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$\mathcal{W}_{1+\infty}$ algebra as a symmetry behind the Alday-Gaiotto-Tachikawa relation

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We give some evidences which imply that $\mathcal{W}_{1+\infty}$ algebra describes the symmetry behind the Alday-Gaiotto-Tachikawa(-Wyllard) conjecture: a correspondence between the partition function of $\mathcal{N} = 2$ supersymmetric quiver gauge theories and the correlators of Liouville (Toda) field theory.

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I. INTRODUCTION

The purpose of this paper is to give a proposal on the symmetry behind the correspondence between four-dimensional $\mathcal{N} = 2$ supersymmetric quiver gauge theory and two-dimensional conformal Liouville (Toda) system, which was discovered by seminal papers by Seiberg-Witten [1,2] and later has been deepened by recent breakthroughs [3,4].

In particular in [4], an explicit relation between the two—Nekrasov's partition function [5] for $SU(2)$ quiver gauge theories and the conformal block of Liouville theory [6]—was given. It was then generalized to the $SU(N)$ case, where the Liouville theory is replaced by A_{N-1} Toda theory [7].

Such correspondence is interesting because (1) it implies a nontrivial relation between two-dimensional and four-dimensional physics which may be explained by the strong coupling physics of M-theory 5-brane; and (2) it apparently relates the formulas with very different mathematical origin, i.e. one in the geometry of instanton moduli space of the gauge theories and the other in the representation theory of infinite dimensional Lie algebra such as Virasoro algebra or W_N algebra.

At this stage, this correspondence (called the Alday-Gaiotto-Tachikawa [AGT] relation or the Alday-Gaiotto-Tachikawa-Wyllard [AGT-W] relation for its generalization), has two issues in different levels to be fully explored. First, one needs to know the precise definition of the statement

$$Z^{\text{Nekrasov}} = \langle V_1 \ldots V_n \rangle, \tag{1}$$

where the left-hand side is the partition function of super Yang-Mills theory and the right-hand side is the chiral correlator of Liouville (Toda) field theory. While the left-hand side is well-known for linear quiver gauge theories [5], the corresponding chiral correlation function of the Liouville (Toda) theory is only known in the special case [8]. Furthermore, there remain some open issues for the choice of vertex operators and intermediate states.

Second, one needs to understand more profound issues about such correspondence exists. This is certainly much more important for the future development but it is out of reach of this paper since we would like to focus on the symmetry of two-dimensional conformal field theory (CFT).

Recently, a major step to understand the right-hand side of Eq. (1) was undertaken [9–11]. The key issue here is how to understand the factorized form of Z^{Nekrasov} from the CFT viewpoint. For $SU(2)$ quiver, the authors of [9] proposed a basis with two Young diagram indices $|Y_1, Y_2\rangle$, where the factors in Nekrasov's partition function are reproduced through the norm and three-point functions in terms of them. When one of the Y_i's is null, the basis coincides with the Jack symmetric polynomial [12]. They also presented an algorithm to construct such a basis for general cases. Later, for the simpler case (where the central charge $c = 1$ or $Q = 0$), it was conjectured in [11] that the state $|Y_1, Y_2\rangle$ is given by the direct product of Schur polynomials $|Y_1, Y_2\rangle = s_{Y_1} s_{Y_2}$.

Other than these technical improvements toward the proof of the AGT-W relation, these studies reveal the importance of integration of somewhat mysterious "$U(1)$ factor" [4] to construct these useful bases. This implies that the original symmetry such as Virasoro or W algebra should be properly enhanced to include the $U(1)$ factor. In this paper, we propose that $\mathcal{W}_{1+\infty}$ algebra, whose representation was studied long ago [13–16], should be the proper symmetry behind the AGT-W relation, at least for $Q = 0$.

As its strange name implies, $\mathcal{W}_{1+\infty}$ algebra contains a $U(1)$ current operator together with the infinite number of higher spin generators. With appropriate choice of representation, we will show that the algebra reduces to W_N algebra and $U(1)$ current as expected from above constructions. It is known further that a Schur polynomial gives an appropriate basis which diagonalizes all the commuting charges. Therefore, it is very natural that this $U(1)$ current corresponds to the $U(1)$ factor, and this discussion gives some indirect evidences why $\mathcal{W}_{1+\infty}$ algebra is relevant to the AGT-W relation.

This paper is organized as follows. In Sec. II, we briefly review the definition of $\mathcal{W}_{1+\infty}$ algebra and its

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representation. In particular, we emphasize the importance of "quasifinite" representation [13]. This section is a brief summary of [16]. In Sec. III, we demonstrate explicitly how the representation of $\mathcal{W}_{1+\infty}$ algebra with the central charge $C = N$ reduces to that of W_N algebra together with $U(1)$ current. In Sec. IV, we discuss that the AGT-W relation is reduced to the problem of computation of three-point function of $\mathcal{W}_{1+\infty}$ algebra. After presenting our conjecture, we show some evidences by generalizing the computation of [11] to W_3 cases. In [11], the proof of the AGT-W conjecture was reduced to the so-called "chain vector." In Sec. V, we review what a chain vector is and derive its explicit form for Virasoro and W_3 cases. Compared with [11], our novelty is the use of a free boson variable from the beginning (this helps to simplify the computation) and derivation of a W_3 chain vector. After this preparation, in Sec. VI, we combine the $U(1)$ factor as predicted by $\mathcal{W}_{1+\infty}$ algebra and reproduce Nekrasov's formula. In Sec. VII, we illustrate future directions. In the Appendix, we give a summary of our notation.

II. A BRIEF REVIEW OF $\mathcal{W}_{1+\infty}$ ALGEBRA AND ITS REPRESENTATION

In the following, we briefly review some relevant material of the representation theory of $\mathcal{W}_{1+\infty}$ algebra. We follow the description of [16].

A. $\mathcal{W}_{1+\infty}$ algebra

$\mathcal{W}_{1+\infty}$ algebra is a quantum realization of algebra generated by higher order differential operators $z^m D^n$ ($D := z\frac{\partial}{\partial z}$, $m \in \mathbb{Z}$, $n = 0, 1, 2, \ldots$). We define a map from the differential operators to quantum operators through $z^m D^n \to W(z^m D^n)$. $\mathcal{W}_{1+\infty}$ algebra is most compactly expressed in the following form:

$$[W(z^n e^{xD}), W(z^m e^{yD})] = (e^{mx} - e^{ny})W(z^{n+m}e^{(x+y)D}) - C\frac{e^{mx} - e^{ny}}{e^{x+y} - 1}\delta_{n+m,0}, \quad (2)$$

where C is the central extension parameter and $W(z^n e^{xD}) := \sum_{m=0}^{\infty} \frac{x^m}{m!} W(z^n D^m)$.

The algebra contains $U(1)$ current operators $J_m = W(z^m)$. There is some ambiguity in the choice of Virasoro operators. One may take, for example, $-W(z^n D)$, which satisfies Virasoro algebra with central charge $-2C$. For this choice, however, the $U(1)$ currents have anomalous transformation $[J_n, W(z^m D)] = -nJ_{n+m} + \frac{C}{2}n(n-1)\delta_{n+m,0}$. Then a better choice is

$$L_n = -W(z^n D) - \frac{n+1}{2}W(z^n), \quad (3)$$

with which J_n transforms as the primary field with spin 1. This operator satisfies the Virasoro algebra with central charge C. Together with these familiar ones, $\mathcal{W}_{1+\infty}$ algebra also contains an infinite number of higher spin operators $W(z^n D^m)$ whose commutation relation with the Virasoro operator is

$$[L_n, W(z^m D^l)] = (ln - m)W(z^{m+n}D^l) + \cdots. \quad (4)$$

The first term implies that these operators transform as spin $l + 1$ fields, but the algebra contains extra terms $\cdots$ which implies that they should be modified to be primary fields. We will come back to this problem for the spin 3 case in Sec. III B.

The algebra has an infinite number of commuting charges $W(D^n)$ ($n = 0, 1, 2, \ldots$) and we need their eigenvalues to specify the representation. As usual, the highest weight state (HWS) $|\Delta\rangle$ is defined by

$$W(z^n D^m)|\Delta\rangle = 0 \qquad (n > 0, m \geq 0)$$
$$W(D^n)|\Delta\rangle = \Delta_n|\Delta\rangle \qquad (n \geq 0), \quad (5)$$

where Δ_n ($n = 0, 1, \ldots$) are complex number parameters to specify the representation. They are more conveniently expressed in the form of a generating function,

$$W(e^{xD})|\Delta\rangle = -\Delta(x)|\Delta\rangle, \qquad \Delta(x) := -\sum_{n=0}^{\infty}\frac{x^n}{n!}\Delta_n. \quad (6)$$

The Hilbert space is generated from HWS by applying $W(z^{-n}D^m)$ ($n>0$). Since $[L_0, W(z^{-n}D^m)] = nW(z^{-n}D^m)$, the inner products of such states are block diagonal with respect to the eigenvalue of L_0 which we call "level" of the state.

