Black hole partition function using the hybrid formalism of superstrings

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The IIA superstring partition function Z_{IIA} , on a Euclidean $\text{AdS}_2 \times S^2 \times \text{CY}_3$ computes the modified elliptic genus Z_{BH} of the associated black hole. The hybrid formalism of superstrings on $\text{AdS}_2 \times S^2$, defined as a sigma model on the coset supermanifold $PSU(1, 1|2)/U(1) \times U(1)$ with a Wess-Zumino term, together with Calabi-Yau and chiral boson conformal field theories, is used to calculate the partition function of IIA superstrings on the Euclidean attractor geometry $\text{AdS}_2 \times S^2 \times \text{CY}_3$. Instead of the kappa symmetry analysis used by Beasely *et al.* in Ref. [33], we use world-sheet superconformal invariance to construct a nilpotent Becchi, Rouet, Stora, Tyutin (BRST) operator. The sigma model action is explicitly shown to be closed under this BRST operator. Localization arguments are then used to deform the world-sheet path integral with the addition of a BRST exact term, where contributions arise only from the center of AdS_2 and the north and south poles of S^2 . This leads to the Ooguri, Strominger, and Vafa result $Z_{\text{BH}} = Z_{\text{IIA}} = |Z_{\text{top}}|^2$, where $|Z_{\text{top}}|^2$ is the square of the topological string partition function.

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I. INTRODUCTION

In [1], Ooguri, Strominger, and Vafa (OSV) conjectured a relationship of the form

$$Z_{\rm BH} = |Z_{\rm top}|^2,\tag{1}$$

where $Z_{\rm BH}$ is the (indexed) entropy of four-dimensional Bogomol'nyi-Prasad-Sommerfeld (BPS) black holes in type II Calabi-Yau compactifications and $Z_{\rm top}$ is the square of the topological string partition function evaluated at the attractor point on the associated Calabi-Yau. One way to view the relationship is to think of it as an asymptotic expansion in the limit of large black hole charges. In this limit, $Z_{\rm BH}$ receives all order perturbative contributions from the *F*-term corrections in the low energy effective action of the $\mathcal{N} = 2$ supergravity. For the type IIA superstrings on Calabi-Yau 3-folds, these are of the general form

$$\int d^4x d^4\theta (W_{\alpha\beta}W^{\alpha\beta})^g F_g(X^\Lambda), \tag{2}$$

where X^{Λ} are the vector multiplet fields and the Weyl superfield $W^{\alpha\beta}$ involves the graviphoton field strength and the Weyl tensor. These *F*-term corrections are in turn captured by the topological string amplitudes F_g [2,3]. Taking into account the fact that the BPS black hole entropy gets corrected in the presence of these terms via attractor mechanism [4–9], and that the supergravity partition function defines a mixed thermodynamic ensemble, provides the link (1) [1]. Several refinements of the relationship (1) have been suggested in the literature [10–27].

In particular, in [28], an *M*-theory lift was used to calculate the black hole partition function at low temperatures as a dilute-gas sum over BPS wrapped membranes in $AdS_3 \times S^2 \times CY_3$, via the Maldacena, Strominger, Witten (MSW) conformal field theory (CFT) [29] and agrees with the Gopakumar-Vafa partition

function [30,31] (see related discussion in [32]). The relation (1) is then obtained after doing a modular transformation to go to high temperatures. Further, in [33] a computation of the black hole partition function without the need for a modular transformation was presented. In this method, first an *M*-theory lift of the IIA attractor geometry leads to a quotient of $AdS_3 \times S^2 \times CY_3$, whose asymptotic boundary is a torus. Then, using AdS_3/CFT_2 duality, it is shown that the partition function of IIA theory on the attractor geometry $AdS_2 \times S^2 \times CY_3$, denoted as Z_{IIA} , is equivalent to the partition function of the black hole. Starting from the Euclidean Calabi-Yau attractor geometry for the black hole, carrying D0-D2-D4 charges q_0 , q_A (electric), p^A (magnetic), respectively, it was argued that [33]

$$Z_{\text{IIA}}(\phi^{0}, \phi^{A}, p^{A}) = Z_{\text{BH}}$$

= $\text{Tr}_{R}(-)^{F} e^{(-(4\pi^{2}/\phi^{0}))L_{0}-q_{A}(\phi^{A}/\phi^{0})},$ (3)

where ϕ^0 , ϕ^A are potentials conjugate to the D0-D2 charges. The trace on the right-hand side runs over the black hole microstates computed in the CFT on the boundary torus dual to the *M*-theory lift of the IIA attractor geometry. In essence, the black hole partition function can be evaluated in terms of the type IIA partition function on $AdS_2 \times S^2 \times CY_3$ geometry [34].

In [33], the Green-Schwarz formalism of superstrings was used to set up such a calculation, based on the general procedure for computing instanton generated superpotential and higher derivative *F*-terms in the low energy effective action of string theory [35]. The idea is to evaluate the IIA partition function on Euclidean $AdS_2 \times S^2 \times \mathcal{M}$, in a perturbative string loop expansion, where \mathcal{M} is the moduli space of world-sheet instantons which wrap isolated holomorphic curves inside the Calabi-Yau geometry. At genus *g*, the $AdS_2 \times S^2$ path integral receives contributions from genus *g* (anti-)instantons which wrap (anti)holomorphic cycles in the Calabi-Yau geometry and sit at the center of AdS₂ and (north) south poles of S^2 . The instantons typically break some supersymmetries and one has to integrate over the resulting zero modes in the path integral. The analysis of [33] involved regularizing the divergence coming from such integrals using κ -symmetry (see [36–38] for a different approach and [39–41] for recent work). Furthermore, the Calabi-Yau part of the instanton partition function was assumed to produce the topological string partition function and using some input from supergravity, the relation (1) was obtained.

The purpose of this paper is to use the hybrid formalism [42] to calculate the partition function of IIA superstrings on $AdS_2 \times S^2$ with Ramond-Ramond (RR) flux. There are several advantages of this approach over the Green-Schwarz formalism. First, the calculation of scattering amplitudes in the hybrid formalism can be performed in a super-Poincaré covariant manner and is relevant for describing d = 4, N = 2 theories. The calculation of superspace low energy effective actions in Calabi-Yau compactifications and their connection to topological string amplitudes can be derived nicely [43,44]. Another important advantage is that studying string propagation in Ramond-Ramond backgrounds is in general difficult in the traditional Ramond-Neveu-Schwarz (RNS) formalism due to the need to introduce spin fields. In the Green-Schwarz formalism, covariant quantization is problematic, except in light-cone gauge. To address these perennial issues, several covariant quantization techniques have been developed over the last few years by Berkovits and collaborators. In ten dimensions, the pure spinor formalism [45] has led to a remarkable simplification of the calculation of superstring loop amplitudes in flat space and their equivalence to the RNS formalism has been fully demonstrated (see e.g. [46-50] and the nonminimal approach [51-54]). The formulation of the pure spinor superstring in $AdS_5 \times S^5$ background [45,55,56], following earlier studies on hybrid formalism in $AdS_3 \times S^3$ [57,58] and in $AdS_2 \times S^2$ [42] backgrounds, is being actively used in a world-sheet approach to the Maldacena conjecture [59–61].

There have been innumerable applications of the hybrid formalism of superstrings in the presence of RR backgrounds; to list a few, scattering amplitudes in lower dimensions [44,62], in obtaining C-deformation [63] from superstrings in graviphoton backgrounds [64], emergence of non(anti)commutative superspace [65,66], for Calabi-Yau compactifications to two dimensions [67], and more recently in the study of flux vacua from the world-sheet point of view [68,69].

We use the formulation of hybrid superstrings in the near horizon geometry of extremal black holes in four dimensions, presented in [42]. It was shown that two-dimensional sigma models based on the coset supermanifold $\frac{PSU(1,1|2)}{U(1)\times U(1)}$ with the Wess-Zumino term provide an elegant description of superstrings propagating on AdS₂ × S² background with Ramond-Ramond flux. It is also desirable to have a world-sheet perspective on our understanding of the derivation of OSV relation (1); as mentioned earlier, the world-sheet approaches to large N topological string dualities are playing a special role in our understanding of the Maldacena conjecture [61]. For the present case, the connection of low energy F-terms and the topological string amplitudes can be understood nicely in the hybrid approach [44] (see [70,71] for a review). More importantly, in the hybrid approach, the dilaton in type II theories couples to the N = (2, 2) world-sheet supercurvature via the Fradkin-Tseytlin term in the action [43]. This has the advantage that the scattering amplitudes have a welldefined dependence on the string coupling constant. This will be important while trying to get the right factors of topological string coupling in the IIA partition function.

The rest of the paper is organized as follows. In Sec. II, we review the hybrid formalism of superstrings in flat space. In Sec. III, we discuss the $\frac{PSU(1,1|2)}{U(1)\times U(1)}$ sigma model and write down the lowest order terms in the $AdS_2 \times S^2$ background in the $U(1) \times U(1)$ notation, using the expressions for left-invariant currents. In Sec. IV, we present various partition functions in $AdS_2 \times S^2 \times CY_3$ related to the black hole partition function. In Sec. V, we present the computation of type IIA partition function over $AdS_2 \times S^2 \times CY_3$ by embedding the theory in N = 4topological strings. We provide localization arguments using the Becchi, Rouet, Stora, Tyutin (BRST) method and obtain the OSV relation.

II. HYBRID FORMALISM OF SUPERSTRINGS

The fundamental definition of N = 2 superconformal theory describing hybrid superstrings in four dimensions is by a field redefinition of N = 1 RNS matter and ghost variables [70,72–76]. The N = 2, c = 6 system obtained after field redefinition is twisted and splits in to a fourdimensional c = -3, N = 2 superconformal field theory, coupled to an internal six-dimensional c = 9, N = 2 superconformal theory. Now, one can proceed in two ways to define physical states and calculate scattering amplitudes. The c = 6, N = 2 generators can be untwisted and coupled to a c = -6, N = 2 ghost system to write down a BRST operator. Another way is to embed this N = 2 system in small N = 4 algebra and use the N = 4 topological method, where there is no need to introduce ghost variables. The general four-dimensional and six-dimensional actions in arbitrary curved backgrounds with Ramond-Ramond flux have been discussed in [43,57], respectively. Below, we review the flat four-dimensional case and consider the $AdS_2 \times S^2$ case in the next section.

Flat space-time

For the case of type IIA superstrings compactified on Calabi-Yau 3-folds, the hybrid variables are as follows

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[42]. The four-dimensional part consists of four bosonic and 16 fermionic space-time variables given as X^m for m = 0 to 3 and $(\theta_L^{\alpha}, p_{L\alpha}), (\bar{\theta}_L^{\dot{\alpha}}, \bar{p}_{L\dot{\alpha}}) (\theta_R^{\alpha}, p_{R\alpha}), (\bar{\theta}_R^{\dot{\alpha}}, \bar{p}_{R\dot{\alpha}}),$ where α , $\dot{\alpha} = 1, 2$. The six-dimensional part has six bosonic variables Y^j, \bar{Y}_j for j = 1 to 3 and their associated superpartners, left-moving fermions $(\psi_L^j, \bar{\psi}_{Lj})$ and rightmoving fermions $(\psi_R^j, \bar{\psi}_{Rj})$ corresponding to the Calabi-Yau CFT. Finally, one has a chiral boson ρ_L and an antichiral boson ρ_R . The world-sheet action corresponding to these fields in the superconformal gauge can be divided into three parts as

$$S_{d=4} + S_{CY} + S_{\rho_L, \rho_R}.$$
 (4)

The four-dimensional part of the action in flat superspace is

$$S_{d=4} = \frac{1}{\alpha'} \int dz d\bar{z} \bigg[\frac{1}{2} \bar{\partial} X^m \partial X_m + p_{L\alpha} \bar{\partial} \theta_L^\alpha + \bar{p}_{L\dot{\alpha}} \bar{\partial} \bar{\theta}_L^{\dot{\alpha}} + p_{R\alpha} \partial \theta_R^\alpha + \bar{p}_{R\dot{\alpha}} \partial \bar{\theta}_R^{\dot{\alpha}} \bigg].$$
(5)

