0_d \to K^0\pi^0\pi^0) = A(B^+ \to K^+\pi^0\pi^0),$$

up to possible relative signs. They find that, on the whole, the data respect these relations.

The expression of the amplitudes in terms of diagrams allows us to go beyond these results. Using the method of Sec. III to distinguish $I_{\pi\pi}^{\text{sym}}$ and $I_{\pi\pi}^{\text{anti}}$, it is possible to consider the two cases separately, under the condition that only the diagram $\tilde{P}'_{tc}$ is retained in the amplitudes.

In the symmetric scenario, we have the following predictions:

$$A(B^+ \to K^0\pi^+\pi^0) = A(B^0_d \to K^+\pi^0\pi^-) = 0,$$
$$A(B^+ \to K^+\pi^+\pi^-) = A(B^0_d \to K^0\pi^+\pi^-)$$
$$= -\sqrt{2}A(B^0_d \to K^0\pi^0\pi^0)$$
$$= -\sqrt{2}A(B^+ \to K^+\pi^0\pi^0). \tag{29}$$

And in the antisymmetric scenario, we have

$$A(B^0_d \to K^0\pi^0\pi^0) = A(B^+ \to K^+\pi^0\pi^0) = 0,$$
$$A(B^+ \to K^0\pi^+\pi^0) = -A(B^0_d \to K^+\pi^0\pi^-)$$
$$= -\sqrt{2}A(B^+ \to K^+\pi^+\pi^-)$$
$$= \sqrt{2}A(B^0_d \to K^0\pi^+\pi^-). \tag{30}$$

These provide further tests of the SM.

In fact, several of these decays have been measured: $B^+ \to K^+\pi^+\pi^-$ [14,16], $B^0_d \to K^0\pi^+\pi^-$ [17], and $B^0_d \to K^+\pi^0\pi^-$ [18]. We can therefore test some of the above relations. Specifically, in terms of branching ratios (integrated over the entire Dalitz plot), the predictions are

$$\mathcal{B}(K^+\pi^0\pi^-)_{\text{sym}} = 0,$$

$$\mathcal{B}(K^+\pi^+\pi^-)_{\text{sym}} = (\tau_+/\tau_0)\mathcal{B}(K^0\pi^+\pi^-)_{\text{sym}},$$

$$\tfrac{1}{2}(\tau_+/\tau_0)\mathcal{B}(K^+\pi^0\pi^-)_{\text{anti}} = \mathcal{B}(K^+\pi^+\pi^-)_{\text{anti}}$$

$$= (\tau_+/\tau_0)\mathcal{B}(K^0\pi^+\pi^-)_{\text{anti}}. \quad (31)$$

We determine the symmetric and antisymmetric amplitudes for the three decays using the Dalitz-plot method described in Sec. III. Consider first $B^+ \to K^+\pi^+\pi^-$. We write this amplitude in terms of $x \equiv (p_{K^+} + p_{\pi^+})^2$ and $y \equiv (p_{K^+} + p_{\pi^-})^2$. Given the decay amplitude $f(x, y)$, the symmetric amplitude is taken to be $f_{\text{sym}} = \frac{1}{\sqrt{2}} \times (f(x, y) + f(y, x))$, and we compute the integral of $|f_{\text{sym}}|^2$ and $|f|^2$ over the Dalitz plot.[3] A similar procedure is carried out for the antisymmetric amplitude $f_{\text{anti}} = \frac{1}{\sqrt{2}} \times (f(x, y) - f(y, x))$. The other two decays are treated in the same way.

Although the full amplitudes for $B^+ \to K^+\pi^+\pi^-$ and $B_d^0 \to K^0\pi^+\pi^-$ are split roughly equally between symmetric and antisymmetric, the same is not true for $B_d^0 \to K^+\pi^0\pi^-$:

$$\Gamma(K^+\pi^+\pi^-)_{\text{sym}} = 0.65\Gamma(K^+\pi^+\pi^-),$$

$$\Gamma(K^0\pi^+\pi^-)_{\text{sym}} = 0.68\Gamma(K^0\pi^+\pi^-), \quad (32)$$

$$\Gamma(K^+\pi^0\pi^-)_{\text{sym}} = 0.11\Gamma(K^+\pi^0\pi^-).$$

With these, we obtain

$$\mathcal{B}(K^+\pi^0\pi^-)_{\text{sym}} = (4.0 \pm 0.3) \times 10^{-6},$$

$$\mathcal{B}(K^+\pi^+\pi^-)_{\text{sym}} = (33.3 \pm 2.0) \times 10^{-6},$$

$$(\tau_+/\tau_0)\mathcal{B}(K^0\pi^+\pi^-)_{\text{sym}} = (36.4 \pm 1.5) \times 10^{-6},$$

$$\tfrac{1}{2}(\tau_+/\tau_0)\mathcal{B}(K^+\pi^0\pi^-)_{\text{anti}} = (17.1 \pm 1.3) \times 10^{-6}, \quad (33)$$

$$\mathcal{B}(K^+\pi^+\pi^-)_{\text{anti}} = (17.6 \pm 1.0) \times 10^{-6},$$

$$(\tau_+/\tau_0)\mathcal{B}(K^0\pi^+\pi^-)_{\text{anti}} = (17.0 \pm 0.7) \times 10^{-6}.$$

(Note that the above errors do not include the errors in the parameters obtained from the Dalitz-plot analyses of the three decays.) We therefore see that the data agree with the predictions of Eq. (31). In particular, $\mathcal{B}(K^+\pi^0\pi^-)_{\text{sym}}$ is indeed greatly suppressed, in agreement with the SM.

C. Weak-phase information

Since the expressions for the decay amplitudes include the weak phase γ, it is natural to ask whether γ can be extracted from measurements of $B \to K\pi\pi$ decays. The answer is "yes" if the number of unknown theoretical parameters in the amplitudes is less than or equal to the

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[3]Note that, because of the coefficient $\frac{1}{\sqrt{2}}$ in f_{sym}, one must integrate over only half of the Dalitz plot to avoid double counting. Alternatively, f_{sym} can be defined with a factor $\frac{1}{2}$, and one integrates over the entire Dalitz plot. There are no such issues with f.

number of observables. In performing this comparison, we examine separately the $I_{\pi\pi}^{\text{sym}}$ and $I_{\pi\pi}^{\text{anti}}$ scenarios.

Consider first the $I_{\pi\pi}^{\text{sym}}$ case. Here there are six $B \to K\pi\pi$ decays. On the other hand, the first relation in Eq. (6) shows that the amplitudes for $B^+ \to K^0\pi^+\pi^0$ and $B_d^0 \to K^+\pi^0\pi^-$ are equal (up to a sign), so that there are only five independent decays. The Dalitz-plot analyses of these decays allow one to obtain the momentum-dependent branching ratios and direct CP asymmetries of $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^+\pi^0\pi^0$, $B_d^0 \to K^+\pi^0\pi^-$, $B_d^0 \to K^0\pi^+\pi^-$, and $B_d^0 \to K^0\pi^0\pi^0$. In addition, one can measure the momentum-dependent indirect CP asymmetry of $B_d^0 \to K^0\pi^+\pi^-$. (The indirect CP asymmetry of $B_d^0 \to K^0\pi^0\pi^0$ will be very difficult, if not impossible, to measure.) Thus, there are essentially 11 (momentum-dependent) observables in $I_{\pi\pi}^{\text{sym}}$ $B \to K\pi\pi$ decays.

For the case of $I_{\pi\pi}^{\text{anti}}$, there are four decays, yielding 9 observables: the momentum-dependent branching ratios and direct CP asymmetries of $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^0\pi^+\pi^0$, $B_d^0 \to K^+\pi^0\pi^-$, and $B_d^0 \to K^0\pi^+\pi^-$, and the momentum-dependent indirect CP asymmetry of $B_d^0 \to K^0\pi^+\pi^-$. Since this is fewer than above, we conclude that the $I_{\pi\pi}^{\text{sym}}$ scenario is the more promising for extracting γ.

