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Two-center black holes, qubits, and elliptic curves

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We relate the U-duality invariants characterizing two-center extremal black-hole solutions in the stu, st^2, and t^3 models of $N = 2$, $d = 4$ supergravity to the basic invariants used to characterize entanglement classes of four-qubit systems. For the elementary example of a D0D4-D2D6 composite in the t^3 model we illustrate how these entanglement invariants are related to some of the physical properties of the two-center solution. Next we show that it is possible to associate elliptic curves to charge configurations of two-center composites. The hyperdeterminant of the hypercube, a four-qubit polynomial invariant of order 24 with 2 894 276 terms, is featuring the j invariant of the elliptic curve. We present some evidence that this quantity and its straightforward generalization should play an important role in the physics of two-center solutions.

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I. INTRODUCTION

The aim of the present paper is to show that it is possible to relate the entanglement measures usually used in studies concerning four-qubit systems to the U-duality invariants found recently by Ferrara *et al.* [1] characterizing extremal two-center black-hole solutions in the stu, st^2, and t^3 models. Interestingly as a byproduct of establishing this correspondence one can also come across an interesting connection between charge configurations of such black holes and a special class of elliptic curves.

Multicenter black-hole solutions provide an interesting research direction within the rapidly evolving field of black-hole solutions in supergravity, string, and M theory. For such solutions the attractor mechanism [2] has been generalized giving rise to split attractor flows [3,4]. For two-center solutions the latter term refers to the situation, when in moduli space after crossing walls of marginal stability the attractor flows are separately evolving to the attractor points of the constituent single center solutions. Recently these developments have triggered activity in a variety of new research fields such as attractor flow trees, entropy enigmas, microstate counting, and bound state recombination [5–10].

For single center solutions it has become obvious that the notion of duality charge orbits and their invariants [11] are useful concepts for classifying black-hole solutions together with their supersymmetry properties. Except for a special case [12] for multicenter solutions the corresponding structure of orbits and invariants is still unknown. In order the remedy this situation in a recent paper Ferrara *et al.* conducted [1,13] a systematic study on the structure of invariants characterizing the charge configurations and invariants of two-center solutions. In the case of the stu, st^2, and t^3 models looking at the structure of such invariants one immediately notices structural similarities to the well-known sets of four-qubit invariants [14,15] discussed in the seemingly unrelated field of quantum information.

Since the advent of the black-hole qubit correspondence [16] such coincidences should not come as a surprise. It is well-known by now that few qubit entangled systems are capable of providing interesting new insight into the structure of black-hole solutions and their attractor flows [17]. The occurrence of these qubits is related to the presence of tensor products of the spin $\frac{1}{2}$ irreducible representations of $SL(2)$ groups. Such products of $SL(2)$s show up as subgroups of U-duality groups governing the entangled web of dualities of supergravity models giving rise to black-hole solutions. Initially mathematical coincidences were recorded merely for two- and three-qubit systems and the corresponding axion-dilaton and stu black holes [16], however, evidence for n-qubit systems with $n > 3$ to make their presence in this context started to accumulate.

In this line of development the relevance of four-qubit systems to black-hole solutions in supergravity was first pointed out in our paper [18] where the isomorphism

$$so(4,4) \simeq sl(2)^4 \oplus (2,2,2,2) \qquad (1)$$

has been used to describe 7×16 of the 133 $E_{7(7)}$ generators of $N = 8$, $d = 4$ supergravity. These generators, describing seven copies of four-qubit states, do not belong to the $sl(2)^7$ subalgebra of $E_{7(7)}$. A suitable incidence geometry accounting for the relationship between the 7 groups of 16 generators is that of the dual Fano plane giving rise to a geometry, dual to the one describing the "tripartite entanglement of seven qubits" interpretation [18,19] of the quartic $E_{7(7)}$ black-hole entropy formula. The isomorphism of Eq. (1) was also discussed in the review paper of Borsten *et al.* [20] providing further interesting examples of simple qubit systems.

In our next paper [21] in a four-qubit entanglement based formalism the structure of extremal stationary spherically symmetric black-hole solutions in the STU model of $N = 2$, $d = 4$ supergravity was described. The basic idea facilitating this interpretation was the fact that stationary solutions in $d = 4$ supergravity can be described by dimensional reduction along the time direction [22]. In this $d = 3$ picture the global symmetry group $SL(2, \mathbb{R})^3$ of the STU model is extended by the Ehlers $SL(2, \mathbb{R})$ accounting for the fourth qubit. One can then introduce a four-qubit state depending on the charges, the moduli, and the warp factor. Here it was also noticed that in the terminology of four-qubit entanglement extremal black-hole solutions should correspond to nilpotent, and nonextremal ones to semisimple states. The upshot of these considerations was the emerging possibility of relating the entanglement properties of such and similar states to different classes of black-hole solutions in the STU model. The challenge of elaborating on this idea was recently taken up in the papers of Borsten *et al.* [23]. In these papers the authors applied the black-hole qubit correspondence to the problem of classifying four-qubit entanglement. The key technical ingredient was the Kostant-Sekiguchi theorem which establishes the link between nilpotent orbits of extremal black holes and four-qubit entanglement types. The emerging picture is: we have 31 entanglement families which reduce to nine up to permutations of the qubits. These nice papers confirmed once again that the input coming from string theory can be useful in establishing results in a different field, since the literature until now on four-qubit entanglement classification was confusing and seemingly contradictory.

In this paper we would like to show that the charge orbit classification of two-center black-hole solutions in the *stu* model is another arena where four-qubit systems naturally make their appearance. As a first possible step in this direction here we establish a correspondence between the U-duality invariants of Ferrara *et al.* [1] and the four-qubit invariants showing up in classification schemes of entanglement types in quantum information. Establishing this correspondence simplifies some of the invariants proposed so far, clarifies their geometric and algebraic roles, and provides hints for further generalizations outside the framework of $N = 2$, $d = 4$ supergravity. As an extra bonus the four-qubit picture also hints at a basic physical role these invariants are playing in the theory of two-center solutions. For one of the invariants not fully appreciated yet, our considerations establish a special role. It is the $SL(2)^{\times 4}$ and permutation invariant hyperdeterminant of type $2 \times 2 \times 2 \times 2$. This is a polynomial of order 24 in the 16 amplitudes of the four-qubit state. Mapping the 16 amplitudes to the 16 charges characterizing two-center solutions in the *stu* model, for a special case we show that the structure of this hyperdeterminant seem to govern issues of consistency in the realm of two-center solutions. These ideas also suggest a natural way for associating an elliptic curve of a special kind to a particular charge configuration. The coefficients of our elliptic curve are the algebraically independent four-qubit invariants, and its discriminant is just the hyperdeterminant. We also present some evidence for the conjecture that the structure of the j invariant of the elliptic curve should play an important role in the physical properties of the two-center solution. The idea that elliptic functions and the j invariant might play some role in four-qubit systems and the black-hole qubit correspondence was first suggested by P. Gibbs [24] some related discussion appeared in the paper of Bellucci *et al.* [25].

The organization of this paper is as follows. In Sec. II we summarize the background material on four-qubit invariants, reduced density matrices, and the structure of the hyperdeterminant of the hypercube. We introduce a quartic polynomial featuring the algebraically independent four-qubit $SL(2, \mathbb{C})$ invariants. In Sec. III we are discussing extremal two-center black-hole charge states in a four-qubit based picture. Here we work out a dictionary between the invariants found by Ferrara *et al.* [1] in the so called Calabi-Visentini basis and the algebraically independent four-qubit ones in the "special coordinates" basis. We give some of the invariants used in Ref. [1] a simpler appearance, and connect other invariants of physical meaning to properties of four-qubit reduced density matrices. Here we also show that the set of algebraically independent polynomial invariants in both the four-qubit and the Ferrara *et al.* description are based on two seemingly different quartic polynomials however, with the *same* resolvent cubic.

Section IV is devoted to a case study featuring Bogomolny-Prasad-Sommerfield (BPS) D0D4-D2D6 composites in the t^3 model. In the paper of Bates and Denef [4] this elementary example has already turned out to be a good playing ground for investigating the basic properties of two-center solutions, hence we opted for illustrating the physical role of our four-qubit invariants in the very same setting. These considerations relate the (necessary) consistency condition, guaranteeing the BPS composite to exist, to the positivity of the hyperdeterminant and to the extra constraint that the two nonzero invariants of the t^3 model are having the *same sign*. It turns out that precisely these conditions are the ones guaranteeing the fundamental quartic polynomial to have real roots. We then associate an elliptic curve of the Weierstrass canonical form to the resolvent cubic of this quartic and show how physical properties are nicely encapsulated in the structure of its j invariant.

In Sec. V we examine the status of our rather *ad hoc* assignment: the two-center charge configuration-elliptic curve, more thoroughly. By switching to the most general Tate form of an elliptic curve we show that our association of elliptic curves to two-center black-hole charge configurations in the *stu* model is a natural one. This means that the nonzero coefficients a_j with $j = 1, 2, 3, 4$ appearing in the Tate form are algebraically independent four-qubit invariant homogeneous polynomials of order $2j$. These coefficients have important physical meaning: a_1 is just the canonical symplectic pairing between the charge vectors of the two centers. The vanishing of a_4 gives rise to the st^2, and a further vanishing of a_2 results in the t^3 truncation. One can also see that in the st^2 and t^3 models sending a_3 to zero corresponds to the limit when our elliptic curve degenerates. One of the nontrivial coefficients a_6 in the Tate form is always zero for the stu model. This is related to the vanishing of a nontrivial polynomial constraint of homogeneous degree 12 valid in the stu model, already observed by Ferrara *et al.* [1]. Based on the latest results of Andrianopoli *et al.* [13] we conjecture that we should be able to generalize our correspondence between charge orbits and elliptic curves also for the case of maximal $N = 8$, $d = 4$ supergravity. In this case the stu model should arise as an $a_6 = 0$ truncation implemented by the vanishing of a polynomial of order 12.

The aim of our last speculative section, VI, is to draw the readers attention to some interesting structural similarities showing up in a variety of physical contexts where our four-qubit invariants parametrizing elliptic curves might play a crucial role. Here we give a new look and interpretation to a triality invariant curve originally introduced by Seiberg and Witten [26]. Now this curve is parametrized by four-qubit invariants also displaying permutation invariance. In this new setting we also invoke the F-theory interpretation of this curve as was given by Sen [27]. Finally, our conclusions and some comments are left for Sec. VI.

II. FOUR-QUBIT SYSTEMS

In order to facilitate a four-qubit description of the two-center charge configurations in the stu model our aim in this subsection is to review the background material on four-qubit states and their entanglement measures. A four-qubit state can be written in the form

$$|\Lambda\rangle = \sum_{i_0 i_1 i_2 i_3 = 0,1} \Lambda_{i_0 i_1 i_2 i_3} |i_0 i_1 i_2 i_3\rangle,$$

$$|i_0 i_1 i_2 i_3\rangle \equiv |i_0\rangle \otimes |i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle \in V_0 \otimes V_1 \otimes V_2 \otimes V_3 \qquad (2)$$

where $V_{0,1,2,3} \equiv \mathbb{C}^2$. Let the subgroup of stochastic local operations and classical communication [28] representing admissible four-partite manipulations on the qubits be just $SL(2,\mathbb{C})^{\otimes 4}$ acting on $|\Lambda\rangle$ as

$$|\Lambda\rangle \mapsto (S_0 \otimes S_1 \otimes S_2 \otimes S_3)|\Lambda\rangle, \qquad S_\alpha \in SL(2,\mathbb{C}), \quad \alpha = 0, 1, 2, 3. \qquad (3)$$

Our aim is to give a unified description of four-qubit states taken together with their SLOCC transformations and their associated invariants. As we will see states and transformations taken together can be described in a unified manner using the group $SO(8,\mathbb{C})$.

Let us discuss the structure of four-qubit $SL(2,\mathbb{C})^{\times 4}$ invariants [14,15,29,30]. The number of algebraically independent four-qubit invariants is four. We have one quadratic, two quartic, and one sextic invariant. In our recent paper [15] we investigated the structure of these invariants in the special frame where two of our qubits played a distinguished role. As we will see this scenario is just the one needed in the two-center STU black-hole context since in this setting one of the special qubits (the one labeled by the number 0) will be associated to the horizontal $SL_h(2,\mathbb{R})$ of Ferrara *et al.* [1] and the other (the one labeled by the number 1) is arising as the first factor from the structure $SL(2,\mathbb{R}) \times SO(2,2)$ known from the STU model. Indeed, such a structure is the one arising as a special case of the infinite Jordan symmetric sequence of $N = 2$ $d = 4$ supergravity theories [31].

To an arbitrary state $|\Lambda\rangle$ we can also associate a 4×4 matrix

$$\mathcal{L} \equiv \begin{pmatrix} \Lambda_{0000} & \Lambda_{0001} & \Lambda_{0010} & \Lambda_{0011} \\ \Lambda_{0100} & \Lambda_{0101} & \Lambda_{0110} & \Lambda_{0111} \\ \Lambda_{1000} & \Lambda_{1001} & \Lambda_{1010} & \Lambda_{1011} \\ \Lambda_{1100} & \Lambda_{1101} & \Lambda_{1110} & \Lambda_{1111} \end{pmatrix} \equiv \begin{pmatrix} A^1 & A^2 & A^3 & A^4 \\ B^1 & B^2 & B^3 & B^4 \\ C^1 & C^2 & C^3 & C^4 \\ D^1 & D^2 & D^3 & D^4 \end{pmatrix}, \qquad (4)$$

or four four-vectors. The splitting of the amplitudes of $|\Lambda\rangle$ into such four-vectors reflects our special choice for the distinguished qubits compatible with our conventions. We will also need the matrices

$$\mathcal{M} = \begin{pmatrix} A^1 & A^2 & B^1 & B^2 \\ C^1 & C^2 & D^1 & D^2 \\ A^3 & A^4 & B^3 & B^4 \\ C^3 & C^4 & D^3 & D^4 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} A^1 & A^3 & B^1 & B^3 \\ A^2 & A^4 & B^2 & B^4 \\ C^1 & C^3 & D^1 & D^3 \\ C^2 & C^4 & D^2 & D^4 \end{pmatrix}. \qquad (5)$$

Notice that in four-qubit notation the matrices $\mathcal{M}$ and $\mathcal{N}$ are arising from $\mathcal{L}$ by the permutations (012)(3) and (0)(123) meaning that the index structure of these matrices is

$$\mathcal{L} \leftrightarrow \Lambda_{i_0 i_1 i_2 i_3}, \quad \mathcal{M} \leftrightarrow \Lambda_{i_1 i_2 i_0 i_3}, \quad \mathcal{N} \leftrightarrow \Lambda_{i_0 i_2 i_3 i_1}. \qquad (6)$$

These matrices are entering in the reduced density matrices as

$$\varrho_{01} \equiv \mathrm{Tr}_{23}|\Lambda\rangle\langle\Lambda| = \mathcal{L}\mathcal{L}^\dagger, \quad \varrho_{12} \equiv \mathrm{Tr}_{03}|\Lambda\rangle\langle\Lambda| = \mathcal{M}\mathcal{M}^\dagger, \quad \varrho_{02} \equiv \mathrm{Tr}_{13}|\Lambda\rangle\langle\Lambda| = \mathcal{N}\mathcal{N}^\dagger, \qquad (7)$$

$$\bar{\varrho}_{23} \equiv \mathrm{Tr}_{01}|\Lambda\rangle\langle\Lambda| = \mathcal{L}^\dagger\mathcal{L}, \quad \bar{\varrho}_{03} \equiv \mathrm{Tr}_{12}|\Lambda\rangle\langle\Lambda| = \mathcal{M}^\dagger\mathcal{M}, \quad \bar{\varrho}_{13} \equiv \mathrm{Tr}_{02}|\Lambda\rangle\langle\Lambda| = \mathcal{N}^\dagger\mathcal{N}. \qquad (8)$$

where an overline denotes complex conjugation.