The Calabi-Yau part is

$$S_{\rm CY} = \frac{1}{\alpha'} \int dz d\bar{z} (\bar{\partial} Y^j \partial \bar{Y}_j + \psi_L^j \bar{\partial} \bar{\psi}_{Lj} + \psi_R^j \bar{\partial} \bar{\psi}_{Rj}), \quad (6)$$

and S_{ρ_L,ρ_R} stands for the action of the chiral and antichiral bosons. For the present case, all the world-sheet fields satisfy periodic boundary conditions. The free field operator-product expansions (OPEs) corresponding to the above action are

$$\begin{aligned} X^{m}(y)X^{n}(z) &\to -\alpha'\eta^{mn}\ln|y-z|^{2}, \\ p_{L\alpha}(y)\theta_{L}^{\beta}(z) &\to \frac{\alpha'\delta_{\alpha}^{\beta}}{y-z}, \quad \bar{p}_{L\dot{\alpha}}(y)\bar{\theta}_{L}^{\dot{\beta}}(z) \to \frac{\alpha'\delta_{\dot{\alpha}}^{\dot{\beta}}}{y-z}, \\ p_{R\alpha}(y)\theta_{R}^{\beta}(z) &\to \frac{\alpha'\delta_{\alpha}^{\beta}}{\bar{y}-\bar{z}}, \quad \bar{p}_{R\dot{\alpha}}(y)\bar{\theta}_{R}^{\dot{\beta}}(z) \to \frac{\alpha'\delta_{\dot{\alpha}}^{\dot{\beta}}}{\bar{y}-\bar{z}}, \\ \rho_{L}(y)\rho_{L}(z) \to -\ln(y-z), \quad \rho_{R}(y)\rho_{R}(z) \to -\ln(\bar{y}-\bar{z}). \end{aligned}$$

$$(7)$$

The world-sheet action (4) is manifestly conformally invariant from the conformal dimensions of the fields. The action (4) further has an N = 2 superconformal invariance realized nonlinearly, the left-moving generators of which are¹

$$T^{L} = \frac{1}{2} \partial X^{m} \partial X_{m} + p_{L\alpha} \partial \theta_{L}^{\alpha} + \bar{p}_{L\dot{\alpha}} \partial \bar{\theta}_{L}^{\dot{\alpha}} + \frac{\alpha'}{2} \partial \rho_{L} \partial \rho_{L} + T^{L}_{CY}$$
$$G^{-L} = \frac{1}{\sqrt{\alpha'}} e^{\rho_{L}} (d_{L})^{2} + G^{-L}_{CY} \quad G^{+L} = \frac{1}{\sqrt{\alpha'}} e^{-\rho_{L}} (\bar{d}_{L})^{2} + G^{+L}_{CY}$$
$$J^{L} = \alpha' \partial \rho_{L} + J^{L}_{CY} \tag{8}$$

where

$$d_{L\alpha} = p_{L\alpha} + i\sigma^{m}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}_{L}\partial X_{m} - \frac{1}{2}(\bar{\theta}_{L})^{2}\partial\theta_{L\alpha} + \frac{1}{4}\theta_{L\alpha}\partial(\bar{\theta}_{L})^{2},$$

$$\bar{d}_{L\dot{\alpha}} = \bar{p}_{L\dot{\alpha}} + i\sigma^{m}_{\alpha\dot{\alpha}}\theta^{\alpha}_{L}\partial X_{m} - \frac{1}{2}(\theta_{L})^{2}\partial\bar{\theta}_{L\dot{\alpha}} + \frac{1}{4}\bar{\theta}_{L\dot{\alpha}}\partial(\theta_{L})^{2},$$

(9)

and $(d_L)^2$ means $\frac{1}{2} \epsilon^{\alpha\beta} d_{L\alpha} d_{L\beta}$. The right-moving N = 2 generators are given as

$$T^{R} = \frac{1}{2} \bar{\partial} X^{m} \bar{\partial} X_{m} + p_{R\alpha} \bar{\partial} \theta^{\alpha}_{R} + \bar{p}_{R\dot{\alpha}} \bar{\partial} \bar{\theta}^{\dot{\alpha}}_{R} + \frac{\alpha'}{2} \bar{\partial} \rho_{R} \bar{\partial} \rho_{R} + T^{R}_{CY}$$

$$G^{-R} = \frac{1}{\sqrt{\alpha'}} e^{\rho_{R}} (d_{R})^{2} + G^{+R}_{CY},$$

$$G^{+R} = \frac{1}{\sqrt{\alpha'}} e^{-\rho_{R}} (\bar{d}_{R})^{2} + G^{-R}_{CY}, \quad J^{R} = \alpha' \bar{\partial} \rho_{R} - J^{R}_{CY} \quad (10)$$

with

$$d_{R\alpha} = p_{R\alpha} + i\sigma^{m}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}_{R}\bar{\partial}X_{m} - \frac{1}{2}(\bar{\theta}_{R})^{2}\bar{\partial}\theta_{R\alpha} + \frac{1}{4}\theta_{R\alpha}\bar{\partial}(\bar{\theta}_{R})^{2},$$

$$\bar{d}_{R\dot{\alpha}} = \bar{p}_{R\dot{\alpha}} + i\sigma^{m}_{\alpha\dot{\alpha}}\theta^{\alpha}_{R}\bar{\partial}X_{m} - \frac{1}{2}(\theta_{R})^{2}\bar{\partial}\bar{\theta}_{R\dot{\alpha}} + \frac{1}{4}\bar{\theta}_{R\dot{\alpha}}\bar{\partial}(\theta_{R})^{2}.$$
(11)

The \pm signs on the fermionic generators $G^{L,R}$ represent their U(1) charges with respect to $J^{L,R}$. Also, $[T_{CY}^{L,R}, G_{CY}^{L,R\pm}, G_{CY}^{L,R\pm}, J_{CY}^{L,R}]$ are the left- and right-moving N = 2 c = 9 generators describing the Calabi-Yau 3-fold. We have written the left- and right-moving algebras explicitly for the type IIA superstring. The type IIB generators can be obtained by noting that under mirror symmetry, $J_{CY}^R \rightarrow -J_{CY}^R$, resulting in interchange $G_{CY}^{+R} \rightarrow$ G_{CY}^{-R} . The space-time part remains unaffected by this change.

The set of operators $d_{L\alpha}$, $\bar{d}_{L\dot{\alpha}}$, as shown in [77], satisfy the algebra

$$d_{L\alpha}(y)\bar{d}_{L\dot{\alpha}}(z) \to 2i\alpha' \frac{\prod_{L\alpha\dot{\alpha}}}{y-z}, \qquad d_{L\alpha}(y)d_{L\beta}(z) \to \text{reg},$$

$$\bar{d}_{L\dot{\alpha}}(y)\bar{d}_{L\dot{\beta}}(z) \to \text{reg}, \qquad (12)$$

with similar expressions for the right-moving part. Here

$$\Pi_{L}^{m} = \partial X^{m} - \frac{i}{2} \sigma_{\alpha\dot{\alpha}}^{m} (\theta_{L}^{\alpha} \partial \bar{\theta}_{L}^{\dot{\alpha}} + \bar{\theta}_{L}^{\dot{\alpha}} \partial \theta_{L}^{\alpha}),$$

$$\Pi_{R}^{m} = \bar{\partial} X^{m} - \frac{i}{2} \sigma_{\alpha\dot{\alpha}}^{m} (\theta_{R}^{\alpha} \bar{\partial} \bar{\theta}_{R}^{\dot{\alpha}} + \bar{\theta}_{R}^{\dot{\alpha}} \bar{\partial} \theta_{R}^{\alpha}).$$
(13)

The operators $d_{L\alpha}$, $\bar{d}_{L\dot{\alpha}}$ also commute with the space-time supersymmetry generators of the action

¹The superscripts *L*, *R* on various generators stand for left- and right-moving parts of the algebra. When these superscripts are not indicated explicitly, we always mean the left-moving part. We also adopt the notation $|A|^2 = A_L A_R$

$$q_{L\alpha} = \int dz Q_{L\alpha}$$

where $Q_{L\alpha} = p_{L\alpha} - i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}_L \partial X_m - \frac{1}{4}(\bar{\theta}_L)^2 \partial \theta_{L\alpha},$
 $\bar{q}_{L\dot{\alpha}} = \int dz \bar{Q}_{L\dot{\alpha}}$
where $\bar{Q}_{\dot{\alpha}} = \bar{p}_{L\dot{\alpha}} - i\sigma^m_{\alpha\dot{\alpha}}\theta_{L\alpha}\partial X_m - \frac{1}{4}(\theta_L)^2 \partial \bar{\theta}_{L\dot{\alpha}}.$ (14)

Further, $q_{R\alpha}$ and $\bar{q}_{R\dot{\alpha}}$ are obtained by replacing *L* with *R* and ∂ with $\bar{\partial}$. Thus, Eqs. (9) and (14) contain the world-sheet versions of the space-time superspace covariant derivatives and supersymmetry generators, respectively.

III. HYBRID FORMALISM IN $AdS_2 \times S^2$

The four-dimensional part of the action $S_{d=4}$ in Eq. (4) can be replaced with the world-sheet action for the sigma model on $AdS_2 \times S^2$. Below, we discuss the $AdS_2 \times S^2$ model in detail. In [42] it was shown that the twodimensional sigma models based on the coset supermanifold $PSU(1, 1|2)/U(1) \times U(1)$ can be used to quantize superstrings on $AdS_2 \times S^2$ in RR backgrounds. One of the main advantages of the hybrid approach is that fourdimensional super-Poincaré invariance and the N = 2 target space-time supersymmetry can be made manifest even in the presence of RR fields. For instance, like in Green-Schwarz formalism, the model has fermions which are spinors in target space and scalars on the world sheet allowing a simple treatment of RR fields. Second, like in RNS formalism, the action is quadratic in flat space allowing for quantization. As mentioned earlier, a first principle definition of the world-sheet variables in hybrid formalism is through a redefinition of RNS world-sheet variables. This follows from the fact that any critical N = 1 string can be embedded in an N = 2 string [73].

For type II superstrings, the four κ -symmetries of the Green-Schwarz action are replaced by a critical worldsheet N = (2, 2) superconformal invariance. These local N = 2 superconformal generators are nothing but the twisted RNS world-sheet generators. The important difference with the RNS approach is that the vertex operators for RR fields do not have square-root cuts with supersymmetry generators and there is no need to sum over spin structures on the world sheet [70].

A. $AdS_2 \times S^2$ sigma model

The $AdS_2 \times S^2$ sigma model action is constructed as a gauged principle chiral field [42,78]. Let $g(x) \in G$ denote a map from the world sheet into the supergroup *G*, with the current $J = g^{-1}dg$ valued in the Lie algebra *G*. The sigma model action on the coset space is constructed by gauging a subgroup *H*, whose Lie algebra is \mathcal{H}_0 . To construct a string theory on $AdS_2 \times S^2$, one has to quotient the group $G \equiv PSU(1, 1|2)$ by (the right action of) bosonic

subgroups $H \equiv U(1) \times U(1)$. This subgroup is in fact the invariant locus of a \mathbb{Z}_4 automorphism, the existence of which plays a crucial role in the construction. In effect, a \mathbb{Z}_4 decomposition of the PSU(1, 1|2) Lie algebra \mathcal{G} can be written as [42]

$$\mathcal{G} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$$

where the subspace \mathcal{H}_k is the eigenspace of the \mathbb{Z}_4 generator Ω with eigenvalue $(i)^k$. The subspaces \mathcal{H}_1 and \mathcal{H}_3 contain all the fermionic generators of the algebra \mathcal{G} and \mathcal{H}_0 , \mathcal{H}_2 contain the bosonic generators of \mathcal{G} .