The six $I_{\pi\pi}^{\text{sym}}$ amplitudes are given in Eq. (7). Although there are a large number of diagrams in these amplitudes, they can be combined into a smaller number of effective diagrams:

$$\sqrt{2}A(B^+ \to K^0\pi^+\pi^0)_{\text{sym}} = -T_a'e^{i\gamma} - T_b'e^{i\gamma} + P_{\text{EW},a}' + P_{\text{EW},b}',$$

$$A(B_d^0 \to K^0\pi^+\pi^-)_{\text{sym}} = -T_a'e^{i\gamma} - P_a'e^{i\gamma} + P_b',$$

$$\sqrt{2}A(B_d^0 \to K^0\pi^0\pi^0)_{\text{sym}} = -T_b'e^{i\gamma} + P_a'e^{i\gamma} - P_b'$$
$$+ P_{\text{EW},a}' + P_{\text{EW},b}',$$

$$A(B^+ \to K^+\pi^+\pi^-)_{\text{sym}} = -P_a'e^{i\gamma} + P_b' - P_{\text{EW},a}',$$

$$\sqrt{2}A(B^+ \to K^+\pi^0\pi^0)_{\text{sym}} = T_a'e^{i\gamma} + T_b'e^{i\gamma} + P_a'e^{i\gamma}$$
$$- P_b' - P_{\text{EW},b}',$$

$$\sqrt{2}A(B_d^0 \to K^+\pi^0\pi^-)_{\text{sym}} = T_a'e^{i\gamma} + T_b'e^{i\gamma} - P_{\text{EW},a}' - P_{\text{EW},b}',$$

$$(34)$$

where

$$T_a' \equiv T_1' - T_2',$$

$$T_b' \equiv C_2' + T_2',$$

$$P_a' \equiv \tilde{P}_{uc}' + T_2' + C_1',$$

$$P_b' \equiv \tilde{P}_{tc}' + \tfrac{1}{3}P_{\text{EW1}}' + \tfrac{2}{3}P_{\text{EW1}}'^C - \tfrac{1}{3}P_{\text{EW2}}'^C, \quad (35)$$

$$P_{\text{EW},a}' \equiv P_{\text{EW1}}'^C - P_{\text{EW2}}'^C,$$

$$P_{\text{EW},b}' \equiv P_{\text{EW2}}' + P_{\text{EW2}}'^C.$$

The amplitudes can therefore be written in terms of 6 effective diagrams. This corresponds to 12 theoretical

parameters[4]: 6 magnitudes of diagrams, 5 relative (strong) phases, and γ. We remind the reader that the diagrams are momentum dependent. This does not pose a problem. They will be determined via a fit to the data. But since the experimental observables are themselves momentum dependent, the fit will yield the momentum dependence of each diagram.

Unfortunately, as noted above, there are only 11 experimental observables. Therefore, in order to extract weak-phase information (γ), one requires additional input.

A previous analysis made an attempt in this direction. In 2003, Deshpande, Sinha, and Sinha (DSS) wrote schematic expressions for the symmetric $B \rightarrow K\pi\pi$ amplitudes, including tree and EWP contributions [19]. Now, in $B \rightarrow \pi K$ decays, it was shown that, under flavor SU(3) symmetry, the EWP diagrams are proportional to the tree diagrams (apart from their weak phases) [20]. DSS assumed that the EWP and tree contributions to $B^+ \rightarrow K^0 \pi^+ \pi^0$ are related in the same way. This gives additional input and allows the measurement of γ. Unfortunately, it was subsequently noted that the assumed EWP-tree relation in $K\pi\pi$ does not hold [21], so that γ cannot be extracted. This is the present situation.

In fact, the situation can be remedied. Referring to the $B_d^0 \rightarrow K^0 \pi^+ \pi^0$ amplitude in Eq. (7), DSS made the assumption that $T_1' + C_2'$ is related to $P_{EW2}' + P_{EW1}'^C$, and this was shown not to be true. We agree with this. However, there are other EWP-tree relations which do hold, and their inclusion does allow the extraction of γ. The full derivation is rather complicated, and so we present this in a separate paper [22].

Finally, we note that there is another method for obtaining γ from $B \rightarrow K\pi\pi$ decays. In two-body $\bar{b} \rightarrow \bar{s}$ B decays, the diagrams are expected to obey the approximate hierarchy [6]

$$1: P_{tc}', \qquad \bar{\lambda}: T', P_{EW}', \qquad \bar{\lambda}^2: C', P_{uc}', P_{EW}'^C, \qquad (36)$$

where $\bar{\lambda} \simeq 0.2$. If the three-body decay diagrams obey a similar hierarchy, one can neglect C_1', C_2', $\tilde{P}_{uc}'$, $P_{EW1}'^C$, and $P_{EW2}'^C$, and incur only a $\sim5\%$ theoretical error. But if these diagrams are neglected, then two of the effective diagrams vanish: $P_{EW,a}' \rightarrow 0$ and $T_b' - P_a' \rightarrow 0$ [Eq. (35)]. In this case, the amplitudes can be written in terms of 4 effective diagrams, corresponding to 8 theoretical parameters: 4 magnitudes of diagrams, 3 relative (strong) phases, and γ. Given that there are 11 experimental observables, the weak phase γ can be extracted.[5]

The down side of this method is that it is difficult to test the assumption that certain diagrams are negligible. Indeed, the presence of resonances may change the hierarchy. In light of this, the theoretical error is uncertain, and this must be addressed if this method is used.

V. $B \rightarrow KK\bar{K}$ DECAYS

We now turn to $B \rightarrow KK\bar{K}$ decays, also a $\bar{b} \rightarrow \bar{s}$ transition. The four processes are $B^+ \rightarrow K^+ K^+ K^-$, $B^+ \rightarrow K^+ K^0 \bar{K}^0$, $B_d^0 \rightarrow K^+ K^0 K^-$, and $B_d^0 \rightarrow K^0 K^0 \bar{K}^0$. Here the overall wave function of the final KK pair must be symmetrized. If the relative angular momentum is even, the isospin state must be symmetric ($I = 1$); if it is odd, the isospin state must be antisymmetric ($I = 0$).

For the symmetric case, the final state has $I = \frac{1}{2}$ or $\frac{3}{2}$, so there are three different ways of reaching it. There should therefore be one relation among the four decay amplitudes. From the Wigner-Eckart theorem, it is

$$A(B^+ \rightarrow K^+ K^+ K^-)_{\text{sym}} + \sqrt{2}A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{\text{sym}}$$
$$= \sqrt{2}A(B_d^0 \rightarrow K^+ K^0 K^-)_{\text{sym}} + A(B_d^0 \rightarrow K^0 K^0 \bar{K}^0)_{\text{sym}}. \qquad (37)$$

In terms of diagrams, the amplitudes are given by

$$A(B^+ \rightarrow K^+ K^+ K^-)_{\text{sym}} = -T_{2,s}'e^{i\gamma} - C_{1,s}'e^{i\gamma} - \hat{P}_{uc}'e^{i\gamma} + \hat{P}_{tc}' + \frac{2}{3}P_{EW1,s}' - \frac{1}{3}P_{EW1}' + \frac{2}{3}P_{EW2,s}'^C - \frac{1}{3}P_{EW1}'^C,$$

$$\sqrt{2}A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{\text{sym}} = \hat{P}_{uc}'e^{i\gamma} - \hat{P}_{tc}' + \frac{1}{3}P_{EW1,s}' + \frac{1}{3}P_{EW1}' + \frac{1}{3}P_{EW2,s}'^C + \frac{1}{3}P_{EW1}'^C,$$

$$\sqrt{2}A(B_d^0 \rightarrow K^+ K^0 K^-)_{\text{sym}} = -T_{2,s}'e^{i\gamma} - C_{1,s}'e^{i\gamma} - \hat{P}_{uc}'e^{i\gamma} + \hat{P}_{tc}' + \frac{2}{3}P_{EW1,s}' - \frac{1}{3}P_{EW1}' + \frac{2}{3}P_{EW2,s}'^C - \frac{1}{3}P_{EW1}'^C, \qquad (38)$$

$$A(B_d^0 \rightarrow K^0 K^0 \bar{K}^0)_{\text{sym}} = \hat{P}_{uc}'e^{i\gamma} - \hat{P}_{tc}' + \frac{1}{3}P_{EW1,s}' + \frac{1}{3}P_{EW1}' + \frac{1}{3}P_{EW2,s}'^C + \frac{1}{3}P_{EW1}'^C,$$

where $\hat{P}' \equiv P_{2,s}' + P_1'$. It is straightforward to verify that the relation of Eq. (37) is reproduced. On the other hand, one sees that there are, in fact, two relations:

$$A(B^+ \rightarrow K^+ K^+ K^-)_{\text{sym}} = \sqrt{2}A(B_d^0 \rightarrow K^+ K^0 K^-)_{\text{sym}}, \qquad \sqrt{2}A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{\text{sym}} = A(B_d^0 \rightarrow K^0 K^0 \bar{K}^0)_{\text{sym}}. \qquad (39)$$

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[4]In fact, there is another theoretical parameter—the phase of B_d^0-$\bar{B}_d^0$ mixing. β enters in the expression for the indirect *CP* asymmetry. However, the value for β can be taken from the indirect *CP* asymmetry in $B_d^0 \rightarrow J/\psi K_S$ [12].

[5]This technique does not work when the $\pi\pi$ pair is in an antisymmetric state of isospin. In this case, there are still more theoretical unknowns than observables, so that γ cannot be extracted.

What is happening is the following. Equation (37) is exact. However, when annihilation/exchange-type diagrams are neglected—as is done in our diagrammatic expressions of amplitudes—then one finds the two relations above. This is an example of how one can go beyond the exact relations if certain negligible diagrams are dropped.

In order to test these relations, it is necessary to isolate the symmetric piece of the decay amplitudes. $B^+ \to K^+ K^+ K^-$ and $B_d^0 \to K^0 K^0 \bar{K}^0$ are automatically symmetric since the final states contain truly identical particles. On the other hand, for $B_d^0 \to K^+ K^0 K^-$ and $B^+ \to K^+ K^0 \bar{K}^0$, the symmetric amplitude can be obtained using the Dalitz-plot method of Sec. III. Now, the Dalitz plot of $B_d^0 \to K^+ K^0 K^-$ has already been measured [23,24]. This allows us to test the first relation in Eq. (39).

