

Thermodynamical inequivalence of quantum stress-energy and spin tensors

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It is shown that different couples of stress-energy and spin tensors of quantum-relativistic fields, which would be otherwise equivalent, are in fact inequivalent if the second law of thermodynamics is taken into account. The proof of the inequivalence is based on the analysis of a macroscopic system at full thermodynamical equilibrium with a macroscopic total angular momentum and a specific instance is given for the free Dirac field, for which we show that the canonical and Belinfante stress-energy tensors are not equivalent. For this particular case, we show that the difference between the predicted angular momentum densities for a rotating system at full thermodynamical equilibrium is a quantum effect, persisting in the nonrelativistic limit, corresponding to a polarization of particles of the order of $\hbar\omega/KT$ (ω being the angular velocity) and could in principle be measured experimentally. This result implies that specific stress-energy and spin tensors are physically meaningful even in the absence of gravitational coupling and raises the issue of finding the thermodynamically right (or the right class of) tensors. We argue that the maximization of the thermodynamic potential theoretically allows us to discriminate between two different couples, yet for the present we are unable to provide a theoretical method to single out the best couple of tensors in a given quantum field theory. The existence of a nonvanishing spin tensor would have major consequences in hydrodynamics, gravity and cosmology.

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I. INTRODUCTION

It is commonly known that stress-energy and spin tensors are not uniquely defined in field theory as long as gravity is disregarded. In quantum field theory, distinct stress-energy tensors differing by the divergence of a rank-3 tensor provide, once integrated in three-dimensional space, the same generators of space-time translations provided that the flux of the additional rank-3 tensor field (hereafter referred to as superpotential) vanishes at the boundary. Correspondingly, in classical field theory, the spatial integrals of $T^{0\nu}$ yield the same values of total energy and momentum. Within field theory on a flat spacetime, it is then possible to generate apparently equivalent stress-energy tensors which are, e.g., symmetric or nonsymmetric. Indeed, gravitational coupling provides an unambiguous way of defining the stress-energy tensor; in general relativity, it is symmetric by construction and the spin tensor vanishes. However, in a likely extension known as Einstein-Cartan theory (not excluded by present observations) the spin tensor is nonvanishing and the stress-energy tensor is nonsymmetric.

Can we say something more? In classical physics we have a stronger requirement with respect to a quantum theory: we would like the energy, momentum and angular momentum content of *any* arbitrary macroscopic spatial region to be well-defined concepts; otherwise stated, we would like to have objective values for the energy, momentum and angular momentum densities. If these quantities are to be the components of the stress-energy and spin tensors, such a requirement strongly limits the freedom to change these tensors. It is crucial to emphasize, from the

very beginning, the difference between quantum and classical tensors. The quantum stress-energy and spin tensors, henceforth denoted with a hat $\hat{\cdot}$, are operatorial expressions depending on the microscopic quantum field operators Ψ , whereas the classical ones are c-numbers. The relation between them is [1]

$$T^{\mu\nu}(x) = \text{tr}[\hat{\rho}:\hat{T}^{\mu\nu}(x):] \quad \mathcal{S}^{\lambda,\mu\nu}(x) = \text{tr}[\hat{\rho}:\hat{\mathcal{S}}^{\lambda,\mu\nu}(x):] \quad (1)$$

where $\hat{\rho}$ is the density operator describing the (mixed or pure) quantum state and $:$ denotes normal ordering; the latter is usually introduced in the mean value definition in order to subtract the zero-point infinities.¹ According to (1), a change of quantum stress-energy and spin tensors could induce a change of the corresponding classical ones in an undesirable fashion, meaning that energy or momentum or angular momentum density get changed. However, the change that classical mean values undergo as a reflection of a variation of quantum tensors crucially depends on the physical state $\hat{\rho}$. Particularly, we will see that the freedom of varying the stress-energy and spin tensors at a quantum level depends on the symmetry features of the physical state: a highly symmetric state allows more changes of quantum tensors than a state with little symmetry does.

In this paper, we prove that a system at full thermodynamical equilibrium with a macroscopic value of angular momentum, then rigidly rotating [3], allows us to discriminate between different quantum spin tensors, and,

¹For a discussion of the meaning of normal ordering for interacting fields see, e.g., Ref. [2]. We stress that the results obtained in this work, particularly in Sec. VI are anyhow independent of the use of normal ordering in Eq. (1).

consequently, between different quantum stress-energy tensors. This kind of inequivalence shows up only for a rotating system whereas all quantum tensors are equivalent for a system at the more familiar thermodynamical equilibrium with vanishing macroscopic angular momentum. The paper is organized as follows: in Sec. II we will discuss the general class of stress-energy tensor transformations ensuring the invariance of conservation equations; in Sec. III we will discuss the usual thermodynamical equilibrium distribution and its symmetries and in Sec. IV we will do the same for a system at full thermodynamical equilibrium with angular momentum; in Sec. V we will obtain the most general form of mean stress-energy and spin tensor for a system at full thermodynamical equilibrium with angular momentum and show that the equivalence between different quantum tensors no longer applies unless peculiar conditions are met; in Sec. VI we will present and prove a concrete instance of inequivalence for the free Dirac field; finally, in Sec. VII we will summarize and further illustrate the obtained result and discuss the possible consequences thereof.