The $AdS_2 \times S^2$ sigma model action is a sum of the action on the coset supermanifold $PSU(1, 1|2)/U(1) \times U(1)$ and a Wess-Zumino term and is given as² [42]

$$S_{\text{AdS}_2 \times S^2} = \frac{1}{\pi \lambda^2} \int d^2 x \operatorname{Str}\left(\frac{1}{2}J_2 \bar{J}_2 + \frac{3}{4}J_1 \bar{J}_3 + \frac{1}{4}\bar{J}_1 J_3\right), \quad (15)$$

where the left-invariant currents are given as

$$J_{2} = (g^{-1}\partial g)^{m}T_{m},$$

$$J_{1} = (g^{-1}\partial g)^{\alpha}_{R}T^{R}_{\alpha} + (g^{-1}\partial g)^{\dot{\alpha}}_{R}T^{R}_{\dot{\alpha}},$$

$$J_{3} = (g^{-1}\partial g)^{\alpha}_{L}T^{L}_{\alpha} + (g^{-1}\partial g)^{\dot{\alpha}}_{L}T^{L}_{\dot{\alpha}},$$

(16)

where T_m , $T_{\alpha,\dot{\alpha}}^{R,L}$ denote the PSU(1, 1|2) generators corresponding to translations and supersymmetry. \overline{J} 's can be obtained by replacing ∂ with $\overline{\partial}$ and Str stands for supertrace over the PSU(1, 1|2) matrices, the algebra of which is given in Eq. (28). The action in Eq. (15) has invariance under global PSU(1, 1|2) transformations, which are realized by left-multiplication as $\delta g = (\Sigma^A T_A)g$. There is also an invariance under local $U(1) \times U(1)$ gauge transformations, which is realized on $g(x, \theta)$ by a right-multiplication as $\delta g = g \Sigma(x)$. The supersymmetry transformations (up to a local Lorentz rotation) of the superspace coordinates are achieved by a global left action of the superalgebra on the coset superspace, whereas the κ -supersymmetry corresponds to a local right action on the same coset superspace. The κ -supersymmetry is explicitly broken in the present case and is replaced by world-sheet superconformal invariance [79].

It is useful to write the action by introducing auxiliary fields as

$$S_{\text{AdS}_{2}\times S^{2}} = \frac{1}{\alpha'} \int d^{2}z \bigg[\frac{1}{2} \eta_{cd} J^{c} \bar{J}^{d} - \frac{1}{4Ng_{s}} \delta_{\alpha_{L}\beta_{R}} (J^{\alpha_{L}} \bar{J}^{\beta_{R}} - \bar{J}^{\alpha_{L}} J^{\beta_{R}}) - \frac{1}{4Ng_{s}} \delta_{\dot{\alpha}_{L}\dot{\beta}_{R}} (J^{\dot{\alpha}_{L}} \bar{J}^{\dot{\beta}_{R}} - \bar{J}^{\dot{\alpha}_{L}} J^{\dot{\beta}_{R}}) + d_{\alpha_{L}} \bar{J}^{\alpha_{L}} + \bar{d}_{\dot{\alpha}_{L}} \bar{J}^{\dot{\alpha}_{L}} + d_{\alpha_{R}} J^{\alpha_{R}} + \bar{d}_{\dot{\alpha}_{R}} J^{\dot{\alpha}_{R}} + Ng_{s} d_{\alpha_{L}} d_{\beta_{R}} \delta^{\alpha_{L}\beta_{R}} + Ng_{s} \bar{d}_{\dot{\alpha}_{L}} \bar{d}_{\dot{\beta}_{R}} \delta^{\dot{\alpha}_{L}\dot{\beta}_{R}} \bigg].$$
(17)

²We use the notation $J \equiv J_z$, $\bar{J} \equiv J_{\bar{z}}$.

This form of the action can be obtained by writing the flat space-time action in (5) in terms of manifestly supersymmetric variables of Eqs. (13) and (21), and then covariantizing in a curved background [42]. Using the following constraint equations

$$d_{\alpha_L} = \frac{1}{Ng_s} \delta_{\alpha_L \beta_R} J^{\beta_R}, \qquad d_{\alpha_R} = -\frac{1}{Ng_s} \delta_{\beta_L \alpha_R} \bar{J}^{\beta_L}, \quad (18)$$

and performing the scaling

$$E_M^c \to (Ng_s)^{-1} E_M^c, \quad E_M^{\alpha_{L,R}}, E_M^{\dot{\alpha}_{L,R}} \to (Ng_s)^{-(1/2)} E_M^{\alpha_{L,R}}, E_M^{\dot{\alpha}_{L,R}},$$
(19)

one arrives at the action

$$S_{\text{AdS}_{2}\times S^{2}} = \frac{1}{\alpha' N^{2} g_{s}^{2}} \int d^{2}z \bigg[\frac{1}{2} \eta_{cd} J^{c} \bar{J}^{d} - \frac{1}{4} \delta_{\alpha_{L}\beta_{R}} (J^{\alpha_{L}} \bar{J}^{\beta_{R}} + 3 \bar{J}^{\alpha_{L}} J^{\beta_{R}}) - \frac{1}{4} \delta_{\dot{\alpha}_{L}\dot{\beta}_{R}} (J^{\dot{\alpha}_{L}} \bar{J}^{\dot{\beta}_{R}} + 3 \bar{J}^{\dot{\alpha}_{L}} J^{\dot{\beta}_{R}}) \bigg].$$
(20)

The above PSU(1, 1|2) invariant action can be written in a more familiar form for type IIA superstrings on $AdS_2 \times S^2$ with Ramond-Ramond fluxes in a manifestly space-time supersymmetric manner using the following identifications [42]:

$$J^{c} \equiv \Pi^{c}, \qquad J^{\alpha_{L,R}} \equiv \Pi^{\alpha_{L,R}}, \qquad J^{\dot{\alpha}_{L,R}} \equiv \Pi^{\dot{\alpha}_{L,R}}, \quad (21)$$

and similarly for \bar{J} 's as well. Here, $\Pi_j^A = E_M^A \partial_j Z^M$, $Z^M = (X^m, \theta_L^{\mu}, \bar{\theta}_L^{\dot{\mu}}, \theta_R^{\hat{\mu}}, \bar{\theta}_R^{\dot{\mu}})$, supervierbein E_M^A and A takes the tangent-superspace values $[c, \alpha_L, \dot{\alpha}_L, \alpha_R, \dot{\alpha}_R]$. The full action is now

$$S = S_{\mathrm{AdS}_2 \times S^2} + S_{\mathrm{CY}} + S_{\rho_L, \rho_R}.$$
 (22)

We now collect the N = 2 generators in this notation

$$T = T_{d=4} + T_{CY}, \qquad G^{-} = G_{d=4}^{-} + G_{CY}^{-},$$

$$G^{+} = G_{d=4}^{+} + G_{CY}^{+}, \qquad J = J_{d=4} + J_{CY},$$
(23)

where

$$\begin{split} T_{d=4} &= \Pi_{\bar{z}}^{c} \Pi_{c\bar{z}} + d_{\alpha} \Pi_{\bar{z}}^{\alpha} + \bar{d}_{\dot{\alpha}} \Pi_{\bar{z}}^{\dot{\alpha}} + \frac{1}{2} \bar{\partial} \rho \bar{\partial} \rho, \\ G_{d=4}^{-} &= e^{\rho} (d)^{2}, \quad G_{d=4}^{+} = e^{-\rho} (\bar{d})^{2}, \quad J_{d=4} = \bar{\partial} \rho \quad (24) \end{split}$$

represent the generators of c = -3, N = 2 algebra describing $AdS_2 \times S^2$, and

$$T_{\rm CY} = \bar{\partial} Y^j \bar{\partial} \bar{Y}_j + \frac{1}{2} (\psi^j \bar{\partial} \bar{\psi}_j + \bar{\psi}_j \bar{\partial} \psi^j),$$

$$G_{\rm CY}^- = \psi^j \bar{\partial} \bar{Y}_j, \quad G_{\rm CY}^+ = + \bar{\psi}_j \bar{\partial} Y^j, \quad J_{\rm CY} = \psi^j \bar{\psi}_j, \quad (25)$$

represent the c = 9, N = 2 generators describing the Calabi-Yau 3-fold. The generators of $AdS_2 \times S^2$ algebra, given in Eq. (24), are similar to the ones of flat space-time given in Eqs. (8) and (10), except for the constraints

satisfied by d in Eq. (18). One can use the equations of motion in (18) to write the four-dimensional generators explicitly in terms of the left-invariant currents. We also note that the fermionic generators in (24) remain holomorphic even at the quantum level [42]. The generators in Eq. (23) correspond to a c = 6, N = 2 algebra. In Sec. IV, we topologically twist this algebra and embed it in an N = 4 algebra.

B. AdS₂ × S² action in $U(1) \times U(1)$ notation

For the localization arguments to be presented in Sec. V C, it will further be useful to write the action (20) in a manner in which the $U(1) \times U(1)$ charges are manifest. This notation is quite useful, as most of the algebraic relations can be easily obtained just from constructing $U(1) \times U(1)$ invariant expressions. The sigma model on the coset space G/H with the Wess-Zumino term is oneloop conformally invariant provided that the group G is Ricci flat and H is the fixed locus of a Z_4 automorphism of G. For instance, J_1 decomposes under $H = U(1) \times U(1)$ as³

$$J_1 \equiv J_R^{++}, J_R^{--}, J_R^{-+}, J_R^{+-}.$$
 (26)

Now, in terms of the decomposition given in (26), the fermionic generators of the N = 2 superconformal algebra in AdS₂ × S² can be constructed as [42]

$$G_{d=4}^{-R} = \int dz (e^{\rho_R} J_R^{++} J_R^{--}), \quad G_{d=4}^{+R} = \int dz (e^{-\rho_R} J_R^{+-} J_R^{-+})$$
(27)

and also the left-moving part involving J^3 . The transformations of various fields under these generators are obtained by a global right-multiplication of g. Note that unlike in flat space, the currents have dependence on both holomorphic and antiholomorphic coordinates.

Let us start by writing down the PSU(1, 1|2) algebra relations. The generators of the bosonic subgroup $SO(2, 1) \times SO(3)$ are $M, T_{+0}^{+}, T_{-0}^{-}, N, T_{0^+_{+}}^{+}, T_{0^-_{-}}^{-}$, where the subscripts denotes charges under $U(1) \times U(1)$. The fermionic generators are T_{++}^i, T_{--}^i and T_{+-}^i, T_{-+}^i , where *i*, j = L, R. Thus, the relevant relations of the PSU(1, 1|2)algebra are

$$[T_{+0}^{i}, T_{-0}^{i}] = M, \qquad [T_{0_{+}^{i}}, T_{0_{-}^{i}}^{i}] = N,$$

$$\{T_{++}^{i}, T_{+-}^{j}\} = \delta^{ij}T_{+0}^{i}, \qquad \{T_{++}^{i}, T_{-+}^{j}\} = \delta^{ij}T_{0_{+}^{i}},$$

$$\{T_{--}^{i}, T_{+-}^{j}\} = \delta^{ij}T_{0_{-}^{i}}, \qquad \{T_{--}^{i}, T_{-+}^{j}\} = \delta^{ij}T_{-0}^{i},$$

$$\{T_{++}^{i}, T_{--}^{j}\} = \epsilon^{ij}(N-M), \qquad \{T_{-+}^{i}, T_{+-}^{j}\} = \epsilon^{ij}(N+M),$$

$$(28)$$

³The fermionic generators have appropriate flat-space limits; for instance, $J_{++,--} \rightarrow d_{\alpha}$, $J_{+-,-+} \rightarrow d_{\dot{\alpha}}$. We take the ++, -- subscripts to follow from undotted indices of fermions and +-, -+ indices from dotted ones.