Now we introduce on the vector space $\mathbb{C}^4 \simeq \mathbb{C}^2 \times \mathbb{C}^2$ corresponding to the third and fourth qubit a symmetric bilinear form $\mathbf{g}$: $\mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$ with matrix representation

$$g = \varepsilon \otimes \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9}$$

This means that we have an $SL(2,\mathbb{C})^{\times 2}$ invariant quantity with the explicit form

$$\begin{aligned} g(A,B) \equiv A \cdot B = g_{\alpha\beta}A^\alpha B^\beta = A_\alpha B^\alpha \\ = A^1B^4 - A^2B^3 - A^3B^2 + A^4B^1. \end{aligned} \tag{10}$$

We can also introduce a *dual four-qubit state*

$$|\lambda\rangle = \sum_{i_0i_1i_2i_3=0,1} \lambda_{i_0i_1i_2i_3}|i_0i_1i_2i_3\rangle \tag{11}$$

with the associated matrix

$$\begin{aligned} l \equiv \begin{pmatrix} \lambda_{0000} & \lambda_{0001} & \lambda_{0010} & \lambda_{0011} \\ \lambda_{0100} & \lambda_{0101} & \lambda_{0110} & \lambda_{0111} \\ \lambda_{1000} & \lambda_{1001} & \lambda_{1010} & \lambda_{1011} \\ \lambda_{1100} & \lambda_{1101} & \lambda_{1110} & \lambda_{1111} \end{pmatrix} \\ \equiv \begin{pmatrix} a^1 & a^2 & a^3 & a^4 \\ b^1 & b^2 & b^3 & b^4 \\ c^1 & c^2 & c^3 & c^4 \\ d^1 & d^2 & d^3 & d^4 \end{pmatrix}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} a^\alpha = \epsilon^{\alpha\beta\gamma\delta}B_\beta C_\gamma D_\delta, \qquad b^\beta = \epsilon^{\alpha\beta\gamma\delta}A_\alpha C_\gamma D_\delta, \\ c^\gamma = \epsilon^{\alpha\beta\gamma\delta}A_\alpha B_\beta D_\delta, \qquad d^\delta = \epsilon^{\alpha\beta\gamma\delta}A_\alpha B_\beta C_\gamma. \end{aligned} \tag{13}$$

Here $\epsilon^{1234} = +1$, and indices are lowered by the matrix of g. Notice that the amplitudes of the dual four-qubit state are *cubic* in the original ones.

Using these definitions we define the quadratic and sextic invariants as

$$I_1 \equiv \tfrac{1}{2}(A \cdot D - B \cdot C), \qquad I_3 \equiv \tfrac{1}{2}(a \cdot d - b \cdot c). \tag{14}$$

The explicit form of the sextic invariant in terms of the dot product of Eq. (10) is

$$\begin{aligned} 2I_3 = \mathrm{Det}\begin{pmatrix} A\cdot A & A\cdot B & A\cdot D \\ A\cdot C & B\cdot C & C\cdot D \\ A\cdot D & B\cdot D & D\cdot D \end{pmatrix} \\ - \mathrm{Det}\begin{pmatrix} A\cdot B & B\cdot B & B\cdot C \\ A\cdot C & B\cdot C & C\cdot C \\ A\cdot D & B\cdot D & C\cdot D \end{pmatrix}. \end{aligned} \tag{15}$$

We also recall that the explicit form of I_1 is hiding its permutation invariance. Moreover, though the expression of I_3 of Eq. (14) is similar to the one of I_1 the invariant I_3 is *not* invariant under the permutation of the qubits.

Now we turn to the structure of quartic invariants. We have two independent of such invariants [14] and the simplest of them is the obvious expression

$$I_4 \equiv \mathrm{Det}\mathcal{L}, \tag{16}$$

i.e., the determinant of the 4×4 matrix of Eq. (4). In order to present the definition of the second one we define separable bivectors of the form

$$\Pi_{\mu\nu\alpha\beta} \equiv \Lambda_{\mu\alpha}\Lambda_{\nu\beta} - \Lambda_{\mu\beta}\Lambda_{\nu\alpha}, \qquad \alpha,\beta,\mu,\nu = 1,2,3,4. \tag{17}$$

Here our labeling convention $\Lambda_{\mu\alpha}$ indicates that $\mu = 1, 2, 3, 4$ identifies the four-vector in question [i.e. A, B, C, or D of Eq. (4)], and the label $\alpha = 1, 2, 3, 4$ refers to the component of the particular vector. Now our last invariant is the quartic combination

$$I_2 = \tfrac{1}{6}\Pi_{\mu\nu\alpha\beta}\Pi^{\mu\nu\alpha\beta}. \tag{18}$$

Obviously the symmetric nondegenerate bilinear form g of Eq. (10) acting on four vectors like $A, B, C, D \in \mathbb{C}^4$ is inducing a corresponding symmetric nondegenerate bilinear form on the space of bivectors $\bigwedge^2\mathbb{C}^4$. By an abuse of notation we use again the symbol $\cdot$ for this new bilinear form with the definition [15]

$$(A\wedge B)\cdot(C\wedge D) \equiv 2((A\cdot C)(B\cdot C) - (A\cdot D)(B\cdot C)). \tag{19}$$

Now I_2 can also be written in the equivalent form

$$\begin{aligned} I_2 = \tfrac{1}{6}[(A\wedge B)\cdot(C\wedge D) + (A\wedge C)\cdot(B\wedge D) \\ - \tfrac{1}{2}(A\wedge D)^2 - \tfrac{1}{2}(B\wedge C)^2]. \end{aligned} \tag{20}$$

An important comment here is in order. Let us have a look at I_4 and also at the determinants of the matrices of Eq. (5)

$$L \equiv I_4 = \mathrm{Det}\mathcal{L}, \qquad M \equiv \mathrm{Det}\mathcal{M}, \qquad N \equiv \mathrm{Det}\mathcal{N}. \tag{21}$$

Then one can prove [14,32]

$$L + M + N = 0. \tag{22}$$

One also has the constraint

$$M - N = 3I_2 - 2I_1^2, \tag{23}$$

that we will need later.

It is known [14] that the minimal set of algebraically independent $SL(2)^{\times 4}$ invariants is consisting of a quadratic, two quartic and one sextic invariant. Our choice for this set will be [15]: I_1, I_2, I_4, and I_3. Let us now present the reason for this choice. Let us consider the matrix

$$\Omega \equiv \mathcal{L}g\mathcal{L}^T g. \tag{24}$$

Then its characteristic polynomial is

$$\begin{aligned} \Sigma_4(\Lambda_{i_0i_1i_2i_3}, t) \equiv \mathrm{Det}(\mathbf{1}t - \Omega) \\ = t^4 - 4I_1t^3 + 6I_2t^2 - 4I_3t + I_4^2. \end{aligned} \tag{25}$$

Clearly by Newton's identities we have

$$I_1 = \frac{1}{4}\mathrm{Tr}\Omega, \qquad I_2 = \frac{1}{12}[(\mathrm{Tr}\Omega)^2 - \mathrm{Tr}\Omega^2], \tag{26}$$

$$I_3 = \frac{1}{24}[(\mathrm{Tr}\Omega)^3 - 3\mathrm{Tr}\Omega\,\mathrm{Tr}\Omega^2 + 2\mathrm{Tr}\Omega^3], \quad (I_4)^2 = \mathrm{Det}\Omega. \tag{27}$$

This form of writing our invariants is related to the fact that there is a $1-1$ correspondence between the $SL(2,\mathbb{C})^{\otimes 4}$ orbits of four-qubit states and the $SO(4,\mathbb{C})\times SO(4,\mathbb{C})$ ones of 4×4 matrices.

The polynomial of Eq. (25) in the four-qubit context appeared in our recent paper [15]. its role as a characteristic polynomial has been emphasized in Ref. [29]. The discriminant of this fourth order polynomial is the hyperdeterminant [33] D_4 of the $2\times 2\times 2\times 2$ hypercube $\Lambda_{i_0 i_1 i_2 i_3}$. It is a polynomial of degree 24 in the 16 amplitudes and has 2 894 276 terms [34]. D_4 can be expressed [15] in terms of our fundamental invariants as

$$256 D_4 = S^3 - 27T^2, \tag{28}$$

where

$$\begin{aligned} S &= (I_4^2 - I_2^2) + 4(I_2^2 - I_1 I_3), \\ T &= (I_4^2 - I_2^2)(I_1^2 - I_2) + (I_3 - I_1 I_2)^2. \end{aligned} \tag{29}$$

For an alternative form of D_4 see the papers of Refs. [14,30].

In closing this section we briefly discuss some results on the classification of entanglement classes for four qubits [29,35]. By entanglement classes we mean orbits under $SL(2,\mathbb{C})^{\times 4}\cdot \mathrm{Sym}_4$ where Sym_4 is the symmetric group on four symbols. The basic result states that four qubits can be entangled in nine different ways [29,35]. It is to be contrasted with the two entanglement classes [28] obtained for three qubits. For a refined classification of four-qubit entanglement motivated by the black-hole qubit correspondence see the papers of Borsten *et al.* [23].

Let us consider the matrix

$$\mathcal{R}_\Lambda \equiv \begin{pmatrix} 0 & \Lambda g \\ -\Lambda^T g & 0 \end{pmatrix}, \tag{30}$$

which now can be regarded as an element of the Lie algebra of $SO(8,\mathbb{C})$. If the matrix $\mathcal{R}_\Lambda$ is diagonalizable under the action

$$\mathcal{R}_\Lambda \mapsto S\mathcal{R}_\Lambda S^{-1}, \qquad S = \begin{pmatrix} S_0\otimes S_1 & 0 \\ 0 & S_2 \otimes S_3 \end{pmatrix}, \qquad S_\alpha \in SL(2,\mathbb{C}), \tag{31}$$

we say that the corresponding four-qubit state $|\Lambda\rangle$ is *semisimple*. If $\mathcal{R}_\Lambda$ is *nilpotent* then we call the corresponding state $|\Lambda\rangle$ nilpotent too. It is known that a nilpotent orbit is *conical*, i.e., if $|\Lambda\rangle$ is an element of the orbit then $\lambda|\Lambda\rangle$ is also an element for all nonzero complex numbers λ. Hence, a nilpotent orbit is also a $GL(2,\mathbb{C})^{\times 4}$ orbit. It is clear that for nilpotent states all of our algebraically independent invariants are zero.

A semisimple state of four qubits can always be transformed to the form [35]

$$\begin{aligned} |G_{abcd}\rangle = {}& \frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle \\ &+ |1100\rangle) + \frac{b+c}{2}(|0101\rangle + |1010\rangle) \\ &+ \frac{b-c}{2}(|0110\rangle + |1001\rangle), \end{aligned} \tag{32}$$

where a, b, c, d are complex numbers. This class corresponds to the so called Greenberger-Horne-Zeilinger (GHZ) class found in the three-qubit case [28]. For this state the reduced density matrices obtained by tracing out all but one of the qubits are proportional to the identity. This is the state with maximal four-partite entanglement. Another interesting property of this state is that it does not contain true three-partite entanglement. A straightforward calculation shows that the values of our invariants (I_1, I_2, I_3, I_4) occurring for the state $|G_{abcd}\rangle$ representing the generic class are

$$\begin{aligned} I_1 &= \frac{1}{4}[a^2+b^2+c^2+d^2], \\ I_2 &= \frac{1}{6}[(ab)^2+(ac)^2+(ad)^2+(bc)^2+(bd)^2+(cd)^2], \end{aligned} \tag{33}$$

$$\begin{aligned} I_3 &= \frac{1}{4}[(abc)^2+(abd)^2+(acd)^2+(bcd)^2], \\ I_4 &= abcd, \end{aligned} \tag{34}$$

hence the values of the invariants $(4I_1, 6I_2, 4I_3, I_4^2)$ are given in terms of the elementary symmetric polynomials in the variables $(t_1,t_2,t_3,t_4) = (a^2,b^2,c^2,d^2)$. For the semisimple states $|G_{abcd}\rangle$ the value of D_4 can be expressed as [14,15]

$$D_4 = \frac{1}{256}\Pi_{i<j}(t_i - t_j)^2, \quad (t_1,t_2,t_3,t_4) \equiv (a^2,b^2,c^2,d^2). \tag{35}$$

Notice that for the states $|G_{abcd}\rangle$ with D_4 nonvanishing ($t_i \neq t_j$) the corresponding matrix of Eq. (30) belongs to a Cartan subalgebra of $SO(8,\mathbb{C})$. The stabilizer of such states corresponds to the Weyl group of $SO(8,\mathbb{C})$. This stabilizer is the Klein group generated by the four elements $I\otimes I\otimes I\otimes I$ and $\sigma_a\otimes\sigma_a\otimes\sigma_a\otimes\sigma_a$ for $a = 1, 2, 3$.

III. TWO-CENTER EXTREMAL BLACK HOLES AS FOUR-QUBIT SYSTEMS

In the paper [1] of Ferrara *et al.*, in order to describe the structure of the U-duality invariant polynomials associated to the two-center extremal black-hole solutions the Calabi-Visentini (CV) basis has been used. Here by U-duality we mean the continuous limit [1] valid for large values of the charges of the usual-nonperturbative string theory symmetries. First, we describe the connection of the CV basis to the one making the four-qubit structures explicit. Next we turn to an entanglement based understanding of the structure of the two-center U-duality invariants.

In the CV basis the two-center black-hole solutions are characterized by a pair of real charge vectors $\mathcal{Q}_1$, $\mathcal{Q}_2 \in \mathbb{R}^8$,

$$\begin{aligned} \mathcal{Q}_1 &\equiv (P^0, P^1, P^2, P^3, Q_0, Q_1, Q_2, Q_3)^T, \\ \mathcal{Q}_2 &\equiv (p^0, p^1, p^2, p^3, q_0, q_1, q_2, q_3)^T. \end{aligned} \tag{36}$$

As we see, these charge vectors are containing two four-vectors each, namely, P^I, $Q_I \equiv \eta_{IJ} Q^J$, and p^I, $q_I \equiv \eta_{IJ} q^J$ $I, J = 0, 1, 2, 3$ where the raising and lowering of the indices I and J are effected by the metric η_{IJ} and η^{IJ} of $SO(2, 2)$ answering the symmetric bilinear form h acting on the charge four-vectors P^I and Q^J in the CV basis as

$$\begin{aligned} h(P, Q) \equiv P \circ Q &= \eta_{IJ} P^I Q^J \\ &= -P^0Q^0 - P^1Q^1 + P^2Q^2 + P^3Q^3. \end{aligned} \tag{37}$$

Let us now relate the 16 component charge vector in the CV basis characterizing a particular two-center extremal black-hole solution in the STU model to a *real unnormalized* four-qubit pure state $|\Lambda\rangle$ by relating the four-vectors $P^I, Q^I, p^I, q^I \in \mathbb{R}^4$ of Eq. (36) and $A^\alpha, B^\alpha, C^\alpha, D^\alpha \in \mathbb{R}^4$ of Eq. (4) as follows:

$$\begin{pmatrix} P^0 \\ P^1 \\ P^2 \\ P^3 \end{pmatrix}_{\rm CV} = \frac{1}{\sqrt{2}} \begin{pmatrix} A^1 - A^4 \\ A^2 + A^3 \\ -A^1 - A^4 \\ -A^2 + A^3 \end{pmatrix}, \quad \begin{pmatrix} Q^0 \\ Q^1 \\ Q^2 \\ Q^3 \end{pmatrix}_{\rm CV} = \frac{1}{\sqrt{2}} \begin{pmatrix} B^1 - B^4 \\ B^2 + B^3 \\ -B^1 - B^4 \\ -B^2 + B^3 \end{pmatrix}, \tag{38}$$

$$\begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix}_{\rm CV} = \frac{1}{\sqrt{2}} \begin{pmatrix} C^1 - C^4 \\ C^2 + C^3 \\ -C^1 - C^4 \\ -C^2 + C^3 \end{pmatrix}, \quad \begin{pmatrix} q^0 \\ q^1 \\ q^2 \\ q^3 \end{pmatrix}_{\rm CV} = \frac{1}{\sqrt{2}} \begin{pmatrix} D^1 - D^4 \\ D^2 + D^3 \\ -D^1 - D^4 \\ -D^2 + D^3 \end{pmatrix}. \tag{39}$$

Then using the definitions above we clearly have, for example,

$$P \circ Q = A \cdot B = g(A, B) = g_{\alpha\beta} A^\alpha B^\beta, \tag{40}$$

with the bilinear form g defined as in Eq. (10).