We use the Dalitz-plot analysis of $B_d^0 \to K^+ K_S K^-$ given in Ref. [23], with $A(B_d^0 \to K^+ K^0 K^-) = \sqrt{2} A(B_d^0 \to K^+ K_S K^-)$. We find $\Gamma(B_d^0 \to K^+ K^0 K^-)_{\text{sym}} = 0.57 \Gamma(B_d^0 \to K^+ K^0 K^-)$. This then gives

$$2(\tau_+/\tau_0) \mathcal{B}(B_d^0 \to K^+ K^0 K^-)_{\text{sym}} = (30.0 \pm 2.8) \times 10^{-6}. \quad (40)$$

(Note that the above error does not include the errors in the parameters obtained from the Dalitz-plot analysis of Ref. [23].) This is to be compared with [1]

$$\mathcal{B}(B^+ \to K^+ K^+ K^-) = (32.5 \pm 1.5) \times 10^{-6}. \quad (41)$$

We therefore see that the first relation in Eq. (39) is satisfied. This supports our assumption that annihilation/exchange-type diagrams are negligible.

In the antisymmetric case, there are only two decays: $B^+ \to K^+ K^0 \bar{K}^0$ and $B_d^0 \to K^+ K^0 K^-$. $A(B^+ \to K^+ K^+ K^-)$ and $A(B_d^0 \to K^0 K^0 \bar{K}^0)$ vanish because there is no way of antisymmetrizing the $K^+ K^+$ or $K^0 K^0$ pair. Here the final state has $I = \frac{1}{2}$, and there are two different ways of reaching it. We therefore expect no relation between the amplitudes.

In order to write the amplitudes in terms of diagrams, we have to antisymmetrize the K^+-K^0 state. As was done for $K\pi\pi$, we adopt the following rule: all diagrams with the K^+-K^0 in order of decreasing charge from top to bottom are unmodified; all diagrams with the K^+-K^0 in order of increasing charge from top to bottom get an additional factor of -1. The amplitudes (multiplied by $\sqrt{2}$) are then given by

$$\sqrt{2} A(B^+ \to K^+ K^0 \bar{K}^0)_{\text{anti}} = -\hat{P}'_{uc} e^{i\gamma} + \hat{P}'_{tc} - \tfrac{1}{3} P'_{\text{EW1},s} - \tfrac{1}{3} P'_{\text{EW1}} + \tfrac{1}{3} P'^C_{\text{EW2},s} + \tfrac{1}{3} P'^C_{\text{EW1}},$$

$$\sqrt{2} A(B_d^0 \to K^+ K^0 K^-)_{\text{anti}} = -T'_{2,s} e^{i\gamma} + C'_{1,s} e^{i\gamma} - \hat{P}'_{uc} e^{i\gamma} + \hat{P}'_{tc} + \tfrac{2}{3} P'_{\text{EW1},s} - \tfrac{1}{3} P'_{\text{EW1}} - \tfrac{2}{3} P'^C_{\text{EW2},s} + \tfrac{1}{3} P'^C_{\text{EW1}}. \tag{42}$$

As expected, there is no relation between these two amplitudes.

A. Penguin dominance

Assuming penguin dominance, Gronau and Rosner (GR) find that isospin reflection implies the following equalities [3]:

$$A(B^+ \to K^+ K^+ K^-) = -A(B_d^0 \to K^0 K^0 \bar{K}^0), \tag{43}$$
$$A(B^+ \to K^+ K^0 \bar{K}^0) = -A(B_d^0 \to K^+ K^0 K^-).$$

By distinguishing the symmetric and antisymmetric isospin states, it is possible to go beyond these predictions. In the symmetric scenario, if only $\hat{P}'_{tc}$ is retained, we predict

$$A(B^+ \to K^+ K^+ K^-) = -A(B_d^0 \to K^0 K^0 \bar{K}^0)$$
$$= -\sqrt{2} A(B^+ \to K^+ K^0 \bar{K}^0)$$
$$= \sqrt{2} A(B_d^0 \to K^+ K^0 K^-). \tag{44}$$

[Note: the relations given in Eq. (39) actually hold for all diagrams, not just $\hat{P}'_{tc}$.] As discussed above, the present data confirm the relation $A(B^+ \to K^+ K^+ K^-) = \sqrt{2} A(B_d^0 \to K^+ K^0 K^-)$. In the antisymmetric scenario, we have only $A(B^+ \to K^+ K^0 \bar{K}^0) = A(B_d^0 \to K^+ K^0 K^-)$. As with $K\pi\pi$ decays, these provide further tests of the SM.

B. Isospin amplitudes

In Ref. [2], GR write the amplitudes for $B \to KK\bar{K}$ decays in terms of isospin amplitudes. It is instructive to compare this with the diagrammatic description.

As described above, there are five independent isospin amplitudes, denoted by $A_{\Delta I}^{I(KK),I_f} \equiv \langle I(KK), I_f | \Delta I | \tfrac{1}{2} \rangle$, where $I(KK)$ is the isospin of the KK pair [$I(KK) = 1(0)$ is symmetric (antisymmetric)], I_f is the isospin of the final state, and the weak Hamiltonian has $\Delta I = 0$ or 1. They are listed as $A_0^{0,(1/2)}$, $A_0^{1,(1/2)}$, $A_1^{0,(1/2)}$, $A_1^{1,(1/2)}$, and $A_1^{1,(3/2)}$.

As noted by GR, the $B \to KK\bar{K}$ amplitudes depend on the kaons' momenta. The amplitudes for $B^+ \to K^+ K^0 \bar{K}^0$ and $B_d^0 \to K^+ K^0 K^-$ take different values when the K^+ and K^0 momenta are exchanged. Thus, GR obtain expressions for six decay amplitudes in terms of the five isospin amplitudes:

$$A(B^+ \rightarrow K^+ K^+ K^-)_{p_1 p_2 p_3} = 2A_0^{1,1/2} - 2A_1^{1,1/2} + A_1^{1,3/2},$$

$$A(B_d^0 \rightarrow K^0 K^0 \bar{K}^0)_{p_1 p_2 p_3} = -2A_0^{1,1/2} - 2A_1^{1,1/2} + A_1^{1,3/2},$$

$$A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{p_1 p_2 p_3}$$
$$= A_0^{0,1/2} - A_0^{1,1/2} - A_1^{0,1/2} + A_1^{1,1/2} + A_1^{1,3/2},$$

$$A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{p_2 p_1 p_3}$$
$$= -A_0^{0,1/2} - A_0^{1,1/2} + A_1^{0,1/2} + A_1^{1,1/2} + A_1^{1,3/2},$$

$$A(B_d^0 \rightarrow K^+ K^0 K^-)_{p_1 p_2 p_3}$$
$$= A_0^{0,1/2} + A_0^{1,1/2} + A_1^{0,1/2} + A_1^{1,1/2} + A_1^{1,3/2},$$

$$A(B_d^0 \rightarrow K^+ K^0 K^-)_{p_2 p_1 p_3}$$
$$= -A_0^{0,1/2} + A_0^{1,1/2} - A_1^{0,1/2} + A_1^{1,1/2} + A_1^{1,3/2}. \quad (45)$$

The above amplitudes are related to those of Eqs. (38) and (42) as follows:

$$A(B^+ \rightarrow K^+ K^+ K^-)_{\text{sym}} = A(B^+ \rightarrow K^+ K^+ K^-)_{p_1 p_2 p_3},$$

$$A(B_d^0 \rightarrow K^0 K^0 \bar{K}^0)_{\text{sym}} = A(B_d^0 \rightarrow K^0 K^0 \bar{K}^0)_{p_1 p_2 p_3},$$

$$\sqrt{2} A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{\text{sym}}$$
$$= A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{p_1 p_2 p_3} + A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{p_2 p_1 p_3},$$

$$\sqrt{2} A(B_d^0 \rightarrow K^+ K^0 K^-)_{\text{sym}}$$
$$= A(B_d^0 \rightarrow K^+ K^0 K^-)_{p_1 p_2 p_3} + A(B_d^0 \rightarrow K^+ K^0 K^-)_{p_2 p_1 p_3},$$

$$\sqrt{2} A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{\text{anti}}$$
$$= A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{p_1 p_2 p_3} - A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{p_2 p_1 p_3},$$

$$\sqrt{2} A(B_d^0 \rightarrow K^+ K^0 K^-)_{\text{anti}}$$
$$= A(B_d^0 \rightarrow K^+ K^0 K^-)_{p_1 p_2 p_3} - A(B_d^0 \rightarrow K^+ K^0 K^-)_{p_2 p_1 p_3}. \quad (46)$$

Now, because there are six decay amplitudes, but only five isospin amplitudes, there must be a relation between the decay amplitudes. GR give this relation as

$$A(B^+ \rightarrow K^+ K^+ K^-)_{p_1 p_2 p_3} + A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{p_1 p_2 p_3}$$
$$+ A(B^+ \rightarrow K^+ K^0 \bar{K}^0)_{p_2 p_1 p_3} = A(B_d^0 \rightarrow K^0 K^0 \bar{K}^0)_{p_1 p_2 p_3}$$
$$+ A(B_d^0 \rightarrow K^+ K^0 K^-)_{p_1 p_2 p_3} + A(B_d^0 \rightarrow K^+ K^0 K^-)_{p_2 p_1 p_3}$$
$$= 3A_1^{1,3/2}. \quad (47)$$

This is the same as the relation in Eq. (37). However, when one expresses the amplitudes in terms of diagrams, there are, in fact, two relations instead of one [Eq. (39)]. This implies that

$$A_1^{1,1/2} = -\tfrac{1}{4} A_1^{1,3/2}, \quad (48)$$

so that there are really four independent isospin amplitudes instead of five. As described above, the extra relation is a consequence of neglecting the annihilation/exchange-type diagrams. In other words, the above relation among isospin

amplitudes is a good approximation and could not have been deduced without performing a diagrammatic analysis.