a

where $\epsilon^{LR} = 1 = -\epsilon^{RL}$. Using the above generators, the action in (20) can be written as

$$S_{AdS_{2}\times S^{2}} = \frac{1}{\alpha' N^{2} g^{2}} \int d^{2}z \bigg[\frac{1}{2} (J_{+}^{+0} \bar{J}_{-}^{-0} - J_{+}^{0+} \bar{J}_{-}^{0-} + J_{-}^{-0} \bar{J}_{+}^{+0} - J_{-}^{0-} \bar{J}_{-}^{0+}) \\ - \frac{1}{4} (J_{L}^{--} \bar{J}_{R}^{++} + 3 \bar{J}_{L}^{--} J_{R}^{++} + J_{L}^{++} \bar{J}_{R}^{--} + 3 \bar{J}_{L}^{++} J_{R}^{--}) \\ - \frac{1}{4} (J_{L}^{+-} \bar{J}_{R}^{-+} + 3 \bar{J}_{L}^{+-} J_{R}^{-+} + J_{L}^{-+} \bar{J}_{R}^{+-} + 3 \bar{J}_{L}^{-+} J_{R}^{+-}) \bigg].$$

$$(29)$$

This action and its equations of motion are important to the localization arguments to be presented in Sec. VC. Using the variation of a general current J as $\delta J = d\delta X + [J, \delta X]$, we obtain the equations of motion of fermionic currents, following from (29) as

$$\nabla \bar{J}_{L}^{--} = 0, \qquad \nabla \bar{J}_{L}^{++} = 0, \qquad \nabla \bar{J}_{L}^{-+} = 0,$$

$$\nabla \bar{J}_{L}^{+-} = 0, \qquad \bar{\nabla} J_{R}^{--} = 0, \qquad \bar{\nabla} J_{R}^{++} = 0, \qquad (30)$$

$$\bar{\nabla} J_{R}^{-+} = 0, \qquad \bar{\nabla} J_{R}^{+-} = 0,$$

where we have only written the equations for covariantly holomorphic and antiholomorphic currents. In Eq. (30), $\nabla J = \partial J + [J_0, J]$ with J_0 denoting the generators in the subspace \mathcal{H}_0 , stands for the covariant derivative, and similarly for ∇ . Using the parameterization of the group element as $g = \exp(XT + \theta_L T_L + \theta_R T_R)$, the leftinvariant currents are

$$J_{+}^{+0} = \partial X_{+}^{+0} + \frac{1}{2} (\partial \theta_{L}^{++} \theta_{L}^{+-} + \partial \theta_{L}^{+-} \theta_{L}^{++} + L \leftrightarrow R) + \cdots,$$

$$J_{-}^{-0} = \partial X_{-}^{-0} + \frac{1}{2} (\partial \theta_{L}^{--} \theta_{L}^{-+} + \partial \theta_{L}^{-+} \theta_{L}^{--} + L \leftrightarrow R) + \cdots,$$

$$J_{+}^{0^{+}} = \partial X_{+}^{0^{+}} + \frac{1}{2} (\partial \theta_{L}^{++} \theta_{L}^{-+} + \partial \theta_{L}^{-+} \theta_{L}^{++} + L \leftrightarrow R) + \cdots,$$

$$J_{-}^{0^{-}} = \partial X_{-}^{0^{-}} + \frac{1}{2} (\partial \theta_{L}^{--} \theta_{L}^{+-} + \partial \theta_{L}^{+-} \theta_{L}^{--} + L \leftrightarrow R) + \cdots,$$
(31)

and the barred ones can be obtained similarly. The left-handed currents for instance are

$$J_{L}^{++} = \partial \theta_{L}^{++} + \frac{1}{2} (-\partial X_{+}^{+0} \theta_{R}^{-+} + \partial \theta_{R}^{-+} X_{+}^{+0} + \partial X_{+}^{0+} \theta_{R}^{+-} - \partial \theta_{R}^{+-} X_{+}^{0+}) + \cdots,$$

$$J_{L}^{--} = \partial \theta_{L}^{--} + \frac{1}{2} (-\partial X_{-}^{-0} \theta_{R}^{+-} + \partial \theta_{R}^{+-} X_{-}^{-0} + \partial X_{-}^{0-} \theta_{R}^{-+} - \partial \theta_{R}^{-+} X_{-}^{0-}) + \cdots,$$

$$J_{L}^{+-} = \partial \theta_{L}^{+-} + \frac{1}{2} (\partial X_{+}^{+0} \theta_{R}^{--} - \partial \theta_{R}^{--} X_{+}^{+0} - \partial X_{-}^{0-} \theta_{R}^{++} + \partial \theta_{R}^{++} X_{-}^{0-}) + \cdots,$$

$$J_{L}^{-+} = \partial \theta_{L}^{-+} + \frac{1}{2} (\partial X_{-}^{-0} \theta_{R}^{++} - \partial \theta_{R}^{++} X_{-}^{-0} - \partial X_{+}^{0+} \theta_{R}^{--} + \partial \theta_{R}^{--} X_{+}^{0+}) + \cdots,$$
(32)

and the right-handed currents and barred ones can be obtained similarly. Thus, the first few terms relevant for us in the action (29) are

$$S = \int d^2 z (\partial X^{+0}_{+} \bar{\partial} X^{-0}_{-} - \partial X^{0+}_{+} \bar{\partial} X^{0-}_{-} - \partial \theta^{++}_L \bar{\partial} \theta^{--}_R - \partial \theta^{--}_L \bar{\partial} \theta^{++}_R + \partial \theta^{+-}_L \bar{\partial} \theta^{-+}_R + \partial \theta^{-+}_L \bar{\partial} \theta^{+-}_R + \cdots).$$
(33)

The OPEs corresponding to the action (33) are

$$X_{+}^{+0}(y)X_{-}^{-0}(z) \sim -\ln|y-z|, \qquad X_{+}^{0+}(y)X_{-}^{0-}(z) \sim +\ln|y-z|, \qquad \theta_{L}^{++}(y)\theta_{R}^{--}(z) \sim \ln|y-z|, \qquad \theta_{L}^{+-}(y)\theta_{R}^{++}(z) \sim \ln|y-z|, \qquad \theta_{L}^{+-}(y)\theta_{R}^{+-}(z) \sim -\ln|y-z|, \qquad \theta_{L}^{+-}(y)\theta_{R}^{+-}(z) \sim -\ln|y-z|.$$
(34)

Notice that the auxiliary fields have been integrated out in the action in (33). There are new OPEs among the fermionic coordinates in the $AdS_2 \times S^2$ background, as in the $AdS_3 \times S^3$ case [58].

IV. BLACK HOLE AND TOPOLOGICAL STRING PARTITION FUNCTIONS

In this section, we give details of the black hole and topological string partition functions to be used in the next section. We start with the topological string amplitudes of N = 2 and N = 4 topological strings. The partition function of N = 2 topological strings leads to the right-hand side of the relation in Eq. (1) and that of N = 4 topological strings will lead to a method of calculating the partition function of the physical type IIA superstrings, appearing in Eq. (3). In the final subsection, we present the black hole partition function.

BLACK HOLE PARTITION FUNCTION USING THE ...

A. N = 2 topological string partition function

To define the N = 2 topological strings, we use the generators of two copies of the c = 9, $\mathcal{N} = 2$ algebra of Calabi-Yau, given in Eq. (25): T_{CY}^L , G_{CY}^{-L} , G_{CY}^{+L} , J_{CY}^L and T_{CY}^R , G_{CY}^{-R} , G_{CY}^{+R} , J_{CY}^R , where all the fermionic generators $G_{CY}^{L,R\pm}$ have spin $\frac{3}{2}$. Now, we perform a twist which shifts the U(1) charges of the fields, leading to a topological theory. An A-twist corresponds to the choice of shifts, $T_{\text{new,CY}}^L = T_{\text{CY}}^L - \frac{1}{2} \bar{\partial} J_{\text{CY}}^L$ and $T_{\text{new,CY}}^R = T_{\text{CY}}^R + \frac{1}{2} \partial J_{\text{CY}}^R$. This results in the shift of spins h^L , h^R of the generators of left- and right-moving algebras as $h_{\text{new}}^L = h^L - \frac{1}{2}q^L$ and $h_{\text{new}}^R = h^R + \frac{1}{2}q^R$, where q^L , q^R are the U(1) charges, respectively. Note that $T_{\text{new,CY}}(y)T_{\text{new,CY}}(z)$ OPEs have no central charge in the twisted N = 2 system, and all bosonic and fermionic world-sheet fields have integer spin. However, $G^+_{CY}(y)G^-_{CY}(z)$ OPEs have a central charge given as $\hat{c} = 3$ [2]. As a result of this twisting, G_{CY}^{-L} , G_{CY}^{+R} acquire spin 2 and G_{CY}^{+L} , G_{CY}^{-R} acquire spin 1. The N = 2topological string amplitude is then given as [2]

$$F_g = \int_{\mathcal{M}_g} \left\langle \left| \prod_{i=1}^{3g-3} G_{\mathrm{CY}}^-(\mu_i) \right|^2 \right\rangle, \tag{35}$$

where $\langle |G_{CY}^{-}(\mu)|^2 \rangle$ corresponds to $\langle G_{CY}^{-L}(\mu)G_{CY}^{+R}(\mu) \rangle$. The spin 2 generators G_{CY}^{-L} and G_{CY}^{+R} appear in the amplitude and behave similar to the *b*-ghost in the bosonic string. The spin 1 generators can be used to form a fermionic nilpotent operator, to study the cohomology of the theory. The formula (35) should also be understood as coming from coupling the N = 2 theory to topological gravity. One can define the full topological string free energy to be

$$\mathcal{F}_{\text{top}} = \sum_{g=0}^{\infty} g_{\text{top}}^{2g-2} F_g, \qquad (36)$$

where g_{top} is the coupling constant weighing the contributions at different genera. Finally, the topological string partition function is defined as

$$Z_{\rm top} = \exp \mathcal{F}_{\rm top}.$$
 (37)

One can repeat the above discussion for an \bar{A} -twist, corresponding to the choice of shifts, $T_{\text{new}}^L = T^L + \frac{1}{2}\bar{\partial}J^L$ and $T_{\text{new}}^R = T^R - \frac{1}{2}\partial J^R$, leading up to the antitopological string partition function \bar{Z}_{top} . $|Z_{\text{top}}|^2$ in Eq. (1) then stands for the product of the N = 2 topological and its conjugate, the antitopological string partition functions, Z_{top} and \bar{Z}_{top} , respectively.

B. N = 4 topological strings

The N = 2 topological strings discussed in the previous subsection were modeled after the N = 0 bosonic strings. In a similar vein, N = 4 topological strings consisting of a twisted small N = 4 algebra are modeled after N = 2strings. The twisted $\mathcal{N} = 4$ generators are

$$\{T, G^+, \tilde{G}^+, G^-, \tilde{G}^-, J^{++}, J, J^{--}\}$$
(38)

and correspond to an algebra with central charge $\hat{c} = 2$ [44]. The OPEs of the algebra are given in [44] and one sees that there are two doublets (G^+, \tilde{G}^-) and (\tilde{G}^+, G^-) . There is also an $SU(2)_f$ flavor which rotates these doublets and leaves the N = 4 unchanged. Explicitly, the flavor rotation is

$$\tilde{\tilde{G}}^{+}(u) = u_1 \tilde{G}^{+} + u_2 G^{+}, \quad \widehat{G}^{-}(u) = u_1 G^{-} - u_2 \tilde{G}^{-}
\tilde{\tilde{G}}^{-}(u) = u_2^* \tilde{G}^{-} - u_1^* G^{-}, \quad \widehat{G}^{+}(u) = u_2^* G^{+} + u_1^* \tilde{G}^{+}, \quad (39)$$

where $|u_1|^2 + |u_2|^2 = 1$ and the complex conjugate of u_a is $\epsilon^{ab} u_b^*$ [80]. The generators \tilde{G}^{\pm} , appearing in Eq. (39), are defined as

$$\tilde{G}^{-} \equiv \left[\oint J^{--}, G^{+} \right], \qquad \tilde{G}^{+} \equiv \left[\oint J^{++}, G^{-} \right], \quad (40)$$

where

$$J^{++} \equiv \exp\left(\int^{z} J\right), \qquad J^{--} \equiv \exp\left(-\int^{z} J\right). \tag{41}$$

The flavor rotated N = 4 topological string amplitude is given by [44]

$$Z = \sum_{n=-2g+2}^{2g-2} \frac{(4g-4)!}{(2g-2+n)!(2g-2-n)!} F_g^n u_1^{2g-2+n} u_2^{2g-2-n}$$
$$= \int_{\mathcal{M}_g} \left\langle |\widehat{G}^-(\mu_1) \dots \widehat{G}^-(\mu_{3h-3})|^2 \Big[\int |\widehat{G}^+|^2 \Big]^{g-1} \int J \bar{J} \right\rangle,$$
(42)

where μ 's denote the Beltrami differentials and the $\int J$ type of insertions are needed to ensure that the path integral does not vanish trivially. Also, the $\int \widehat{\tilde{G}^+}$ insertions in the path integral allow the use of N = 2 topological strings in the calculation of amplitudes.