B. Quasifinite representation

Unlike the usual two-dimensional chiral algebra, $\mathcal{W}_{1+\infty}$ algebra contains an infinite number of states $W(z^{-n}D^m)|\Delta\rangle$ ($m = 0, 1, 2, \ldots$) at each level n. It makes the handling of Hilbert space quite difficult.

To make the situation better, we require a condition on $\Delta(x)$ (first discussed in [13]) such that most of the states at each level except for a finite set become null. Such representation is called "quasifinite representation." It is realized by requiring conditions of the form

$$W(z^{-n}b_n(D))|\Delta\rangle \sim 0, \quad (7)$$

where $b_n(x)$ is a polynomial of x. If such a condition is imposed, all operators of the form $W(z^{-n}D^m b_n(D))|\Delta\rangle$ become null, thus there remain only a finite number of operators $W(z^{-n}D^l)$ with $l<$ order $(b_n(x))$ for each level n. The polynomials $b_n(x)$ are determined from that for level 1 $b_1(x) := b(x)$ through the consistency with the algebra,

$$b_n(x) = \mathrm{lcm}(b(x), b(x-1), \ldots, b(x-n+1)), \quad (8)$$

where "lcm" means the least common multiple and $b(x)$ is the characteristic polynomial. In order to have such null states, $\Delta(x)$ needs to satisfy

$$b\left(\frac{d}{dx}\right)((e^x - 1)\Delta(x) + C) = 0. \quad (9)$$

In particular, for $b(x) = \prod_{i=1}^{K}(x-\lambda_i)^{m_i}$ with $\lambda_i \neq \lambda_j$, the solution to Eq. (9) is

$$\Delta(x) = \frac{\sum_{i=1}^{K} p_i(x)e^{\lambda_i x} - C}{e^x - 1}, \tag{10}$$

where $p_i(x)$ is a polynomial of degree $m_i - 1$ and satisfies $\sum_i p_i(0) = C$.

We note here that $\mathcal{W}_{1+\infty}$ algebra has a one-parameter family of automorphism which is called "spectral flow." The transformation rule is

$$\tilde{W}(z^n e^{xD}) = W(z^n e^{x(D+\lambda)}) - C\frac{e^{\lambda x}-1}{e^x-1}\delta_{n0}, \tag{11}$$

where $\tilde{W}$ satisfies the same algebra as W, but their eigenvalues for $|\Delta\rangle$ are modified. For the representation (10), this transformation is realized as a shift $\lambda_i \longrightarrow \lambda_i + \lambda$. It implies that the representation obtained by the shift of λ_i by λ has exactly the same property as the original one.

Unitary representations.—In order to make the Hilbert space unitary, we need to impose further constraints on $\Delta(x)$ [14,15]. It may be summarized as follows.

First, the multiplicity indices m_i in $b(x)$ should be 1. Then the solution (10) for $m_i = 1$ becomes

$$\Delta(x) = \sum_{i=1}^{K} C_i \frac{e^{\lambda_i x}-1}{e^x-1}, \qquad \sum_{i=1}^{K} C_i = C. \tag{12}$$

Second, the parameters C_i in Eq. (12) must be a positive integer. In particular, for $C_i = 1$ (for all i) and $\lambda_i - \lambda_j \neq$ integer (for all pairs $i \neq j$), we have a free fermion representation

$$\begin{aligned} b^{(i)}(z) &= \sum_{r\in\mathbb{Z}} b_r^{(i)} z^{-r-\lambda_i-1}, \\ c^{(i)}(z) &= \sum_{r\in\mathbb{Z}} c_r^{(i)} z^{-r+\lambda_i}, \\ b^{(i)}(z)c^{(j)}(w) &\sim \frac{\delta_{ij}}{z-w}, \\ b_r^{(i)}|\Delta\rangle &= c_s^{(i)}|\Delta\rangle = 0 \quad (r\geq 0, s\geq 1), \\ c_s^{(i)\dagger} &= b_{-s}^{(i)}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} W(z^n e^{xD}) &= \sum_{i=1}^{K}\bigg(\sum_{r+s=n} e^{x(\lambda_i - s)}E^{(i)}(r,s) - \frac{e^{\lambda_i x}-1}{e^x-1}\delta_{n,0}\bigg), \\ E^{(i)}(r,s) &= {}^{\star}_{\star} b_r^{(i)} c_s^{(i)} {}^{\star}_{\star}, \end{aligned} \tag{14}$$

where the normal ordering

$${}^{\star}_{\star} \quad {}^{\star}_{\star}$$

is defined as ${}^{\star}_{\star} b_r^{(i)} c_s^{(i)} {}^{\star}_{\star} = b_r^{(i)} c_s^{(i)}$ if $r \leq -1$ and as $-c_s^{(i)} b_r^{(i)}$ if $r \geq 0$. Note that the parameter K in Eq. (12) equals C in this case. The Hilbert space is the tensor product of free fermions with fermion number $= 0$ for each i. A convenient basis of such states is labeled by K Young diagrams $\vec{Y} = (Y_1, \ldots, Y_K)$. For example, for $K = 1$ case, the state associated with $Y = [f_1, \ldots, f_r]$ (*i.e.* length of rows are $f_1 \geq \ldots \geq f_r \geq 1$) is given by[1]

$$\begin{aligned} |Y\rangle &= b_{-\bar{f}_1} b_{-\bar{f}_2} \ldots b_{-\bar{f}_r} |-r\rangle, \\ \bar{f}_i &= f_i - i - 1, \\ |-r\rangle &= c_{-r+1}\ldots c_{-1}c_0|\Delta\rangle. \end{aligned} \tag{15}$$

The basis $|\vec{Y}\rangle$ for general K is a tensor product of such states. After bosonization, such basis is written as the product of Schur polynomials, as we will see later. We note that good characterization of such states is that they are diagonal with respect to $W(D^n)$ action as shown in Sec. 3.1 of [16].

We note that if some of the λ_i's satisfy $\lambda_i - \lambda_j =$ integer, the free fermion basis does not give the Hilbert space of $\mathcal{W}_{1+\infty}$ algebra. To see this, we should remember the definition of the polynomial $b_n(x)$. For $\lambda_i - \lambda_j \neq$ integer, $b_n(x) = \prod_{i=1}^{n} b(x-i)$. If $\lambda_i - \lambda_j =$ integer, however, the order of polynomial $b_n(x)$ becomes lower, since we have lcm in Eq. (8). It implies that we have extra null states. Thus the Hilbert space of $\mathcal{W}_{1+\infty}$ becomes in general smaller than those spanned by free fermions.

III. REDUCTION OF $\mathcal{W}_{1+\infty}$ TO W_N ALGEBRA AND $U(1)$ FACTOR

A. Explicit form of some unitary representations

Before we start, we explain the structure of representations with $C = 1, 2, \ldots$ to some detail.

$C = 1$.—In this case, $\Delta(x) = \frac{e^{\lambda x}-1}{e^x-1}$. By using the spectral flow, one may shift $\lambda \longrightarrow 0$, which means $\Delta(x) \longrightarrow 0$. In this sense, we have only one highest weight state. The Hilbert space of $\mathcal{W}_{1+\infty}$ algebra coincides with that of one free fermion pair with fermion number zero. Therefore, the partition function becomes

$$Z(q) = \sum_{\mathcal{H}} q^{L_0} = q^{\lambda^2/2}\prod_{n=1}^{\infty}\frac{1}{1-q^n}. \tag{16}$$

$C = 2$.—In this case, the generating function of the weights Δ_n becomes

$$\begin{aligned} \Delta(x) &= \frac{e^{\lambda_1 x}-1}{e^x-1} + \frac{e^{\lambda_2 x}-1}{e^x-1} \\ &= \sum_{i=1,2}\bigg(\lambda_i + \frac{1}{2}(\lambda_i^2 - \lambda_i)x \\ &\quad + \frac{1}{12}(2\lambda_i - 1)(\lambda_i - 1)x^2 + \cdots\bigg), \end{aligned} \tag{17}$$

[1] For other representations, see for example Appendix B of [16].

which implies

$$J_0|\Delta\rangle = -(\lambda_1 + \lambda_2)|\Delta\rangle,\quad L_0|\Delta\rangle = \frac{1}{2}(\lambda_1^2 + \lambda_2^2)|\Delta\rangle, \ldots. \tag{18}$$

The spectral flow may be used to set J_0 eigenvalue to be zero, and so one may put $\lambda_1 = \lambda/2$, $\lambda_2 = -\lambda/2$. Then the conformal weight of $|\Delta\rangle$ becomes $\lambda^2/4$, which looks like the Virasoro conformal weight for the vertex operator of a free boson $e^{\pm\lambda\phi/\sqrt{2}}$. This will be confirmed in the next subsection.