It is straightforward to express the remaining isospin amplitudes in terms of diagrams:

$$A_0^{1,1/2} = \tfrac{1}{4}[-T'_{2,s} e^{i\gamma} - C'_{1,s} e^{i\gamma} - 2\hat{P}'_{uc} e^{i\gamma} + 2\hat{P}'_{tc}$$
$$+ \tfrac{1}{3} P'_{\text{EW1},s} - \tfrac{2}{3} P'_{\text{EW1}} + \tfrac{1}{3} P'^C_{\text{EW2},s} - \tfrac{2}{3} P'^C_{\text{EW1}}],$$

$$A_1^{1,3/2} = \tfrac{1}{3}[-T'_{2,s} e^{i\gamma} - C'_{1,s} e^{i\gamma} + P'_{\text{EW1},s} + P'^C_{\text{EW2},s}],$$

$$A_0^{0,1/2} = \tfrac{1}{4}[-T'_{2,s} e^{i\gamma} + C'_{1,s} e^{i\gamma} - 2\hat{P}'_{uc} e^{i\gamma} + 2\hat{P}'_{tc} \quad (49)$$
$$+ \tfrac{1}{3} P'_{\text{EW1},s} - \tfrac{2}{3} P'_{\text{EW1}} - \tfrac{1}{3} P'^C_{\text{EW2},s} + \tfrac{2}{3} P'^C_{\text{EW1}}],$$

$$A_1^{0,1/2} = \tfrac{1}{4}[-T'_{2,s} e^{i\gamma} + C'_{1,s} e^{i\gamma} + P'_{\text{EW1},s} - P'^C_{\text{EW2},s}].$$

(Recall that, despite their having the same name, the diagrams which contribute to the $A_{\{0,1\}}^{1,\{1/2,3/2\}}$ and $A_{\{0,1\}}^{0,1/2}$ isospin amplitudes are not the same—they can have different sizes.) In the limit of penguin dominance, $A_1^{1,3/2}$ and $A_0^{1,1/2}$ vanish. This is consistent with what is found in the previous section.

C. Weak-phase information

As was the case for $B \rightarrow K\pi\pi$ decays, the amplitudes contain the weak phase γ, and so one wonders if it can be measured in $B \rightarrow KK\bar{K}$ decays. Here the answer is "perhaps."

When the isospin state of the KK pair is symmetric, there are four decays. However, due to the equality relations in Eq. (39), two of these have the same amplitudes as the other two. There are therefore 6 observables: the momentum-dependent branching ratios, direct CP asymmetries and indirect CP asymmetries of $B_d^0 \rightarrow K^+ K^0 K^-$ and $B_d^0 \rightarrow K^0 K^0 \bar{K}^0$. In the antisymmetric scenario, there are 5 observables: the momentum-dependent branching ratios and direct CP asymmetries of $B^+ \rightarrow K^+ K^0 \bar{K}^0$ and $B_d^0 \rightarrow K^+ K^0 K^-$, and the momentum-dependent indirect CP asymmetry of $B_d^0 \rightarrow K^+ K^0 K^-$. [Note: CP interchanges $K \leftrightarrow \bar{K}$. Thus, unlike $B \rightarrow K\pi\pi$, the separation of symmetric and antisymmetric KK states does not fix the CP of the final $KK\bar{K}$ state for the indirect CP asymmetries. This can be achieved by constructing states of the three final mesons which are totally symmetric or antisymmetric. (These states are discussed in more detail in Sec. VII and Ref. [22].) These have the same diagrammatic decomposition as the $(KK)_{\text{sym}}$ and $(KK)_{\text{antisym}}$ states [Eqs. (38) and (42)].]

However, in either case, the amplitudes [Eqs. (38) and (42)] are written in terms of 4 effective diagrams, corresponding to 8 theoretical parameters: 4 magnitudes of diagrams, 3 relative (strong) phases, and γ. This is larger than the number of observables, and so the weak phase γ cannot be extracted from $B \rightarrow KK\bar{K}$ decays.

The best that one can do is to assume the hierarchy of Eq. (36) and neglect all C', $\hat{P}'_{uc}$, and P'^C_{EW} diagrams. This

reduces the number of effective diagrams to 3, which corresponds to 6 theoretical parameters. This is equal to the number of observables in the symmetric case, so that γ can be extracted here, albeit with discrete ambiguities. And, as described above, the theoretical error is uncertain.

VI. $B \to K\bar{K}\pi$ DECAYS

We now consider $B \to K\bar{K}\pi$ decays, which are $\bar{b} \to \bar{d}$ transitions. Here there are seven processes:

$B^+ \to K^+ K^- \pi^+$, $B^+ \to K^+ \bar{K}^0 \pi^0$, $B^+ \to K^0 \bar{K}^0 \pi^+$, $B_d^0 \to K^+ K^- \pi^0$, $B_d^0 \to K^+ \bar{K}^0 \pi^-$, $B_d^0 \to K^0 \bar{K}^0 \pi^0$, and $B_d^0 \to K^0 K^- \pi^+$. There are no identical particles in the final state, so here we do not have to distinguish symmetric and anti-symmetric isospin states.

In $B \to K\bar{K}\pi$, the final state has $I = 0$, $I = 1$ (twice) or $I = 2$. The weak Hamiltonian has $\Delta I = \frac{1}{2}$ or $\frac{3}{2}$, so there are six paths to the final state. This implies that there is one relation among the seven decay amplitudes. It is

$$\sqrt{2}A(B_d^0 \to K^+ K^- \pi^0) + A(B_d^0 \to K^0 K^- \pi^+) - A(B^+ \to K^+ K^- \pi^+) + \sqrt{2}A(B_d^0 \to K^0 \bar{K}^0 \pi^0)$$
$$+ A(B_d^0 \to K^+ \bar{K}^0 \pi^-) - A(B^+ \to K^0 \bar{K}^0 \pi^+) - \sqrt{2}A(B^+ \to K^+ \bar{K}^0 \pi^0) = 0. \tag{50}$$

In terms of diagrams, the amplitudes are given by

$$A(B^+ \to K^+ K^- \pi^+) = [T_{2,s} + C_{1,s} + P_{a;uc}]e^{-i\alpha} - P_{a;tc} + \tfrac{1}{3}P_{\text{EW1}} - \tfrac{2}{3}P_{\text{EW1},s} + \tfrac{1}{3}P_{\text{EW1}}^C - \tfrac{2}{3}P_{\text{EW2},s}^C,$$
$$\sqrt{2}A(B^+ \to K^+ \bar{K}^0 \pi^0) = [T_{1,s} + C_{2,s} - P_{a;uc} + P_{b;uc}]e^{-i\alpha} + P_{a;tc} - P_{b;tc} - P_{\text{EW2},s} - \tfrac{1}{3}P_{\text{EW1}}^C - \tfrac{2}{3}P_{\text{EW1},s}^C + \tfrac{1}{3}P_{\text{EW2}}^C - \tfrac{1}{3}P_{\text{EW2},s}^C,$$
$$A(B^+ \to K^0 \bar{K}^0 \pi^+) = -P_{b;uc}e^{-i\alpha} + P_{b;tc} - \tfrac{1}{3}P_{\text{EW1}} - \tfrac{1}{3}P_{\text{EW1},s} - \tfrac{1}{3}P_{\text{EW1},s}^C - \tfrac{1}{3}P_{\text{EW2}}^C,$$
$$\sqrt{2}A(B_d^0 \to K^+ K^- \pi^0) = C_{1,s}e^{-i\alpha} + \tfrac{1}{3}P_{\text{EW1}} - \tfrac{2}{3}P_{\text{EW1},s},$$
$$A(B_d^0 \to K^+ \bar{K}^0 \pi^-) = [T_{1,s} + P_{b;uc}]e^{-i\alpha} - P_{b;tc} - \tfrac{2}{3}P_{\text{EW1},s}^C + \tfrac{1}{3}P_{\text{EW2}}^C,$$
$$\sqrt{2}A(B_d^0 \to K^0 \bar{K}^0 \pi^0) = [C_{2,s} - P_{a;uc} - P_{b;uc}]e^{-i\alpha} + P_{a;tc} + P_{b;tc} - \tfrac{1}{3}P_{\text{EW1}} - \tfrac{1}{3}P_{\text{EW1},s} - P_{\text{EW2},s} - \tfrac{1}{3}P_{\text{EW1}}^C - \tfrac{1}{3}P_{\text{EW1},s}^C$$
$$\qquad - \tfrac{1}{3}P_{\text{EW2}}^C - \tfrac{1}{3}P_{\text{EW2},s}^C,$$
$$A(B_d^0 \to K^0 K^- \pi^+) = [T_{2,s} + P_{a;uc}]e^{-i\alpha} - P_{a;tc} + \tfrac{1}{3}P_{\text{EW1}}^C - \tfrac{2}{3}P_{\text{EW2},s}^C, \tag{51}$$

where $P_a \equiv P_1 + P_{2,s}$, $P_b \equiv P_{1,s} + P_2$, and all amplitudes have been multiplied by $e^{i\beta}$. With these expressions, the relation of Eq. (50) is reproduced.