Now, one constructs unintegrated and integrated vertex operators and calculates scattering amplitudes for any critical N = 2 strings. A case relevant for the present situation is the scattering of 2g - 2 chiral graviphotons and two gravitons in flat four-dimensional space-time. This calculation was performed in [44] and contributes terms of the kind in Eq. (2) for the low energy effective action of type II strings [2,3]. In [35] (see also [81–83]), a physical gauge formalism was used to study, in a general situation, what kind of terms in the low energy effective action of superstrings can be generated by world-sheet instantons which wrap holomorphic curves C in the Calabi-Yau geometry. The A-model computation consists of counting the bosonic and fermionic zero modes contributing to the instanton partition function. In this case, the instanton that contributes is isolated and of genus $g \ge 1$. One also needs to bring down an appropriate number of insertions of interaction terms involving Weyl supermultiplets from the world volume of the instanton. The final result is [35]

$$\delta S = \int d^4 x d^4 \theta (W^2)^g \exp\left(-\frac{A(C)}{2\pi\alpha'} + i \int_C B\right),$$

where the sum over holomorphic curves C, accompanied by the exponential factor denoting the area of the world sheet, defines the A-model amplitude F_g at genus g, given in Eq. (35). This is equivalent to the computation in [44], since in flat space-time, one needs graviton-graviphoton insertions in the path integral to generate low energy F-terms.

C. Z_{BH}

 $Z_{\rm BH}$ stands for the partition function of extremal black holes in four dimensions, which are solutions of N = 2supergravity coupled to n_V vector multiplets with magnetic and electric charges p^{Λ} , q_{Λ} . The asymptotic values of the moduli fields X^{Λ} , $\Lambda = 0, 1, \dots, n_V$ of the vector multiplets are arbitrary in the black hole solution. Near the horizon of the black hole, the values of the moduli fields are constant and fixed by the attractor equations [4]

$$p^{\Lambda} = \operatorname{Re}[CX^{\Lambda}], \qquad q_{\Lambda} = \operatorname{Re}[CF_{\Lambda}], \qquad (43)$$

where $F_{\Lambda} = \frac{\partial F}{\partial X^{\Lambda}}$ and *C* is a complex constant. In the presence of higher derivative *F*-terms in the action, encoded in the prepotential

$$F(X^{\Lambda}, W^2) = \sum_{g=0}^{\infty} F_g(X^{\Lambda}) W^{2g},$$
 (44)

where F_g are computed by Eq. (35) and W^2 contains the square of the anti-self-dual graviphoton field strength, the attractor equations are modified due to [7–9]

$$C^2 W^2 = 256,$$
 (45)

in the gauge K = 0, C = 2Q. Here, K denotes the Kähler potential and Q is a complex combination of electric and magnetic graviphoton charges. Thus, the black hole partition function takes the form [1]

$$\ln Z_{\rm BH} = -4\pi Q^2 \bigg[\sum_{g} F_g \bigg(\frac{p^{\Lambda} + i \frac{\phi^{\Lambda}}{\pi}}{2Q} \bigg) \bigg(\frac{8}{Q} \bigg)^{2g} \bigg], \quad (46)$$

where one uses $X^{\Lambda} = (p^{\Lambda} + i \frac{\phi^{\Lambda}}{\pi})/2Q$. Here, ϕ^{Λ} are continuous electric potentials, which are conjugate to integer electric charges q^{Λ} . As argued in [33,34], the partition function in Eq. (46) is related to Z_{IIA} as in Eq. (3). The computation of Z_{IIA} involves taking in to account the contributions of world-sheet instantons which wrap nontrivial curves in the Calabi-Yau 3-fold [33–35].

V. TYPE IIA PARTITION FUNCTION ON AdS₂ × S^2 × CY₃

In this section, we calculate the partition function of type IIA superstrings on $AdS_2 \times S^2 \times CY_3$ and show its connection to the N = 2 topological string partition

function. There are at least two methods to compute the partition function of type IIA superstrings on $AdS_2 \times S^2 \times CY_3$. In the first method, using the fact that c = 6 is the critical central charge for an N = 2 matter system given in Eq. (23), one introduces a set of c = -6, N = 2 ghosts, constructs an N = 2 BRST operator, and calculates the partition function using standard N = 2 techniques [84]. The second method is to twist the original c = 6, N = 2 algebra of Eq. (23), embed it in a small N = 4 algebra and compute the partition function using the N = 4 topological method we follow and is based on the N = 4 topological strings discussed in Sec. IV B. The N = 4 topological method can be used to calculate the partition function function of any N = 2 system which has a critical central charge of c = 6 [44].

A. Embedding $AdS_2 \times S^2 \times CY_3$ in N = 4 topological strings

As mentioned above, to use the N = 4 topological method in the present case, one has to embed the AdS₂ × $S^2 \times CY_3$ generators in the twisted N = 4 algebra.

We start by explicitly denoting the c = 6, N = 2 algebra generators given in Eq. (23) as T^L , G^{-L} , G^{+L} , J^L and T^R , G^{-R}, G^{+R}, J^{R} , where all the fermionic generators $G^{L,R\pm}$ have spin $\frac{3}{2}$. Now, there are two possible embeddings of the $AdS_2 \times S^2 \times CY_3$ algebra in the N = 4 topological algebra. In one case, we end up with an A-model in the Calabi-Yau geometry discussed in Sec. IVA, and in the other case, the conjugate A-model. We now discuss the A-model embedding. This can be achieved by the shifts $T_{new}^L = T^L - T_{new}^L$ $\frac{1}{2}\partial J^L$ and $T^R_{\text{new}} = T^R - \frac{1}{2}\partial J^R$. This results in the shift of spins h^L , h^R of the generators of left- and right-moving algebras as $h_{\text{new}}^L = h^L - \frac{1}{2}q^L$ and $h_{\text{new}}^R = h^R - \frac{1}{2}q^R$, where q^L , q^R are the U(1) charges, respectively. As a result, the generators which acquire spin 1 and spin 2 in the full $AdS_2 \times S^2 \times CY_3$ geometry are G^{+L} , G^{+R} and G^{-L} , G^{-R} , respectively. Using Eqs. (8), (10), and (23), this translates to G_{CY}^{+L} , G_{CY}^{-R} acquiring spin 1 and G_{CY}^{-L} , G_{CY}^{+R} acquiring spin 2 in the Calabi-Yau theory. This matches exactly with the A-twist discussed in Sec. IVA. Thus, we have an A-model topological string in the Calabi-Yau geometry. On the other hand, in the $AdS_2 \times S^2$ geometry, it is $G_{d=4}^{+L}$, $G_{d=4}^{+R}$ that acquire spin 1 and $G_{d=4}^{-L}$, $G_{d=4}^{-R}$ acquire spin 2. We will have more to say on this when we discuss the construction of BRST operator in Sec. VC. To summarize, the twisting of the algebra in the $AdS_2 \times S^2$ is opposite to that in the Calabi-Yau geometry.

Now again the $T_{\text{new}}(y)T_{\text{new}}(z)$ OPEs have no central charge in the twisted N = 2 system, and all bosonic and fermionic world-sheet fields have integer spin. However, $G^+(y)G^-(z)$ OPEs have a central charge given as $\hat{c} = 2$. Since, $\hat{c} = 2$ is the critical central charge of an N = 4 topological algebra discussed in Sec. IV B, one can enlarge this twisted N = 2 algebra, with two additional currents of charge ± 2 denoted by J^{++} and J^{--} , which together with J

satisfy an $SU(2)_c$ algebra, where the subscript *c* stands for color. Under this $SU(2)_c$, G^- and G^+ generate two new supercurrents \tilde{G}^+ and \tilde{G}^- . Using the definitions in Eq. (23) and the formulas in Eq. (40), we can explicitly write

$$\tilde{G}^{-} = e^{-2\rho - J_{\rm CY}} (\bar{d})^2 + e^{-\rho} \tilde{G}_{\rm CY}^{--},$$

$$\tilde{G}^{+} = e^{2\rho + J_{\rm CY}} (d)^2 + e^{\rho} \tilde{G}_{\rm CY}^{++},$$
(47)

where \tilde{G}_{CY}^{--} and \tilde{G}_{CY}^{++} are residues of the pole in the OPE of $e^{J_{CY}}$ with G_{CY}^{-} and $e^{-J_{CY}}$ with G_{CY}^{+} , respectively. Their explicit form will not be needed here, but can be constructed using the methods in [68]. The full set of generators for this N = 4 system is already given in Eq. (38).

B. Z_{IIA}

In the last subsection, we showed how the hybrid formalism in $AdS_2 \times S^2 \times CY_3$ is a critical N = 2 string theory and can be embedded in the N = 4 topological algebra. This embedding, in particular, implies that the partition function of IIA superstrings on $AdS_2 \times S^2 \times CY_3$ can be calculated using Eq. (42).

One of the consequences of topological twisting is the existence of a nilpotent generator Q. In terms of the twisting procedure discussed in last subsection, this generator can be constructed from G^{+L} and G^{+R} . The explicit form of this generator will not be needed here. However, the $AdS_2 \times S^2$ part of this generator is important for localization arguments and will be discussed in Sec. V C. Because of the twisting, the type IIA path integral Z_{IIA} reduces to a sum of local contributions from the fixed points of the nilpotent generator Q [85]. In the Calabi-Yau geometry, these contributions in the A-model and anti-instantons in the conjugate \bar{A} -model. It is well known that the world-sheet instantons wrap nontrivial curves in the Calabi-Yau geometry and lead to holomorphic maps [86], given as

$$\bar{\partial}Y^i = 0. \tag{48}$$

The anti-instantons come from antiholomorphic maps in the Calabi-Yau geometry. In $AdS_2 \times S^2$ geometry, such world-sheet instanton configurations break translational symmetries and associated supersymmetries. After a Euclidean rotation in the world-sheet action in Eq. (17), the instantonic configuration can be obtained by setting the world-sheet variables X^m , θ^{α} , $\theta^{\dot{\alpha}}$, and their right-moving counterparts to zero.⁴ The only nontrivial terms remaining in the $AdS_2 \times S^2$ action (17) are

$$\int (d_{\alpha_L} d_{\beta_R} W^{\alpha_L \beta_R} + \bar{d}_{\dot{\alpha}_L} \bar{d}_{\dot{\beta}_R} W^{\dot{\alpha}_L \dot{\beta}_R}).$$
(49)

To summarize, Z_{IIA} reduces to a sum of two contributions, coming from world-sheet instantons and anti-intantons,⁵ which wrap holomorphic and antiholomorphic curves in the Calabi-Yau geometry, respectively, and have legs in AdS₂ × S². The action in Eq. (49) will be useful in the evaluation of these contributions to the partition function.

We now discuss the instanton contribution to Z_{IIA} and denote it as Z_{IIA}^{I} . Toward the end, we briefly discuss the anti-instanton contribution to the type IIA partition function, denoted as Z_{IIA}^{AI} . This corresponds to an opposite twist to the one in Sec. VA. The final answer for Z_{IIA} will be obtained by joining Z_{IIA}^{I} and Z_{IIA}^{AI} .

1. World-sheet instantons

With the choice of twisting in Sec. VA, Eq. (50) below corresponds to the instanton contribution to the type IIA partition function, Z_{IIA}^{I} . One way to see this is that in the Calabi-Yau geometry, we have an *A*-model topological string. The spin 2 generators G^- , \tilde{G}^- are like the *b*-ghost of the bosonic strings and appear in the partition function in Eq. (50), integrated against the 3g - 3 Beltrami differentials. Thus using Eq. (42), the instanton contribution to the partition function of type IIA superstrings on $AdS_2 \times S^2 \times CY_3$ is

$$Z_{\text{IIA}}^{I} = \left| \int du \sum_{n_{I}=2-2g}^{2g-2} (u_{2}^{*})^{2g-2+n_{I}} (u_{1}^{*})^{2g-2-n_{I}} \right|^{2} \\ \times \prod_{j=1}^{3g-3} \int d^{2}m_{j} \prod_{i=1}^{g} \int d^{2}v_{i} \left\langle \left| \prod_{i=1}^{g-1} \widehat{G}^{+}(v_{i}) J(v_{g}) \right. \right. \\ \left. \times \left(\int \mu_{j} \widehat{G}^{-} \right) \right|^{2} \right\rangle,$$
(50)

where the \hat{G}^+ and \hat{G}^- are defined in Eq. (39). Just as in flat space [44], the integration over *u* can be done using

$$\int du (u_1^{2g-2+m} u_2^{2g-2-n}) (u_2^*)^{2g-2+n} (u_1^*)^{2g-2-n}$$

= $\delta_{mn} (2g-2+m)! (2g-2-n)!.$ (51)

To calculate $Z_{\text{IIA}}^{\text{I}}$ in hybrid formalism, the world-sheet fields which we need to integrate over in the path integral are (a) $\text{AdS}_2 \times S^2$ fields: X^m , $(\theta^{\alpha}, d_{\alpha})$, $(\bar{\theta}^{\dot{\alpha}}, \bar{d}_{\dot{\alpha}})$ and their right-moving counterparts, for $m = 0, \dots, 3$ and $\alpha, \dot{\alpha} =$ 1, 2; (b) Calabi-Yau variables; and (c) the chiral boson ρ and its right-moving counterpart.