Notation

In this paper we adopt the natural units, with $\hbar = c = K = 1$. The Minkowskian metric tensor is $\text{diag}(1, -1, -1, -1)$; for the Levi-Civita symbol we use the convention $\epsilon^{0123} = 1$. We will use the relativistic notation with repeated indices assumed to be saturated. Operators in Hilbert space will be denoted by an upper hat, e.g., \hat{R} , with the exception of the Dirac field operator which is denoted with a capital Ψ .

II. TRANSFORMATIONS OF STRESS-ENERGY AND SPIN TENSORS

The conservation equations ensuing from the relativistic translational and Lorentz invariance are the well-known continuity equations of energy-momentum and total angular momentum

$$\begin{aligned}\partial_\mu \hat{T}^{\mu\nu} &= 0 \\ \partial_\lambda \hat{\mathcal{J}}^{\lambda,\mu\nu} &= \partial_\lambda (\hat{\mathcal{S}}^{\lambda,\mu\nu} + x^\mu \hat{T}^{\lambda\nu} - x^\nu \hat{T}^{\lambda\mu}) \\ &= \partial_\lambda \hat{\mathcal{S}}^{\lambda,\mu\nu} + \hat{T}^{\mu\nu} - \hat{T}^{\nu\mu} = 0\end{aligned}\quad (2)$$

However, stress-energy \hat{T} and spin tensor $\hat{\mathcal{S}}$ are not uniquely defined in quantum field theory; once a particular couple $(\hat{T}, \hat{\mathcal{S}})$ of these tensors is found, e.g., applying Noether's theorem to some Lagrangian density (the so-called *canonical* tensors), it is possible to generate new couples $(\hat{T}', \hat{\mathcal{S}}')$ through the following pseudogauge transformation [4]:

$$\begin{aligned}\hat{T}'^{\mu\nu} &= \hat{T}^{\mu\nu} + \frac{1}{2} \partial_\alpha (\hat{\Phi}^{\alpha,\mu\nu} - \hat{\Phi}^{\mu,\alpha\nu} - \hat{\Phi}^{\nu,\alpha\mu}) \\ \hat{\mathcal{S}}'^{\lambda,\mu\nu} &= \hat{\mathcal{S}}^{\lambda,\mu\nu} - \hat{\Phi}^{\lambda,\mu\nu},\end{aligned}\quad (3)$$

where $\hat{\Phi}$ is an arbitrary tensor of rank-three antisymmetric in the last two indices depending on the fields $\hat{\psi}$. It is easy to check that the new couple fulfills the same continuity Eqs. (2) as the original one and that the new total angular momentum tensor, like the stress-energy tensor, differs from the original one by a divergence

$$\begin{aligned}\hat{\mathcal{J}}'^{\lambda,\mu\nu} &= \hat{\mathcal{J}}^{\lambda,\mu\nu} + \frac{1}{2} \partial_\alpha [x^\mu (\hat{\Phi}^{\alpha,\lambda\nu} - \hat{\Phi}^{\lambda,\alpha\nu} - \hat{\Phi}^{\nu,\alpha\lambda}) \\ &\quad - x^\nu (\hat{\Phi}^{\alpha,\lambda\mu} - \hat{\Phi}^{\lambda,\alpha\mu} - \hat{\Phi}^{\mu,\alpha\lambda})].\end{aligned}\quad (4)$$

The spatial integrals over the domain Ω

$$\begin{aligned}\hat{P}^\nu &= \int_\Omega d^3x \hat{T}^{0\nu} \\ \hat{J}^{\mu\nu} &= \int_\Omega d^3x \hat{\mathcal{J}}^{0,\mu\nu} = \int d^3x \hat{\mathcal{S}}^{0,\mu\nu} + x^\mu \hat{T}^{0\nu} - x^\nu \hat{T}^{0\mu}\end{aligned}\quad (5)$$

are conserved (and are generators of translations and Lorentz transformations if the domain is the whole space) provided that the fluxes at the boundary vanish

$$\begin{aligned}\int_{\partial\Omega} dS \hat{T}^{i\nu} n_i &= 0 \\ \int_{\partial\Omega} dS (\hat{\mathcal{S}}^{i,\mu\nu} + x^\mu \hat{T}^{i\nu} - x^\nu \hat{T}^{i\mu}) n_i &= 0,\end{aligned}\quad (6)$$

n_i being the surface normal versor. The Eqs. (6) are usually ensured by enforcing special boundary conditions for the fields; for instance, the familiar periodic boundary conditions for a box, or requiring some expression involving the field or its normal derivatives to vanish at the boundary (this will be the case in this paper, see Sec. VI). It is very important to stress here that one can obtain conserved stress-energy and angular momentum tensors and a corresponding set of conserved generators even within a finite or bounded region, thus breaking global translational and rotational symmetry, provided that suitable boundary conditions are imposed on the fields; this point will be further discussed in Sec. VI.