Let us also give the connection between the Calabi-Visentini basis and the one usually used in special geometry, i.e., the special coordinates (SC) symplectic frame. This frame yields the usual set of electric and magnetic charges i.e. $(P^I, Q_I)_{\rm SC}$ and $(p^I, q_I)_{\rm SC}$. In the following we will use these charges so it is important to clarify their relationship to the components of our four-qubit state $|\Lambda\rangle$ (see Eq. (4)).

$$\begin{pmatrix} \Lambda_{0000} \\ \Lambda_{0001} \\ \Lambda_{0010} \\ \Lambda_{0011} \end{pmatrix} = \begin{pmatrix} A^1 \\ A^2 \\ A^3 \\ A^4 \end{pmatrix} = \begin{pmatrix} P^0 \\ P^2 \\ P^3 \\ Q_1 \end{pmatrix}_{\rm SC}, \quad \begin{pmatrix} \Lambda_{0100} \\ \Lambda_{0101} \\ \Lambda_{0110} \\ \Lambda_{0111} \end{pmatrix} = \begin{pmatrix} B^1 \\ B^2 \\ B^3 \\ B^4 \end{pmatrix} = \begin{pmatrix} P^1 \\ Q_3 \\ Q_2 \\ -Q_0 \end{pmatrix}_{\rm SC} \tag{41}$$

$$\begin{pmatrix} \Lambda_{1000} \\ \Lambda_{1001} \\ \Lambda_{1010} \\ \Lambda_{1011} \end{pmatrix} = \begin{pmatrix} C^1 \\ C^2 \\ C^3 \\ C^4 \end{pmatrix} = \begin{pmatrix} p^0 \\ p^2 \\ p^3 \\ q_1 \end{pmatrix}_{\rm SC}, \quad \begin{pmatrix} \Lambda_{1100} \\ \Lambda_{1101} \\ \Lambda_{1110} \\ \Lambda_{1111} \end{pmatrix} = \begin{pmatrix} D^1 \\ D^2 \\ D^3 \\ D^4 \end{pmatrix} = \begin{pmatrix} p^1 \\ q_3 \\ q_2 \\ -q_0 \end{pmatrix}_{\rm SC}. \tag{42}$$

We note here however, that our conventions are slightly different from the ones used in Ref. [1]. The charges p^1, p^2, p^3, q_0 and P^1, P^2, P^3, Q_0 in the SC basis used by us are the negatives of the corresponding ones in the SC basis as used in Ref. [1]; see Eqs. (38) and (39), Eqs. (41) and (42), and Eq. (B.3) of that paper.

Notice also that in our four-qubit state $|\Lambda\rangle$ with amplitudes $\Lambda_{0i_1i_2i_3}$, and $\Lambda_{1i_1i_2i_3}$ the first set of amplitudes labeled by $i_0 = 0$ corresponds to the charge configuration of the first black-hole and the second labeled by $i_0 = 1$ describes the second black-hole. As we see the first label plays a distinguished role with an extra $SL(2, \mathbb{R})_0$ (dubbed by Ferrara et al. the horizontal one) acting on. This group represents the generalized exchange symmetry between the centers.

Let us now define three bivectors

$$X \equiv A \wedge B, \quad Y \equiv C \wedge D, \quad Z \equiv \tfrac{1}{2}(A \wedge D - B \wedge C). \tag{43}$$

In component notation we have for example

$$X_{\alpha\beta} = A_\alpha B_\beta - A_\beta B_\alpha. \tag{44}$$

Switching to the Calabi-Visentini basis these objects are the T tensors used in the paper of Ferrara *et al.* [1]. As we can see the bivectors X and Y are separable, i.e., they are precisely the ones satisfying the Plücker relations

$$\begin{aligned} X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23} = 0, \\ Y_{12}Y_{34} - Y_{13}Y_{24} + Y_{14}Y_{23} = 0; \end{aligned} \tag{45}$$

on the other hand, the bivector Z is *entangled*, i.e., in the nomenclature of fermionic entanglement [36], it has Slater rank two.

Now we introduce the shorthand notation for the product of two separable bivectors as defined in Eq. (19)

$$X \cdot Y \equiv (A \wedge B) \cdot (C \wedge D). \tag{46}$$

Notice that in the notation of Eq. (41) and (42)

$$X^2 \equiv X \cdot X = 2(A^2B^2 - (A \cdot B)^2), \qquad Y^2 \equiv Y \cdot Y = 2(C^2D^2 - (C \cdot D)^2), \tag{47}$$

are just 2 times the quartic invariants of the charges characterizing the two black holes

$$\tfrac{1}{2}X^2 = I_4(\mathcal{Q}_1) = -D_3(\Lambda_{0i_1i_2i_3}), \qquad \tfrac{1}{2}Y^2 = I_4(\mathcal{Q}_2) = -D_3(\Lambda_{1i_1i_2i_3}). \tag{48}$$

Here $D_3(\Lambda_{i_1i_2i_3})$ is Cayley's hyperdeterminant [37].

With these definitions we can define the quantities

$$I_{+2} = \tfrac{1}{2}X^2, \qquad I_{+1} = \tfrac{1}{2}X \cdot Z, \qquad I_0 = \tfrac{1}{6}(2Z^2 - X \cdot Y), \qquad I_{-1} = \tfrac{1}{2}Y \cdot Z, \qquad I_{-2} = \tfrac{1}{2}Y^2, \tag{49}$$

and the ones

$$I' = \tfrac{1}{2}X \cdot Y, \qquad I'' = \tfrac{3}{2}Z^2. \tag{50}$$

When reinterpreted in the CV basis these are precisely the $SL(2,\mathbb{R})_1 \times SL(2,\mathbb{R})_2 \times SL(2,\mathbb{R})_3$ invariants of Ferrara *et al.* [1]. The important property of the invariants of Eq. (49) is that they are *covariants* with respect to $SL(2,\mathbb{R})_0$ acting on the distinguished (horizontal) qubit. Indeed, they are sitting in the 5 (spin 2) irreducible representation of this group.

Now we elucidate another aspect of this important property of the set of invariants of Eq. (49). In order to do this we first recall that for two-qubits the canonical measure of pure state entanglement is the concurrence [38]

$$\mathcal{C} = 2|D_2(\Lambda_{i_1i_2})| = 2|\Lambda_{00}\Lambda_{11} - \Lambda_{01}\Lambda_{10}|, \tag{51}$$

which is related to the determinant of an ordinary 2×2 matrix. For three-qubits the basic quantity characterizing genuine three-qubit entanglement [28] is the three-tangle

$$\tau = 4|D_3(\Lambda_{i_0i_1i_2})|, \tag{52}$$

where now D_3 is Cayley's hyperdeterminant [37,38]. According to the method of Schläfli [33] D_3 is related to the discriminant Δ_2 of the quadratic polynomial

$$\Pi_2(\Lambda_{i_0i_1i_2}, t) \equiv D_2(\Lambda_{0i_1i_2}t + \Lambda_{1i_1i_2}) = (\Lambda_0 \cdot \Lambda_0)t^2 + 2(\Lambda_0 \cdot \Lambda_1)t + (\Lambda_1 \cdot \Lambda_1), \tag{53}$$

where Λ_0 and Λ_1 are four-vectors with components $(\Lambda_{000}, \Lambda_{001}, \Lambda_{010}, \Lambda_{011})$ and $(\Lambda_{100}, \Lambda_{101}, \Lambda_{110}, \Lambda_{111})$ and the $\cdot$ product is the usual one of Eq. (10). Moreover, due to permutation invariance of D_3 we obtain the same expression whenever the first or the second qubit plays a distinguished role. Obviously the quantities $J_{+1} = (\Lambda_0 \cdot \Lambda_0)$, $J_0 = (\Lambda_0 \cdot \Lambda_1)$, and $J_{-1} = (\Lambda_1 \cdot \Lambda_1)$ are $SL(2)_1 \times SL(2)_2$ invariants, however, they are *covariants* under the "horizontal" $SL(2)_0$. Indeed, the triple J_{+1}, J_0, J_{-1} transforms according to the irreducible representation **3** of spin 1 of this horizontal group which can be regarded as some sort of generalized exchange symmetry working between the *two* two-qubit systems. It is also clear that searching singlets with respect to this horizontal symmetry group can reveal some new properties of our pair of two-qubit systems, namely, that they are secretly comprising a system having a higher degree of symmetry.

Now, proceeding by analogy we define the polynomial

$$\Pi_4(\Lambda_{i_0i_1i_2i_3}, t) \equiv D_3(\Lambda_{0i_1i_2i_3}t + \Lambda_{1i_1i_2i_3}) = I_{+2}t^4 + 4I_{+1}t^3 + 6I_0t^2 + 4I_{-1}t + I_{-2}. \tag{54}$$

A straightforward calculation shows that the coefficients of this polynomial are precisely the covariants of Eq. (49). Now, according to theorem 14.4.1 and corollary 14.2.10 of Ref. [33] the discriminant of this quartic polynomial Δ_4 divided by 256 is just the hyperdeterminant $D_4(\Lambda_{i_0i_1i_2i_3})$ of the hypercube of format $2 \times 2 \times 2 \times 2$. Besides being a singlet with respect to the horizontal $SL(2)_0$, D_4 is also an invariant under S_4 the permutation group of the four-qubit system.

At this point one can notice [15] that the polynomials Σ_4 and Π_4 of Eqs. (25) and (54) are having the same discriminants Δ_4, hence, both can be used to obtain an expression for D_4. Notice that Σ_4 is a polynomial with its coefficients I_1, I_2, I_3, I_4 also being $SL(2)_0$ singlets, however, Π_4 is a polynomial with coefficients $I_{\pm 2}$, $I_{\pm 1}$, I_0 being merely $SL(2)_0$ covariants. Using this observation we can construct new $SL(2)_0$ singlets from the quantities of Eq. (49) by relating them to the known algebraically independent four-qubit ones, namely, I_1, I_2, I_3, and I_4.

In order to relate the $SL(2)_0$ singlets found by Ferrara *et al.* to our four algebraically independent four-qubit invariants one just has to compare the relevant expressions. In fact many of their invariants and the constraints satisfied by them take in this four-qubit setting a much simpler and instructive form. In particular their complete set of invariants with corresponding degrees 2, 4, 6, 8 denoted by $\mathcal{W}$, χ, I_6, $\mathrm{Tr}(\mathcal{J}^2)$ is related to ours as

$$\mathcal{W} = 2I_1, \qquad \chi = 3I_2 - 2I_1^2, \qquad I_6 = -I_3. \tag{55}$$

By virtue of Eqs. (23) and (50) we also have the relations

$$I' - I'' = \tfrac{3}{2}I_2, \qquad \chi = M - N. \tag{56}$$

We still have to account for the invariant $\mathrm{Tr}(\mathcal{J}^2)$ of order 8 built from $\mathcal{J}$ the symmetric traceless matrix comprising the covariants $I_{\pm 2}, I_{\pm 1}, I_0$. The 5 independent components of this matrix are transforming according to the **5** of $SL(2)_0$. In order to reveal the meaning of this invariant and also an extra one $\mathrm{Tr}(\mathcal{J}^3)$ of order 12 let us reconsider the awkward looking polynomial constraint of Eq. (5.6) of Ref. [1].

$$\mathcal{P}_{12} \equiv I_6^2 + \mathcal{W}\chi I_6 + \mathrm{Tr}(\mathcal{J}^3) + \frac{\mathrm{Tr}(\mathcal{J}^2)\mathcal{W}^2}{12} - \frac{\mathrm{Tr}(\mathcal{J}^2)\chi}{3} - \frac{\mathcal{W}^6}{432} + \frac{\mathcal{W}^4\chi}{36} + \frac{5\mathcal{W}^2\chi^2}{36} + \frac{4\chi^3}{27} = 0. \quad (57)$$

Using the dictionary of Eq. (55) we can cast this constraint in the nice form

$$\mathrm{Tr}\,(\mathcal{J}^3) = [4(I_2^2 - I_1 I_3) - \mathrm{Tr}(\mathcal{J}^2)](I_1^2 - I_2) - (I_3 - I_1 I_2)^2. \quad (58)$$

There is one more invariant [1] of order 8 which is directly related to $\mathrm{Tr}(\mathcal{J}^2)$,

$$\mathcal{P}_8 \equiv -12\,\mathrm{Tr}(\mathcal{J}^2) + 24 I_6 \mathcal{W} + (\mathcal{W}^2 + 2\chi)^2. \quad (59)$$

Recall that the constraint $\mathcal{P}_8 = 0$ implements the reduction of the *stu* model to the st^2 model [1] in a manifestly $SL(2)_0$ invariant manner. Using again Eq. (55) the new form of $\mathcal{P}_8$ is

$$\mathcal{P}_8 = 12(3I_2^2 - 4I_1 I_3 - \mathrm{Tr}(\mathcal{J}^2)). \quad (60)$$

Putting this into Eq. (58) and recalling Eq. (29) one obtains the simple expressions

$$\mathrm{Tr}\,(\mathcal{J}^2) = S, \qquad \mathrm{Tr}(\mathcal{J}^3) = -T, \quad (61)$$

provided

$$\mathcal{P}_8 = -12 I_4^2 = -12L^2. \quad (62)$$

The first result of these considerations is that $-\frac{1}{12}\mathcal{P}_8$ is really the square of the basic fourth order invariant $L = I_4$. According to Eqs. (4) and (21) L is just the determinant of the matrix $\mathcal{L}$ we have started our four-qubit considerations with. Moreover, according to Eqs. (7) and (8) we also see that for *unnormalized* four-qubit states we have

$$-\tfrac{1}{12}\mathcal{P}_8 = \mathrm{Det}\varrho_{01} = \mathrm{Det}\varrho_{12}, \quad (63)$$

hence this invariant is related to the determinant of the reduced density matrices of our four-qubit state $|\Lambda\rangle$ corresponding to the bipartite split of the form: (01)(23). We can also conclude that in the four-qubit picture the reduction of the *stu* model to the st^2 is effected by sending one of the eigenvalues of the reduced density matrices corresponding to the (01)(23) split to zero. Notice however, that the remaining density matrices corresponding to the remaining two splits (02)(13) and (03)(12) are generally not sharing this property. This means that for the st^2 model we have

$$L = 0, \qquad M \neq 0, \qquad N \neq 0. \quad (64)$$

Recall now that a suitable further reduction to the t^3 model is obtained by employing the following two $SL(2)_0$ invariant constraints [1]:

$$\chi = 0, \qquad \mathcal{P}_8 = 0. \quad (65)$$

By virtue of Eq. (56) and the constraint $L + M + N = 0$ these constraints can be described in the compact form

$$L = M = N = 0. \quad (66)$$

This means that for the t^3 reduction of the *stu* model *all of the* reduced density matrices of Eqs. (7) and (8) of the four-qubit state $|\Lambda\rangle$ have a zero eigenvalue.