Let us start by choosing $n_I = g - 1$ in Eq. (50), which was argued in [44], to be the piece relevant for computing

⁴For the four-dimensional part, we sometimes use the flatspace notation of Sec. , as the left-invariant currents of $AdS_2 \times S^2$ can always be written in terms of them, using the expansions in Eqs. (31) and (32).

⁵Let us note that, to reproduce the *F*-terms in Eq. (2), one has to insert 2g - 2 chiral graviphoton and two graviton vertex operators in the amplitude in flat space. In $AdS_2 \times S^2$, the relevant insertions are provided by the action of the world-sheet instantons in Eq. (49). Also, the Calabi-Yau part of the type IIA partition function, will come out similar to the one in flat space [44].

the low energy *F*-terms in Eq. (2). Next, it is needed to write the partition function (50), transforming variables from the hatted generators, \tilde{G}^+ and \hat{G}^- , using the definitions in Eq. (39). This can be done as follows. There are $3g - 3 \hat{G}^-$'s and $g - 1 \hat{G}^+$'s in the partition function (50). Thus, using (39), \hat{G}^- 's can contribute $3g - 3 - l G^-$'s and $+l \tilde{G}^-$'s, whereas \hat{G}^+ 's can contribute $g - 1 - l G^+$'s and $+l \tilde{G}^+$'s, in the partition function. These contributions are further subject to the constraints coming from the background charges of the U(1) current *J*. For the partition function to not vanish trivially, one has to ensure that G^{\pm} and \tilde{G}^{\pm} contributions contain 1 - g units of ρ charge and 3g - 3 units of J_{CY} charge [44]. This is only possible if each G^- contributes G^-_{CY} , each \tilde{G}^- contributes $e^{-2\rho - \int J_{CY}(\bar{d})^2}$, each G^+ contributes $(\bar{d})^2 e^{-\rho}$, and each \tilde{G}^+ contributes $\tilde{G}^{++}_{CY} e^{\rho}$. Making these choices, we have⁶

$$Z_{\text{IIA}}^{\text{I}} = \prod_{j=1}^{3g-5} \int d^2 m_j \det(\text{Im}\tau) |\det\omega^k(v_i)|^2 \\ \times \left\langle \left| \prod_{i=l+1}^{g-1} e^{-\rho}(\bar{d})^2(v_i) J(v_g) \right. \right. \\ \left. \times \prod_{j=1}^l \left(e^{-\rho}(\bar{d})^2 \int \mu_j G_{\text{CY}}^- \right) \prod_{k=l+1}^{3g-3} \left(\int \mu_k G_{\text{CY}}^- \right) (.1) \left| \right|^2 \right\rangle.$$
(52)

Here, τ is the period matrix and ω^k are g holomorphic oneforms on Σ_g . We have also inserted $1 = \int (G^+, \bar{\theta}^2 e^{\rho})$ in Eq. (52). This makes the functional integral well defined and can be used to remove the factor of $J(v_g)$, as follows. Using the definitions in Eq. (23), notice that G^+ has OPEs only with $J(v_g)$ in Eq. (52). Thus, one can pull the $\int G^+$ counter off the term $\bar{\theta}^2 e^{\rho}$, till it hits $J(v_g)$, giving

$$Z_{\text{IIA}}^{\text{I}} = \prod_{j=1}^{3g-3} \int d^2 m_j \det(\text{Im}\tau) |\det\omega^k(v_i)|^2 \\ \times \left\langle \left| \prod_{i=l+1}^g e^{-\rho}(\bar{d})^2(v_i) \prod_{j=1}^l \left(e^{-\rho}(\bar{d})^2 \int \mu_j G_{\text{CY}}^- \right) \right. \right. \\ \left. \times \prod_{k=l+1}^{3g-3} \left(\int \mu_k G_{\text{CY}}^- \right) (\bar{\theta}^2 e^{\rho}) \left| \right|^2 \right\rangle.$$
(53)

In Eq. (53), apart from functional integral over various world-sheet coordinates, one also has to perform integra-

tions over the zero modes of these variables. The counting of zero modes is as follows. There are four bosonic zero modes of X^m and eight fermionic zero modes corresponding to θ , $\bar{\theta}$'s. Further, on a genus g Riemann surface, each variable d contributes g-zero modes, giving in total 8g-zero modes. These zero modes are saturated as follows. In Eq. (53), four zero modes of $\bar{\theta}$'s are explicitly present together with 4g-zero of \bar{d} 's. Another 4g-zero modes of d's can also be pulled out from 2g powers of the first term in the instanton action in Eq. (49) as

$$\int (d_{\alpha_L} d_{\beta_R} \delta^{\alpha_L \beta_R} N g_s)^{2g}, \tag{54}$$

where in an extremal black hole background, the expectation value of the graviphoton field strength is $W^{\alpha_L \beta_R} = \delta^{\alpha_L \beta_R} Ng_s$. These exactly saturate the required zero modes of *d*'s, in the partition function in Eq. (53). Thus we have

$$Z_{\text{IIA}}^{I} = \prod_{j=1}^{3g-3} \int d^{2}m_{j} \det(\text{Im}\tau) |\det\omega^{k}(v_{i})|^{2} \\ \times \left\langle \left| \prod_{i=l+1}^{g} e^{-\rho}(\bar{d})^{2}(v_{i}) \prod_{j=1}^{l} \left(e^{-\rho}(\bar{d})^{2} \int \mu_{j} G_{\text{CY}}^{-} \right) \right. \\ \left. \times \prod_{k=l+1}^{3g-3} \left(\int \mu_{k} G_{\text{CY}}^{-} \right) (\bar{\theta}^{2} e^{\rho}) \right|^{2} (g_{\text{top}})^{2g} \left(\prod_{r=1}^{2g} \int d^{2} z_{r} d^{2} \right) \right\rangle$$
(55)

$$= \prod_{j=1}^{3g-3} \int d^2 m_j \det(\operatorname{Im}\tau) |\det\omega^k(v_i)|^2$$
$$\times \left\langle \left| \prod_{i=l+1}^g e^{-\rho}(\bar{d})^2(v_i) \prod_{j=1}^l (e^{-\rho}(\bar{d})^2)(\bar{\theta}^2 e^{\rho}) \right|^2 (g_{\text{top}})^{2g} \\\times \left(\prod_{r=1}^{2g} \int d^2 z_r d^2 \right) \right\rangle \left\langle \left| \left(\prod_{j=1}^{3g-3} \int \mu_j G_{\text{CY}}^- \right) \right|^2 \right\rangle,$$
(56)

where $g_{top} = Ng_s$ and in the last line of Eq. (56) the Calabi-Yau part is separated.

In Eq. (56), the integrations over various world-sheet coordinates can now be done exactly as in flat space-time [44]. The functional integrals over the ρ and $(\bar{d}_1, \bar{\theta}^1)$ fields contribute $|Z_1|^{-2}[\det(\mathrm{Im}\tau)]^{-1}$, $(\bar{p}_2, \bar{\theta}^2)$ fields contributes $|Z_1 \det \omega_j(v_k)|^2$, $(d_\alpha, \theta^\alpha)$ contribute $|Z_1|^4 (\det(\mathrm{Im}\tau))^2$, and x^m 's give a factor of $|Z_1|^{-4} (\det(\mathrm{Im}\tau))^{-2}$, where $(Z_1)^{-(1/2)}$ is the partition function for a chiral boson [87]. Thus, we have

$$Z_{\text{IIA}}^{1} = g_{\text{top}}^{2g} \int d^{4}X d^{2}\theta_{L} d^{2}\theta_{R} \prod_{j=1}^{3g-3} \\ \times \int d^{2}m_{j} \Big\langle \Big| \Big(\prod_{j=1}^{3g-3} \int \mu_{j} G_{\text{CY}}^{-} \Big) \Big|^{2} \Big\rangle, \quad (57)$$

⁶We omit the details of the regularization over the negative energy of the chiral boson, which is same as in flat space and discussed in detail in [44]. The regularization fixes the poles of $e^{\rho}\tilde{G}_{CY}^{++}$ coming from \tilde{G}^+ to be sewn in with those of $e^{-2\rho}\int_{J_{CY}}^{J_{CY}}(\bar{d})^2$ coming from \tilde{G}^- , giving the term $\prod_{j=1}^{l} (e^{-\rho}(\bar{d})^2 \int \mu_j G_{CY}^-)$ in Eq. (52). It also contributes a factor of det(Im τ)| det $\omega^k(v_i)$ |², from the Jacobian of change of variables, allowing the v_i 's to be any g points on Σ_g .

where, explicitly, the A-model topological string amplitude as given in Eq. (35) is

$$F_{g} = \prod_{j=1}^{3g-3} \int d^{2}m_{j} \left\langle \prod_{j=1}^{3g-3} \left(\int \mu_{j}^{L} G_{\rm CY}^{-L} \right) \left(\int \mu_{j}^{R} G_{\rm CY}^{+R} \right) \right\rangle.$$
(58)

Let us emphasize that, out of the eight fermionic zero modes required for a nonzero result, the $|\bar{\theta}|^2$ term in Eq. (56) has been used to absorb the $(\bar{\theta}_{L\dot{\alpha}}, \bar{\theta}_{R\dot{\alpha}})$ integrals. One is left with the integrations over four bosonic and fermionic variables in (57) as there is no action for these variables. In the following subsection, localization arguments are used to do the integrations over these variables.

2. Anti-instantons

A similar analysis as above can be repeated for the contribution of anti-instantons, which results in an \bar{A} -topological model in the Calabi-Yau geometry. In this case, the anti-instanton contribution to the IIA partition function with \bar{A} -twist is

$$Z_{\text{IIA}}^{\text{AI}} = \left| \int du \sum_{n_{I}=2-2g}^{2g-2} (u_{2}^{*})^{2g-2+n_{I}} (u_{1}^{*})^{2g-2-n_{I}} \right|^{2} \\ \times \prod_{j=1}^{3g-3} \int d^{2}m_{j} \prod_{i=1}^{g} \int d^{2}v_{i} \\ \times \left\langle \left| \prod_{i=1}^{g-1} \widehat{\tilde{G}}^{-}(v_{i}) J(v_{g}) (\int \mu_{j} \widehat{G}^{+}) \right|^{2} \right\rangle,$$
(59)

where the \tilde{G}^- and \widehat{G}^+ are defined in Eq. (39) In this case, to do the integration over the $(\theta_{L\alpha}, \theta_{R\alpha})$ zero modes, $1 = \int (G^-, \theta^2 e^{-\rho})$ is inserted in the path integral and the counter-pulling argument over $J(v_g)$ is repeated. The absorption of 4*g*-zero modes of \bar{d} 's requires pulling 2*g* powers of the second term in the instanton action in Eq. (49). Finally, one is left with the integration over four X_m and $(\bar{\theta}_{L\dot{\alpha}}, \bar{\theta}_{R\dot{\alpha}})$ zero modes as

$$\bar{Z}_{\text{IIA}}^{\text{AI}} = g_{\text{top}}^{2g} \int d^4 X d^2 \bar{\theta}_L d^2 \bar{\theta}_R \prod_{j=1}^{3g-3} \int d^2 m_j \\ \times \left\langle \left| \left(\prod_{j=1}^{3g-3} \int \mu_j G_{\text{CY}}^+ \right) \right|^2 \right\rangle, \tag{60}$$

where the \bar{A} -model topological string amplitude is

$$\bar{F}_{g} = \prod_{j=1}^{3g-3} \int d^{2}m_{j} \left\langle \prod_{j=1}^{3g-3} \left(\int \mu_{j}^{L} G_{\mathrm{CY}}^{+L} \right) \left(\int \mu_{j}^{R} G_{\mathrm{CY}}^{-R} \right) \right\rangle.$$
(61)

For this case, the opposite twisting leads to G_{CY}^{+L} and G_{CY}^{-R} acquiring spin 2. A nilpotent operator in this case in the Calabi-Yau geometry is

$$\bar{Q}_{\rm CY} = G_{\rm CY}^{-L} + G_{\rm CY}^{+R}.$$
 (62)

The full partition function of type IIA superstrings on $AdS_2 \times S^2 \times CY_3$ is obtained from Eqs. (57) and (60).