The spatial integrals (5) of the new tensors are invariant, thus yielding the same generators, if the tensor $\hat{\Phi}$ is such that the following boundary integrals vanish, according to Eqs. (3) and (4)

$$\begin{aligned}\int_{\partial\Omega} dS (\hat{\Phi}^{i,0\nu} - \hat{\Phi}^{0,i\nu} - \hat{\Phi}^{\nu,i0}) n_i &= 0 \\ \int_{\partial\Omega} dS [x^\mu (\hat{\Phi}^{i,0\nu} - \hat{\Phi}^{0,i\nu} - \hat{\Phi}^{\nu,i0}) \\ &\quad - x^\nu (\hat{\Phi}^{i,0\mu} - \hat{\Phi}^{0,i\mu} - \hat{\Phi}^{\mu,i0})] n_i = 0.\end{aligned}\quad (7)$$

In fact, if the above conditions are met, the flux integrals (6) of the new primed tensors vanish because the new tensors also fulfill the continuity equations. In conclusion, a pseudogauge transformation like (3) is always possible provided that the boundary conditions (7) are ensured; in this case the couple $(\hat{T}, \hat{\mathcal{S}})$ and $(\hat{T}', \hat{\mathcal{S}}')$ are regarded as

equivalent in quantum field theory because they give the same total energy, momentum and angular momentum, in the operatorial sense.

The classical counterpart of transformation (3) can be calculated by applying Eq. (1) to both sides and this obviously leads to

$$\begin{aligned} T'^{\mu\nu} &= T^{\mu\nu} + \frac{1}{2} \partial_\alpha (\Phi^{\alpha,\mu\nu} - \Phi^{\mu,\alpha\nu} - \Phi^{\nu,\alpha\mu}) \\ S'^{\lambda,\mu\nu} &= S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} \end{aligned} \quad (8)$$

If the system is macroscopic, we would like the mean values of those tensors to be invariant under a transformation (8), and not just their integrals. This is because energy, momentum and total angular momentum densities classically must take on objective values, independent of the particular quantum tensors. A minimal requirement would be the invariance of the aforementioned densities, that is,

$$T'^{0\nu} = T^{0\nu} \quad \mathcal{J}'^{0,\mu\nu} = \mathcal{J}^{0,\mu\nu}.$$

However, this is a frame-dependent requirement; a Lorentz-boosted frame would measure a different energy-momentum density if only the first row of the stress-energy tensor was invariant under transformation (8) in one particular frame. We are thus to enforce a stricter requirement, namely,

$$T'^{\mu\nu} = T^{\mu\nu} \quad (9)$$

whereas, for the rank-3 angular momentum tensor, we can make a looser request

$$\mathcal{J}'^{\lambda,\mu\nu} = \mathcal{J}^{\lambda,\mu\nu} + g^{\lambda\mu} K^\nu - g^{\lambda\nu} K^\mu \quad (10)$$

where K is a vector field. Indeed, if we limit ourselves to spatial indices $\mu, \nu = 1, 2, 3$, the above equation is enough to ensure that the angular momentum densities, with $\lambda = 0$, are the same in *any* inertial frame. Comparing Eq. (9) with Eq. (8), we get

$$\partial_\alpha (\Phi^{\alpha,\mu\nu} - \Phi^{\mu,\alpha\nu} - \Phi^{\nu,\alpha\mu}) = 0, \quad (11)$$

while comparing Eq. (10) with the mean of Eq. (4) and taking (11) into account, we obtain a simple condition for the superpotential to meet

$$\begin{aligned} &\frac{1}{2} \partial_\alpha [x^\mu (\Phi^{\alpha,\lambda\nu} - \Phi^{\lambda,\alpha\nu} - \Phi^{\nu,\alpha\lambda}) \\ &\quad - x^\nu (\Phi^{\alpha,\lambda\mu} - \Phi^{\lambda,\alpha\mu} - \Phi^{\mu,\alpha\lambda})] = g^{\lambda\mu} K^\nu - g^{\lambda\nu} K^\mu \\ \Rightarrow &\frac{1}{2} (\Phi^{\mu,\lambda\nu} - \Phi^{\lambda,\mu\nu} - \Phi^{\nu,\mu\lambda} - \Phi^{\nu,\lambda\mu} \\ &\quad + \Phi^{\lambda,\nu\mu} + \Phi^{\mu,\nu\lambda}) = -\Phi^{\lambda,\mu\nu} = g^{\lambda\mu} K^\nu - g^{\lambda\nu} K^\mu. \end{aligned} \quad (12)$$