The second result of our considerations finally clarifies the role played by the invariants $\mathrm{Tr}(\mathcal{J}^2)$ and $\mathrm{Tr}(\mathcal{J}^3)$. In particular, looking at Eqs. (28), (29), and (61) we see that the four-qubit hyperdeterminant $256D_4$, which is just the discriminant of our polynomial Σ_4 of Eq. (25), can be expressed with the help of these invariants of order 8 and 12 as $S^3 - 27T^2$. Moreover, since the discriminant of a quartic is the same as the discriminant of its resolvent cubic one can also show that the two quartic equations $\Sigma_4 = 0$ and $\Pi_4 = 0$ of Eqs. (25) and (54) are having the *same* resolvent cubics. Indeed, a straightforward calculation shows that the corresponding resolvent cubics in both cases are of the form

$$u^3 - Su - 2T = 0. \quad (67)$$

In the first case we get back to the known expressions for S and T of Eq. (29), and in the second one we get [1]

$$S = 3I_0^2 - 4I_{+1}I_{-1} + I_{+2}I_{-2},$$
$$T = I_0^3 + I_{+1}^2 I_{-2} + I_{-1}^2 I_{+2} - I_{+2}I_0 I_{-2} - 2I_{+1}I_0 I_{-1}. \quad (68)$$

IV. THE D2D6-D0D4 SPLIT

A. Invariants

In order to uncover the role of our four-qubit invariants playing in the theory of two-center black-hole solutions let us consider a special class of two-center black-hole solutions in the t^3 model featuring a D0D4-D2D6 split in the type IIA duality frame. In this case, the vectors of Eqs. (41) and (42) in the special coordinate (SC) basis are

$$\begin{pmatrix} A^1 \\ A^2 \\ A^3 \\ A^4 \end{pmatrix} = \begin{pmatrix} 0 \\ P \\ P \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} B^1 \\ B^2 \\ B^3 \\ B^4 \end{pmatrix} = \begin{pmatrix} P \\ 0 \\ 0 \\ -U \end{pmatrix}, \quad (69)$$

corresponding to the first black hole with charge configuration $\mathcal{Q}_1$ of a BPS D0D4 system and

$$\begin{pmatrix} C^1 \\ C^2 \\ C^3 \\ C^4 \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ 0 \\ q \end{pmatrix}, \qquad \begin{pmatrix} D^1 \\ D^2 \\ D^3 \\ D^4 \end{pmatrix} = \begin{pmatrix} 0 \\ q \\ q \\ 0 \end{pmatrix}, \quad (70)$$

corresponding to the second black hole with charge configuration $\mathcal{Q}_2$ of a BPS D2D6 system. For BPS configurations in the first case we should have

$$-D_3(\mathcal{Q}_1) = (A\cdot A)(B\cdot B) - (A\cdot B)^2 = 4UP^3 > 0, \tag{71}$$

and in the second the corresponding constraint is

$$-D_3(\mathcal{Q}_2) = (C\cdot C)(D\cdot D) - (C\cdot D)^2 = -4vq^3 > 0. \tag{72}$$

Notice that our charge splits for the special values of $U = 4$, $P = q = 1$ and $v = -4$ incorporates the illustrative example of Bates and Denef [4] (in that paper v is related to ours via a sign flip). The four-vectors A^α, B^β, C^γ, and D^δ are comprising the 16 amplitudes of a real four-qubit state as displayed in Eqs. (2) and (4). Now using Eqs. (14)–(16) and (20) the algebraically independent four-qubit invariants I_1, I_2, I_3, and I_4 can be calculated. The explicit forms of these invariants are

$$I_1 = \tfrac{1}{2}(Uv - 3Pq), \qquad I_2 = \tfrac{2}{3}I_1^2, \qquad I_3 = -Pq(Pq + Uv)^2, \qquad I_4 = 0. \tag{73}$$

Notice that by virtue of Eqs. (22) and (23) our D0D4-D2D6 example illustrates the constraints we have already discussed in connection with the t^3 model, namely, the ones $L = M = N = 0$. Since $L = I_4 = 0$ and $3I_2 = 2I_1^2$ ($\chi = 0$) the quartic equation $\Sigma_4 = 0$ arising from the polynomial of Eq. (25) now reduces to a cubic one of the following form:

$$t^3 - 4I_1t^2 + 4I_1^2t - 4I_3 = 0. \tag{74}$$

The discriminant of this cubic equation is

$$\Delta = 4I_3\left(I_3 - \left(\frac{2I_1}{3}\right)^3\right). \tag{75}$$

According to Eqs. (28) and (29) the hyperdeterminant D_4 is also related to this discriminant and is of the form

$$256D_4 = 27I_3^3\left(\left(\frac{2I_1}{3}\right)^3 - I_3\right) = -(UvPq)(Pq)^2(Uv + 9Pq)^2(Uv + Pq)^6. \tag{76}$$

B. Consistency condition

Let us now consider the necessary condition [3,4] for our two-center charge configuration supporting a corresponding two-center stationary extremal BPS black-hole solution. As it is well-known this condition is of the form

$$|\mathbf{x}_1 - \mathbf{x}_2| = \langle \mathcal{Q}_1, \mathcal{Q}_2\rangle \frac{|\mathcal{Z}_1 + \mathcal{Z}_2|_{r=\infty}}{2\,\mathrm{Im}(\bar{\mathcal{Z}}_2\mathcal{Z}_1)_{r=\infty}}, \tag{77}$$

where $\mathbf{x}_{1,2}$ are the locations of the centers, $r = |\mathbf{x}|$, $\mathcal{Z}_{1,2}$ are the central charges corresponding to the charges $\mathcal{Q}_{1,2}$, and the symplectic product of the charge vectors $\langle \mathcal{Q}_1, \mathcal{Q}_2\rangle$ is related to our quadratic four-qubit invariant as

$$I_1 = \tfrac{1}{2}\langle \mathcal{Q}_1, \mathcal{Q}_2\rangle. \tag{78}$$

The explicit forms of the central charges for our centers are

$$\mathcal{Z}_1 = e^{K/2}(U - 3P\tau^2), \qquad \mathcal{Z}_2 = e^{K/2}(3q\tau + v\tau^3). \tag{79}$$

Here τ is as usual the complex scalar field of the t^3 model

$$\tau = x - iy, \qquad y > 0 \tag{80}$$

with negative imaginary part [39,40], and $K = -\log(8y^3)$ is the Kähler potential.

Now the constraint dictated by Eq. (77) is

$$I_1\,\mathrm{Im}(\bar{\mathcal{Z}}_2\mathcal{Z}_1)_\infty > 0. \tag{81}$$

Explicitly we have

$$8y_\infty^2\,\mathrm{Im}(\bar{\mathcal{Z}}_2\mathcal{Z}_1)_\infty = -3Pv(x_\infty^2 + y_\infty^2)^2 + (9Pq + 3Uv)x_\infty^2 + (9Pq - Uv)y_\infty^2 + 3Uq. \tag{82}$$

We are interested in the structure of BPS D0D4-D2D6 composites, hence, according to Eqs. (71) and (72) we should have

$$UP > 0, \qquad vq < 0. \tag{83}$$

For all possible sign combinations satisfying these constraints we have

$$(-Pv)I_1 < 0. \tag{84}$$

Let us now write our consistency condition as

$$8y_\infty^2\left(-\frac{I_1}{Pv}\right)(-Pv\,\mathrm{Im}(\bar{\mathcal{Z}}_2\mathcal{Z}_1)_\infty) \equiv \left(-\frac{I_1}{Pv}\right)\mathcal{P} > 0. \tag{85}$$

Now, a calculation shows that by virtue of Eq. (84) consistency demands that

$$\mathcal{P} = 3(Pv(x_\infty^2 + y_\infty^2) + I_1 - Uv)^2 + UP(2vy_\infty)^2 - \tfrac{3}{4}(Uv + Pq)(Uv + 9Pq) < 0. \tag{86}$$

Let us now look at the expression for our hyperdeterminant of format $2\times2\times2\times2$ as given by Eq. (76). Clearly, positivity of D_4 implies that $-UvPq > 0$ and neither $Uv = -Pq$ nor $Uv = -9Pq$. Notice that all of these conditions are compatible with our physically interesting situation. For $-UvPq > 0$ is compatible with our choice of signs supporting a pair of BPS configurations, and in order to talk about a D0D4-D2D6 split none of the four charges can be zero. Moreover, since the first term of Eq. (86) is positive, the second one for BPS configurations is positive as well ($UP > 0$), in order to satisfy this condition $(Uv + Pq)(Uv + 9Pq)$ has to be positive. Hence we see that *all* of the physically relevant conditions are encoded into the structure of the hyperdeterminant D_4. In particular $D_4 > 0$ and $-PvI_1 < 0$ of Eq. (84) gives a necessary condition for the consistency condition to hold. Unfortunately the second of our conditions is featuring $-Pv$ which is not coming from any of our four-qubit invariants.

In order to eliminate this shortcoming let us now take another look at the form of our polynomial of Eq. (25). By employing the substitution

$$t = I_1 + y, \tag{87}$$

this polynomial can be transformed to the reduced form

$$y^4 + ay^2 + by + c, \tag{88}$$

where

$$a = 6(I_2 - I_1^2), \qquad b = 12I_1I_2 - 4I_3 - 8I_1^3, \qquad c = -3(I_1^2 - I_2)^2 + S, \tag{89}$$

and S is our well-known quantity defined by Eq. (29) or alternatively by Eq. (68). Now the conditions

$$\Delta_4 > 0, \qquad a < 0, \qquad c < \frac{a^2}{4} \tag{90}$$

imply [41] that our original polynomial Eq. (25) featuring the fundamental 4-qubit invariants is having only *real* roots. Note that here Δ_4 is the discriminant of Eq. (25), and we also recall that $\Delta_4 = 256D_4$. A calculation for the degenerate cases ($I_4 = 0$) shows that these conditions are

$$D_4 > 0, \qquad I_1^2 > M, \qquad I_1I_3 > M(I_1^2 - M), \tag{91}$$

yielding the st^2 model and the further specialization $M = 0$ results in

$$D_4 > 0, \qquad I_1I_3 > 0, \tag{92}$$

corresponding to the t^3 model. Now, by virtue of Eq. (73) it is easy to see that a further specification to the case of our BPS D0D4-D2D6 split renders our condition $-P\upsilon I_1 < 0$ equivalent to the one $I_1I_3 > 0$. Hence we obtained the nice result that for D0D4-D2D6 splits the conditions encapsulated in the positivity of three-qubit (i.e. U-duality) invariants of Eqs. (71) and (72) and the positivity of the four-qubit ones of Eq. (92) provide a necessary condition for the consistency condition for such two-center composites to hold.

Let us also verify explicitly that the aforementioned criteria indeed provide real roots of our polynomial (25) featuring the algebraically independent four-qubit invariants. For the t^3 model one of the roots is zero due to the vanishing of the invariant I_4. For the remaining three roots we have to look at the solutions of Eq. (74). After the substitution

$$s = t - 2\left(\frac{2I_1}{3}\right) \tag{93}$$

and the definitions

$$\mu = -3\left(\frac{2I_1}{3}\right)^2, \qquad \nu = 2\left(\frac{2I_1}{3}\right)^3 - 4I_3, \tag{94}$$

this cubic equation and its discriminant takes the form

$$s^3 + \mu s + \nu = 0, \qquad \Delta = \left(\frac{\mu}{2}\right)^2 + \left(\frac{\nu}{3}\right)^3, \tag{95}$$

where for the explicit form of Δ see Eq. (75). According to Eq. (76) $\Delta < 0$, this yields for Cardano's formula the case of "casus irreducibilis"[42] with explicit solutions s_{j+1}, $j = 0, 1, 2$. Transforming back these solutions by using Eq. (93) to the variables t_{j+1} we obtain the final solutions

$$t_{j+1} = 4\left(\frac{2I_1}{3}\right)\sin^2\left(\frac{\Phi}{6} + \frac{\pi}{3}j\right), \qquad j = 0, 1, 2, \tag{96}$$

where

$$\sin^2\left(\frac{\Phi}{2}\right) = \frac{I_3}{(\frac{2I_1}{3})^3}, \tag{97}$$

a quantity clearly *positive* by virtue of Eqs. (73), i.e., $I_1I_3 > 0$ for all sign combinations compatible with Eq. (83). Since, according to Eqs. (75) and (76), the sign of D_4 is just the opposite of the sign of Δ we see that in the special case of the BPS D0D4-D2D6 split in the t^3 model the real roots we have obtained are in accord with our conditions of Eq. (92) used in a more general context.

C. Real roots and canonical forms

Our explicit formulas for the real roots of the fundamental polynomial of Eq. (25) enable an explicit construction of the canonical forms of the four-qubit states associated to the charge configurations describing two-center solutions. Let us give just a few examples for the BPS D0D4-D2D6 split. First, we write the roots of the fundamental polynomial of Eq. (25) for the BPS D0D4-D2D6 splits of the t^3 model in the following form:

$$t_{j+1} = 4(e-f)\sin^2\left(\frac{\Phi}{6} + \frac{\pi}{3}j\right), \quad j = 0, 1, 2 \quad t_4 = 0, \tag{98}$$

$$\cos\Phi = \frac{e(e+3f)^2 + f(f+3e)^2}{e(e+3f)^2 - f(f+3e)^2}, \quad e = \frac{U\upsilon}{3}, \quad f = Pq. \tag{99}$$

Note that for BPS solutions we have $ef < 0$. Notice also that for our special case $2I_1 = \langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle = 3(e - f)$ cannot be zero for BPS splits so this charge configuration is mutually nonlocal.

As our first example let us consider the nontrivial cases $f \neq 0$ and $e \neq 0$ when $D_4 = 0$. The first case is characterized by the constraint $e + 3f = 0$. In this case $I_3 \neq 0$ and from Eq. (99) we get $\Phi = \pi$, hence,

$$t_1 = t_3 = \frac{2I_1}{3}, \quad t_2 = 4\left(\frac{2I_1}{3}\right), \quad t_4 = 0, \quad I_1 = \frac{4U\upsilon}{3}. \tag{100}$$

Using now Eq. (35) up to permutations one can associate a canonical form to this configuration as shown in Eq. (32). This situation arises, for example, when $U = -\upsilon = 3$ and $P = q = 1$. Notice that this is a highly degenerate case since now $D_3(\mathcal{Q}) = 0$, and $W = 3(1-\tau)(1+\tau^2)$, hence, $Z = 0$ for $\tau = \pm 1$ which is on the boundary of the lower half plane. One can also see that for $e + 3f =$ the superpotential W can be written in a factorized form.

Our second example is associated with the case when $f + 3e = 0$ producing the other nontrivial zero for D_4. In this case $I_3 = 0$ and $\Phi = 0$, hence, we have

$$t_1 = 0, \quad t_2 = t_3 = 3\left(\frac{2I_1}{3}\right), \quad t_4 = 0, \quad I_1 = 2Uv. \quad (101)$$

From Eq. (32) one can see that the canonical state is proportional to the one $|0101\rangle + |1010\rangle$.

Our last example is a one with $\Phi = \frac{\pi}{2}$. In this case $e(e+3f)^2 + f(f+3e)^2 = 0$ which is of the form

$$\lambda^3 + 5\lambda^2 + \frac{5}{3}\lambda + \frac{1}{27} = 0, \qquad \lambda = \frac{Pq}{Uv}. \quad (102)$$

It is easy to check that $\lambda = -1/3$, i.e., $e + f = 0$ is a solution, hence, $(\lambda + 1/3)(\lambda^2 + 14\lambda/3 + 1/9)$ is a factorized form of our polynomial, yielding the solutions

$$\lambda_1 = -\tfrac{1}{3}, \qquad \lambda_{2,3} = \tfrac{1}{3}(-7 \pm 4\sqrt{3}). \quad (103)$$

Let us consider the rational solution $\lambda = -\frac{1}{3}$. In this case we get

$$t_1 = \frac{4I_1}{3}, \quad t_{2,3} = \frac{I_1}{3}(1 \mp \sqrt{3})^2, \quad t_4 = 0, \quad I_1 = Uv. \quad (104)$$

Now the real roots are all different, hence, $D_4 \neq 0$. However, now $Uv = -3Pq$, hence, $(Uv + Pq) \times (Uv + 9Pq) < 0$. A consequence of this is that the consistency condition of Eq. (86) cannot be satisfied so no charge configuration of this kind supports a two-center solution. The canonical form can again be read off from Eqs. (32)–(35). Notice also that for all three cases $t_1 + t_2 + t_3 + t_4 = 4I_1$ as it has to be, moreover, the corresponding values for λ can be written in the form $\lambda_k = -1/3^k$ for $k = 0, 1, 2$.