C. BRST localization

In this section, our aim is perform the integration over X_m , $(\theta_{L\alpha}, \theta_{R\alpha})$ and $(\bar{\theta}_{L\dot{\alpha}}, \bar{\theta}_{R\dot{\alpha}})$, appearing in Eqs. (57) and (60). In the present situation, the world-sheet instanton in AdS₂ × S² contributes four bosonic zero modes coming from the breaking of translation isometries and their supersymmetric partners. From the *PSU*(1, 1|2) sigma model point of view, the left-multiplication symmetry corresponding to the generators T_m and $T_{\alpha,\dot{\alpha}}$ is broken by the instanton expectation value. Since there is no action for X_m in Eqs. (57) and (60), the integration gives rise to an infinite space-time volume of AdS₂. Also, since there is no action for $(\theta_{L\alpha}, \theta_{R\alpha})$ and $(\bar{\theta}_{L\dot{\alpha}}, \bar{\theta}_{R\dot{\alpha}})$, the result is zero. Below, we use a localization procedure similar to [33] to solve this problem.

Let us note that, in the Green-Schwarz treatment [33], due to the presence of κ -symmetry, a world-sheet instanton preserves four out of eight space-time supersymmetries. In particular, instantons and anti-instantons at the opposite poles of S^2 preserve same amount of supersymmetry, but a different set, namely, chiral and antichiral, respectively. This reflects in our analysis in Eqs. (57) and (60). In Eq. (57), one is left with integration over the chiral set of variables $(\theta_{L\alpha}, \theta_{R\alpha})$ of the instantons and in Eq. (60), over the antichiral variables corresponding to anti-instantons. In the hybrid description of the superstring such (anti-) instantons break all space-time supersymmetries, since one cannot perform a compensating κ -transformation to be in the gauge. Instead, one has superconformal invariance, which can be used to write down a operator which renders the partition function finite. The advantage of the hybrid description is that a unified BRST treatment can be given for $AdS_2 \times S^2$ and the Calabi-Yau parts, using the nilpotent operator Q discussed at the beginning of Sec. V. Since the $AdS_2 \times S^2$ and the Calabi-Yau CFTs decouple, Q contains four- and six-dimensional parts. Below, we explicitly give the $AdS_2 \times S^2$ part of this operator denoted as $Q_{d=4}$ for localizing world-sheet instantons and $\bar{Q}_{d=4}$ for anti-instantons. These operators turn out to be BRST operators of the $AdS_2 \times S^2$ sigma model.

In the present case, the essence of the localization procedure is to use these BRST operators to make a semiclassical expansion of the path integral around the saddle points. This is possible because the PSU(1, 1|2) sigma model action can be shown to be exact under these BRST operators, constructed from the left-invariant currents. Thus, if the action $S_{AdS_2 \times S^2}$ is $\bar{Q}_{d=4}$ -closed and $\bar{Q}_{d=4}^2 = 0$, then one can deform the action by adding another BRST exact term as

$$S_{\text{AdS}_2 \times S^2} = S_{\text{AdS}_2 \times S^2} + t\bar{I}, \tag{63}$$

where $\bar{I} = \bar{Q}_{d=4}\bar{V}$ is an arbitrary parameter for some choice of \bar{V} . Adding such a term does not change the cohomology

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and the parameter t can be varied freely [88,89]. In particular, as $t \to \infty$, the theory localizes to the critical points of $\bar{Q}_{d=4}\bar{V}$. Thus, one can integrate over these critical points to render the path integral in Eq. (60) finite, by localizing the genus g anti-instantons at the north pole of S^2 . The analysis can be repeated using $Q_{d=4}$ for integration over variables in Eq. (60).

We now write $\bar{Q}_{d=4}$ and then show that the sigma model action is $\bar{Q}_{d=4}$ -closed. For this purpose, it is useful to use the $U(1) \times U(1)$ form of the AdS₂ × S² action derived in Sec. III B and write⁷

$$\bar{Q}_{d=4} = G_{d=4}^{-L} + G_{d=4}^{-R}.$$
(64)

In terms of left-invariant currents, we have

$$G_{d=4}^{-L} = \int e^{\rho_L} \bar{J}_L^{++} \bar{J}_L^{--}, \quad G_{d=4}^{-R} = \int e^{\rho_R} J_R^{++} J_R^{--}.$$
 (65)

The operator $\bar{Q}_{d=4}$ in Eq. (64) is nilpotent, i.e., $\bar{Q}_{d=4}^2 = 0$ as it is constructed from a sum of $G_{d=4}^{-L}$ and $G_{d=4}^{-R}$, which have regular OPEs of the N = 2 algebra. Also, using the equations of motion in (30), one can check that $G_{d=4}^{-L}$ is

antiholomorphic and $G_{d=4}^{-R}$ is holomorphic. The BRST operator is constructed from covariantly holomorphic and antiholomorphic *H*-invariant combination of currents [42]. The action of $\bar{Q}_{d=4}$ on the group element *g* is to transform it by a right-multiplication as

$$\Lambda \bar{Q}_{d=4}(g) = g(\Lambda(e^{\rho}\bar{J}_3) + \Lambda(e^{\rho}J_1)), \tag{66}$$

where Λ is a constant anticommuting parameter. The leftinvariant currents transform under (66) as

$$\bar{Q}_{d=4}(J_{j}) = \delta_{j+1,0}\partial(\Lambda e^{\rho}\bar{J}_{3}) + [J_{j+1}, \Lambda e^{\rho}\bar{J}_{3}]
+ \delta_{j+3,0}\partial(\Lambda e^{\rho}J_{1}) + [J_{j+3}, \Lambda e^{\rho}J_{1}],
\bar{Q}_{d=4}(\bar{J}_{j}) = \delta_{j+1,0}\bar{\partial}(\Lambda e^{\rho}\bar{J}_{3}) + [\bar{J}_{j+1}, \Lambda e^{\rho}\bar{J}_{3}]
+ \delta_{j+3,0}\bar{\partial}(\Lambda e^{\rho}J_{1}) + [\bar{J}_{j+3}, \Lambda e^{\rho}J_{1}],$$
(67)

where *j* is defined modulo 4, i.e. $J_j \equiv J_{j+4}$. Now, one can show that the *PSU*(1, 1|2) sigma model action in (29) is closed under the BRST operator (64). To check this, we follow the procedure used in [56], and start by rewriting the sigma model action in Eq. (29) as

$$S_{\text{AdS}_{2}\times S^{2}} = \int d^{2}z \bigg[\frac{1}{2} (J_{+}^{+0}\bar{J}_{-}^{-0} - J_{+}^{0+}\bar{J}_{-}^{0-} + J_{-}^{-0}\bar{J}_{+}^{+0} - J_{-}^{0-}\bar{J}_{-}^{0+}) + \frac{1}{2} (-J_{L}^{--}\bar{J}_{R}^{++} - \bar{J}_{L}^{--}J_{R}^{++} - J_{L}^{++}\bar{J}_{R}^{--} - \bar{J}_{L}^{++}J_{R}^{--}) \\ + \frac{1}{2} (J_{L}^{+-}\bar{J}_{R}^{-+} + \bar{J}_{L}^{+-}J_{R}^{-+} + J_{L}^{-+}\bar{J}_{R}^{+-} + \bar{J}_{L}^{-+}J_{R}^{+-}) + \frac{1}{4} (J_{L}^{--}\bar{J}_{R}^{++} - \bar{J}_{L}^{--}J_{R}^{++} + J_{L}^{++}\bar{J}_{R}^{--} - \bar{J}_{L}^{++}J_{R}^{--}) \\ + \frac{1}{4} (-J_{L}^{+-}\bar{J}_{R}^{-+} + \bar{J}_{L}^{+-}J_{R}^{-+} - J_{L}^{-+}\bar{J}_{R}^{+-} + \bar{J}_{L}^{-+}J_{R}^{+-}) \bigg].$$
(68)

Now, under the BRST operator (64), the variations of the terms in first and the third brackets in (68) can be shown to cancel each other. The variation of the terms in second bracket is

$$\left\langle -\frac{1}{2} \left[\nabla (\Lambda e^{\rho_L} \bar{J}_L^{--}) \bar{J}_R^{++} + J_L^{--} \bar{\nabla} (\Lambda e^{\rho_R} J_R^{++}) + \bar{\nabla} (\Lambda e^{\rho_L} \bar{J}_L^{--}) J_R^{++} + \bar{J}_L^{--} \nabla (\Lambda e^{\rho_R} J_R^{++}) \right. \\ \left. + \nabla (\Lambda e^{\rho_L} \bar{J}_L^{++}) \bar{J}_R^{--} + J_L^{++} \bar{\nabla} (\Lambda e^{\rho_R} J_R^{--}) + \bar{\nabla} (\Lambda e^{\rho_L} \bar{J}_L^{++}) J_R^{--} + \bar{J}_L^{++} \nabla (\Lambda e^{\rho_R} J_R^{--}) \right] \right\rangle,$$
(69)

and the variations of the terms in the fourth and fifth brackets are respectively

$$\left\langle \frac{1}{4} \left[\nabla (\Lambda e^{\rho_L} \bar{J}_L^{--}) \bar{J}_R^{++} + J_L^{--} \bar{\nabla} (\Lambda e^{\rho_R} J_R^{++}) - \bar{\nabla} (\Lambda e^{\rho_L} \bar{J}_L^{--}) J_R^{++} - \bar{J}_L^{--} \nabla (\Lambda e^{\rho_R} J_R^{++}) + \nabla (\Lambda e^{\rho_L} \bar{J}_L^{++}) \bar{J}_R^{--} \right. \\ \left. + J_L^{++} \bar{\nabla} (\Lambda e^{\rho_R} J_R^{--}) - \bar{\nabla} (\Lambda e^{\rho_L} \bar{J}_L^{++}) J_R^{--} - \bar{J}_L^{++} \nabla (\Lambda e^{\rho_R} J_R^{--}) \right] \right\rangle,$$

$$(70)$$

and

$$\left\langle \frac{1}{4} \left[-(J_{+0}^{+0}\Lambda e^{\rho_{R}}J_{R}^{--} - J_{-0}^{0-}\Lambda e^{\rho_{R}}J_{R}^{++})\bar{J}_{R}^{-+} - J_{L}^{+-}(\bar{J}_{+}^{0+}\Lambda e^{\rho_{L}}\bar{J}_{L}^{--} - \bar{J}_{-}^{-0}\Lambda e^{\rho_{L}}\bar{J}_{L}^{++}) - (-J_{+}^{0+}\Lambda e^{\rho_{R}}J_{R}^{--} + J_{-}^{-0}\Lambda e^{\rho_{R}}J_{R}^{++})\bar{J}_{R}^{+-} - J_{L}^{--}(\bar{J}_{-}^{0+}\Lambda e^{\rho_{L}}\bar{J}_{L}^{--}) + (\bar{J}_{+}^{+0}\Lambda e^{\rho_{R}}J_{R}^{--} - \bar{J}_{-}^{0-}\Lambda e^{\rho_{R}}J_{R}^{++})J_{R}^{-+} + \bar{J}_{L}^{+-}(J_{+}^{0+}\Lambda e^{\rho_{L}}\bar{J}_{L}^{--} - J_{-}^{-0}\Lambda e^{\rho_{L}}\bar{J}_{L}^{++}) + (-\bar{J}_{+}^{0+}\Lambda e^{\rho_{R}}J_{R}^{--} + \bar{J}_{-}^{-0}\Lambda e^{\rho_{R}}J_{R}^{++})J_{R}^{+-} + \bar{J}_{L}^{-+}(J_{-}^{0-}\Lambda e^{\rho_{L}}\bar{J}_{L}^{+-} - J_{+}^{+0}\Lambda e^{\rho_{L}}\bar{J}_{L}^{--})] \right\rangle.$$

$$(71)$$

⁷The choice of this operator coincides with the fact that the instanton and the anti-instanton require integration over fermionic variables with a different set of indices, namely, undotted and dotted. This is also related to the fact that in Euclidean AdS₂ × S², the symplectic Majorana condition on fermions, does not take dotted into undotted indices. For instance, $(\theta_i^{\alpha})^{\dagger} = \epsilon^{\alpha\beta}\epsilon_{ij}\theta_{\beta}^{j}$ and $(\theta_i^{\dot{\alpha}})^{\dagger} = \epsilon^{\alpha\beta}\epsilon_{ij}\theta_{\beta}^{j}$, where α , $\dot{\alpha}$, β , $\dot{\beta} = 1, 2$ and i, j = L, R (see [33] for related discussion).