If $\lambda \notin \mathbb{Z}$, the partition function is that for two free bosons,

$$Z = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2}. \tag{19}$$

For $\lambda \in \mathbb{Z}$, however, the Hilbert space is in general smaller than that of fermionic representation, since $b_n(x) \neq \prod_{i=1}^{n} b(x-i)$ for $n > |\lambda|$. In particular for $\lambda = 0$, $\Delta(x) = 0$ with the characteristic polynomial $b(x) = x$. This implies that $W(z^{-1}D^m)|\Delta\rangle = 0$ for $m = 1, 2, \ldots$. More explicitly, $W(z^{-1})|\Delta\rangle = \sum_{i=1,2} E^{(i)}(-1,0)|\Delta\rangle$ and $W(z^{-1}D)|\Delta\rangle = \sum_{i=1,2}\lambda_i E^{(i)}(-1,0)|\Delta\rangle$. They are not independent if $\lambda_1 = \lambda_2$, since the second state is a linear function of the first one. Therefore, the partition function for $\lambda = 0$ becomes

$$Z = \frac{1}{1-q}\prod_{n=2}^{\infty}\frac{1}{(1-q^n)^2}. \tag{20}$$

$C > 2$.—Up to spectral flow symmetry, the representation contains $C - 1 (= N - 1)$ parameters. The vertex operator of W_N algebra has the same number of independent parameters. In fact, we can identify them as we see in next subsection.

B. Reduction from $\mathcal{W}_{1+\infty}$ to W_N

In order to see the connection with the AGT-W relation, we need to see the explicit relation with $\mathcal{W}_N$ algebra. To see it, we start from the free fermion realization (14) with $C = K = N$. We introduce free fermion fields

$$b^{(i)}(z) = \sum_n b_n^{(i)} z^{-n-1}, \qquad c^{(i)}(z) = \sum_n c_n^{(i)} z^{-n}. \tag{21}$$

We note that after replacing $c_n = \psi_{n-1/2}$ and $b_n = \bar{\psi}_{n+1/2}$, this definition agrees with the standard Dirac fermion $\psi(z)$, $\bar{\psi}(z)$ in the Neveu-Schwarz (NS) sector. We define a generating function of $W(z^n e^{xD})$ as

$$\begin{aligned} W(\zeta, x) &:= \sum_n W(z^n e^{xD})\zeta^{-n-1} \\ &= \sum_{i=1}^{N}\sum_{r,s}\left(e^{x(\lambda_i - s)} {}^\star_\star b_r^{(i)}\zeta^{-r-1}c_s^{(i)}\zeta^{-s}{}^\star_\star - \frac{e^{\lambda_i x}-1}{e^x - 1}\zeta^{-1}\right) \\ &= \sum_{i=1}^{N} e^{x\lambda_i}{}^\star_\star b^{(i)}(\zeta)e^{xD_\zeta}c^{(i)}(\zeta){}^\star_\star - \zeta^{-1}\Delta(x) \\ &= \sum_{i=1}^{N} e^{x\lambda_i}{}^\star_\star b^{(i)}(\zeta)c^{(i)}(e^x\zeta){}^\star_\star - \zeta^{-1}\Delta(x), \end{aligned} \tag{22}$$

where $D_\zeta := \zeta\partial_\zeta$. We apply the standard bosonization rule to the fermions

$$b^{(i)}(\zeta) =: e^{-\phi^{(i)}(\zeta)}:, \qquad c^{(i)}(\zeta) =: e^{\phi^{(i)}(\zeta)}:, \tag{23}$$

where :: refers to the normal ordering of bosonic oscillator and

$$\phi^{(i)}(\zeta) = x^{(i)} + \alpha_0^{(i)}\log\zeta - \sum_n \frac{\alpha_n^{(i)}}{n}\zeta^{-n},\quad [\alpha_n^{(i)}, \alpha_m^{(j)}] = n\delta_{n+m}\delta^{ij}. \tag{24}$$

The fermionic normal ordering means

$${}^\star_\star b(\zeta)c(e^x\zeta){}^\star_\star = b(\zeta)c(e^x\zeta) - \frac{1}{\zeta(e^x - 1)}, \tag{25}$$

then the generating function $W(\zeta, x)$ can be written in a simplified form as

$$\begin{aligned} W(\zeta, x) &= \sum_{i=1}^{N} e^{x\lambda_i}\frac{-1}{\zeta(e^x-1)}(:e^{\phi^{(i)}(e^x\zeta)-\phi^{(i)}(\zeta)}: - 1) - \frac{1}{\zeta}\Delta(x) \\ &= -\sum_{i=1}^{N}\frac{1}{\zeta(e^x-1)}(:e^{\phi^{(i)}(e^x\zeta)-\phi^{(i)}(\zeta)+x\lambda_i}: - 1). \end{aligned} \tag{26}$$

Here the exponent function can be written as

$$\phi^{(i)}(e^x\zeta) - \phi^{(i)}(\zeta) + x\lambda_i = (\alpha_0^{(i)} + \lambda_i)x + (\text{oscillator part}), \tag{27}$$

so we can see that λ_i plays a role of shifting momentum $p^{(i)} = \alpha_0^{(i)}$ in free boson. Therefore, one may rewrite the vertex operator part as

$$:e^{\phi^{(i)}(e^x\zeta)-\phi^{(i)}(\zeta)+x\lambda_i}: = :e^{\varphi^{(i)}(e^x\zeta)-\varphi^{(i)}(\zeta)+x\lambda_i}::e^{\varphi^U(e^x\zeta)-\varphi^U(\zeta)}:, \tag{28}$$

where $\varphi^U(\zeta) := \frac{1}{N}\sum_i\phi^{(i)}(\zeta)$ and $\varphi^{(i)}(\zeta) = \phi^{(i)}(\zeta) - \varphi^U(\zeta)$. By using this expression, we can separate $W(\zeta, x)$ into W_N and $U(1)$ parts,

$$W(\zeta, x) = -\frac{1}{\zeta(e^x-1)}(\Xi(\zeta, x)\Xi^U(\zeta, x) - N), \tag{29}$$

$$\Xi(\zeta, x) := \sum_{i=1}^{N} :e^{\varphi^{(i)}(e^x\zeta)-\varphi^{(i)}(\zeta)}:,$$
$$\Xi^U(\zeta, x) := :e^{\varphi^U(e^x\zeta)-\varphi^U(\zeta)}:. \tag{30}$$

Here the factor $x\lambda_i$ is absorbed into the redefinition of zero mode of $\varphi^{(i)}$ field. Note that Eq. (29) tells us how to decompose the operator into $U(1)$ factor and W_N generators.

Since $\Xi(\zeta)$ is invariant under Weyl reflection of $\varphi^{(i)}$, we conjecture that the module generated by such operators should be rewritten in terms of $W_n^{(m)}$ ($m = 2, 3, \ldots, N$). Let us confirm it for $N = 2, 3$. We expand $W(\zeta, x)$ as

$$\begin{aligned} W(\zeta,x) &= \sum_n \zeta^{-n-1} W(z^n e^{xD}) \\ &= \sum_n \sum_{m=0}^{\infty} \frac{x^m}{m!} \zeta^{-n-1} W(z^n D^m) \\ &= J(\zeta) - \zeta x(T(\zeta) - \frac{1}{2}\partial_\zeta J(\zeta)) + \frac{1}{2}\zeta^2 x^2 \tilde{W}_3(\zeta) + \cdots, \end{aligned} \tag{31}$$

where $J(\zeta) = \sum_n J_n \zeta^{-n-1}$, $T(\zeta) = \sum_n L_n \zeta^{-n-2}$, and $\tilde{W}_3(\zeta) = \sum_n W(z^n D^2)\zeta^{-n-3}$. Now we write explicit form of operators in terms of free bosons.