Plugging this last result back into Eq. (11) one obtains

$$2\partial_\alpha (K^\mu g^{\alpha\nu} - K^\alpha g^{\mu\nu}) = 2\partial^\nu K^\mu - 2g^{\mu\nu} \partial \cdot K = 0.$$

Contracting the indices μ and ν we obtain at once that the divergence of the vector field K vanishes and so, because of the above equation

$$\partial^\nu K^\mu = 0 \quad (13)$$

Therefore the Eqs. (11) and (12) imply that the vector field K is a constant field. The possible directions of this field will be dictated by the symmetry properties of the system under consideration, as we will see in the next two sections.

It should be emphasized that the conditions (11) and (12) do not need to apply to the quantum tensor $\hat{\Phi}$, which only has to meet the boundary conditions (7), as has been seen. On the other hand, if we take the mean values of (7) applying $\text{tr}(\hat{\rho})$ on both sides, the ensuing equation is a trivial consequence of the Eq. (11). In fact, it may happen that the mean value of the superpotential Φ fulfills Eqs. (11) and (12) even though its quantum correspondent $\hat{\Phi}$ does not, because of specific features of the density operator $\hat{\rho}$. In this case, the couples (\hat{T}, \hat{S}) and (\hat{T}', \hat{S}') are to be considered equivalent only with regard to a particular density operator, that is, for a specific quantum state.

We will see in the next two sections that the equivalence between couples of tensors crucially depends on the symmetry properties of the physical state $\hat{\rho}$ (either mixed or pure). Particularly, we shall see that if $\hat{\rho}$ is the usual thermodynamical equilibrium operator, proportional to $\exp[-\hat{H}/T + \mu\hat{Q}/T]$, any quantum tensor $\hat{\Phi}$ will result in a mean value Φ fulfilling Eq. (11) and (12). This means that *all* possible quantum microscopic stress-energy and spin tensors will yield the same physics in terms of macroscopically observable quantities.

III. THERMODYNAMICAL EQUILIBRIUM

The familiar thermodynamical equilibrium distribution (in the thermodynamical limit $V \rightarrow \infty$)

$$\hat{\rho} = \frac{1}{Z} \exp(-\hat{H}/T + \mu\hat{Q}/T) \quad (14)$$

where \hat{Q} denotes a conserved charge, and Z is the grand-canonical partition function

$$Z = \text{tr}[\exp(-\hat{H}/T + \mu\hat{Q}/T)]$$

is remarkably symmetric. It is space-time translationally invariant, since both \hat{Q} and \hat{H} commute with translation operators $\hat{T}(a) = \exp[ia \cdot \hat{P}]$. This entails that the mean value of any space-time-dependent operator $\hat{A}(x)$, including stress-energy and spin tensor, are independent of the space-time position

$$\begin{aligned}
\text{tr}[\hat{\rho}:\hat{A}(x+a):] &= \text{tr}[\hat{\rho}:\hat{T}(a)\hat{A}(x)\hat{T}(a)^{-1}:] \\
&= \text{tr}[\hat{\rho}:\hat{T}(a):\hat{A}(x):\hat{T}(a)^{-1}] \\
&= \text{tr}[\hat{T}(a)^{-1}\hat{\rho}\hat{T}(a):\hat{A}(x):] = \text{tr}[\hat{\rho}:\hat{A}(x):]
\end{aligned} \tag{15}$$

where the cyclicity of the trace and the transparency of the normal ordering with respect to translations have been used.² As a consequence, the mean value of any space-time derivative vanishes, and so will the divergences on the right-hand side of Eq. (3). Therefore, the mean stress-energy tensor will be the same regardless of the particular microscopic quantum tensor used. For instance, for the Dirac field, the three tensors

$$i\bar{\Psi}\gamma^\mu\partial^\nu\Psi \quad \frac{i}{2}\bar{\Psi}\gamma^\mu\overleftrightarrow{\partial}^\nu\Psi \quad \frac{i}{4}[\bar{\Psi}\gamma^\mu\overleftrightarrow{\partial}^\nu\Psi + (\mu \leftrightarrow \nu)] \tag{16}$$

will result in the same mean stress-energy tensor.