It is interesting to realize that all these special charge configurations giving rise to special four-qubit canonical forms are outside the domain of legitimate two-center solutions. Later when we connect the special values of e and f to properties of the j function we have something more to say about this phenomenon. In order to get charge configurations supporting BPS two-center solutions we have to chose the asymptotic moduli from the eligible region bounded by the usual wall of marginal stability [4]. In our case this wall is given by the locus

$$4\xi x_\infty^2 = (x_\infty^2 + y_\infty^2 + \xi)(x_\infty^2 + y_\infty^2 + \eta), \qquad \xi = \frac{U}{3P}, \qquad \eta = -\frac{3q}{v}. \quad (105)$$

D. Splitting of invariants

Notice that the quartic invariant for the t^3 model has the explicit form

$$-D_3(\mathcal{Q}_1 + \mathcal{Q}_2) = -(Uv)^2 + 3(Pq)^2 - 6UvPq + 4UP^3 - 4vq^3. \quad (106)$$

Now $I_{+2} = 4UP^3$ and $I_{-1} = -4vq^3$, hence, according to Eq. (54) we can write

$$-D_3(\mathcal{Q}_1 + \mathcal{Q}_2) = \Pi_4(\Lambda_{i_0 i_1 i_2 i_3}, 1) = [4I_{+1} + 6I_0 + 4I_{-1}] + I_{+2} + I_{-2}. \quad (107)$$

From this expression we see that the relationship between the quantities $-D_3(\mathcal{Q}_1 + \mathcal{Q}_2)$, $-D_3(\mathcal{Q}_{1,2})$ is governed by the combination $4I_{+1} + 6I_0 + 4I_{-1}$ of $SL(2)_0$ covariants. Such relationships are needed for studying situations when the two-center solution is BPS, but the corresponding single center one is not.

For D0D4-D2D6 splits the quantity above can also be written in two equivalent forms featuring quantities related to factors of $SL(2)_0$ invariants

$$-D_3(\mathcal{Q}_1 + \mathcal{Q}_2) = -8UvPq - 2I_1(Uv + Pq) - D_3(\mathcal{Q}_1) - D_3(\mathcal{Q}_2), \quad (108)$$

or

$$-D_3(\mathcal{Q}_1 + \mathcal{Q}_2) = -24(Pq)^2 - 2I_1(Uv + 9Pq) - D_3(\mathcal{Q}_1) - D_3(\mathcal{Q}_2). \quad (109)$$

Hence, if both of the constituents are BPS, i.e., $-D_3(\mathcal{Q}_1) > 0$ and $-D_3(\mathcal{Q}_2) > 0$, the conditions for the single centered system to be BPS too, i.e., $-D_3(\mathcal{Q}_1 + \mathcal{Q}_2) > 0$ are again governed by a four-qubit invariant I_1 or by the factors of another four-qubit one, namely, D_4. It is amusing to realize that all of the factors of D_4 are appearing in these two equations.

As an example one can see that if $I_1 < 0$ and $D > 0$ and moreover $Uv + Pq > 0$ (which is a factor of D_4), then two BPS configurations again yield a BPS one. In the example of Bates and Denef [4] $U = -v = 4$ and $P = q = 1$, $-D_3(\mathcal{Q}_1) > 0$ and $-D_3(\mathcal{Q}_2) > 0$, hence, $D_4 > 0$ and $I_1 < 0$ but $Uv + Pq < 0$, hence, $-D_3(\mathcal{Q}_1 + \mathcal{Q}_2) = -125 < 0$. In this case a single center BPS solution does not exist, although the corresponding two-center one does.

E. The j invariant

It should be obvious by now that the basic mathematical object giving rise to our observations is the fundamental polynomial of Eq. (25) featuring all of our algebraically independent four-qubit invariants. For no matter what kind of two-center charge configuration we have we can construct this quartic polynomial. As a next step we can calculate its resolvent qubic of Eq. (67). After the substitution $u = 2x$ this polynomial takes the form $2(4x^3 - Sx - T)$. As a next step to this resolvent cubic we associate an elliptic curve of Weierstrass canonical form as

$$y^2 = 4x^3 - Sx - T. \tag{110}$$

The j invariant of this curve is defined as

$$j = 1728\frac{S^3}{S^3 - 27T^2}. \tag{111}$$

In particular for the special case of the t^3 model we get

$$j = \frac{(2I_1)^3((2I_1)^3 - 24I_3)^3}{I_3^3((2I_1)^3 - 27I_3)}. \tag{112}$$

After recalling the definitions of Eq. (99) a further specification to our case of the BPS D0D4-D2D6 split one obtains

$$\begin{aligned} 1728S^3 &= (2I_1)^3((2I_1)^3 - 24I_3)^3 \\ &= 27(e-f)^3[27e(3f+e)^2 - 3f(3e+f)^2]^3, \end{aligned} \tag{113}$$

$$256D_4 = S^3 - 27T^2 = -27[e(3f+e)^2][f(3e+f)^2]^3. \tag{114}$$

Now

$$\begin{aligned} &27e(3f+e)^2 - 3f(3e+f)^2 \\ &= (3e - \xi_1 f)(3e - \xi_2 f)(3e - \xi_3 f), \end{aligned} \tag{115}$$

where

$$\xi_{j+1} = -5 + 4\sqrt[3]{2}\omega^j, \qquad \omega_j = e^{2\pi j/3}, \qquad j = 0, 1, 2. \tag{116}$$

Interestingly the number ξ_1 can also be written in the form

$$\xi_1 = -5 + 4\sqrt[3]{2} = 3\left(\frac{1-\sqrt[3]{2}}{1+\sqrt[3]{2}}\right)^2, \tag{117}$$

moreover, one can check that $\xi_1 + \xi_2 + \xi_3 = -15$, $\xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3 = 75$, and $\xi_1\xi_2\xi_3 = 3$.

Finally, one obtains for the j invariant the expression

$$\begin{aligned} j = &\frac{(f-e)^3}{[e(3f+e)^2][f(3e+f)^2]^3} \\ &\times [(3e - \xi_1 f)(3e - \xi_2 f)(3e - \xi_3 f)]^3. \end{aligned} \tag{118}$$

Clearly the denominator of j is zero precisely when D_4 is vanishing. According to the results of the previous subsections for BPS ($ef < 0$) composites, when this happens the consistency condition cannot be satisfied. The numerator on the other hand can vanish either when $2I_1 = \langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle = 0$, i.e., the charge configuration is local, or when the remaining real factor vanishes, i.e., when $Uv = \xi_1 Pq$. Since, according to Eq. (117), ξ_1 is positive, for BPS composites the latter two conditions cannot be satisfied. Hence, the j function is not having any pathological behavior for the physically legitimate cases of BPS D0D4-D2D6 composites.

Notice however, that the j function can be made to vanish for non-BPS D0D4-D2D6 composites. In this case we have $ef > 0$, and one can try to combine the elementary non-BPS solutions obtained for the D0D4 and D2D6 systems [43]. For such non-BPS composites with local charge configurations a generalization of the consistency condition has recently been given [44].

V. ELLIPTIC CURVES

A. Elliptic curves and four-qubit states

Let us try to justify that our association of an elliptic curve to the resolvent cubic Eq. (67) of our fundamental polynomial of Eq. (25) is a natural one. For this it is useful to regard our three-qubit charge states $|\mathcal{Q}_1\rangle$ and $|\mathcal{Q}_2\rangle$ with amplitudes given by Eqs. (36), (41), and (42) as the ones embedded into more general unnormalized four-qubit ones with *complex* amplitudes. (We will have something more to say about this embedding later.) Hence, we can use the algebraically closed field of complex numbers to put our considerations in a more general setting.

Let us note that according to Luque and Thibon [14] we can write our key quantity $256D_4 = S^3 - 27T^2$ in yet another form by rewriting the polynomial invariants S and T as

$$12S = U^2 - 2V, \qquad 216T = U^3 - 3UV + 216D^2, \tag{119}$$

where

$$U = H^2 + 4(M - L), \qquad V = 12(HD - 2LM). \tag{120}$$

The algebraically independent invariants [14] H, L, M, D showing up in these expressions are related to our set I_1, I_2, I_3, I_4 as [15]

$$\begin{aligned} I_1 &= \tfrac{1}{2}H, \qquad I_2 = \tfrac{1}{6}(H^2 + 2L + 4M), \\ I_3 &= D + \tfrac{1}{2}HL, \qquad I_4 = L. \end{aligned} \tag{121}$$

It is known that a nonsingular projective curve of genus 1 is isomorphic to a plane cubic curve of the Tate form [45]

$$\begin{aligned} &x_0x_2^2 + a_1x_0x_1x_2 + a_3x_0^2x_2 - x_1^3 - a_2x_0x_1^2 \\ &\quad - a_4x_0^2x_1 - a_6x_0^3 = 0. \end{aligned} \tag{122}$$

Here the coefficients a_1, a_2, a_3, a_4, a_6 can be taken from an algebraically closed field $\mathbb{F}$ of an arbitrary characteristic and (x_0, x_1, x_2) are homogeneous coordinates of the projective plane $\mathbb{F}P^2$. (Of course for qubits we are merely interested in the complex case, i.e., our concern will be curves in $\mathbb{C}P^2$.) Using inhomogeneous coordinates $(x, y) \equiv (x_1/x_0, x_2/x_0)$ this can be rewritten as

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{123}$$

Now the j invariant and discriminant Δ of a plane cubic curve of this form is defined [45] as

$$j = \frac{c_4^3}{\Delta} = 1728\frac{c_4^3}{c_4^3 - c_6^2}, \tag{124}$$

where

$$c_4 = b_2^2 - 24b_4, \qquad c_6 = -b_2^3 + 36b_2b_4 - 216b_6, \tag{125}$$

$$b_2 = a_1^2 + 4a_2, \quad b_4 = a_1a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6. \tag{126}$$

A necessary and sufficient condition for our curve to be nonsingular is $\Delta \neq 0$.

Now looking at Eqs. (119) and (120) one can easily check that four-qubit states provide a particularly nice parametrization for a special class of plane cubic curves with $a_6 \equiv 0$. Indeed setting

$$b_2 = U = H^2 + 4(M - L), \qquad b_4 = \tfrac{1}{12}V = HD - 2LM, \qquad b_6 = D^2, \tag{127}$$

yields

$$c_4 = 12S, \qquad c_6 = -216T. \tag{128}$$

Moreover, one can also see that

$$a_1 = H = 2I_1, \quad a_2 = M - L, \quad a_3 = D = I_3 - I_1I_4, \quad a_4 = -LM, \quad a_6 = 0, \tag{129}$$

hence, the family of such cubic curves for the stu model ($H \neq 0$, $D \neq 0$, $M \neq 0$, $L \neq 0$) is

$$y^2 + Hxy + Dy = x^3 + (M - L)x^2 - LMx \tag{130}$$

for the st^2 model ($H \neq 0$, $D \neq 0$, $M \neq 0$, $L = 0$)

$$y^2 + Hxy + Dy = x^3 + Mx^2 \tag{131}$$

and finally for the t^3 model ($H \neq 0$, $D \neq 0$, $L = 0$, $M = 0$) it is

$$y^2 + Hxy + Dy = x^3. \tag{132}$$

For all of these curves the hyperdeterminant is given by the usual formula $256D_4 = S^3 - 27T^2$ with the j invariant having the form of Eq. (111). As an example let us take a look at the hyperdeterminant D_4 for the st^2 model which is given by the expression

$$256D_4 = I_3^2(I_3[(2I_1)^2 - 27I_3] + 36M[I_2^2 - 2I_1I_3]), \qquad 2M = 3I_2 - 2I_1^2. \tag{133}$$

One can see that in accordance with our association of cubic curves to the stu model and its truncations for $M = 0$ we get back to Eq. (76), i.e., the expression for the t^3 model.

Recall also that if the characteristic of the field is not 2 or 3 via further projective transformations [i.e. completing the square on the left hand side and then the cube on the right hand side of Eq. (130)] one obtains the Weierstrass canonical form of Eq. (110) which is simply related to the resolvent cubic of our polynomial Σ_4 of Eq. (25) we have started our considerations with. Notice, however, that although due to the fact that now $\mathbb{F} = \mathbb{C}$ the Weierstrass form in our case can always be reached, this form is not showing the basic differences between the structures of the cubic curves associated to the stu, st^2 and t^3 models, hence we prefer the Tate form.

In the following, as usual, by an elliptic curve we will mean a nonsingular plane cubic curve as given by Eq. (122) and the "point at infinity" $(x_0, x_1, x_2) = (0, 0, 1)$. Since for these curves we have only $(0, 0, 1)$ as a point at infinity, by an abuse of notation one can call Eq. (123) as the defining equation for the corresponding elliptic curve. An elliptic curve is defined over a subfield $\mathbb{F}'$ of $\mathbb{F}$ if all the coefficients a_1, a_2, a_3, a_4, a_6 are taken from $\mathbb{F}'$. In the case of the discrete version of the continuous U-duality group, as nonperturbative string symmetries, the coefficients of this curve are the algebraically independent four-qubit invariants H, L, M, D [related to the ones I_1, I_2, I_3, I_4 via Eq. (121)] taken from the ring of integers of the subfield $\mathbb{Q}$. In this case the arithmetic aspects of the theory of elliptic curves within the context of two-center black-hole solutions become important [46].

The upshot of these considerations is that to a particular charge configuration of two-center black holes of the stu model we can indeed associate quite naturally an elliptic curve of a *special kind* (i.e. $a_6 = 0$). This elliptic curve is of the form of Eq. (130).

Notice that the invariants H, L, M play an important physical role. Since $H = \langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle$ the vanishing of this invariant gives rise to local charge configurations, in this case the xy term of Eq. (130) is missing. The vanishing conditions on L and M give rise to truncations to the st^2 and t^3 models. Notice also that since for $L = 0$ we have $D = I_3$, hence in the st^2 and t^3 models sending D to zero corresponds to taking the limit when our elliptic curve degenerates [see Eqs. (76), (121), and (133)].

In establishing the correspondence between two-center charge configurations and elliptic curves we tacitly assumed that our unnormalized charge states with real amplitudes (transforming according to the corresponding representation of the *continuous* U-duality group valid in the supergravity approximation) are regarded as ones embedded in unnormalized four-qubit states with complex amplitudes. However, until this point we did not specify what kind of complex states we are having in mind.