For $N = 2$, we write $\varphi^{(1)} = \frac{1}{2}(\phi^{(1)} - \phi^{(2)}) =: \varphi^V$ and $\varphi^{(2)} = -\varphi^V$. Then Eq. (29) gives

$$\begin{aligned} J(\zeta) &= -2\partial\varphi^U T \\ (\zeta) &= T^V(\zeta) + (\partial\varphi^U)^2 \\ \tilde{W}_3(\zeta) &= -2(\partial\varphi^U)T^V - \partial T^V - \frac{1}{\zeta}T^V - \frac{2}{3}(\partial\varphi^U)^3 \\ &\quad - 2(\partial^2\varphi^U)(\partial\varphi^U) - \frac{2}{3}(\partial^3\varphi^U) \\ &\quad - \frac{1}{\zeta}((\partial^2\varphi^U) + (\partial\varphi^U)^2), \end{aligned} \tag{32}$$

where $T^V(\zeta) := (\partial\varphi^V)^2$. The $U(1)$ current operator and Virasoro operator take the standard form with $C = 2$. The expression for $\tilde{W}_3$ is complicated, but the dependence on φ^V can be written in terms of only T^V and its derivative. In this sense, we can expect that the Hilbert space of $\mathcal{W}_{1+\infty}$ algebra can be expressed in terms of the reduced set, i.e. the Virasoro operator T^V and $U(1)$ current $J = -2\partial\varphi^U$.

For $N = 3$, we write

$$\begin{aligned} \varphi^{(1)} &= \frac{1}{3}(2\phi^{(1)} - \phi^{(2)} - \phi^{(3)}) =: -\frac{1}{\sqrt{6}}(\varphi_1^V + \sqrt{3}\varphi_2^V) \\ \varphi^{(2)} &= \frac{1}{3}(-\phi^{(1)} + 2\phi^{(2)} - \phi^{(3)}) =: -\frac{1}{\sqrt{6}}(\varphi_1^V - \sqrt{3}\varphi_2^V) \\ \varphi^{(3)} &= \frac{1}{3}(-\phi^{(1)} - \phi^{(2)} + 2\phi^{(3)}) = \sqrt{\frac{2}{3}}\varphi_1^V. \end{aligned} \tag{33}$$

Then the expression for generators becomes

$$\begin{aligned} J(\zeta) &= -3\partial\varphi^U \\ T(\zeta) &= T^V(\zeta) + \frac{3}{2}(\partial\varphi^U)^2 \\ \tilde{W}_3(\zeta) &= -\sqrt{\frac{2}{3}}W_3^V - (\partial\varphi^U)T^V - \frac{1}{2}\partial T^V - \frac{1}{2\zeta}T^V \\ &\quad - (\partial\varphi^U)^3 - 3(\partial\varphi^U)(\partial^2\varphi^U) - \partial^3\varphi^U \\ &\quad - \frac{3}{2\zeta}((\partial\varphi^U)^2 + \partial^2\varphi^U), \end{aligned} \tag{34}$$

where

$$\begin{aligned} T^V(\zeta) &:= \frac{1}{2}(\partial\varphi_1^V)^2 + \frac{1}{2}(\partial\varphi_2^V)^2, \\ W_3^V(\zeta) &:= \frac{1}{6}((\partial\varphi_2^V)^3 - 3(\partial\varphi_1^V)^2\partial\varphi_2^V). \end{aligned} \tag{35}$$

The $U(1)$ current and Virasoro generator are again the standard ones for $C = 3$. In the expression of $\tilde{W}_3(\zeta)$, $W_3^V(\zeta)$ is a spin 3 primary field with respect to $T^V(\zeta)$ and coincides with the W_3 generator for the central charge $c = 2$. The other terms are written in terms of $T^V(\zeta)$, $\partial\varphi^U$, and their derivatives. We conjecture that the higher terms in Eq. (31) can be also written in terms of only $W_3(\zeta)$, $T^V(\zeta)$, $\partial\varphi^U$, and their derivatives. If it is true, the Hilbert space for a $C = 3$ system is described by the W_3 operators and the $U(1)$ part.

IV. CONJECTURE AND SOME EVIDENCES

A. General strategy

We hope that we have convinced the readers who followed Sec. II and III of the following fact: $\mathcal{W}_{1+\infty}$ algebra contains an infinite number of two-dimensional chiral fields with spin $1, 2, \ldots, \infty$. When we limit ourselves to the unitary quasifinite representations, the central charge C must be a finite positive integer N and the independent chiral fields must be limited to those with spin $1, 2, \ldots, N$. Among these fields, those with spin $2, 3, \ldots, N$ coincide with the chiral fields of W_N algebra. We have shown it explicitly for $N = 2, 3$ in Sec. III B, but its generalization for $N > 3$ would be clear through our arguments. A novelty here is that we also have $U(1)$ current $J(\zeta)$. While we may decouple it from W_N generators in the Hilbert space, we need it to realize the larger symmetry $\mathcal{W}_{1+\infty}$.

We may compare the situation in the AGT-W relation [4,7]. In these works, the chiral symmetry in the two-dimensional side is described by W_N algebra with central charge $c = N - 1 + Q^2N(N^2 - 1)$ in order to be compared with $SU(N)$ quiver gauge theories. Here the parameter $Q = b + 1/b$ corresponds to a set of deformation parameters $\epsilon_{1,2}$ appearing in Nekrasov's partition function as $Q = \frac{\epsilon_1+\epsilon_2}{\sqrt{\epsilon_1\epsilon_2}}$. In order to compare the correlation function of Liouville (more generally, Toda) field theory with

Nekrasov's partition function, we need extra an $U(1)$ factor for the former function [4].

In the $\mathcal{W}_{1+\infty}$ approach with the quasifinite unitary representation, we need to restrict ourselves to $Q = 0$ and $C = (N-1) + 1$, where the former $N-1$ part is described by W_N algebra and the latter one is from a free boson which describes the $U(1)$ factor. While it has limitation to the background charge Q, it shows how to integrate the $U(1)$ factor with W_N algebra or Toda fields.

As we mentioned in the introduction, some efforts had been made to integrate the $U(1)$ factor with Virasoro current in [9,11] for the $N = 2$ case. Let us briefly review some relevant materials in [9].

In order to describe the chiral correlators, the authors introduce two chiral algebras, i.e. Virasoro algebra described by L_n and the $U(1)$ current described by a free boson a_n. They use an additional free boson c_n to describe L_n as

$$L_n = \sum_{k\neq 0,n} c_k c_{n-k} + i(2P - nQ)c_n,$$
$$L_0 = \frac{Q^2}{4} - P^2 + 2\sum_{k>0} c_{-k}c_k, \tag{36}$$

where P is the momentum of the boson c_n which describes the vertex operator. Then they propose to introduce a particular basis $|P\rangle_{\vec{Y}}$ with Young tableaux $\vec{Y} = (Y_1, Y_2)$ for the Hilbert space described by a_n and c_n such that (i) the inner product with vertex operator insertion coincides with $Z_{\rm bf}$ in [4] (the factor of Nekrasov's partition function for a bifundamental field),

$$\frac{{}_{\vec{Y}'}\langle P'|V_\alpha|P\rangle_{\vec{Y}}}{\langle P'|V_\alpha|P\rangle} = \mathcal{F}^{\vec{Y}'}_{\vec{Y}}(\alpha|P,P')$$
$$= \prod_{i,j=1,2}\prod_{s\in Y_i}(Q - E_{Y_i,Y'_j}(P_i - P'_j|s) - \alpha)$$
$$\times \prod_{t\in Y_j}(E_{Y'_j,Y_i}(P'_j - P_i|t) - \alpha), \tag{37}$$

with $\vec{P} = (P, -P)$, $\vec{P}' = (P', -P')$, and $E_{X,Y}(P|s) := P - bA_Y(s) + b^{-1}(L_X(s) + 1)$, where $A(s)/L(s)$ is the arm/leg length of a Young tableau; and (ii) the inner product of these states is diagonal and equals $1/Z_{\rm vec}$ in [4] (the inverse of the factor of Nekrasov's partition function for a vector field),

$${}_{\vec{Y}'}\langle P|P\rangle_{\vec{Y}} = N_{\vec{Y}}\delta_{\vec{Y},\vec{Y}'}, \qquad N_{\vec{Y}} = \mathcal{F}^{\vec{Y}}_{\vec{Y}}(0|P,P). \tag{38}$$

Once one finds such basis, one may decompose any correlator as

$$\langle\Phi_1\ldots\Phi_n\rangle = \langle\Phi_1|\Phi_2\sum_{\vec{Y}_1}|\tilde{P}_1\rangle_{\vec{Y}_1}$$
$$\times \frac{1}{N_{\vec{Y}_1}}{}_{\vec{Y}_1}\langle P_1|\Phi_3\ldots\Phi_{n-2}\sum_{\vec{Y}_{n-3}}|\tilde{P}_{n-3}\rangle_{\vec{Y}_{n-3}}$$
$$\times \frac{1}{N_{\vec{Y}_{n-3}}}{}_{\vec{Y}_{n-3}}\langle P_{n-3}|\Phi_{n-1}|\Phi_n\rangle, \tag{39}$$

which coincides with Nekrasov's partition function by construction after replacing Φ_i to vertex operators.