Also, the density operator (14) manifestly enjoys rotational symmetry, for \hat{H} and \hat{Q} commute with rotation operators \hat{R} . This implies that most components of tensors vanish. To show that, it is sufficient to choose suitable rotational operators and repeat the same reasoning as in Eq. (15). For instance, choosing the $\hat{R}_2(\pi)$ operator, i.e., the rotation of 180 degrees around the 2 (or y) axis, changing the sign of 1 (or x) and 3 (or z) components and leaving 2 and 0 unchanged, in the $x = (t, \mathbf{0})$ one has

$$\begin{aligned}
T^{12}(x) &= \text{tr}[\hat{\rho}:\hat{T}^{12}(x):] = \text{tr}[\hat{R}_2(\pi)\hat{\rho}\hat{R}_2(\pi)^{-1}:\hat{T}^{12}(x):] \\
&= \text{tr}[\hat{\rho}\hat{R}_2(\pi)^{-1}:\hat{T}^{12}(x):\hat{R}_2(\pi)] \\
&= \text{tr}[\hat{\rho}\mathbf{R}_2(\pi)_\mu^1\mathbf{R}_2(\pi)_\nu^2:\hat{T}^{\mu\nu}(\mathbf{R}_2(\pi)^{-1}(x)):] \\
&= -\text{tr}[\hat{\rho}:\hat{T}^{12}(\mathbf{R}_2(\pi)^{-1}(x)):] \\
&= -\text{tr}[\hat{\rho}:\hat{T}^{12}(x):] = -T^{12}(x)
\end{aligned} \tag{17}$$

where, in the last equality, we have taken advantage of the homogeneity of all mean values shown in Eq. (15); thus, $T^{12}((t, \mathbf{0})) = 0$ and, in view of the translational invariance $T^{12}(x) = 0 \forall x$. Similarly, by choosing other rotation operators, it can be shown that all off-diagonal elements of a tensor vanish. The only nonvanishing components are the diagonal ones, which, again owing to the rotational

²Here a comment is in order. The transparency of the normal ordering with respect to a conjugation transformation, that is, $:AF(\Psi)\mathbf{A}^{-1} := \mathbf{A}:F(\Psi):\mathbf{A}^{-1}$ where \mathbf{A} is a translation or a Lorentz transformation and F a function of the fields and its derivatives, is guaranteed for free fields provided that the vacuum $|0\rangle$ is an eigenstate of the same transformation, which is always the case. For interacting fields, we will assume that the definition of normal ordering (for this problem, see e.g., Ref. [2]) is such that transparency for conjugation holds; anyhow, for the examined case in Sec. VI we will just need transparency for a free field.

symmetry (choose $\hat{R}_i(\pi/2)$ and repeat the above reasoning), are equal

$$T^{11}(x) = T^{22}(x) = T^{33}(x).$$

The component $T^{00}(x)$ can also be nonvanishing and its value is unrelated to the other diagonal ones. Altogether, the mean stress-energy tensor can only have the diagonal (symmetric) form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} = (\rho + p)\hat{t}^\mu\hat{t}^\nu - pg^{\mu\nu},$$

where \hat{t} is the unit time vector with components $(1, \mathbf{0})$ and p have the physical meaning of proper energy density and pressure. It should be stressed that, for a system at full thermodynamical equilibrium described by $\hat{\rho}$ in Eq. (14), they would be the same regardless of the particular form of the quantum stress-energy tensor, e.g., those in Eq. (16) for the free Dirac field.

As far as the superpotential is concerned, it is easy to convince oneself, by using suitable rotations, that the only nonvanishing components are

$$\begin{aligned}
\Phi^{1,01}(x) &= \Phi^{2,02}(x) = \Phi^{3,03}(x) = -\Phi^{1,10}(x) \\
&= -\Phi^{2,20}(x) = -\Phi^{3,30}(x).
\end{aligned}$$

Hence, one scalar function B , independent of x , is sufficient to determine the spin tensor for a system at full thermodynamical equilibrium

$$\Phi^{\lambda,\mu\nu} = B(g^{\lambda\nu}\hat{t}^\mu - g^{\lambda\mu}\hat{t}^\nu).$$

This tensor has exactly the form for a ‘‘good’’ superpotential derived in Eq. (12) fulfilling condition (13). In conclusion, *any* transformation of the kind (3) will yield the same energy, momentum and angular momentum density for all inertial frames and so, all quantum stress-energy and spin tensors are equivalent as far as the density operator (14) is concerned.

IV. THERMODYNAMICAL EQUILIBRIUM WITH ANGULAR MOMENTUM

The situation is remarkably different for a thermodynamical system having a macroscopic nonvanishing total angular momentum. In this case, in its rest frame (defined as the one where the total momentum vanishes) the density operator reads [3,5]

$$\hat{\rho} = \frac{1}{Z_\omega} \exp(-\hat{H}/T + \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T + \mu\hat{Q}/T), \tag{18}$$

where $\boldsymbol{\omega}$ has the physical meaning of a constant, fixed angular velocity around which the system rigidly rotates. The factor Z_ω is the *rotational* grand-canonical partition function

$$Z_\omega = \text{tr}[\exp(-\hat{H}/T + \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T + \mu \hat{Q}/T)]. \quad (19)$$

The density operator (18) is much less symmetric than that in (14) and this has remarkable and interesting consequences on the allowed transformations of stress-energy and spin tensor. The surviving symmetries in (18) are time-translations $\mathbb{T}(t)$ and translations along the $\boldsymbol{\omega}$ axis $\mathbb{T}(z)$, rotations around the $\boldsymbol{\omega}$ axis $\mathbb{R}_{\hat{\boldsymbol{\omega}}}(\varphi)$ and reflection $\Pi_{\hat{\boldsymbol{\omega}}}$ with respect to planes orthogonal to $\boldsymbol{\omega}$ (if \hat{H} is parity-invariant).