In a recent series of papers it was shown [17,47] that for single center extremal stationary spherically symmetric

BPS and non-BPS black-hole solutions apart from charge states, i.e., unnormalized three-qubit states with real amplitudes, we can also introduce unnormalized complex three-qubit states which are depending on the charges and also on the moduli fields. (For a more general class of such states even the warp factor can be included [47].) It has been shown that the complex amplitudes $\psi_{000}, \psi_{001}, \ldots, \psi_{111}$ of such states $|\psi\rangle$ are related to the central charge and its covariant derivatives with respect to the moduli. For example the amplitudes featuring the central charge have the form [17]

$$\psi_{000} = \sqrt{2}\bar{Z}, \qquad \psi_{111} = -\sqrt{2}Z. \tag{134}$$

Moreover, recently it has also been demonstrated that such complex three-qubit states are just special types of four-qubit ones [21]. It is natural to expect that for two-center solutions complex four-qubit states of similar type should show up. For example in the consistency condition of Eq. (77) the term $\mathrm{Im}(\bar{Z}_2 Z_1)$ can be reinterpreted as a truncation of a pairing between the two complex amplitudes $\Psi_{0000} \equiv \sqrt{2}Z_1$ and $\Psi_{1111} = -\sqrt{2}\bar{Z}_2$ of such four-qubit states. In this respect recall the formalism developed by Denef [3] in the type IIB duality frame where apart from the usual antisymmetric, topological moduli independent intersection product (our four-qubit invariant I_1 in the *stu* model) the importance of the symmetric, positive definite, moduli dependent Hodge product is emphasized (in the *stu* model giving rise to the interpretation of the black-hole potential as the norm of a three-qubit state [17]). Such structures are the natural ones appearing in the multicenter black-hole context. In the *stu* case it is easy to establish a correspondence between this formalism and the one where complex three and four-qubit states appear [48]. We conjecture that such moduli dependent complex states and their associated invariants featuring elliptic curves could be the relevant mathematical objects hiding behind the considerations of this paper based merely on charge states. Note that moduli dependent curves of that kind would also display explicit dependence on extra complex parameters giving rise to elliptic fibrations. We will have something more to say about this possibility in Sec. V.

B. Degeneracies

The discriminant Δ of our elliptic curve is related to the hyperdeterminant D_4 as $256D_4 = \Delta$. For nonvanishing D_4 the curves are nonsingular. Since our curves are of genus 1 they are topologically tori. The vanishing of D_4 results in different degeneracies of these tori (for example one of their homologically nontrivial cycles can contract to a point). One can illustrate this in the *stu* and st^2 models when both of the two BPS constituents have vanishing Bekenstein-Hawking entropy, i.e., $I_{\pm 2} = 0$ for both centers.

Indeed, let us use the parametrization as introduced by Sen [8] for such two-center composites $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$,

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} adQ - abP \\ cdQ - bcP \end{pmatrix} + \begin{pmatrix} abP - bcQ \\ adP - cdQ \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{135}$$

Here P and Q are four-vectors related to the ones of Eqs. (38) and (42) as follows:

$$A = a(dQ - bP), \qquad B = c(dQ - bP), \qquad C = b(aP - cQ), \qquad D = d(aP - cQ). \tag{136}$$

Since the vectors A and B and C and D are now proportional, the bivectors of X and Y of Eq. (43) are zero, hence, $I_{\pm 2} = I_{\pm 1} = 0$ though $I_0 \neq 0$. Now Eq. (68) shows that

$$S = 3I_0^2, \qquad T = I_0^3, \tag{137}$$

hence, $S^3 - 27T^2 = 0$ i.e. $D_4 = 0$. The nonseparable bivector is $Z = \frac{1}{2}Q \wedge P$ hence,

$$I_0 = \tfrac{1}{3}Z^2 = \tfrac{1}{12}(Q \wedge P) \cdot (Q \wedge P) = -\tfrac{1}{6}D_3(\mathcal{Q}). \tag{138}$$

Alternatively, one can directly check that $I_3 = I_4 = 0$ and $6I_2 = -6I_0 = D_3(\mathcal{Q})$ in accordance with Eq. (56), hence, $S = 3I_2^2$ and $T = I_2^3$ yielding again $D_4 = 0$.

In order to obtain the Tate form of our elliptic curve we have to calculate the quantities H, L, M, D. One can show that

$$H = \hat{P} \cdot \hat{Q}, \qquad L = 0, \qquad M = -\tfrac{1}{4}\hat{P}^2\hat{Q}^2, \qquad D = 0, \tag{139}$$

where we introduced the notation

$$\begin{pmatrix} \hat{Q} \\ \hat{P} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}. \tag{140}$$

Now the elliptic curve of Eq. (130) is of the form

$$y^2 + \hat{P} \cdot \hat{Q}xy = x^3 - \tfrac{1}{4}\hat{P}^2\hat{Q}^2x^2. \tag{141}$$

Completing the square on the left-hand side (and by an abuse of notation using y for the new variable again) yields

$$y^2 = x^2(x + \tfrac{1}{4}D_3(\mathcal{Q})). \tag{142}$$

Note the quantities $\hat{P}$ and $\hat{Q}$, hence, the Tate form of the cubic curve is featuring the parameters a, b, c, d characterizing the two-center split, however, the latter form is not. This is due to the fact that $D_3(\mathcal{Q}) = (P \cdot Q)^2 - P^2Q^2 = (\hat{P} \cdot \hat{Q})^2 - \hat{P}^2\hat{Q}^2$. This leads us to the important observation that at least for this singular example two-center charge configurations labeled by elements of $SL(2, \mathbb{Z})$ correspond to the same singular curve with the (142) canonical form. Hence, apart from the four copies of $SL(2)$'s related to four-qubit systems there seems to be a *fifth* $SL(2)$ parametrizing a hidden torus. We will return to this important point later.

Equation (142) represents a cubic curve with a *node* at the point (0, 0). The tangent lines at (0, 0) are obtained from the equation $y^2 - (D_3/4)x^2 = 0$. Such lines are of the form

$$(y - \tfrac{1}{2}\sqrt{D_3})(y + \tfrac{1}{2}\sqrt{D_3}) = 0. \tag{143}$$

It is important to recall that we assumed that all of our curves are over the complex numbers, i.e., $\mathbb{F} = \mathbb{C}$. Hence, for a cubic in the canonical form $y^2 = x^3 + \alpha x + \beta$ we are still free to use transformations of the form $(x, y) \mapsto (\mu^2 x, \mu^3 y)$ and $(\alpha, \beta) \mapsto (\mu^4\alpha, \mu^6\beta)$, where $\mu \in \mathbb{C}^\times$. Using the transformation $(x, y) \mapsto (-x, -iy)$, i.e., $\mu = i$ transforms Eq. (142) to the form

$$y^2 = x^2(x - \tfrac{1}{4}D_3(\mathcal{Q})). \tag{144}$$

As we see, such transformations are similar to the ones changing a BPS charge configuration ($D_3 < 0$) to a non-BPS one ($D_3 > 0$). Interestingly, as shown by Eq. (143) [and a similar one obtained from Eq. (144)] the pair of lines at the node are parametrized by $\sqrt{-D_3}$ for the BPS case or $\sqrt{D_3}$ for the non-BPS one, i.e., quantities proportional to the Bekenstein-Hawking entropy.

Notice also that Eq. (144) can also be parametrized as

$$y^2 = 3t(x' - t)^2 + (x' - t)^3 = (x')^3 - 3t^2(x') - 2t^3, \qquad t = -\frac{D_3}{12}. \tag{145}$$

Using Eqs. (137) and (138) we can write this as

$$y^2 = x'^3 - \tfrac{1}{4}Sx' - \tfrac{1}{4}T. \tag{146}$$

Now, a further transformation $(u, v) = (x/2, x/8)$, i.e., $\mu^2 = 1/2$ transforms this to the form $v^2 = u^3 - Su - 2T$ of Eq. (67), with the left-hand side being just the resolvent cubic of $\Sigma_4 = 0$. The procedure discussed here illustrates how the general Tate form of our cubic curve of Eq. (141) can be transformed to a resolvent cubic form.

It is important to realize that if the domain of definition of our curve is the subfield $\mathbb{Q}$ from the general Tate form of Eq. (130) merely the form

$$y^2 = x^3 - \tfrac{1}{4}Sx + \tfrac{1}{4}T \tag{147}$$

can be reached. Further reduction is possible only if μ is a square in $\mathbb{Q}$. In the case of our singular curve of Eq. (142), if D_3 is not a square we cannot represent the tangent lines in the form of Eq. (143), hence, the tangent lines in this case are not rational.

Because of permutation symmetry we have no parametrization of the (135) type for the t^3 model available. However, as a degenerate example in this case we can consider the example of Bates and Denef [4] instead by sending one of the charges, e.g., P to zero. In this case one of the black holes is a small one, and the two-center composite is characterized by the charges U, v, q. Now one checks that $I_3 = 0$, hence, according to Eq. (76) we get again $D_4 = 0$. The explicit form of the corresponding degenerate elliptic curve is

$$y^2 + Hxy = x^3, \qquad H = \langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle = Uv, \tag{148}$$

with $j = \infty$ just like in the previous example.

C. Invariance properties

Since the coefficients a_j of the elliptic curve of Eq. (130) are invariants the same curve is associated with equivalent two-center charge configurations in the *stu* model. In other words our curve is clearly invariant under the action of the group $SL(2, \mathbb{R})_0 \times G_4$, where $SL(2, \mathbb{R})$ is the horizontal symmetry group of generalized exchange transformations of the centers and G_4 is the $d = 4$ continuous U-duality group of the *stu* model which is $SL(2, \mathbb{R})^{\times 3}$.

However, as far as physics is concerned, in a special case of the t^3 model we have also found some connection between the structure of $SL(2, \mathbb{R})_0 \times G_4$ (four-qubit) invariants and the structure of the consistency condition of Eq. (77). This connection at first sight is not surprising since the consistency condition contains a four-qubit invariant $2I_1 = \langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle$, however, more importantly it is also featuring the quantity $\mathrm{Im}(\bar{\mathcal{Z}}_2 \mathcal{Z}_1)$, which is not invariant with respect to the full group of $SL(2, \mathbb{R})_0 \times G_4$ transformations.

In order to show this let us observe, however, that this quantity is invariant under transformations belonging to the horizontal subgroup $SL(2, \mathbb{R})_0$. In order to prove this let us recall our definition of Eq. (36) and then write the central charges as

$$\mathcal{Z}_1 = e^{K/2}(Q_0 + Q_1\tau_1 + Q_2\tau_2 + Q_3\tau_3 - P^1\tau_2\tau_3 - P^2\tau_1\tau_3 - P^3\tau_1\tau_2 + P^0\tau_1\tau_2\tau_3), \tag{149}$$

$$\mathcal{Z}_2 = e^{K/2}(q_0 + q_1\tau_1 + q_2\tau_2 + q_3\tau_3 - p^1\tau_2\tau_3 - p^2\tau_1\tau_3 - p^3\tau_1\tau_2 + p^0\tau_1\tau_2\tau_3), \tag{150}$$

then for a, b, c, $d \in \mathbb{R}$ we have $\mathcal{Z}_1' = a\mathcal{Z}_1 + b\mathcal{Z}_2$, and $\bar{\mathcal{Z}}_2' = c\bar{\mathcal{Z}}_1 + d\bar{\mathcal{Z}}_2$, hence, $\mathrm{Im}(\bar{\mathcal{Z}}_2'\mathcal{Z}_1') = \mathrm{Im}(\bar{\mathcal{Z}}_2\mathcal{Z}_1)$ due to $ad - bc = 1$. However, this quantity is invariant merely under a special subgroup [49] of G_4. Indeed, we have

$$\mathcal{Z}(K\tau_1, K\tau_2, K\tau_3; K\mathcal{Q}_i) = \mathcal{Z}(\tau_1, \tau_2, \tau_3; \mathcal{Q}_i), \tag{151}$$

where K is the subgroup of G_4 transformations of the form

$$\begin{pmatrix} 1 & 0 \\ k_1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ k_2 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ k_3 & 1 \end{pmatrix} |\mathcal{Q}_i\rangle, \qquad \tau_a \mapsto \tau_a + k_a, \quad k_a \in \mathbb{R}, \quad a = 1, 2, 3. \tag{152}$$

Notice that the group K with $k_a \in \mathbb{Z}$ is precisely the stabilizer of the cusps in the three copies of the fundamental domain of the modular group, i.e., the stabilizer of $\tau_a = -i\infty$ for $a = 1, 2, 3$.

Since the walls of marginal stability are determined by the central charges and for the four-qubit invariant $2I_1$ we have $\langle K\mathcal{Q}_1, K\mathcal{Q}_2\rangle = \langle \mathcal{Q}_1, \mathcal{Q}_2\rangle$ clearly wall crossing does not obstruct the K subgroup. Therefore, we see that unlike our elliptic curves the consistency condition is left invariant (in the above sense) merely with respect to the subgroup $SL(2,\mathbb{R})_0 \times K$. Hence, the correspondence between the physics of two-center solutions and our special class of elliptic curves should be refined (see, in this respect, the comments at the end of Sec. VA).

Now we turn to another important issue we have not discussed yet. Based on our experience with the degenerate case studied in the previous subsection we expect that for the nondegenerate case our mapping of two-center charge configurations characterized by four-qubit invariants to elliptic curves should be many to one. Moreover, since in the Tate form we should have $a_6 = 0$ our mapping in the *stu* case cannot be onto either. We also know from the theory of elliptic functions that if we work in the algebraic closure of our field $\mathbb{F}$, i.e., $\mathbb{F}^{\text{alg}}$ two elliptic curves are isomorphic if and only if their j invariants are the same. Hence, in the case of $\mathbb{R}$ working with the field of complex numbers charge configurations with different four-qubit invariants could be mapped to isomorphic elliptic curves with the same j invariant. In the singular case according to Eqs. (139) and (140) the values of H and M are clearly different for different splits, however, $j = \infty$ in all cases.

Now over $\mathbb{C}$ every elliptic curve E is isomorphic to an elliptic curve $E(\Lambda)$ where Λ is a lattice in the complex plane and the mapping between $\mathbb{C}/\Lambda$ (a torus) and $E(\Lambda)$ is an isomorphism provided by the usual map defined by the Weierstrass $\mathcal{P}$ function. Explicitly, $E(\Lambda)$ is of the form

$$x_0 x_2^2 = 4x_1^3 - g_2(\hat{\tau})x_1 x_0^2 - g_3(\hat{\tau})x_0^3, \qquad (153)$$

where $g_2(\hat{\tau}) = 60G_4(\hat{\tau})$ and $g_3(\hat{\tau}) = 140G_6(\hat{\tau})$ with $G_{2k}(\hat{\tau})$ are the Eisenstein series

$$G_{2k}(\hat{\tau}) = \sum_{(m,n)\in\mathbb{Z}^2,(m,n)\neq(0,0)} \frac{1}{(m+n\hat{\tau})^{2k}}, \qquad (154)$$

and if $x \equiv x_1/x_0$ and $y = x_2/x_0$ then $x = \mathcal{P}(z)$ and $y = \mathcal{P}'(z)$. Here the lattice vectors of Λ in $\mathbb{C}$ are $\omega_1 = 1$ and $\omega_2 = \hat{\tau}$ with $\hat{\tau} \in \mathbb{H}$ being the modular parameter of the torus and the Weierstrass $\mathcal{P}$ function is defined as

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2,(m,n)\neq(0,0)} \left[\frac{1}{(z+m+n\hat{\tau})^2} - \frac{1}{(m+n\hat{\tau})^2}\right]. \qquad (155)$$

Here we have used the notation $\hat{\tau}$ for the modular parameter in order not to confuse it with the moduli τ_1, τ_2, τ_3 of the *stu* and with the moduli τ of the t^3 model. Now the j invariant gives rise to the j function

$$j(\hat{\tau}) = 1728 \frac{g_2(\hat{\tau})^3}{g_2(\hat{\tau})^3 - 27g_3(\hat{\tau})^2}. \qquad (156)$$

Comparing this with the expression of the j invariant as given in terms of the four-qubit invariants S and T of Eq. (111) to a two-center charge configuration we can associate a torus with modular parameter $\hat{\tau}$. Now $j: \mathbb{H} \to \mathbb{C}$ is an automorphic function, i.e.,

$$j\left(\frac{a\hat{\tau}+b}{c\hat{\tau}+d}\right) = j(\hat{\tau}), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \qquad (157)$$

It can be regarded [50] as a map providing a holomorphic universal covering of the orbifold $\mathcal{H}/PSL(2,\mathbb{Z})$ [i.e. the fundamental domain for $PSL(2,\mathbb{Z})$]. Its Fourier series at $\hat{\tau} = i\infty$ is provided by the variable $u \equiv \exp 2\pi i\hat{\tau}$ as

$$j(\hat{\tau}) - 744 = \frac{1}{u} + \sum_{n=1}^{\infty} c_n u^n, \qquad c_n \in \mathbb{Z}^+ \cup \{0\}. \qquad (158)$$

From this we see that the two-center charge configurations with discriminant $D_4 = 0$ of the previous subsection with $j = \infty$ should correspond to the modular parameter $\hat{\tau} = i\infty$, i.e., the cusp of the Riemann surface $\mathcal{H}/PSL(2,\mathbb{Z})$. The different possible two-center splits labeled by elements of $SL(2,\mathbb{Z})$ not changing the value of the j invariant seem to be related to the modular property of the j function as shown by Eq. (157). It is also intriguing to recall that the tangent lines at the node of the degenerate elliptic curve are parametrized by the Bekenstein-Hawking entropy [see Eq. (143)].