In [9], the authors gave the explicit form of the basis $|P\rangle_{\vec{Y}}$ when one of Y_i is null ($\varnothing$). For such cases, it is given as the Jack symmetric polynomial ${\rm Jac}_Y(x_1,\ldots,x_{|Y|})$ with its coupling constant $-b^2$ or $-1/b^2$. Here the power symmetric polynomials of arguments $x_1,\ldots,x_{|Y|}$ are given in terms of linear combination of oscillators $\sum_n(x_n)^k \propto a_{-k} \pm c_{-k}$, where $\pm$ depends on which Y_i is null. For generic $\vec{Y}$, the explicit construction of the states $|P\rangle_{\vec{Y}}$ is difficult and the authors gave the algorithm for the construction.

Later in [11], Belavin and Belavin found that the construction of the basis is simplified when $Q = 0$. Namely, the basis can be defined by the product of two Schur polynomials

$$|P\rangle_{\vec{Y}} = s_{Y_1}(x^{(1)})s_{Y_2}(x^{(2)}), \tag{40}$$

where the power symmetric polynomials of x and y are

$$x^{(1)}_k \propto a_{-k} + c_{-k}, \qquad x^{(2)}_k \propto a_{-k} - c_{-k}. \tag{41}$$

Now let us compare their construction with ours. It is well-known that a Schur polynomial can be interpreted as the natural diagonal basis of free fermion system (see, for example, Appendix B in [16], where a concise review is given). Therefore, the state (40) is a basis of two-fermion system. It is natural to compare it to $N = 2$ case in our setup.

In [9,11], the authors did not provide why particular combinations (41) are needed to construct the basis. On the other hand, in our approach, this exactly corresponds to how Virasoro symmetry is obtained from $\mathcal{W}_{1+\infty}$ when $U(1)$ factor is separated,

$$\phi^{(1)} = \varphi^U + \varphi^V, \qquad \phi^{(2)} = \varphi^U - \varphi^V, \tag{42}$$

where φ^U gives a_n and φ^V gives c_n. In Sec. V, we give a detailed study to derive the chain vector by using free fields.

While such coincidence might seem to be accidental, one can proceed to consider the $N > 2$ case as well. The next nontrivial case is $N = 3$, where the Fock space of W_3 algebra is generated by L_{-n} and W_{-n}. In our description, the orthogonal basis $\phi^{(i)}$ $(i = 1, 2, 3)$ are provided from free bosons as

$$\phi^{(1)} = \frac{1}{\sqrt{3}}\tilde{\varphi}^U + \frac{1}{\sqrt{2}}\varphi_1^V + \frac{1}{\sqrt{6}}\varphi_2^V,$$
$$\phi^{(2)} = \frac{1}{\sqrt{3}}\tilde{\varphi}^U - \frac{1}{\sqrt{2}}\varphi_1^V + \frac{1}{\sqrt{6}}\varphi_2^V, \tag{43}$$
$$\phi^{(3)} = \frac{1}{\sqrt{3}}\tilde{\varphi}^U - \sqrt{\frac{2}{3}}\varphi_2^V,$$

where we have changed normalization of the free boson for the $U(1)$ part $\varphi^U \to \frac{1}{\sqrt{3}}\tilde{\varphi}^U$ compared with Eq. (33). In this normalization, the operator product expansion becomes $\tilde{\varphi}^U(z)\tilde{\varphi}^U(0) \sim \ln z$, as those for $\varphi^V_{1,2}$. Therefore, we would like to see such linear combinations give a generalization of the diagonal basis as

$$|\vec{P}\rangle_{\vec{Y}} \sim s_{Y_1}(x^{(1)})s_{Y_2}(x^{(2)})s_{Y_3}(x^{(3)}), \tag{44}$$

where $x^{(1,2,3)}$ are the polynomial representation of $\phi^{(1,2,3)}$. In the following sections, we show that the chain vector, once expanded by this basis, have coefficients which will reproduce Nekrasov's formula correctly as the AGT-W conjecture predicts.

V. CHAIN VECTORS

Definition of level n chain vector.—Let $\mathcal{H}_n$ be the level n states generated from highest weight state $|\Delta\rangle$ by the action of generators of a chiral algebra. For example, for W_3 algebra, it is generated by L_{-n} and W_{-n} from $|\vec{p}\rangle$.

Let $|u_i\rangle$ be a basis of $\mathcal{H}_n$ $(i = 1, \ldots, \dim\mathcal{H}_n)$. We define a projector onto level n states as

$$\Pi_\Delta^{(n)} := \sum_{i,j} |u_i\rangle S_{ij}^{-1}\langle u_j|, \tag{45}$$

where $S_{ij} = \langle u_i|u_j\rangle$ is Shapovalov matrix. It satisfies $O_n\Pi_\Delta^{(N)} = \Pi_\Delta^{(N-n)}O_n$ for any element O_n in the chiral algebra. Then the chain vector at level n is defined as

$$|n\rangle_{\Delta,\Delta_1,\Delta_2} := \Pi_\Delta^{(n)}V_{\Delta_1}(1)|\Delta_2\rangle, \tag{46}$$

where the expression on the right-hand side should be determined by the conformal Ward identities.

We note that with the chain vector, one can express the four-point function as their inner product,

$$\langle\Delta_1|V_{\Delta_2}(z)V_{\Delta_3}(1)|\Delta_4\rangle = \sum_\Delta\sum_{n=0}^\infty z^n_{\Delta,\Delta_2,\Delta_1}\langle n|n\rangle_{\Delta,\Delta_3,\Delta_4}. \tag{47}$$

For the higher correlator, one has to define a generalization of chain vector as

$$\Pi_{\Delta_1}^{(n)}V_{\Delta_2}(1)\Pi_{\Delta_3}^{(m)} =: O(n,m)_{\Delta_1,\Delta_2,\Delta_3} \tag{48}$$

and compute the product

$$\sum_{n_1,\ldots,n_r}\sum_\Delta\langle n_1|O(n_1,n_2)\ldots O(n_{r-1},n_r)|n_r\rangle, \tag{49}$$

where we omit the weight Δ in the operators/vectors. Since this kind of correlator corresponds to instanton contribution of Nekrasov's partition function, the chain vector gives a building block to prove the AGT conjecture.

A. Chain vector for free boson

In this case, the highest weight state is $|p\rangle$ and chiral algebra is generated by a_{-n}. Since the basis of the oscillator Hilbert space $\{a_{-n_1}\ldots a_{-n_r}|p\rangle\}$ is orthogonal, the projector becomes very simple: for example, $\Pi_p^{(1)} = a_{-1}|p\rangle\langle p|a_1$.

Therefore, the evaluation of Eq. (46) involves the calculation of correlators of the form $\langle p|a_{n_1}\ldots a_{n_s}V_r(1)|q\rangle$, but they are also very simple: for example, $\langle p|a_n^l V_r(1)|q\rangle = r\langle p|V_r(1)|q\rangle$. By solving the recursion formula $a_n|N\rangle_{p,r,q} = r|N-n\rangle_{p,r,q}$ which can be proved as

$$a_n|N\rangle_{p,r,q} = a_n\Pi_p^{(N)}V_r(1)|q\rangle = \Pi_p^{(N-n)}a_nV_r(1)|q\rangle$$
$$= r\Pi_p^{(N-n)}V_r(1)|q\rangle = r|N-n\rangle_{p,r,q}, \tag{50}$$

one may obtain a generating function of chain vectors in a closed form,

$$\sum_{n=1}^\infty|n\rangle_{p,r,q}\zeta^n = e^{r\sum_{n=1}^\infty(1/n)a^l_{-n}\zeta^n}|p\rangle, \tag{51}$$

from which one may extract $|n\rangle_{p,r,q}$, for example,

$$|1\rangle_{p,r,q} = ra_{-1}|p\rangle, \qquad |2\rangle_{p,r,q} = \frac{1}{2}(ra_{-2} + (ra_{-1})^2)|\vec{p}\rangle. \tag{52}$$

We note that a chain vector for a free boson depends only on the momentum of $V_r(1)$. This is the characteristic feature for a free boson which is not shared by the chain vector for Virasoro or W_3.

B. Virasoro algebra

The recursion formula for the chain vector is

$$L_k|N\rangle_{\Delta,\Delta_1,\Delta_2} = (\Delta + k\Delta_1 - \Delta_2 + N - k)|N-k\rangle_{\Delta,\Delta_1,\Delta_2}. \tag{53}$$

It may be derived by combining $L_k\Pi_\Delta^{(N)} = \Pi_\Delta^{(N-k)}L_k$ and a conformal Ward identity

$$\langle u|L_kV_{\Delta_1}|\Delta_2\rangle = (\Delta + N - k - \Delta_2 + k\Delta_1)\langle u|V_{\Delta_1}|\Delta_2\rangle,$$

which holds for any level $N-k$ state $\langle u|$ from $\langle\Delta|$.