The density operator (18) can be obtained in several fashions: by maximizing the entropy with the constraint of fixed mean value of angular momentum [6], generalizing to the quantum-relativistic case an argument used by Landau for classical systems [3], or as the limiting macroscopic case of a quantum statistical system with finite volume and fixed angular momentum in its rest frame in an exact quantum sense, i.e., belonging to a specific representation of the rotation group [7]. It should be pointed out that \hat{H} , \hat{Q} and the angular momentum operator along the $\boldsymbol{\omega}$ direction commute with each other, so that the exponential in (18) also factorizes.

The density operator (18) implies that, in its rest frame, the system is rigidly rotating with a velocity field $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$. The classical, nonrelativistic derivation by Landau [3] shows this in a very simple fashion by assuming that the system is made of macroscopic cells. To show the same thing within a quantum formalism implies a little more effort, which is nevertheless quite enlightening. Consider a vector field $\hat{V}(x)$ and calculate its mean value at a point $x + a$ by using space-time translation operators. Evidently

$$\begin{aligned} \text{tr}[\hat{\rho}:\hat{V}^\nu(x+a):] &= \text{tr}[\hat{\rho} \hat{\mathbb{T}}(a):\hat{V}^\nu(x):\hat{\mathbb{T}}(a)^{-1}] \\ &= \text{tr}[\hat{\mathbb{T}}(a)^{-1} \hat{\rho} \hat{\mathbb{T}}(a):\hat{V}^\nu(x):] \\ &= \frac{1}{Z_\omega} \text{tr}[\hat{\mathbb{T}}(a)^{-1} e^{-\hat{H}/T + \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T + \mu \hat{Q}/T} \hat{\mathbb{T}}(a):\hat{V}^\nu(x):] \\ &= \frac{1}{Z_\omega} \text{tr}[e^{-\hat{H}/T + \boldsymbol{\omega} \cdot \hat{\mathbf{J}}(a)^{-1}/T + \mu \hat{Q}/T}:\hat{V}^\nu(x):], \quad (20) \end{aligned}$$

where known commutation properties $[\hat{Q}, \hat{P}^\mu] = 0$ and $[\hat{H}, \hat{P}^\mu] = 0$ have been used. Now, from the theory of Poincaré group it is known that

$$\hat{\mathbb{T}}(a)^{-1} \hat{\mathbf{J}} \hat{\mathbb{T}}(a) = \hat{\mathbf{J}} + \mathbf{a} \times \hat{\mathbf{P}}, \quad (21)$$

whence, from (20)

$$\begin{aligned} \text{tr}[\hat{\rho}:\hat{V}^\nu(x+a):] &= \frac{1}{Z_\omega} \text{tr}[e^{-\hat{H}/T + \boldsymbol{\omega} \cdot (\hat{\mathbf{J}} + \mathbf{a} \times \hat{\mathbf{P}})/T + \mu \hat{Q}/T}:\hat{V}^\nu(x):] \\ &= \frac{1}{Z_\omega} \text{tr}[e^{-\hat{H}/T + (\boldsymbol{\omega} \times \mathbf{a}) \cdot \hat{\mathbf{P}}/T + \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T + \mu \hat{Q}/T}:\hat{V}^\nu(x):]. \quad (22) \end{aligned}$$

We can now define the temperature four-vector

$$\beta = \frac{1}{T}(1, \boldsymbol{\omega} \times \mathbf{a}),$$

which can also be rewritten as:

$$\beta = \frac{1}{T_0} u = \frac{1}{T_0}(\gamma, \gamma \mathbf{v}), \quad (23)$$

where $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{a}$, $\gamma = 1/\sqrt{1-v^2}$ and $T_0 = \gamma T$. The vector \mathbf{v} is manifestly a rigid velocity field, while T_0 is the inverse modulus of β , i.e., the comoving temperature which differs by the constant uniform T by a γ factor [8,9]. The mean value of $V^\nu(x+a)$ in Eq. (22) becomes

$$\begin{aligned} \text{tr}[\hat{\rho}:\hat{V}^\nu(x+a):] &= \frac{1}{Z_\omega} \text{tr}[\exp(-\beta(a) \cdot \hat{P} + \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T + \mu \hat{Q}/T):\hat{V}^\nu(x):]. \quad (24) \end{aligned}$$

Since β is timelike (provided that $v < 1$), it is possible to find a Lorentz transformation Λ such that

$$T_0 \beta_\mu = u_\mu = \Lambda_{0\mu} = g_{0\lambda} \Lambda_\mu^\lambda. \quad (25)$$