However, there are, in fact, two relations:

$$\sqrt{2}A(B_d^0 \to K^+ K^- \pi^0) + A(B_d^0 \to K^0 K^- \pi^+)$$
$$= A(B^+ \to K^+ K^- \pi^+),$$
$$\sqrt{2}A(B_d^0 \to K^0 \bar{K}^0 \pi^0) + A(B_d^0 \to K^+ \bar{K}^0 \pi^-)$$
$$= A(B^+ \to K^0 \bar{K}^0 \pi^+) + \sqrt{2}A(B^+ \to K^+ \bar{K}^0 \pi^0). \tag{52}$$

As was the case in $B \to KK\bar{K}$ decays, the (justified) neglect of certain annihilation/exchange-type diagrams breaks the relation in Eq. (50) into two.

A. T Dominance

In two-body B decays, T is the dominant diagram in $\bar{b} \to \bar{d}$ transitions. Assuming this also holds in three-body B decays, we have the following predictions:

$$A(B^+ \to K^+ K^- \pi^+) = A(B_d^0 \to K^0 K^- \pi^+),$$
$$\sqrt{2}A(B^+ \to K^+ \bar{K}^0 \pi^0) = A(B_d^0 \to K^+ \bar{K}^0 \pi^-),$$
$$A(B^+ \to K^0 \bar{K}^0 \pi^+) = A(B_d^0 \to K^+ K^- \pi^0)$$
$$= A(B_d^0 \to K^0 \bar{K}^0 \pi^0) \simeq 0. \tag{53}$$

These are tests of the SM which can be carried out once these decays are measured.

B. Weak-phase information

There are seven $B \to K\bar{K}\pi$ decays, which yield 16 observables: the branching ratios and direct CP asymmetries of $B^+ \to K^+ K^- \pi^+$, $B^+ \to K^+ \bar{K}^0 \pi^0$, $B^+ \to K^0 \bar{K}^0 \pi^+$, $B_d^0 \to K^+ K^- \pi^0$, $B_d^0 \to K^+ \bar{K}^0 \pi^-$, $B_d^0 \to K^0 \bar{K}^0 \pi^0$, and $B_d^0 \to K^0 K^- \pi^+$, and the indirect CP asymmetries of $B_d^0 \to K^+ K^- \pi^0$, $B_d^0 \to K^0 \bar{K}^0 \pi^0$.

The $B \to K\bar{K}\pi$ amplitudes in Eq. (51) can be written in terms of 10 effective diagrams:

$$A(B^+ \to K^+ K^- \pi^+) = [D_1 + D_3]e^{-i\alpha} + D_2 + D_4,$$

$$\sqrt{2}A(B^+ \to K^+ \bar{K}^0 \pi^0) = D_9 e^{-i\alpha} + D_{10},$$

$$A(B^+ \to K^0 \bar{K}^0 \pi^+) = D_7 e^{-i\alpha} + D_8,$$

$$\sqrt{2}A(B_d^0 \to K^+ K^- \pi^0) = D_1 e^{-i\alpha} + D_2,$$

$$A(B_d^0 \to K^+ \bar{K}^0 \pi^-) = D_5 e^{-i\alpha} + D_6,$$

$$\sqrt{2}A(B_d^0 \to K^0 \bar{K}^0 \pi^0) = [-D_5 + D_7 + D_9]e^{-i\alpha}$$
$$- D_6 + D_8 + D_{10},$$

$$A(B_d^0 \to K^0 K^- \pi^+) = D_3 e^{-i\alpha} + D_4, \qquad (54)$$

where

$$D_1 \equiv C_{1,s},$$

$$D_2 \equiv \tfrac{1}{3}P_{EW1} - \tfrac{2}{3}P_{EW1,s},$$

$$D_3 \equiv T_{2,s} + P_{a;uc},$$

$$D_4 \equiv -P_{a;tc} + \tfrac{1}{3}P^C_{EW1} - \tfrac{2}{3}P^C_{EW2,s},$$

$$D_5 \equiv T_{1,s} + P_{b;uc},$$

$$D_6 \equiv -P_{b;tc} + \tfrac{1}{3}P^C_{EW2} - \tfrac{2}{3}P^C_{EW1,s},$$

$$D_7 \equiv -P_{b;uc},$$

$$D_8 \equiv P_{b;tc} - \tfrac{1}{3}P_{EW1} - \tfrac{1}{3}P_{EW1,s} - \tfrac{1}{3}P^C_{EW2} - \tfrac{1}{3}P^C_{EW1,s},$$

$$D_9 \equiv T_{1,s} + C_{2,s} - P_{a;uc} + P_{b;uc},$$

$$D_{10} \equiv P_{a;tc} - P_{b;tc} - P_{EW2,s} - \tfrac{1}{3}P^C_{EW1} - \tfrac{2}{3}P^C_{EW1,s}$$
$$+ \tfrac{1}{3}P^C_{EW2} - \tfrac{1}{3}P^C_{EW2,s}. \qquad (55)$$

This corresponds to 20 theoretical parameters: 10 magnitudes of diagrams, 9 relative (strong) phases, and α. With only 16 observables, α cannot be extracted.

We therefore need additional input. Fortunately, we have some, similar to that in Secs. IV C and V C. In two-body $\bar{b} \to \bar{d}$ B decays, the diagrams obey the approximate hierarchy [6]

$$1: T, \qquad \bar{\lambda}: C, P_{tc}, P_{uc}, \qquad \bar{\lambda}^2: P_{EW}, \qquad \bar{\lambda}^3: P^C_{EW}. \qquad (56)$$

If the three-body decay diagrams obey a similar hierarchy, all EWP diagrams can be neglected, leading to an error of only ∼5%. In this limit, we have $D_2 = 0$, $D_8 = -D_6$, and $D_{10} = -D_4 + D_6$. So the number of independent diagrams is reduced to 7, i.e. 14 theoretical parameters.[6] Thus, by measuring the observables in $B \to K\bar{K}\pi$ decays, weak-phase information can be obtained. In fact, not all 16 observables are necessary. Experimentally, this is not easy, but it is at least theoretically possible. Of course, as in Secs. IV C and V C, the theoretical error is uncertain, since it is difficult to test the hierarchy of diagrams.

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[6]We assume that, for the indirect *CP* asymmetries, the *CP* of the final state can be fixed as for the decays in previous sections. Otherwise there are 2 additional theoretical parameters.

VII. $B \to \pi\pi\pi$ DECAYS

Finally, we examine $B \to \pi\pi\pi$ decays, also a $\bar{b} \to \bar{d}$ transition. There are four processes: $B_d^0 \to \pi^0\pi^0\pi^0$, $B^+ \to \pi^+\pi^0\pi^0$, $B^+ \to \pi^-\pi^+\pi^+$, and $B_d^0 \to \pi^+\pi^0\pi^-$. In contrast to the other decays, here the final state includes three identical particles under isospin, so that the six permutations of these particles (the group S_3) must be considered. Numbering the particles 1, 2, and 3, the six possible orders are 123, 132, 312, 321, 231, and 213. Under S_3, there are six possibilities for the isospin state of the three π's: a totally symmetric state $|S\rangle$, a totally antisymmetric state $|A\rangle$, or one of four mixed states $|M_i\rangle$ ($i = 1$–4). These can be defined as

$$|S\rangle \equiv \frac{1}{\sqrt{6}}(|123\rangle + |132\rangle + |312\rangle + |321\rangle + |231\rangle + |213\rangle),$$

$$|M_1\rangle \equiv \frac{1}{\sqrt{12}}(2|123\rangle + 2|132\rangle - |312\rangle - |321\rangle$$
$$- |231\rangle - |213\rangle),$$

$$|M_2\rangle \equiv \frac{1}{\sqrt{4}}(|312\rangle - |321\rangle - |231\rangle + |213\rangle),$$

$$|M_3\rangle \equiv \frac{1}{\sqrt{4}}(-|312\rangle - |321\rangle + |231\rangle + |213\rangle),$$

$$|M_4\rangle \equiv \frac{1}{\sqrt{12}}(2|123\rangle - 2|132\rangle - |312\rangle + |321\rangle$$
$$- |231\rangle + |213\rangle),$$

$$|A\rangle \equiv \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |312\rangle - |321\rangle + |231\rangle - |213\rangle). \qquad (57)$$

This choice of mixed states implies that two truly identical particles go in positions 2 and 3. Under the exchange $2 \leftrightarrow 3$, $|M_1\rangle$ and $|M_2\rangle$ are symmetric, while $|M_3\rangle$ and $|M_4\rangle$ are antisymmetric.