From these investigations it is natural to conjecture that similar invariance properties of other two-center splits with $D_4 \neq 0$ should hold. It would be nice to uncover the physical role of this hidden torus and the extra $SL(2)$ symmetry associated with it. Some speculations on how the extra modular parameter $\hat{\tau}$ should be implemented into a four-qubit picture will be given in Sec. VI.

D. A conjecture

Let us consider the elliptic curve of Eq. (130) associated to two-center charge configurations of the *stu* model. As we have noticed, this curve is of the Tate form satisfying the special constraint $a_6 = 0$. The coefficients a_j are four-qubit polynomials with a definite degree of homogeneity $2j$. Indeed, $a_1 = H$ is a quadratic, $a_3 = D$ is a sextic, $a_2 = M - L$ is a quartic, and $a_4 = -LM$ is an octic polynomial in the amplitudes of the four-qubit charge state. (The variables x and y should be assigned the degrees four and six, respectively.) Now, according to this observation the constraint $a_6 = 0$ should be arising from the vanishing condition of a polynomial of degree 12. It is easy to identify this polynomial constraint. It is just the constraint [1] $\mathcal{P}_{12} = 0$ of Eq. (57), or alternatively of Eq. (58), hence, in a redundant notation for the *stu* model we can write

$$y^2 + Hxy + Dy = x^3 + (M-L)x^2 - LMx + \mathcal{P}_{12}. \qquad (159)$$

Since the stu model can be regarded as a consistent truncation of $N = 8$, $d = 4$ maximal supergravity with the continuous [13] U-duality group $G_4 \equiv E_{7(7)}$, one might conjecture that it should be possible to substantially generalize our considerations concerning two-center charge configurations by studying elliptic curves of the form

$$y^2 + P_2 yx + P_6 y = x^3 + P_4 x^2 + P_8 x + P_{12}, \qquad (160)$$

where P_{2j} are polynomial invariants of the group $SL(2)_0 \times G_4$ where $SL(2)_0$ refers to the horizontal symmetry group [1] acting as a generalized exchange symmetry group on the two centers. Now we have two sets of 56 component charge vectors where each of these vectors is transforming according to the symplectic irreducible representation of G_4 which is now the fundamental of $E_{7(7)}$. According to an analysis of two-centered magical charge orbits [13] working in the complexification of G_4 one discovers that the two-centered charge orbits correspond to different real forms of the quotient of the complex groups $E_7/SO(8)$. In particular we have two $\frac{1}{8}$-BPS two-centered charge orbits with one of them being $E_{7(7)}/SO(4,4)$. Recall now Eq. (1) displaying in its structure the group theoretical reason for our occurrence of four-qubit states. According to this equation one should be able to obtain our four-qubit invariants as special cases of $SL(2)_0 \times G_4$ ones. Hence, we conjecture that an elliptic curve of the (160) form should display this reduction procedure via implementing the constraint $\mathcal{P}_{12} = 0$ via a suitable reduction of some unknown $SL(2)_0 \times G_4$ polynomial invariant P_{12}.

In fact, many pieces of such invariants are already at our disposal. For example, the $N = 8$ analogues $J_0, J_{\pm 1}, J_{\pm 2}$ of the four-qubit covariants I_0, $I_{\pm 1}$, $I_{\pm 2}$ of Eq. (49) arising from the polarization of Cayley's hyperdeterminant can now be obtained from the polarization of Cartan's quartic invariant [13]. Using these explicit expressions one can construct the quantities answering the similar ones of Eq. (68)

$$\mathcal{S} = 3J_0^2 - 4J_{+1}J_{-1} + J_{+2}J_{-2},$$
$$\mathcal{T} = J_0^3 + J_{+1}^2 J_{-2} + J_{-1}^2 J_{+2} - J_{+2}J_0 J_{-2} - 2J_{+1}J_0 J_{-1}. \qquad (161)$$

Indeed precisely these polynomials of order 8 and 12 have been suggested as obvious candidates for members of a complete basis for $SL(2)_0 \times G_4$ invariants. Now one can define an elliptic curve

$$y^2 = x^3 - \tfrac{1}{4}\mathcal{S}x + \tfrac{1}{4}\mathcal{T} \qquad (162)$$

as the one corresponding to Eq. (147). As was commented, there this form can always be obtained from the Tate form if the characteristic of the field is neither 2 nor 3. Hence, we conclude that our Tate form, Eq. (160), featuring the unknown invariants P_{2j} should reduce to Eq. (162) after completing the square on the left- and the cube on the right-hand side. Explicitly this process amounts to the transition from using the coefficients $a_j = P_{2j}$ to using the ones $c_2 = -\frac{1}{4}\mathcal{S}$ and $c_4 = \frac{1}{4}\mathcal{T}$ of Eqs. (125) and (126).

As a solid piece of evidence it is also obvious that $P_2 = \langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle$ with $\langle , \rangle$ being the usual symplectic product of charge vectors which is a singlet with respect to $SL(2)_0 \times G_4$. Hence, for mutually local charge configurations our cubic curve of Eq. (160) falls short of the term proportional to xy just like in the stu truncation. Moreover, as shown by Eqs. (12)–(14) in the four-qubit case our quadratic and sextic invariants I_1 and I_3 are duals of each other. Hence, we expect that the invariant P_6 should be related to a quadratic combination of the Freudenthal duals of the corresponding charge vectors which are cubic in terms of the original charges [51]. Moreover, these dual quantities should be antisymmetric with respect to the exchange of the centers. Luckily a quantity J_6 satisfying these criteria is also available [see Eq. (3.23) of Andrianopoli *et al.* [13]].

Unfortunately since according to Eq. (121) $D = I_3 - I_1 I_4$ in the stu case, this quantity is probably not directly related to our invariant P_6. Hence, one is still left with the problem of finding the invariants P_4, P_8, and P_{12}.

In order to gain some insight into the physical meaning of the invariant P_4 let us look again at the stu (four-qubit) case. In this case we know that the corresponding invariant is $M - L$. As a first step we would like to somehow relate this quantity to some invariant known within the context of $N = 8$, $d = 4$ supergravity.

For the stu truncation let us suppose that we have chosen a particular two-center charge configuration parametrized by the four real numbers a, b, c, d of the canonical form of Eq. (32). Now the explicit form of the invariants H, M, L, D is

$$H = \frac{1}{2}(a^2 + b^2 + c^2 + d^2),$$
$$M = \left[\left(\frac{c-d}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2\right]\left[\left(\frac{a+b}{2}\right)^2 - \left(\frac{c+d}{2}\right)^2\right], \qquad (163)$$

$$D = \tfrac{1}{4}(ad - bc)(cd - ab)(ac - bd), \qquad L = abcd. \qquad (164)$$

Consider now the quartic $E_{7(7)}$ invariant [52] expressed in terms of the matrix of central charge $\mathcal{Z}$ of $N = 8$ supergravity in $d = 4$. This matrix is an 8×8 complex antisymmetric one which can be brought to the canonical form after using a suitable $U(8)$ transformation $\mathcal{Z} \mapsto U^T \mathcal{Z} U$. This canonical form is

$$U^T \mathcal{Z} U = \begin{pmatrix} z_0 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_3 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (165)$$

Now Cartan's quartic invariant is

$$J_4(Z) \equiv \mathrm{Tr}(Z\bar{Z}Z\bar{Z}) - \tfrac{1}{4}(\mathrm{Tr}(Z\bar{Z}))^2 + 4(\mathrm{Pf}(Z) + \mathrm{Pf}(\bar{Z})), \tag{166}$$

where the Pfaffian is

$$\mathrm{Pf}(Z) = \frac{1}{2^4 4!}\varepsilon^{ABCDEFGH} Z_{AB}Z_{CD}Z_{EF}Z_{GH}, \quad A, B, \ldots = 1, 2, \ldots 8. \tag{167}$$

Subscripts $A, B \ldots$ label an of **8** $SU(8)$. Using the canonical form one gets the well-known expression

$$J_4 = (|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2)^2 - 4\left(\sum_{n<m}|z_n z_m|^2\right) + 8\,\mathrm{Re}(z_0 z_1 z_2 z_3). \tag{168}$$

It is also known that by an $SU(8)$ transformation it is possible to remove three phases, so for instance z_1, z_2, z_3 can be chosen to be real (or else having the same phase). Hence, the canonical form can be characterized by four real numbers $r_n = |z_n|$ and a phase ϕ. The form of J_4 reflecting these considerations is

$$\begin{aligned} J_4 &= [(r_0 + r_1)^2 - (r_2 + r_3)^2][(r_0 - r_1)^2 - (r_1 - r_2)^2] \\ &\quad + 8r_0 r_1 r_2 r_3(\cos\phi - 1) \\ &= -16M + 8L(\cos\phi - 1) = 8(N - M) + 8L\cos\phi, \end{aligned}$$

where we have used the identity $L + M + N = 0$ of Eq. (22) and we made the identifications $r_0 = a$, $r_1 = b$, $r_2 = c$, and $r_3 = d$. Hence, though we did not manage to get directly to the invariant $M - L$, however, for $\phi = \frac{\pi}{2}$ we get $N - M$ instead, which is related to $M - L$ via a permutation of the qubits labeled by $i_1 i_2 i_3$ carrying the G_4 labels. Notice also that the permutation group S_3 responsible for this triality symmetry at the stu model level is related to triality of the complex group $SO(8, \mathbb{C})$ whose real form $SO(4, 4)$ is featuring the tripartite entanglement of seven qubits interpretation of Cartan's quartic invariant [18,19] of $N = 8$ $d = 4$ supergravity and the corresponding stu truncation. Notice also that according to Eq. (56) $M - N$ is just the invariant χ of Ferrara *et al.* [1]. These considerations show that the unknown invariant P_4 should truncate to $\chi = N - M$ after some suitable permutations.

It is also interesting to notice that for z_n *real*, i.e., $\phi = 0$ we have

$$J_4 = -16M = 16\tilde{r}_0\tilde{r}_1\tilde{r}_2\tilde{r}_3. \tag{169}$$

Here

$$\tilde{r}_n = \sum_{m=0}^{3}(H \otimes H)_{nm} r_m,$$
$$\begin{pmatrix} \tilde{r}_0 \\ \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}, \tag{170}$$

where H is the Hadamard matrix used for implementing discrete Fourier transformation in quantum information theory. Notice that with the definition $Q_n = 2\tilde{r}_n$ we have $J_4 = Q_0 Q_1 Q_2 Q_3$ the standard expression giving rise to the Bekenstein-Hawking entropy $S = \pi\sqrt{4Q_0 Q_1 Q_2 Q_3}$ of the D2-D2-D2-D6 brane configuration in the type IIA duality frame [52].

There still remains the task to understand the meaning of P_8 and P_{12}. Since in the $N = 2$, $d = 4$ context the vanishing of the corresponding invariants identifies the stu, st^2, and t^3 truncations in terms of the structure of the (160) elliptic curve it would be important to know their explicit forms. Finally, the discriminant $\mathcal{S}^3 - 27\mathcal{T}^2$ as a polynomial invariant of order 24 should play the role of some sort of generalization of the hyperdeterminant of type $2 \times 2 \times 2 \times 2$. It would be interesting to clarify what kind of role this discriminant and the associated j function plays in the physics of two-centered black-hole solutions.

VI. A TRIALITY SYMMETRIC CURVE

The aim of this speculative section is to draw the readers attention to some interesting structural similarities showing up in a variety of physical contexts where our four-qubit invariants parametrizing elliptic curves might play a crucial role.

In the previous investigations our basic philosophy for the association of elliptic curves to two-center charge configurations was based on the resolvent cubic of the fundamental polynomial of Eq. (25). Is there any other physically appealing way for this association, which retains the fundamental role of this polynomial and at the same time also features an extra modular parameter $\hat{\tau}$ accounting for a hidden torus? In order to show that the answer to this question is yes let us rewrite Eq. (25) in the form

$$\Sigma_4(\Lambda_{i_0 i_1 i_2 i_3}, \hat{\tau}, -x) = x^4 + (\alpha\beta)(4I_1)x^3 + (\alpha\beta)^2(6I_2)x^2 + (\alpha\beta)^3(4I_3)x + (\alpha\beta)^4 I_4^2. \tag{171}$$

The quantities displaying explicit dependence on $\hat{\tau}$ are defined as

$$\alpha = e_2 - e_1, \qquad \beta = e_3 - e_1, \tag{172}$$

where

$$4(x - e_1)(x - e_2)(x - e_3) = 4x^3 - g_2 x - g_3, \tag{173}$$

with g_2 and g_3 given by Eqs. (153) and (154). Explicitly we have [26,50]

$$e_1 - e_2 = \theta_3^4(0, \hat{\tau}), \qquad e_3 - e_2 = \theta_1^4(0, \hat{\tau}),$$
$$e_1 - e_3 = \theta_2^4(0, \hat{\tau}), \tag{174}$$

where

$$\theta_1(0, \hat{\tau}) = \sum_{n\in\mathbb{Z}} q^{(1/2)(n+1/2)^2}, \tag{175}$$

$$\theta_2(0, \hat{\tau}) = \sum_{n\in\mathbb{Z}} (-1)^n q^{(1/2)n^2}, \tag{176}$$

$$\theta_3(0, \hat{\tau}) = \sum_{n\in\mathbb{Z}} q^{(1/2)n^2}, \tag{177}$$

with $q = e^{2\pi i\hat{\tau}}$. One also has [26,50]

$$e_1 = \tfrac{2}{3} + 16q + 16q^2 + \ldots, \tag{178}$$

$$e_2 = -\tfrac{1}{3} - 8q^{1/2} - 8q - 32q^{3/2} - 8q^2 + \ldots, \tag{179}$$

$$e_3 = -\tfrac{1}{3} + 8q^{1/2} - 8q + 32q^{3/2} - 8q^2 + \ldots. \tag{180}$$

Notice that $e_1 + e_2 + e_3 = 0$ similar to the property of four-qubit quartic invariants $L + M + N = 0$. In the limit $\hat{\tau} \to i\infty$ we have $\alpha\beta \to 1$, hence, from Eq. (171) we get back to our usual polynomial of Eq. (25). Clearly introducing the factor $\alpha\beta$ amounts to rescaling the amplitudes of our four-qubit state of Eq. (2) from Λ to the $\hat{\tau}$ dependent ones $\sqrt{\alpha\beta}\Lambda$.

Let us now introduce an extra complex parameter u to be specified later and consider the following family of elliptic curves:

$$\begin{aligned} y^2 = {} & x(x - \alpha u)(x - \beta u) - (\alpha - \beta)^2 I_1 x^2 \\ & + \tfrac{1}{2}(\alpha - \beta)\alpha\beta[(\alpha + \beta)I_4 - 3(\alpha - \beta)I_2]x \\ & - (\alpha - \beta)\alpha^2\beta^2 I_4 u - (\alpha - \beta)^2\alpha^2\beta^2 I_3. \end{aligned} \tag{181}$$

Now the right-hand side is a quadratic polynomial in u. Its discriminant turns out to be just $(\alpha - \beta)^2\Sigma_4(\Lambda, \hat{\tau}, -x)$. This gives the desired clue for yet another way of associating an elliptic curve to two-center charge configurations.