The chain vector may be derived in terms of Virasoro operators. However, in order to do it, we need to invert the Shapovalov matrix which is complicated. Therefore, instead of doing so, one may solve it more directly in terms of a free boson. For the $c = 1$ case, we have

$$L_n = \frac{1}{2}\sum_k :a_{n-k}a_k:, \tag{54}$$

where $[a_n, a_m] = n\delta_{n+m,0}$. Then we write

$$a_{-n} = nx_n, \qquad a_n = \partial_{x_n} \qquad (n > 0) \tag{55}$$

and express the bosonic Fock space as the polynomials of variables x_n $(n = 1, 2, 3, \ldots)$. For example, we rewrite $a_{-n_1} \ldots a_{-n_r}|p\rangle$ as $n_1 x_{n_1} \ldots n_r x_{n_r}$. Using this correspondence, we denote $\Psi_N(x)$ to represent the chain vector $|N\rangle_{\Delta,\Delta_1,\Delta_2}$. We use the vertex operator representation for primary fields with

$$\Delta = \frac{p^2}{4}, \qquad \Delta_1 = \frac{r^2}{4}, \qquad \Delta_2 = \frac{q^2}{4}, \tag{56}$$

which corresponds to $V_\Delta = e^{p\phi/\sqrt{2}}$, $V_{\Delta_1} = e^{r\phi/\sqrt{2}}$, and $V_{\Delta_2} = e^{q\phi/\sqrt{2}}$. As a result, the recursion relation (53) is written as the differential equation for Ψ_N,

$$\left(\sum_{r=1}^{\infty} r x_r \partial_{x_{k+r}} + \frac{p}{\sqrt{2}}\partial_{x_k} + \frac{1}{2}\sum_{s=1}^{k-1}\partial_{x_s}\partial_{x_{k-s}}\right)\Psi_N = (\Delta + k\Delta_1 - \Delta_2 + N - k)\Psi_{N-k}. \tag{57}$$

Starting from $\Psi_0 = 1$, one may solve it recursively. For example,

$$\Psi_1 = \frac{(p^2 - q^2 + r^2)x_1}{2\sqrt{2}p}, \qquad \Psi_2 = \beta_1 x_1^2 + \beta_2 x_2, \tag{58}$$

where

$$\beta_1 = \frac{(p^2-q^2+r^2)^2 - 4r^2}{16(p^2-1)}$$
$$\beta_2 = \frac{3p^4 - 2p^2(q^2-3r^2+2) - (q-r)(q+r)(q^2-r^2-4)}{8\sqrt{2}p(p^2-1)}. \tag{59}$$

Readers may wonder why the chain vector for Virasoro is rather complicated compared with that of the free boson. Actually if one of the momentum conservation conditions

$$p = \epsilon_q q + \epsilon_r r \qquad (\epsilon_q, \epsilon_r = \pm 1) \tag{60}$$

is satisfied, the chain vector is reduced to that of a free boson,

$$\sum_{n=0}^{\infty} \Psi_n \zeta^n \to \exp\left(-\frac{\epsilon_r r}{\sqrt{2}}\sum_{n=1}^{\infty} x_n \zeta^n\right). \tag{61}$$

If the conservation is violated, we need some screening currents to define the correlator. It explains why such simplification does not generally occur.

C. W_3 algebra

We can derive the chain vector for W_3 algebra similarly, namely, by combining $O_k \Pi_{\bar{\Delta}}^{(N)} = \Pi_{\bar{\Delta}}^{(N-k)} O_k$ with $O_k = L_k$ or W_k and the Ward identity for W_3 algebra. We use the label $\bar{\Delta}$ to represent the eigenvalues (Δ, w) for the highest weight representation.

The recursion formula for L_k is the same as Eq. (53). For the W_k generators, we use Eqs. 38 and 39 of [17]

$$\begin{aligned} W_k|N\rangle_{\bar{\Delta},\bar{\Delta}_1,\bar{\Delta}_2} &= \left(\frac{k(k+3)}{2}w_1 - w_2\right)|N-k\rangle_{\bar{\Delta},\bar{\Delta}_1,\bar{\Delta}_2} \\ &\quad + k\Pi_{\bar{\Delta}}^{(N-k)}(W_{-1}V_{\bar{\Delta}_1})|\bar{\Delta}_2\rangle \\ &\quad + (\Pi_{\bar{\Delta}}^{(N-k)}W_0)V_{\bar{\Delta}_1}|\bar{\Delta}_2\rangle. \end{aligned} \tag{62}$$

In order to make it a closed recursion formula, we need to impose the level 1 null state condition for $V_{\bar{\Delta}_1}$,

$$W_{-1}V_{\bar{\Delta}_1} = \frac{3w_1}{2\Delta_1}L_{-1}V_{\bar{\Delta}_1}. \tag{63}$$

Then the second term of Eq. (62) can be evaluated by the Ward identity for Virasoro. The third term should be left as it is. To summarize, the recursion formula for W_k is given as

$$\begin{aligned} W_k|N\rangle_{\bar{\Delta},\bar{\Delta}_1,\bar{\Delta}_2} &= \left(\frac{k(k+3)}{2}w_1 - w_2 + \frac{3kw_1}{2\Delta_1}(N - k + \Delta - \Delta_1 - \Delta_2)\right)|N-k\rangle_{\bar{\Delta},\bar{\Delta}_1,\bar{\Delta}_2} \\ &\quad + W_0|N-k\rangle_{\bar{\Delta},\bar{\Delta}_1,\bar{\Delta}_2}. \end{aligned} \tag{64}$$

Again, we would like to solve these recursion formulas by free boson representation. If we write

$$\begin{aligned} \partial\phi^1 &= p_1 z^{-1} + \sum_{k=n}^{\infty} x_n z^{n-1} + \sum_{n=1}^{\infty}\frac{z^{-n-1}}{n}\frac{\partial}{\partial x_n}, \\ \partial\phi^2 &= p_2 z^{-1} + \sum_{k=n}^{\infty} y_n z^{n-1} + \sum_{n=1}^{\infty}\frac{z^{-n-1}}{n}\frac{\partial}{\partial y_n}, \end{aligned} \tag{65}$$

the oscillator representation for generators L_k, W_k $(k \geq 0)$ becomes

$$\begin{aligned} L_k &= \sum_{r=1}^{\infty} r x_r \partial_{x_{k+r}} + p_1\partial_{x_k} + \frac{1}{2}\sum_{s=1}^{k-1}\partial_{x_s}\partial_{x_{k-s}} + \sum_{r=1}^{\infty} r y_r \partial_{y_{k+r}} \\ &\quad + p_2\partial_{y_k} + \frac{1}{2}\sum_{s=1}^{k-1}\partial_{y_s}\partial_{y_{k-s}}, \end{aligned} \tag{66}$$

$$6W_k = \sum_{n,m=1}^{n+m<k}\left(\frac{\partial^3}{\partial y_n \partial y_m \partial y_{k-n-m}} - 3\frac{\partial^3}{\partial x_n \partial x_m \partial y_{k-n-m}}\right) + 3\left(\sum_{\substack{n,m=1\\ n+m<k}}(n+m-k)\left(y_{n+m-k}\frac{\partial^2}{\partial y_n \partial y_m} - 2x_{n+m-k}\frac{\partial^2}{\partial x_n \partial y_m}\right.\right.$$
$$\left.\left. - y_{n+m-k}\frac{\partial^2}{\partial x_n \partial x_m}\right)\right) + 3\left(\sum_{n,m=1} nm\left(y_n y_m \frac{\partial}{\partial y_{n+m+k}} - 2x_n y_m \frac{\partial}{\partial x_{n+m+k}} - x_n x_m \frac{\partial}{\partial y_{n+m+k}}\right)\right)$$
$$- 3\left(\sum_{n=1}^{k-1}\left(-p_2\frac{\partial^2}{\partial y_n \partial y_{k-n}} + p_2 \frac{\partial^2}{\partial x_n \partial x_{k-n}} + p_1 \frac{\partial^2}{\partial x_n \partial y_{k-n}}\right)\right) + 6\sum_{n=1}^{\infty} n\left(p_2 y_n \frac{\partial}{\partial y_{n+k}} - p_2 x_n \frac{\partial}{\partial x_{n+k}} - p_1 x_n \frac{\partial}{\partial y_{n+k}}\right.$$
$$\left. - p_1 y_n \frac{\partial}{\partial x_{n+k}}\right) + 3\left((p_2^2 - p_1^2)\frac{\partial}{\partial y_k} - 2p_1 p_2 \frac{\partial}{\partial x_k}\right)(1-\delta_{k0}) + (p_2^3 - 3p_1^2 p_2)\delta_{k0}. \quad (67)$$

For level 1, the recursion formula is

$$L_1|1\rangle = (\Delta + \Delta_1 - \Delta_2)|0\rangle$$
$$W_1|1\rangle = \left(2w_1 - w_2 + w + \frac{3w_1}{2\Delta_1}(\Delta - \Delta_1 - \Delta_2)\right)|0\rangle. \quad (68)$$

If we write $|1\rangle = \alpha_1 x_1 + \alpha_2 y_1$, we obtain

$$L_1|1\rangle = \alpha_1 p_1 + \alpha_2 p_2,$$
$$2W_1|1\rangle = (p_2^2 - p_1^2)\alpha_2 - 2p_1 p_2 \alpha_1. \quad (69)$$

We assign the momentum $(0, m)$ for $V_1(1)$ and (q_1, q_2) for $|V_0\rangle$. We note that V_1 must have a level 1 null state and the assignment for V_1 is one possibility for it.