For the four $B \to \pi\pi\pi$ decays, we have
(1) $B_d^0 \to \pi^0\pi^0\pi^0$: all final-state particles are the same, which means $|123\rangle = |132\rangle = |312\rangle = |321\rangle = |231\rangle = |213\rangle$. In this case, only the state $|S\rangle$ is allowed.
(2) $B^+ \to \pi^+\pi^0\pi^0$: particle 1 is π^+, particles 2 and 3 are π^0. Thus, $|123\rangle = |132\rangle$, $|312\rangle = |213\rangle$, and $|231\rangle = |321\rangle$. This implies that each of $|M_3\rangle$, $|M_4\rangle$, and $|A\rangle$ is not allowed.
(3) $B^+ \to \pi^-\pi^+\pi^+$: particle 1 is π^-, particles 2 and 3 are π^+. Thus, $|123\rangle = |132\rangle$, $|312\rangle = |213\rangle$, and $|231\rangle = |321\rangle$. This implies that each of $|M_3\rangle$, $|M_4\rangle$, and $|A\rangle$ is not allowed.
(4) $B_d^0 \to \pi^+\pi^0\pi^-$: we choose the order such that particle 1 is π^+, particle 2 is π^0, and particle 3 is π^-. All six states are allowed.

The amplitude for a decay with two truly identical particles has an extra factor of $1/\sqrt{2}$; with three truly identical particles, the factor is $1/\sqrt{6}$.

The six elements of S_3 are I (identity), P_{12} (exchanges particles 1 and 2), P_{13} (exchanges particles 1 and 3), P_{23} (exchanges particles 2 and 3), P_{cyclic} (cyclic permutation of particle numbers, i.e. $1 \to 2$, $2 \to 3$, and $3 \to 1$), $P_{\text{anticyclic}}$ (anticyclic permutation of particle numbers, i.e. $1 \to 3$, $2 \to 1$, and $3 \to 2$). Under the group transformations, $|S\rangle \to |S\rangle$ and $|A\rangle \to \pm|A\rangle$. It is easy to see that $|M_1\rangle$ and $|M_3\rangle$ transform among themselves. Writing

$$|M_1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |M_3\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{58}$$

we can represent each group element by a 2×2 matrix:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_{12} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$P_{13} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \qquad P_{23} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$P_{\text{cyclic}} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad P_{\text{anticyclic}} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{59}$$

Similarly, if we write

$$|M_2\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |M_4\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{60}$$

the S_3 matrices take the same form, showing that $|M_2\rangle$ and $|M_4\rangle$ also transform among themselves.

The above allows us to express the amplitudes for all $B \to \pi\pi\pi$ decays in terms of diagrams. We begin with some general comments about diagrams. As an example, consider T_1. In principle, there are six possibilities, T_1^{ijk}, in which the final-state pions i, j, and k run from top to bottom of the diagram in all permutations. Suppose that we want the expression for the amplitude of $B \to \pi_1\pi_2\pi_3$ in a particular $|S_3\rangle$ state, and suppose that the diagram T_1^{ijk} contributes to the decay. For $|S_3\rangle = |S\rangle$, we define T_1^S:

$$T_1^S \equiv \frac{1}{\sqrt{6}}(T_1^{123} + T_1^{132} + T_1^{312} + T_1^{321} + T_1^{231} + T_1^{213}). \tag{61}$$

Each T_1^{ijk} leads to T_1^S in the amplitude. For $|S_3\rangle = |A\rangle$, we have

$$T_1^A \equiv \frac{1}{\sqrt{6}}(T_1^{123} - T_1^{132} + T_1^{312} - T_1^{321} + T_1^{231} - T_1^{213}). \tag{62}$$

Again, each T_1^{ijk} leads to T_1^A in the amplitude, with a coefficient of 1 (-1) if ijk is in cyclic (anticyclic) order.

For the mixed states, one has to take into account the fact that, under group transformations, there is $|M_1\rangle$-$|M_3\rangle$ and $|M_2\rangle$-$|M_4\rangle$ mixing. In order to illustrate how this is done, we focus first on the M_1/M_3 sector. We define

$$T_1^{M_1} \equiv \frac{1}{\sqrt{12}}(2T_1^{123} + 2T_1^{132} - T_1^{312} - T_1^{321} - T_1^{231} - T_1^{213}),$$

$$T_1^{M_3} \equiv \frac{1}{\sqrt{4}}(-T_1^{312} - T_1^{321} + T_1^{231} + T_1^{213}). \tag{63}$$

Suppose $|S_3\rangle = |M_1\rangle$. The contribution to the amplitude of $B \to \pi_1\pi_2\pi_3$ is $[M \times (T_1^{M_1}, T_1^{M_3})^T]_{\text{upper component}}$, where M is the matrix representing the S_3 group element which transforms ijk to 123 [Eq. (59)]. In general, this is a combination of $T_1^{M_1}$ and $T_1^{M_3}$ (though the $T_1^{M_3}$ component can be zero if $M = I$ or P_{23}). Factors of -1 for each $\bar{u}$ and $1/\sqrt{2}$ for each π^0 must also be included. If $|S_3\rangle = |M_3\rangle$, the contribution to the amplitude is $[M \times (T_1^{M_1}, T_1^{M_3})^T]_{\text{lower component}}$. This can be applied analogously to the M_2/M_4 sector, where we define

$$T_1^{M_2} \equiv \frac{1}{\sqrt{4}}(T_1^{312} - T_1^{321} - T_1^{231} + T_1^{213}),$$

$$T_1^{M_4} \equiv \frac{1}{\sqrt{12}}(2T_1^{123} - 2T_1^{132} - T_1^{312} + T_1^{321} - T_1^{231} + T_1^{213}). \tag{64}$$

The entire procedure holds for all diagrams.[7]

With these rules, we can now work out the amplitudes for all decays. We begin first with $|S_3\rangle = |S\rangle$. The amplitudes are

$$\frac{2}{\sqrt{3}}A(B_d^0 \to \pi^0\pi^0\pi^0)_{|S\rangle} = -[C_1^S - C_2^S + P_{uc}^S]e^{-i\alpha} + \left[P_{tc}^S + \frac{1}{3}P_{\text{EW1}}^S - P_{\text{EW2}}^S - \frac{1}{3}P_{\text{EW1}}^{C,S} - \frac{1}{3}P_{\text{EW2}}^{C,S}\right],$$

$$\sqrt{2}A(B^+ \to \pi^+\pi^0\pi^0)_{|S\rangle} = -[T_2^S + C_1^S + P_{uc}^S]e^{-i\alpha} + \left[P_{tc}^S + \frac{1}{3}P_{\text{EW1}}^S - \frac{1}{3}P_{\text{EW1}}^{C,S} + \frac{2}{3}P_{\text{EW2}}^{C,S}\right],$$

$$\frac{1}{\sqrt{2}}A(B^+ \to \pi^-\pi^+\pi^+)_{|S\rangle} = [T_2^S + C_1^S + P_{uc}^S]e^{-i\alpha} - \left[P_{tc}^S + \frac{1}{3}P_{\text{EW1}}^S - \frac{1}{3}P_{\text{EW1}}^{C,S} + \frac{2}{3}P_{\text{EW2}}^{C,S}\right],$$

$$\sqrt{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|S\rangle} = [C_1^S - C_2^S + P_{uc}^S]e^{-i\alpha} - \left[P_{tc}^S + \frac{1}{3}P_{\text{EW1}}^S - P_{\text{EW2}}^S - \frac{1}{3}P_{\text{EW1}}^{C,S} - \frac{1}{3}P_{\text{EW2}}^{C,S}\right], \tag{65}$$

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[7]When applied to the decays in the previous sections, this method produces the same amplitude decomposition as when we used the simple rule of adding a minus sign to diagrams in which the identical particles are exchanged (e.g. in $B \to K\pi\pi$ or $K K \bar{K}$).

where $P \equiv P_1 + P_2$ and all amplitudes have been multiplied by $e^{i\beta}$.