Now one makes use of the Maurer-Cartan equations coming from $\partial \overline{J} - \overline{\partial}J + [J, \overline{J}] = 0$. For instance, we have

$$\nabla \bar{J}_{L}^{--} - \bar{\nabla} J_{L}^{--} + [J_{-}^{-0}, \bar{J}_{R}^{+-}] + [J_{R}^{+-}, \bar{J}_{-}^{-0}] + [J_{0}^{--}, \bar{J}_{R}^{-+}] + [J_{R}^{-+}, \bar{J}_{-}^{0--}] = 0.$$
(72)

Using these relations in (71), one can show that the Eqs. (70) and (71) are exactly equal. Thus, adding Eqs. (69)–(71) and doing some partial integrations, we get the full variation of (68) to be

$$\langle -(\Lambda e^{\rho_L} \bar{J}_L^{--}) \bar{\nabla} J_R^{++} + \nabla \bar{J}_L^{--} (\Lambda e^{\rho_R} J_R^{++})$$

+
$$(\Lambda e^{\rho_L} \bar{J}_L^{++}) \bar{\nabla} J_R^{--} + \nabla \bar{J}_L^{++} (\Lambda e^{\rho_R} J_R^{--}) \rangle.$$
(73)

This is identically zero, using the equations of motion in (30). Thus we have shown that the $AdS_2 \times S^2$ sigma model action in (29) is closed under the BRST operator (64).

Now, to do the X_m and $(\bar{\theta}_{L\dot{\alpha}}, \bar{\theta}_{R\dot{\alpha}})$ integrations in (60), a BRST exact operator is constructed as follows. We require an operator I, such that $\bar{Q}_{d=4}\bar{I} = 0$ and $\bar{I} = \bar{Q}_{d=4}\bar{V}$, which facilitates a simple Gaussian integration over the variables. Below, it is shown that the general form of such an operator is given by

$$\bar{V} = (a(X_m)^2 + b\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}_L^{\dot{\alpha}}\bar{\theta}_R^{\beta})(e^{-\rho_L}\theta_R^2 + e^{-\rho_R}\theta_L^2), \quad (74)$$

where m = 0, 1, 2, 3, and the constants *a* and *b*, are fixed by the BRST procedure. Let us start by noting that the relations

$$\int e^{\rho_L} \bar{J}_L^{++} \bar{J}_L^{--} (e^{-\rho_L} \theta_R^2) = 1,$$

$$\int e^{\rho_R} J_R^{++} J_R^{--} (e^{-\rho_R} \theta_L^2) = 1$$
(75)

render the following choice of the operator,

$$\bar{V} = -\frac{1}{2} [(X_{+}^{+0}X_{-}^{-0} - X_{+}^{0+}X_{-}^{0-} + \theta_{L}^{+-}\theta_{R}^{-+} - \theta_{L}^{-+}\theta_{R}^{+-}) \times (e^{-\rho_{L}}\theta_{R}^{2} + e^{-\rho_{R}}\theta_{L}^{2})],$$
(76)

a trivial deformation of the theory. To see this, one first notes that

$$\bar{Q}_{d=4}\bar{V} = -\frac{1}{2} [\bar{Q}(X_{+}^{+0}X_{-}^{-0} - X_{+}^{0^{+}}X_{-}^{0^{-}} + \theta_{L}^{+-}\theta_{R}^{-+} - \theta_{L}^{-+}\theta_{R}^{+-})] \\ \times (e^{-\rho_{L}}\theta_{R}^{2} + e^{-\rho_{R}}\theta_{L}^{2}) - \frac{1}{2}(X_{+}^{+0}X_{-}^{-0} - X_{+}^{0^{+}}X_{-}^{0^{-}} \\ + \theta_{L}^{+-}\theta_{R}^{-+} - \theta_{L}^{-+}\theta_{R}^{+-})[\bar{Q}(e^{-\rho_{L}}\theta_{R}^{2} + e^{-\rho_{R}}\theta_{L}^{2})].$$

$$(77)$$

Now, it can be shown that,

$$\bar{Q}_{d=4}(X_{+}^{+0}X_{-}^{-0} - X_{+}^{0^{+}}X_{-}^{0^{-}} + \theta_{L}^{+-}\theta_{R}^{-+} - \theta_{L}^{-+}\theta_{R}^{+-}) = 0.$$
(78)

To check this explicitly, one can calculate how the leftinvariant currents rotate various coordinates appearing in (76). Thus, under J_L^{--} and J_L^{++} , we have

$$\delta\theta_L^{--} = \epsilon_L^{--}, \qquad \delta\theta_L^{+-} = \epsilon_L^{--}X^{+0},$$

$$\delta\theta_L^{-+} = -\epsilon_L^{--}X^{0+}, \qquad \delta X^{-0} = -\epsilon_L^{--}\theta_R^{-+}, \quad (79)$$

and

$$\delta\theta_{L}^{++} = \epsilon_{L}^{++}, \qquad \delta\theta_{L}^{+-} = -\epsilon_{L}^{++}X^{0^{-}}, \\\delta\theta_{L}^{-+} = \epsilon_{L}^{++}X^{-0}, \qquad \delta X^{+0}_{+} = -\epsilon_{L}^{++}\theta_{R}^{+-}, \\\delta X^{0^{+}_{+}} = \epsilon_{L}^{++}\theta_{R}^{-+}, \qquad (80)$$

and the transformations with respect to ϵ_R^{++} , ϵ_R^{--} can be obtained similarly. So, the first term in (77) drops out. Using the relations in (75), one gets

$$\bar{I} = -(X_{+}^{+0}X_{-}^{-0} - X_{+}^{0+}X_{-}^{0-} + \theta_{L}^{+-}\theta_{R}^{-+} - \theta_{L}^{-+}\theta_{R}^{+-}).$$
(81)

This is the required operator, as it also satisfies $\bar{Q} \bar{I} = 0$, as already shown above.

The BRST exact term for the case of instantons can be written down in a similar manner. In this case, we need to do integrations over four X_m and $(\theta_{L\alpha}, \theta_{R\alpha})$ zero modes in (57)

$$Q_{d=4} = G_{d=4}^{+L} + G_{d=4}^{+R}.$$
(82)

In terms of left-invariant currents, we have

$$G_{d=4}^{+L} = \int e^{-\rho_L} \bar{J}_L^{+-} \bar{J}_L^{-+}, \quad G_{d=4}^{+R} = \int e^{-\rho_R} J_R^{+-} J_R^{-+}.$$
 (83)

The operator $Q_{d=4}$ in Eq. (82) is also nilpotent, as it is constructed from a sum of $G_{d=4}^{+L}$ and $G_{d=4}^{+R}$, which have regular OPEs of the N = 2 algebra. The action can again be shown to be closed under $Q_{d=4}$ as shown for the case of anti-instantons. Thus, one now adds a term coming from

$$I = -(X_{+}^{+0}X_{-}^{-0} - X_{+}^{0+}X_{-}^{0-} + \theta_{L}^{++}\theta_{R}^{--} - \theta_{L}^{--}\theta_{R}^{++}), \quad (84)$$

to the path integral in (57) as

$$\int d^4 X d^2 \theta_L d^2 \theta_R e^{-tI}.$$
(85)

Note that *I* satisfies $Q_{d=4}I = 0$ and $I = Q_{d=4}V$ for a suitable *V*. To proceed, one makes a Euclidean rotation to write the bosonic part of *I* as $X_{+}^{+0}X_{-}^{-0} + X_{+}^{0+}X_{-}^{0-}$. The Euclidean AdS₂ × S² can be taken as in [33] to be

$$ds^{2} = \frac{4dwd\bar{w}}{(1-w\bar{w})^{2}} + \frac{4dzd\bar{z}}{(1+z\bar{z})^{2}},$$
(86)

and using the embedding

$$X_{+}^{\dagger 0}X^{=0} = \frac{|\omega|^2}{(1-|\omega|^2)^2}, \qquad X_{+}^{0+}X^{0-} = \frac{|z|^2}{(1+|z|^2)^2}, \quad (87)$$

the integral in (85) is

$$\int d^{2}\omega d^{2}z d\theta_{L}^{++} d\theta_{L}^{--} d\theta_{R}^{--} d\theta_{R}^{++} \times e^{-t(((|\omega|^{2}+|z|^{2})((1+|\omega|^{2}|z|^{2})/((1-|\omega|^{2})^{2}(1+|z|^{2})^{2}))-\theta_{L}^{++}\theta_{R}^{--}+\theta_{L}^{--}\theta_{R}^{++})}$$

(88)

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Now, since $Q_{d=4}I = 0$, the expectation value of any $Q_{d=4}$ -invariant operator is zero, as can be seen by differentiating with respect to t. Thus, the parameter t can be varied freely. Consider the $t \rightarrow \infty$ limit; the leading contributions to (88) come from the center of AdS₂ and the south pole of S^2 , i.e., $|\omega| = 0$, |z| = 0. Thus, the bosonic integral becomes $\int d^2 \omega d^2 z e^{-t(|\omega|^2 + |z|^2)}$, giving the regularized value $\sim \pi^2/t^2$. This then also localizes the fermionic path integral to these locations as well, giving the regularized value

$$\sim \int d\theta_L^{++} d\theta_L^{--} d\theta_R^{--} d\theta_R^{++} t^2 (\theta_L^{++} \theta_R^{--} \theta_L^{--} \theta_R^{++}) = t^2.$$
(89)

Thus, the *t*-dependent regularized pieces of the bosonic and fermionic integrations cancel, resulting in a constant. This constant is determined from the supergravity matching of Z_{BH} and Z_{top} as in [33] and comes out as g_{top}^{-2} . Using this in Eq. (57) and summing over genus *g*, the final result is

$$\sum_{g} g_{top}^{2g-2} F_g, \tag{90}$$

which is the full topological string free energy \mathcal{F}_{top} , and F_g are given in Eq. (58). Z_{top} corresponds to the A-model topological string partition function, obtained from \mathcal{F}_{top} in Eq. (37).

For the case of anti-instanton, one uses u = 1/z and localizes the integration region to |w| = 0, |u| = 0, i.e., the center of AdS₂ and the north pole of S^2 . The final result using Eq. (60), and summing over genus g comes out as

$$\sum_{g} g_{\rm top}^{2g-2} \bar{F}_{g},\tag{91}$$

which is the antitopological string free energy $\bar{\mathcal{F}}_{top}$ and \bar{F}_{g} are given in Eq. (61). Its exponentiation gives the partition function \bar{Z}_{top} .

Therefore, the full type IIA partition function and hence the black hole partition function are obtained from \mathcal{F}_{top} in (90) and $\bar{\mathcal{F}}_{top}$ in (91), coming from instantons and antiinstantons, respectively, to be $|Z_{top}|^2$ and one recovers the relation (1) as $Z_{BH} = Z_{IIA} = |Z_{top}|^2$. We note that spacetime supersymmetry was not used explicitly in the calculation of the type IIA partition function and the subsequent emergence of the topological string partition function. The localization procedure relied completely on world-sheet superconformal invariance and the BRST method. It is desirable to extend the computations of this work toward $\mathcal{N} = 4$ and $\mathcal{N} = 8$ superstring theories as well. Applications toward other extensions of the OSV conjecture [90] are also important to explore.

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