By comparing these formulas, one can determine $\alpha_{1,2}$,

$$\alpha_1 = \frac{A_1}{6(p_1^2 - 3p_2^2)}, \qquad \alpha_2 = -\frac{A_2}{6(p_1^2 - 3p_2^2)}, \quad (70)$$

where

$$A_1 := 3p_1^4 - 3(2p_2^2 + q_1^2 + q_2^2)p_1^2 + m^3 p_2 + 3m^2(p_1^2 - p_2^2) + 3mp_2(p_1^2 + p_2^2 - q_1^2 - q_2^2) - p_2(p_2 + 2q_2)(p_2^2 - 2q_2 p_2 - 3q_1^2 + q_2^2)$$
$$A_2 := m^3 + 6p_2 m^2 + 3(p_1^2 + p_2^2 - q_1^2 - q_2^2)m + 2(p_2 - q_2)(4p_2^2 + 4q_2 p_2 - 3q_1^2 + q_2^2). \quad (71)$$

The denominator factor $p_1^3 - 3p_1 p_2^2$ vanishes when $p_1 = 0$ or $p_1 = \pm\sqrt{3}p_2$. This is precisely the correct momentum to have a level 1 null state. Another consistency check is that it reduces to the free boson chain vector once we impose the momentum conservation law

$$p_1 = q_1, \qquad p_2 = m + q_2 \to \alpha_1 = 0, \qquad \alpha_2 = m. \quad (72)$$

VI. COMBINATION WITH THE $U(1)$ PART: COMPARISON WITH GAUGE THEORY

The claim in [11] is that once the chain vector is combined with the $U(1)$ part and reexpanded in terms of a Schur polynomial, its coefficients of expansion implies Nekrasov's formula.

The chain vector for the $U(1)$ part is written in the form

$$\Psi_p^U(\zeta) = \sum_{n=0}^{\infty} \Psi_{p,n}^U \zeta^n = e^{p\sum_{s=1}^{\infty} t_n \zeta^n}. \quad (73)$$

We mix it with the chain vector as

$$\Psi(\zeta) = \Psi^U(\zeta)\Psi^V(\zeta), \quad \text{or} \quad \Psi_N = \sum_{n=0}^{N} \Psi_n^U \Psi_{N-n}^V, \quad (74)$$

where $\Psi^V(\zeta)$ is the generating function for Virasoro or W_3 algebra. Let us first reproduce the results of [11] for Virasoro case.

A. Virasoro vs $SU(2)$ gauge theory

We have already given the explicit form of chain vector for Virasoro algebra in Eq. (58). We have computed the result up to level 3 but do not write it here, since it is complicated and not illuminating.

For $\Psi_{\Delta,\Delta_1,\Delta_2}$ with $\Delta = p^2/4$, $\Delta_1 = r^2/4$, and $\Delta_2 = q^2/4$, we choose the $U(1)$ part to be $\Psi^U_{r/\sqrt{2}}$. In $\mathcal{W}_{1+\infty}$ representation, it implies that we need to use the representation $(\lambda_1, \lambda_2) = (r, 0)$ in Eq. (10) for $C_i = 1$ and $K = 2$. After the combination as Eq. (74) and the change of variables as

$$x_n = (\tilde{x}_n - \tilde{y}_n)/\sqrt{2}, \qquad t_n = (\tilde{x}_n + \tilde{y}_n)/\sqrt{2}, \quad (75)$$

we get an expansion of the form

$$\Psi_N = \sum_{|Y_1|+|Y_2|=N} C(Y_1, Y_2) s_{Y_1}(\tilde{x}) s_{Y_2}(\tilde{y}), \quad (76)$$

where $s_Y(x)$ is the Schur polynomial in terms of a power sum polynomial. For example, up to level 3,

$$s_\varnothing(x) = 1, \qquad s_{[1]}(x) = x_1, \qquad s_{[2]}(x) = \frac{x_1^2}{2} + x_2,$$
$$s_{[1^2]}(x) = \frac{x_1^2}{2} - x_2, \qquad s_{[3]}(x) = \frac{x_1^3}{6} + x_1 x_2 + x_3, \quad (77)$$
$$s_{[1^3]}(x) = \frac{x_1^3}{6} - x_1 x_2 + x_3, \qquad s_{[2,1]}(x) = \frac{x_1^3}{3} - x_3.$$

The coefficient $C(Y_1, Y_2)$ is written in the form

$$C(Y_1, Y_2) = \frac{z(Y_1, p+q+r)z(Y_1, p-q+r)z(Y_2^t, p+q-r)z(Y_2^t, p-q-r)}{D(\vec{Y}, p)}, \tag{78}$$

where

$$\begin{aligned} z(Y, x) &= \prod_{(k,l)\in Y} (x/2 + k - l) \\ D(\vec{Y}, p) &= \prod_{i,j=1}^{2} \prod_{s\in Y_i} (a_i - a_j + A_i(s) + L_j(s) + 1) \end{aligned} \tag{79}$$

and $a_1 = p/2$, $a_2 = -p/2$. $s = (k, l)$ denotes the position of the box in a Young tableau (i.e. the box in k-th column and l-th row). $A(s)/L(s)$ is the arm/leg length of a Young tableau, respectively. In particular, for the level $N = 1, 2, 3$,

$$\begin{aligned} &D(([1], \varnothing), p) = p, \qquad D((\varnothing, [1]), p) = -p, \\ &D(([2], \varnothing), p) = 2p(p+1), \qquad D(([1^2], \varnothing), p) = 2p(p-1), \\ &D(([1], [1]), p) = -(p+1)(p-1), \qquad D((\varnothing, [2]), p) = 2p(p-1), \\ &D((\varnothing, [1^2]), p) = 2p(p+1), D(([3], \varnothing), p) = 6p(p+1)(p+2), \\ &D(([2,1], \varnothing), p) = 3p(p+1)(p-1), \qquad D(([1^3], \varnothing), p) = 6p(p-1)(p-2), \\ &D(([2], [1]), p) = -2p(p-1)(p+2), \qquad D(([1^2], [1]), p) = -2p(p+1)(p-2), \\ &\vdots \end{aligned} \tag{80}$$

Therefore, we can confirm that the coefficients (78) exactly correspond to Nekrasov's partition function with $\epsilon_1/\epsilon_2 = -1$.

B. W_3 vs $SU(3)$ gauge theory

We note that, as in the Virasoro case, the chain vector is constructed out of the free boson $\tilde{\varphi}^V_{1,2}$. Now we need to combine it with the $U(1)$ part $\tilde{\varphi}^U$ and rewrite the combined chain vector in terms of $\phi^{(i)}$. The relation between them is given in Eq. (43) as

$$\begin{aligned} \tilde{\varphi}^V_1 &= \frac{1}{\sqrt{2}}(\phi^{(1)} - \phi^{(2)}), \\ \tilde{\varphi}^V_2 &= \frac{1}{\sqrt{6}}(\phi^{(1)} + \phi^{(2)} - 2\phi^{(3)}), \\ \tilde{\varphi}^U &= \frac{1}{\sqrt{3}}(\phi^{(1)} + \phi^{(2)} + \phi^{(3)}). \end{aligned} \tag{81}$$

We also rewrite the momentum by those for the orthogonal basis,

$$\begin{aligned} &p_1 = \frac{1}{\sqrt{2}}(a_1 - a_2), \qquad p_2 = \frac{1}{\sqrt{6}}(a_1 + a_2 - 2a_3), \qquad p_3 = \frac{1}{\sqrt{3}}(a_1 + a_2 + a_3), \\ &q_1 = \frac{1}{\sqrt{2}}(b_1 - b_2), \qquad q_2 = \frac{1}{\sqrt{6}}(b_1 + b_2 - 2b_3), \qquad q_3 = \frac{1}{\sqrt{3}}(b_1 + b_2 + b_3), \\ &r_1 = \frac{1}{\sqrt{2}}(c_1 - c_2) = 0, \qquad r_2 = \frac{1}{\sqrt{6}}(c_1 + c_2 - 2c_3) = m, \qquad r_3 = \frac{1}{\sqrt{3}}(c_1 + c_2 + c_3), \end{aligned} \tag{82}$$

where p_3, q_3, r_3 are momenta for the $U(1)$ factor. a_i, b_i, c_i are momenta for orthogonal basis $\phi^{(i)}$. We need to impose $r_1 = 0$ for the corresponding vertex to a level 1 null state which is necessary to solve conformal Ward identity.