For the M_1/M_3 sector, the amplitudes are

$$\sqrt{2}A(B^+ \to \pi^+\pi^0\pi^0)_{|M_1\rangle} = \left[\frac{3}{2}T_1^{M_1} - \frac{\sqrt{3}}{2}T_1^{M_3} - T_2^{M_1} - C_1^{M_1} + \frac{3}{2}C_2^{M_1} - \frac{\sqrt{3}}{2}C_2^{M_3} - P_{uc}^{M_1} + \sqrt{3}P_{uc}^{M_3}\right]e^{-i\alpha} + \left[P_{tc}^{M_1} - \sqrt{3}P_{tc}^{M_3}\right.$$
$$\left. - \frac{1}{6}P_{EW1}^{M_1} - \frac{1}{2\sqrt{3}}P_{EW1}^{M_3} + \sqrt{3}P_{EW2}^{M_3} - \frac{1}{3}P_{EW1}^{C,M_1} - \frac{2}{\sqrt{3}}P_{EW1}^{C,M_3} - \frac{5}{6}P_{EW2}^{C,M_1} - \frac{1}{2\sqrt{3}}P_{EW2}^{C,M_3}\right],$$

$$\sqrt{2}A(B^+ \to \pi^-\pi^+\pi^+)_{|M_1\rangle} = \left[-T_2^{M_1} + \sqrt{3}T_2^{M_3} - C_1^{M_1} - \sqrt{3}C_1^{M_3} - P_{uc}^{M_1} + \sqrt{3}P_{uc}^{M_3}\right]e^{-i\alpha}$$
$$+ \left[P_{tc}^{M_1} - \sqrt{3}P_{tc}^{M_3} + \frac{4}{3}P_{EW1}^{M_1} - \frac{2}{\sqrt{3}}P_{EW1}^{M_3} - \frac{1}{3}P_{EW1}^{C,M_1} + \frac{1}{\sqrt{3}}P_{EW1}^{C,M_3} + \frac{2}{3}P_{EW2}^{C,M_1} - \frac{2}{\sqrt{3}}P_{EW2}^{C,M_3}\right],$$

$$6\sqrt{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_1\rangle} = \left[9T_1^{M_1} - 3\sqrt{3}T_1^{M_3} - 3C_1^{M_1} + 3\sqrt{3}C_1^{M_3} + 3C_2^{M_1} - 3\sqrt{3}C_2^{M_3} - 3P_{uc}^{M_1} + 3\sqrt{3}P_{uc}^{M_3}\right]e^{-i\alpha}$$
$$+ \left[3P_{tc}^{M_1} - 3\sqrt{3}P_{tc}^{M_3} - 5P_{EW1}^{M_1} + \sqrt{3}P_{EW1}^{M_3} - 3P_{EW2}^{M_1} + 3\sqrt{3}P_{EW2}^{M_3} - P_{EW1}^{C,M_1} - 5\sqrt{3}P_{EW1}^{C,M_3}\right.$$
$$\left. - P_{EW2}^{C,M_1} + \sqrt{3}P_{EW2}^{C,M_3}\right],$$

$$2\sqrt{6}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_3\rangle} = \left[-3T_1^{M_1} + \sqrt{3}T_1^{M_3} - 4\sqrt{3}T_2^{M_3} + 3C_1^{M_1} + \sqrt{3}C_1^{M_3} + 3C_2^{M_1} + \sqrt{3}C_2^{M_3} + 3P_{uc}^{M_1} - 3\sqrt{3}P_{uc}^{M_3}\right]e^{-i\alpha}$$
$$+ \left[-3P_{tc}^{M_1} + 3\sqrt{3}P_{tc}^{M_3} - P_{EW1}^{M_1} + \sqrt{3}P_{EW1}^{M_3} + 3P_{EW2}^{M_1} + 3\sqrt{3}P_{EW2}^{M_3} + P_{EW1}^{C,M_1} + \sqrt{3}P_{EW1}^{C,M_3}\right.$$
$$\left. - 5P_{EW2}^{C,M_1} + \sqrt{3}P_{EW2}^{C,M_3}\right]. \tag{66}$$

For the M_2/M_4 sector, the amplitudes are

$$\sqrt{2}A(B^+ \to \pi^+\pi^0\pi^0)_{|M_2\rangle} = \left[\frac{3}{2}T_1^{M_2} - \frac{\sqrt{3}}{2}T_1^{M_4} - T_2^{M_2} - C_1^{M_2} + \frac{3}{2}C_2^{M_2} - \frac{\sqrt{3}}{2}C_2^{M_4} - P_{uc}^{M_2} + \sqrt{3}P_{uc}^{M_4}\right]e^{-i\alpha} + \left[P_{tc}^{M_2} - \sqrt{3}P_{tc}^{M_4}\right.$$
$$\left. - \frac{1}{6}P_{EW1}^{M_2} - \frac{1}{2\sqrt{3}}P_{EW1}^{M_4} + \sqrt{3}P_{EW2}^{M_4} - \frac{1}{3}P_{EW1}^{C,M_2} - \frac{2}{\sqrt{3}}P_{EW1}^{C,M_4} - \frac{5}{6}P_{EW2}^{C,M_2} - \frac{1}{2\sqrt{3}}P_{EW2}^{C,M_4}\right],$$

$$\sqrt{2}A(B^+ \to \pi^-\pi^+\pi^+)_{|M_2\rangle} = \left[-T_2^{M_2} + \sqrt{3}T_2^{M_4} - C_1^{M_2} - \sqrt{3}C_1^{M_4} - P_{uc}^{M_2} + \sqrt{3}P_{uc}^{M_4}\right]e^{-i\alpha} + \left[P_{tc}^{M_2} - \sqrt{3}P_{tc}^{M_4}\right.$$
$$\left. + \frac{4}{3}P_{EW1}^{M_2} - \frac{2}{\sqrt{3}}P_{EW1}^{M_4} - \frac{1}{3}P_{EW1}^{C,M_2} + \frac{1}{\sqrt{3}}P_{EW1}^{C,M_4} + \frac{2}{3}P_{EW2}^{C,M_2} - \frac{2}{\sqrt{3}}P_{EW2}^{C,M_4}\right],$$

$$6\sqrt{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_2\rangle} = \left[9T_1^{M_2} - 3\sqrt{3}T_1^{M_4} - 3C_1^{M_2} + 3\sqrt{3}C_1^{M_4} + 3C_2^{M_2} - 3\sqrt{3}C_2^{M_4} - 3P_{uc}^{M_2} + 3\sqrt{3}P_{uc}^{M_4}\right]e^{-i\alpha}$$
$$+ \left[3P_{tc}^{M_2} - 3\sqrt{3}P_{tc}^{M_4} - 5P_{EW1}^{M_2} + \sqrt{3}P_{EW1}^{M_4} - 3P_{EW2}^{M_2} + 3\sqrt{3}P_{EW2}^{M_4} - P_{EW1}^{C,M_2}\right.$$
$$\left. - 5\sqrt{3}P_{EW1}^{C,M_4} - P_{EW2}^{C,M_2} + \sqrt{3}P_{EW2}^{C,M_4}\right],$$

$$2\sqrt{6}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_4\rangle} = \left[-3T_1^{M_2} + \sqrt{3}T_1^{M_4} - 4\sqrt{3}T_2^{M_4} + 3C_1^{M_2} + \sqrt{3}C_1^{M_4} + 3C_2^{M_2} + \sqrt{3}C_2^{M_4} + 3P_{uc}^{M_2} - 3\sqrt{3}P_{uc}^{M_4}\right]e^{-i\alpha}$$
$$+ \left[-3P_{tc}^{M_2} + 3\sqrt{3}P_{tc}^{M_4} - P_{EW1}^{M_2} + \sqrt{3}P_{EW1}^{M_4} + 3P_{EW2}^{M_2} + \sqrt{3}P_{EW2}^{M_4} + P_{EW1}^{C,M_2}\right.$$
$$\left. + \sqrt{3}P_{EW1}^{C,M_4} - 5P_{EW2}^{C,M_2} + \sqrt{3}P_{EW2}^{C,M_4}\right]. \tag{67}$$

Finally, for $|S_3\rangle = |A\rangle$, we have

$$\sqrt{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|A\rangle} = \left[2T_1^A - 2T_2^A - C_1^A - C_2^A - 3P_{uc}^A\right]e^{-i\alpha} + \left[3P_{tc}^A + P_{EW1}^A - P_{EW2}^A - P_{EW1}^{C,A} - P_{EW2}^{C,A}\right]. \tag{68}$$

Now, the final state has isospin $1 \otimes 1 \otimes 1 = 0 \oplus 1 \oplus 1 \oplus 1 \oplus 2 \oplus 2 \oplus 3$. Given that the B meson has $I = \frac{1}{2}$ and the weak Hamiltonian has $\Delta I = \frac{1}{2}$ or $\frac{3}{2}$, there are 9 paths to the final state. We therefore expect four relations among the 13 decay amplitudes. This is indeed what is found:

$$\sqrt{2}A(B_d^0 \to \pi^0\pi^0\pi^0)_{|S\rangle} = -\sqrt{3}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|S\rangle},$$

$$2A(B^+ \to \pi^+\pi^0\pi^0)_{|S\rangle} = -A(B^+ \to \pi^-\pi^+\pi^+)_{|S\rangle},$$

$$\frac{3}{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_1\rangle} + \frac{\sqrt{3}}{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_3\rangle} = A(B^+ \to \pi^+\pi^0\pi^0)_{|M_1\rangle} - A(B^+ \to \pi^-\pi^+\pi^+)_{|M_1\rangle},$$

$$\frac{3}{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_2\rangle} + \frac{\sqrt{3}}{2}A(B_d^0 \to \pi^+\pi^0\pi^-)_{|M_4\rangle} = A(B^+ \to \pi^+\pi^0\pi^0)_{|M_2\rangle} - A(B^+ \to \pi^-\pi^+\pi^+)_{|M_2\rangle}. \tag{69}$$

These relations can also be found using the Wigner-Eckart theorem.