A convenient choice is the pure Lorentz boost along the $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{a}$ direction, which, being orthogonal to $\boldsymbol{\omega}$, leaves the operator $\hat{\mathbf{J}} \cdot \boldsymbol{\omega}$ invariant

$$\Lambda = \exp[-i \text{arccosh}(\gamma) \hat{\mathbf{v}} \cdot \mathbf{K}],$$

where \mathbf{K}_i ($i = 1, 2, 3$) are the generators of pure Lorentz boosts. Thereby, the trace on the right-hand side of the Eq. (24) can be written

$$\begin{aligned} \text{tr}[e^{-\Lambda_{0\mu} \hat{P}^\mu + \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T + \mu \hat{Q}/T}:\hat{V}^\nu(x):] &= \text{tr}[e^{-\hat{\Lambda}^{-1}(\hat{P}^0/T_0 + \gamma \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T_0 + \gamma \mu \hat{Q}/T_0) \hat{\Lambda}}:\hat{V}^\nu(x):] \\ &= \text{tr}[\hat{\Lambda}^{-1} e^{-\hat{P}^0/T_0 + \gamma \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T_0 + \gamma \mu \hat{Q}/T_0} \hat{\Lambda}:\hat{V}^\nu(x):] \\ &= \text{tr}[e^{-\hat{P}^0/T_0 + \gamma \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T_0 + \gamma \mu \hat{Q}/T_0} \hat{\Lambda}:\hat{V}^\nu(x):\hat{\Lambda}^{-1}] \\ &= (\Lambda^{-1})_\mu^\nu \text{tr}[e^{-\hat{P}^0/T_0 + \gamma \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T_0 + \gamma \mu \hat{Q}/T_0}:\hat{V}^\mu(\Lambda(x)):]. \quad (26) \end{aligned}$$

Finally, from (24) and (26) we get

$$\begin{aligned} \text{tr}[\hat{\rho}:\hat{V}^\nu(x+a):] &= \frac{1}{Z_\omega} (\Lambda^{-1})_\mu^\nu \text{tr}[e^{-\hat{P}^0/T_0 + \gamma \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T_0 + \gamma \mu \hat{Q}/T_0}:\hat{V}^\mu(\Lambda(x)):]. \quad (27) \end{aligned}$$

which tells us how to calculate the mean value of a vector field at any space-time point given its value in some other specific point. The most interesting feature of Eq. (27) is that the density operator on the right-hand side the same as $\hat{\rho}$ on the left-hand side with the replacement

$$T \rightarrow T_0 = \gamma(a)T \quad \boldsymbol{\omega} \rightarrow \gamma(a)\boldsymbol{\omega} \quad \mu \rightarrow \gamma(a)\mu. \quad (28)$$

If we choose $x = (0, \mathbf{0})$, i.e., the origin of Minkowski coordinates, and $a = (0, \mathbf{a})$ the Eq. (27) implies

$$\begin{aligned} \text{tr}[\hat{\rho}:\hat{V}^\nu(0, \mathbf{a}):] \\ = \frac{1}{Z_\omega} (\Lambda^{-1})^\nu_\mu \text{tr}[e^{-\hat{P}^0/T_0 + \gamma\boldsymbol{\omega}\cdot\hat{\mathbf{J}}/T_0 + \gamma\mu\hat{Q}/T_0}:\hat{V}^\mu(0, \mathbf{0}):], \end{aligned} \quad (29)$$

that is, the mean value of the vector field at any space-time point (it should be kept in mind that $\hat{\rho}$ is invariant by time translation and so any mean value is stationary) is completely determined by the mean value at the origin of the coordinates, with the same density operator, modulo the replacement of thermodynamical parameters in (28). This particular value is strongly constrained by the symmetries of $\hat{\rho}$. Let us identify the $\boldsymbol{\omega}$ direction as that of the z (or 3) axis (see Fig. 1) and consider the reflection Π_z with respect to $z = 0$ plane and the rotation $\mathbf{R}_3(\pi)$ of an angle π around the z axis; by repeating the same reasoning as for Eq. (17) for $V^\nu(0)$ we can easily conclude that the time component $V^0(0)$ is the only one having a nonvanishing mean value. Note, though, that the mean value on the right-hand side of (29) depends on the distance r from the axis because the density operator is modified by the replacement of the uniform temperature T with a radius-dependent $T_0 = \gamma T$. Therefore, according to Eq. (29) and using (25), the mean value of the vector field can be written