The chain vector is written as

$$\psi_1 = \psi_1^V + \psi_1^U; \qquad \psi_1^V = \alpha_1 x_1 + \alpha_2 x_2, \qquad \psi_1^U = r_3 t, \tag{83}$$

where t is the variable for $U(1)$ boson. Then we need to use the following assignment to proceed:

$$c_1 = c_2 = 0, \qquad c_3 = 3c. \tag{84}$$

From the viewpoint of $\mathcal{W}_{1+\infty}$ representation, this assignment is equivalent to impose $\lambda_1 = \lambda_2 = 0$ in Eq. (12) while leaving λ_3 arbitrary. We note that such assignment was also used for $SU(2)$ case. We guess that similar assignment will be necessary also for higher cases $N > 3$.

Another comment is that we also need to impose $p_3 = q_3 = 0$ in the W_3 chain vector to give the correct formula. This is natural since these parameters are momenta for $U(1)$ which is irrelevant in the representation of W_3 algebra.

We need to rewrite the oscillator similarly,

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}}(X_1 - X_2), \\ x_2 &= \frac{1}{\sqrt{6}}(X_1 + X_2 - 2X_3), \\ t &= \frac{1}{\sqrt{3}}(X_1 + X_2 + X_3). \end{aligned} \tag{85}$$

In terms of this basis, the level 1 chain vector has following factorized form:

$$\psi_1 = \gamma_1 X_1 + \gamma_2 X_2 + \gamma_3 X_3, \tag{86}$$

with

$$\begin{aligned} \gamma_1 &= \frac{(a_1 - b_1 + c)(a_1 - b_2 + c)(a_1 - b_3 + c)}{(a_1 - a_2)(a_1 - a_3)} \\ \gamma_2 &= \frac{(a_2 - b_1 + c)(a_2 - b_2 + c)(a_2 - b_3 + c)}{(a_2 - a_1)(a_2 - a_3)} \\ \gamma_3 &= \frac{(a_3 - b_1 + c)(a_3 - b_2 + c)(a_3 - b_3 + c)}{(a_3 - a_1)(a_3 - a_2)}. \end{aligned} \tag{87}$$

This again takes the expected form, i.e. the denominator factor corresponds to the factor of Nekrasov's partition function for a vector field and the numerator takes the form of that for fundamental or antifundamental matter fields.

We conclude that $\mathcal{W}_{1+\infty}$ symmetry seems to play a critical role in how to recombine free fields. It also seems to be essential in choosing the momentum for the intermediate vertex operator in the form $(0, \ldots, 0, \lambda)$. Also, the denominator factor vanishes when the weight Δ has the form $\lambda_i - \lambda_j =$ integer, which is exactly the null state condition suggested from Eq. (8).

VII. CONCLUSION

In this paper, we argue that $\mathcal{W}_{1+\infty}$ algebra explains the correct inclusion of the $U(1)$ factor to the symmetry of Toda fields. It also gives any W_N symmetry in the same footing, namely, it reduces to choosing correct quasifinite unitary representations. In this sense, it should be regarded as the correct symmetry behind the AGT-W relation.

The reader may have some criticism of our identification of the $U(1)$ factor as merely the enhancement of $SU(N)$ to $U(N)$. We would like to argue, however, that $\mathcal{W}_{1+\infty}$ algebra automatically contains infinite commuting charges $W(D^n)$ which would be helpful to understand the exactly solvable system behind such correspondence.

Of course, the computation made here still depends heavily on the original W symmetry. In this sense, we have not utilized the full machinery of the symmetry. For example, in the computation of the chain vector made in Secs. V and VI, we cannot use $\mathcal{W}_{1+\infty}$ algebra directly. A direct proof in [10], where a Selberg integral is performed, might be helpful.

Since $\mathcal{W}_{1+\infty}$ has much simpler structure than W_N algebra, it is easy to convince ourselves that factorization of Nekrasov's formula may directly come from $\mathcal{W}_{1+\infty}$ symmetry. So far, we have not achieved it since we do not know how to define the three-point functions which seemed not to be studied in the literature.

Such computation would be also useful to give us some inspiration to understand non-Lagrangian strong coupling theories which were conjectured by Gaiotto [3]. In the case of W_N, it was difficult to calculate a corresponding correlation function since the conformal Ward identity could not be solved. For the $\mathcal{W}_{1+\infty}$ case, however, it has much higher symmetry and one may have some hope to define the correlator.

Another material which we cannot study so far is the general case $Q \neq 0$. Since $\mathcal{W}_{1+\infty}$ algebra is limited to describe $C = N$, we need some sort of deformation. Judging from the observation in [9], it will be natural consider the interacting system (Calogero-Sutherland), to guess the symmetry behind it. A generalization of an exactly solvable system in the Appendix of [9] would be a promising direction. We note that the Jack polynomial has an interpretation of null states of W_N algebra [18]. See also the work [19], where general CFT was studied in the context of $\mathcal{W}_{1+\infty}$ algebra.

For the extension of the AGT conjecture to $SU(N)$ linear quiver gauge theory, we have conjectured that general level 1 null state describes the general puncture [3,20]. This correspondence seems to have some subtleties as found later [17]. In [21], the authors gave a proposal which would be a possible solution to the problem. However, the $U(1)$ seems to be involved if we examine levels higher than 2. We hope that $\mathcal{W}_{1+\infty}$ symmetry provides some hints to this issue.

Moreover, the discussion on the surface operator in $SU(N)$ gauge theory is also an interesting topic. For the $SU(2)$ case, it is already known that the corresponding operator in Liouville theory is related to level 2 null states

[22,23]. Then it is natural to expect that the corresponding operator in Toda theory is related to higher level null states. It is very complicated to classify them in W_N algebra, but from the viewpoint of $W_{1+\infty}$ algebra, this discussion may become much simpler.

$\mathcal{W}_{1+\infty}$ symmetry has been applied to many topics, for example, the quantum Hall effect [24], the matrix model [25] (see also a recent development in the context of AGT [26]), topological string [27], and crystal melting [28]. We hope that it is a good time now to develop the representation theory, such as the correlation function, to more detail.

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APPENDIX: SUMMARY OF CONVENTION

Free fields and vertex:

$$\phi^I(z)\phi^J(w) \sim \delta^{IJ}\log(z-w), \qquad \partial_z\phi^I = \sum_n a_n^I z^{-n-1}, \qquad [a_n^I, a_m^J] = n\delta_{n+m,0}\delta^{IJ}, \qquad a_0^I \equiv \hat{p}^I,$$
$$\partial\phi^I(z)V_{\vec{p}}(0) \sim \frac{p^I}{z}V_{\vec{p}}(0), \qquad \lim_{z\to 0}V_{\vec{p}}(z)|0\rangle = |\vec{p}\rangle, \qquad \hat{p}|\lambda\rangle = \lambda|\lambda\rangle. \tag{A1}$$

Noether currents of Virasoro and W_3:

$$T(z) = \frac{1}{2}{:}(\partial\phi^1)^2{:} + \frac{1}{2}{:}(\partial\phi^2)^2{:}, \qquad W(z) = \frac{1}{6}({:}(\partial\phi^2)^3{:} - 3{:}(\partial\phi^1)^2\partial\phi^2{:}). \tag{A2}$$

Conformal and W_3 weight:

$$L_0|\vec{p}\rangle = \Delta(\vec{p})|\vec{p}\rangle, \qquad W_0|\vec{p}\rangle = w(\vec{p})|\vec{p}\rangle, \qquad \Delta(\vec{p}) = \frac{1}{2}((p^1)^2 + (p^2)^2), \qquad w(\vec{p}) = \frac{1}{6}((p^2)^3 - 3p^2(p^1)^2). \tag{A3}$$

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