In passing, we note that, within the SM, the final state with $I = 3$ is unreachable. This then provides a test of the SM. Applying the method of Ref. [25] to $B \to \pi\pi\pi$, one can distinguish the various isospin final states. One can then look for a state with $I = 3$. If one is observed, this will be a smoking-gun signal of new physics.

A. Dalitz plots

Above, we presented the amplitudes for each of the six S_3 states of $B \to \pi\pi\pi$. The obvious question is then whether these states can be distinguished experimentally. Below we show that this can indeed be done.

Consider the decay $B_d^0 \to \pi^+\pi^0\pi^-$. The Dalitz-plot events can be described by $s_+ = (p_{\pi^0} + p_{\pi^+})^2$ and $s_- = (p_{\pi^0} + p_{\pi^-})^2$, so that the decay amplitude, $\mathcal{M}(s_+, s_-)$, can be extracted. We introduce the third Mandelstam variable, $s_0 = (p_{\pi^+} + p_{\pi^-})^2$. It is related to s_+ and s_- as follows:

$$s_+ + s_- + s_0 = m_B^2 + 3m_\pi^2. \tag{70}$$

The totally symmetric SU(3) decay amplitude is then given by

$$|S\rangle = \frac{1}{\sqrt{6}}[\mathcal{M}(s_+, s_-) + \mathcal{M}(s_-, s_+) + \mathcal{M}(s_+, s_0) + \mathcal{M}(s_0, s_+) + \mathcal{M}(s_0, s_-) + \mathcal{M}(s_-, s_0)]. \tag{71}$$

Also,

$$|M_1\rangle = \frac{1}{\sqrt{12}}[2\mathcal{M}(s_+, s_-) + 2\mathcal{M}(s_-, s_+) - \mathcal{M}(s_+, s_0) - \mathcal{M}(s_0, s_+) - \mathcal{M}(s_0, s_-) - \mathcal{M}(s_-, s_0)]. \tag{72}$$

The remaining S_3 states can be found similarly. The method is similar for the other $B \to \pi\pi\pi$ decays.

B. Weak-phase information

In the previous section we showed how all six $B \to \pi\pi\pi$ S_3 states can be experimentally separated. It may then be possible to extract clean information about weak phases. (Note: by measuring the S_3 states, one fixes the CP of the final states, which makes the indirect CP asymmetries well defined.)

Consider $|S_3\rangle = |A\rangle$. Here there is one decay, which yields three observables: the branching ratio, the direct CP asymmetry, and the indirect CP asymmetry of $B_d^0 \to \pi^+\pi^0\pi^-|_{|A\rangle}$. The amplitude is expressed in terms of two effective diagrams: $A(B_d^0 \to \pi^+\pi^0\pi^-)_{|A\rangle} = D_1 e^{-i\alpha} + D_2$, which has four theoretical parameters—the magnitudes of $D_{1,2}$, the relative strong phase, and α. Since the number of theoretical unknowns is greater than the number of observables, one cannot obtain α. Things are similar for $|S_3\rangle = |S\rangle$. Because of the first two relations in Eq. (69), there are only two independent decays, yielding 5 observables. However, there are 8 theoretical parameters, so that, once again, α cannot be extracted.

Things are different for the case of mixed states. Consider the M_1/M_3 sector. There are four decays: (1) $B^+ \to \pi^+\pi^0\pi^0|_{|M_1\rangle}$, (2) $B^+ \to \pi^-\pi^+\pi^+|_{|M_1\rangle}$, (3) $B_d^0 \to \pi^+\pi^0\pi^-|_{|M_1\rangle}$, and (4) $B_d^0 \to \pi^+\pi^0\pi^-|_{|M_3\rangle}$. These yield 10 observables: 4 branching ratios, 4 direct CP asymmetries, and 2 indirect CP asymmetries (of $B_d^0 \to \pi^+\pi^0\pi^-|_{|S_3\rangle}$, $S_3 = M_1$, and M_3). The four decay amplitudes all have the form $D_{1,i}e^{-i\alpha} + D_{2,i}$, $i = 1$–4. The $D_{1,i}$ are related to one another by the third relation in Eq. (69), as are the $D_{2,i}$. The amplitudes are thus a function of 6 effective diagrams, resulting in 12 theoretical parameters: 6 magnitudes, 5 relative strong phases, and α. Since the number of theoretical unknowns exceeds the number of observables, α cannot be extracted. However, if one assumes that the hierarchy of Eq. (56) holds for three-body decays, all EWP diagrams can be neglected, to a good approximation. In this case, all the $D_{2,i}$ are proportional to $P_{tc}^{M_1} - \sqrt{3}P_{tc}^{M_3}$. There are thus only 4 effective diagrams, which yield 8 theoretical parameters. Now the number of theoretical unknowns is smaller than the number of observables, so that α can be obtained from a fit to the data. (It is not even necessary to measure all 10 observables. A difficult-to-obtain quantity, such as the direct CP asymmetry in $B^+ \to \pi^+\pi^0\pi^0|_{|M_1\rangle}$, can be omitted.) A similar method holds for the M_2/M_4 sector. The error on α can be reduced by comparing the two values found.

Now, it must be conceded that the above analysis is quite theoretical—it is far from certain that this can be carried out experimentally [and there is an uncertain theoretical error due to the assumption of Eq. (56)]. Still, it is interesting to see that, in principle, clean weak-phase information can be obtained from $B \to \pi\pi\pi$, or, more generally, from $B \to M_1 M_2 M_3$ decays.

VIII. CONCLUSIONS

In this paper, we have expressed the amplitudes for $B \rightarrow M_1 M_2 M_3$ decays (M_i is a pseudoscalar meson) in terms of diagrams, concentrating on the charmless final states $K\pi\pi$, $KK\bar{K}$, $K\bar{K}\pi$, and $\pi\pi\pi$. The diagrams are similar to those used in two-body decays: the color-favored and color-suppressed tree amplitudes T and C, the gluonic-penguin amplitudes P_{tc} and P_{uc}, and the color-favored and color-suppressed EWP amplitudes $P_{\rm EW}$ and $P_{\rm EW}^C$. Here, because the final state has three particles, there are two types of each diagram, which we call T_1, T_2, C_1, C_2, etc.

We have also demonstrated how to use the Dalitz plots of three-body decays to separate the decay amplitudes into pieces which are symmetric or antisymmetric under the exchange of two of the final-state particles. This is useful for any decay whose final state contains identical particles under isospin. If the relative angular momentum of the two particles is even (odd), the isospin state must be symmetric (antisymmetric). These two possibilities can be distinguished experimentally.

The main advantage of a diagrammatic analysis is that the approximate relative sizes of the diagrams can be estimated. For example, there are annihilation- and exchange-type diagrams which contribute to these decays. However, these are expected to be much smaller than the dominant diagrams (relative size $\lesssim 10\%$), and are not included in our analysis. Previous studies of three-body decays were carried out using isospin amplitudes and gave exact results for the symmetric or antisymmetric states. On the other hand, the neglect of annihilation/exchange-type diagrams can modify these results and can lead to interesting new effects.

As an example, consider $B \rightarrow KK\bar{K}$, which consists of four decays. For the case where the two K's are in a symmetric isospin state, the Wigner-Eckart theorem gives a single relation among the four amplitudes. However, when annihilation/exchange-type diagrams are excluded

from the amplitudes, it is found that this relation actually consists of two equalities, and this leads to new predictions of the SM. Present data allow us to test one of these equalities, and we find agreement with the SM. In the same vein, $B \rightarrow KK\bar{K}$ decays can be written in terms of five isospin amplitudes. The diagrammatic analysis shows that, in fact, only four of these are independent—two of the isospin amplitudes are proportional to one another.

Another consequence of the diagrammatic analysis has to do with weak phases. The CP of a three-particle final state is not fixed, because the relative angular momenta are unknown (i.e. they can be even or odd). For this reason, in the past it was thought that it is not possible to cleanly extract weak-phase information from three-body B decays. In this paper, we demonstrate that this is not true. Using the diagrams, we show that it is possible to cleanly measure the weak phases in some decays, given that it is experimentally possible to distinguish different symmetry combinations of the final-state particles. We explicitly give methods for $K\bar{K}\pi$ and $\pi\pi\pi$, and note that the procedure for $K\pi\pi$ is presented separately. Ways of cleanly extracting the CP phases from other three-body decays will surely be suggested.

There are thus a number of interesting measurements that can be carried out with $B \rightarrow M_1 M_2 M_3$. LHCb is running at present, and the super-B factories will run in the future. Hopefully, these machines will provide interesting data on three-body B decays.

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