Notice that in the limit $\hat{\tau} \to i\infty$ our curve boils down to $y^2 = x(x - u)^2$ a curve similar to the one of Eq. (142) with j invariant ∞. Moreover, via the transformation $v = u + (\alpha - \beta)\frac{1}{2}H$ one can also obtain a new form

$$\begin{aligned} y^2 = {} & x(x - \alpha v)(x - \beta v) + \beta(\alpha - \beta)Hx^2 \\ & - \alpha\beta(\alpha - \beta)(\alpha M + \beta N)x - \alpha\beta(\alpha - \beta)Hvx \\ & - \alpha^2\beta^2(\alpha - \beta)Lv - \alpha^2\beta^2(\alpha - \beta)^2 D. \end{aligned} \tag{182}$$

Though not in the Tate form this curve is displaying similar quantities than our previous curve of Eq. (130). In particular for truncations from stu charge configurations to st^2 and t^3 ones, the structure of the curve is getting simpler step by step. Though this curve now shows an explicit dependence on a modular parameter $\hat{\tau}$ and also featuring a new complex variable v, now its discriminant cannot obviously be related to a hyperdeterminant of type $2 \times 2 \times 2 \times 2$. However, curves like Eq. (182) have other appealing properties which we would like to discuss.

Actually the curve of Eq. (181) is related to a one introduced by Seiberg and Witten in their study of $N = 2$ supersymmetric $SU(2)$ gauge theory with four quark flavors [26]. The parameter u in that case was the gauge invariant modulus representing the square of the Higgs expectation value and $\hat{\tau}$ was the complex coupling constant $\theta/\pi + 8\pi i/g^2$ of Montonen-Olive duality. In that context their curve was parametrized not by algebraically independent four-qubit invariants but by the squares of the four quark masses m_1, m_2, m_3, and m_4. However, after comparing the relevant expressions [see, in particular, Eq. (17.58) of Ref. [26]] it is easy to show that the quark masses squared in that case correspond to the squares of the parameters a, b, c, and d of our four-qubit canonical form of Eqs. (32)–(34).

Apart from the symmetry $SL(2,\mathbb{R})_0 \times G_4$ where $G_4 = SL(2,\mathbb{R})^{\times 3}$ is the continuous U-duality group in the supergravity approximation, the stu model also has an important triality symmetry [53]. In our four-qubit description this S_3 permutation symmetry should manifest itself in representing somehow elliptic curves in a way displaying four-qubit invariants that are also invariant under permutation. In analogy with the Seiberg-Witten curve there is also the possibility to present a triality invariant form of that kind. For this purpose we have to chose from four algebraically independent four-qubit invariants which are also invariant under the full permutation group S_4. Such invariants were first constructed by Schläfli [54] in 1852. Let us denote these invariants as [29,30] H, Γ, Σ, and Π. They are of order 2, 6, 8, and 12, respectively. The new quantities Γ, Σ, and Π are defined as

$$\Gamma = D + E + F, \qquad \Sigma = L^2 + M^2 + N^2,$$
$$\Pi = (L - M)(M - N)(N - L). \tag{183}$$

Here E and F are defined as [14]

$$HL = E - D, \qquad HM = D - F, \qquad HN = F - E. \tag{184}$$

Notice, in particular, that in terms of the quantities H, L, M, D used to describe the Tate form of the elliptic curve of Eq. (130) the permutation invariant version of the sextic invariant reads as

$$\Gamma = 3D + H(L - M). \tag{185}$$

Let us now consider the triality symmetric curve [26]

$$y^2 = w_1w_2w_3 + \frac{\lambda}{3}[(M-N)(e_2-e_3)w_1 + (N-L)(e_3-e_1)w_2 + (L-M)(e_1-e_2)w_3] - \frac{\lambda^2}{3}\Gamma, \tag{186}$$

where

$$w_i = x - e_i\tilde{u} - e_i^2H, \quad \lambda = (e_1-e_2)(e_2-e_3)(e_3-e_1), \qquad i = 1, 2, 3. \tag{187}$$

Notice that after using Eq. (185) and the canonical form for our four-qubit states of Eq. (32) and (35) with $(t_1, t_2, t_3, t_4) = (a^2, b^2, c^2, d^2)$ for the polynomial Γ of sixth order we obtain the expression

$$\frac{1}{3}\Gamma = \frac{3}{16}\sum_{i>j>k} t_it_jt_k - \frac{1}{96}\sum_{i\neq j} t_it_j^2 + \frac{1}{96}\sum_i t_i^3. \tag{188}$$

This is precisely the sixth order invariant of Eq. (16.36) of Seiberg and Witten [26] provided we make the identification $t_i \equiv m_i^2$, i.e., the canonical four-qubit parameters are identified with the quark masses of the four flavors. Similar calculations verify that the remaining invariants of that paper, namely R, T_1, T_2, and T_3 can be identified with the invariants H, $\frac{1}{3}(M-N)$, $\frac{1}{3}(N-L)$, and $\frac{1}{3}(L-M)$ in the canonical parametrization. The important property of the curve of Eq. (186) is that after making the transformations

$$u = \tilde{u} + \tfrac{1}{2}e_1H, \qquad x \to x - \tfrac{1}{2}e_1u + \tfrac{1}{2}e_i^2H, \tag{189}$$

and performing the (weak coupling [26]) limit $\hat{\tau} \to i\infty$ it boils down to the form

$$y^2 = x^2(x-u). \tag{190}$$

Moreover, in order to make contact with our original curve of Eq. (181) one just has to perform the change of variables in Eq. (186)

$$x \mapsto x - e_1\tilde{u} - e_1^2H, \qquad u = \tilde{u} + \tfrac{1}{2}e_1H. \tag{191}$$

It is also interesting to realize that our identification of the quark masses squared with the canonical four-qubit parameters automatically incorporates the special cases when some of the masses are zero with the singular cases in the four-qubit classification scheme of Verstraete [35]. Moreover, switching to the two-center stu context there the canonical parameters would rather be identified with the parameters z_1, z_2, z_3, and z_4 of the canonical form of the central charge matrix of $N = 8$ $d = 4$ supergravity (see also the discussion of the previous seubsection). Recall also in this respect the observation of Seiberg and Witten [26] that triality symmetry of $SO(8)$ is connected to the permutation symmetry S_3 as the mod 2 reduction of $SL(2,\mathbb{Z})$. Comparing Eqs. (17.34)–(17.35) of Ref. [26] with our Eq. (170) related to the structure of the matrix $\mathcal{M}$ and a similar expression related to the matrix $\mathcal{N}$ that are in turn related to the structure of four-qubit reduced density matrices [see Eqs. (7) and (8)], we see that triality symmetry is intrinsically related not only to the structure of four-qubit entanglement but alsoto the structure of the manifold X described by the curve of Eq. (186). In this context it is especially instructive to recall the arguments of Seiberg and Witten on the structure of the cohomology of X. In particular it is tempting to reinterpret the triality invariant expressions for the cohomology classes of the periods [26] as genuine unnormalized four-qubit states in canonical form [see Eqs. (17.36) and (17.37) of that paper]. The redundancy in arriving at such an expression can be attributed to the action of the Weyl group of $SO(8)$ on the canonical parameters (i.e. the action of the Klein group on the canonical form of $|\Lambda\rangle$ in accord with a comment of Luque and Thibon in the four-qubit context [14]. (See also the last paragraph of Sec. II. in this respect.)

Finally, it is amusing to recall yet another context where elliptic curves parametrized by four-qubit invariants reveal some intriguing structural similarities with interesting physics. First of all, it is well-known that in F theory an extra $SL(2,\mathbb{Z})$ and a hidden torus also makes its presence in a spectacular way [52,55]. In this case one considers an elliptic fibration M with some basis manifold B and fiber being a two dimensional torus with modular parameter $\hat{\tau}$. Now F theory is defined on M as type IIB string theory on B with the axion-dilaton modulus of type IIB string theory identified with the modular parameter of the two-torus. In the original setting [55] an elliptically fibered $K3$ surface was considered of the form $y^2 = x^3 + f(u)x + g(u)$ where f and g are polynomials of degree eight and degree 12 in $u \in \mathbb{C}P^1$. This curve describes a torus for each point of the Riemann sphere $\mathbb{C}P^1$ labeled by the complex coordinate u. It is interesting to realize that this elliptically fibered $K3$ surface is of the form of Eq. (147) where S and T are polynomials of order eight and 12. However, according to Eq. (32) when writing S and T in the canonical form we are having four complex coordinates instead of the one u. The points where the torus degenerates corresponds to the vanishing of the discriminant of the cubic which is a polynomial of order 24. In the four-qubit parametrization this just corresponds to the vanishing of our hyperdeterminant of order 24 related to the quantity $S^3 - 27T^2$. In the original F-theory context the compactification described by this elliptical fibration corresponds to a configuration of 24 seven branes of type IIB theory located at the zeros of the discriminant. On the other hand in the four-qubit canonical parametrization the explicit expression for the discriminant of Eq. (35) is similar to the usual ones describing coincident seven brane configurations [56], provided we are regarding one of the canonical parameters to be special.

Notice also that the study of F-theory compactifications [57] is effected by looking at elliptic fibrations described by elliptic curves of the Tate form similar to the one of Eq. (160) where now the coefficients are polynomials in the coordinates of the base manifold B. An important limit studied in these investigations is the Sen limit [56] or orientifold limit. This is achieved by demanding that the axio-dilaton to be constant almost everywhere in type IIB space-time. Explicitly this limit is realized by setting

$$a_3 \mapsto \varepsilon a_3, \qquad a_4 \mapsto \varepsilon a_4, \qquad a_6 \mapsto \varepsilon^2 a_6, \qquad (192)$$

in the Tate form. In the two-center stu context a_i are polynomial invariants of homogeneous degree $2i$ and $a_6 = 0$. For the stu model we have $a_4 = -LM$, $a_3 = D$. For the st^2 and t^3 models $L = 0$ hence $I_3 = D$ and a corresponding limit is achieved by sending the invariant D to zero by, e.g,. scaling down one of the charges to zero. This is precisely what happened in Sec. V B, and also in the situation described by Eq. (148).

Finally, our triality symmetric Seiberg-Witten curve of Eq. (186) also makes its presence in the F-theory context [27]. Here the basic idea is to deform away from the special point in moduli space where the orientifold picture applies. This important deformation is effected by a curve of the form $y^2 = x^3 + \tilde{f}(u)x + \tilde{g}(u)$ where $\tilde{f}$ and $\tilde{g}$ are polynomials in u of degree two and three. Sen has shown that this curve can be cast in the form of Eq. (186) provided we make a suitable mapping of the parameters involved in the two different physical contexts. Notice that in the original papers the relevant curves [26,27] were not parametrized by the four-qubit invariants H, L, M, N, and Γ as in Eq. (186). Instead in these papers four parameters were employed, which are easily identified with the four canonical parameters of four-qubit states of Eq. (32). In particular in the F-theory context the relevant deformation is a one corresponding to splitting of the six coincident zeros of the discriminant away from each other, i.e., in the orientifold picture moving the four coincident seven branes away from the orientifold plane. The collective coordinates of the background describing such a physical situation are that of an $N = 1$ supersymmetric $SO(8)$ gauge theory in eight dimensions with moduli space characterized by a complex scalar field Φ. At a generic point in moduli space the vacuum expectation value of this field has the form similar to the canonical form of the central charge of Eq. (165) we used in Sec. V. in the different context of $N = 8$ supergravity. Now one can check that the four canonical complex parameters c_1, c_2, c_3, and c_4 of $\langle\Phi\rangle$ are behaving exactly like the canonical parameters of a four-qubit state. Again the polynomial expressions used by Seiberg and Witten in their curve of the Eq. (186) form are the ones for the four-qubit invariants H, L, M, N, and Γ now with the simple identification [27] $c_i = m_i$. Our four-qubit analysis as presented in this paper shows similar symmetry properties (permutation symmetry of the stu truncation connected to triality of $SO(8)$ within $N = 8$ supergravity). This might indicate that the physics of two-center black holes could be another arena where this curve plays a basic role. Of course these structural similarities could be superficial, however, in any case the possible physical ramifications should be explored further.

VII. CONCLUSIONS

In this paper we have shown how the U-duality invariants introduced by Ferrara *et al.* [1] characterizing two-center extremal black-hole charge configurations in the stu, st^2, and t^3 models of $N = 2$, $d = 4$ supergravity can be understood as entanglement invariants of four-qubit systems. In this entanglement based picture the geometric and algebraic meaning of these invariants is displayed in a nice and unified manner. For one of the entanglement invariants that have not yet made its debut to the supergravity literature we have found a distinguished role. It is the hyperdeterminant of type $2 \times 2 \times 2 \times 2$ which is the generalization of Cayley's hyperdeterminant featuring the macroscopic black-hole entropy formula in the stu model. For the special example of the BPS D0D4-D2D6 split in the t^3 model we have demonstrated that this polynomial invariant of order 24 seems to govern important issues of consistency for the two-center solutions.

We have also introduced a quartic polynomial featuring the algebraically independent four-qubit invariants. For our simple example we have shown that the property that this polynomial has real roots provides a necessary condition for the consistency condition to hold. The resolvent cubic of this fundamental polynomial can be cast into a cubic of Weierstrass canonical form which in turn can be used to define an elliptic curve associated to the two-center charge configuration. After switching to the more convenient Tate form of this curve we have shown that this association is natural, meaning that the Tate form is displaying combinations of the algebraically independent invariants of physical meaning. Indeed, the Tate form falls short of terms step by step as we perform truncations to the st^2 and t^3 models, restrict attention to mutually local charge configurations, or perform the degenerate limit resulting in splits into small black holes. The discriminant of this elliptic curve is just the hyperdeterminant, a quantity also featuring the j invariant of the curve. For our example the structure of the j invariant nicely encapsulates the basic properties of the D0D4-D2D6 split.

The mapping from two-center black-hole charge configurations to elliptic curves is many to one. Moreover, for the stu model only a special class of curves with $a_6 = 0$ in Eq. (130) can be reached. We observed that the vanishing of a_6 can be attributed to the vanishing of a polynomial of degree 12 characterizing the stu model. Based on this we conjectured that our $N = 2$ picture can be substantially generalized also incorporating two-center charge configurations of $N = 8$ supergravity. Here Cartan's quartic invariant and its polarizations should make their presence together with the usual symplectic invariant and a new invariant of order 6 introduced recently [13]. The discriminant of this curve is of a more general type than would play the role of a generalization of our hyperdeterminant, with probably similar physical meaning than its *stu* descendent.

Finally, we presented other physical contexts where our four-qubit invariants parametrizing elliptic curves also play an important role. Here we have given a new look to the triality invariant curve originally introduced by Seiberg and Witten [26]. In its new form this curve is featuring four-qubit invariants also displaying permutation invariance. In this new setting we have also invoked the F-theory interpretation of this curve as was given by Sen [27]. Taken together with the two-center black-hole context studied in this paper the main unifying theme in these scenarios seems to be the presence of a hidden torus and its extra set of $PSL(2, \mathbb{Z})$ modular transformations. Note that in our investigations elliptic curves appeared merely as natural mathematical objects nicely encapsulating the information on the structure of two-center U-duality invariants. However, apart from this we have also presented some evidence that the hyperdeterminant and the j invariant associated to these curves might contain useful information on issues of marginal stability, domain walls and split attractor flows. The physical background underlying the use of elliptic curves in the F theory and $N = 2$ supersymmetric $SU(2)$ gauge theory context is well-known. Is there any deeper physical reason also accounting for the occurrence of elliptic curves related to the structure of multicenter black-hole solutions?

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