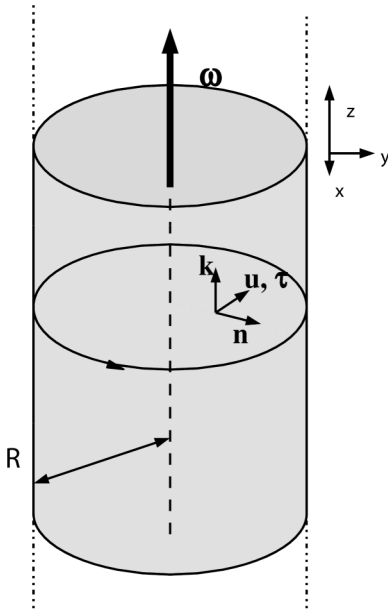


FIG. 1. Rotating cylinder with finite radius R at temperature T . Also shown are the inertial frame axes and the spatial parts of the vectors of tetrad (32).

$$\begin{aligned} V_\nu(x) &= \text{tr}[\hat{\rho}:\hat{V}_\nu(x):] \\ &= \frac{1}{Z_\omega} (\Lambda^{-1})_{\nu 0} \\ &\quad \times \text{tr}[e^{-\hat{P}^0/T_0(r) + \gamma(r)\boldsymbol{\omega}\cdot\hat{\mathbf{J}}/T_0(r) + \gamma(r)\mu\hat{Q}/T_0(r)}:\hat{V}^0(0):] \\ &\equiv \Lambda_{0\nu} V(r) = V(r)u_\nu, \end{aligned} \quad (30)$$

i.e., it must be collinear with the four-velocity field $u = (\gamma, \gamma\mathbf{v})$ in Eq. (23) and, therefore, its field lines are circles centered on the z axis and orthogonal to it. Similarly, we can obtain the general form of tensor fields of various rank and specific symmetry properties as a function of the basic four-velocity field.

However, the previous derivation relies on the fact that the system is infinitely extended in space. Indeed, at a distance from the axis such that $|\boldsymbol{\omega} \times \mathbf{x}| = 1$ the velocity becomes equal to the speed of light and the system has a singularity. We cannot, therefore, take the strict thermodynamical limit $V \rightarrow \infty$ for a system with macroscopic angular momentum. Instead, we have to enforce a spatial cutoff at some distance and figure out how this reflects on the most general forms of vector and tensor fields.

Enforcing a bounded region V for a thermodynamical system implies the replacement of all traces over the full set of states with a trace over a complete set of states $|h_V\rangle$ of the fields for this region V , that we indicate with tr_V ,

$$\text{tr} \rightarrow \text{tr}_V = \sum_{h_V} \langle h_V | \dots | h_V \rangle.$$

The density operator is the same as in (18) with the partition function now obtained by tracing over the localized states. It may sometimes be convenient to introduce the projection operator

$$\mathbf{P}_V = \sum_{h_V} |h_V\rangle\langle h_V|,$$

which allows us to maintain the trace over the full set of states of provided that we replace $\hat{\rho}$ with $\mathbf{P}_V \hat{\rho}$; for a generic operator \hat{A}

$$\text{tr}_V[\hat{\rho} \hat{A}] = \text{tr}[\mathbf{P}_V \hat{\rho} \hat{A}],$$

which amounts to stating that the effective density operator is now $\hat{\rho}_V$

$$\hat{\rho}_V = \frac{1}{Z_\omega} \mathbf{P}_V \exp(-\hat{H}/T + \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T + \mu\hat{Q}/T), \quad (31)$$

where

$$\begin{aligned} Z_\omega &= \text{tr}[\mathbf{P}_V \exp(-\hat{H}/T + \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T + \mu\hat{Q}/T)] \\ &= \text{tr}_V[\exp(-\hat{H}/T + \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T + \hat{Q}/T)]. \end{aligned}$$

In order to maintain the same symmetry of the density operator in (18), \mathbf{P}_V has to commute with \hat{J}_z , \hat{H} , \hat{P}_z , the Lorentz boost along z \hat{K}_z and the reflection operator with respect to any plane parallel to $z = 0$, $\hat{\Pi}_z$ (see Fig. 1). These requirements are met if the region V is a static

longitudinally indefinite cylinder with finite radius R and axis ω , and we will henceforth take this assumption.

There are two important consequences of having a finite radius R . Because of the presence of the projector \mathbf{P}_V , the previous derivation which led us to express the vector field according to the simple formula (30) cannot be carried over to the case of finite (though macroscopic) radius. The reason is that \mathbf{P}_V does not commute with the Lorentz boost along \mathbf{v} or, otherwise stated, a Lorentz boost along a direction other than z will not transform the set of states $|h_V\rangle$ into themselves, as needed for completeness. So, one of the crucial steps in Eq. (26) no longer holds and specifically

$$\begin{aligned} & \text{tr}[\mathbf{P}_V \hat{\Lambda}^{-1} e^{-\hat{P}^0/T_0 + \gamma \omega \cdot \hat{\mathbf{J}}/T_0 + \gamma \mu \hat{Q}/T_0} \hat{\Lambda}: \hat{V}^\nu(x):] \\ & \neq \text{tr}[\mathbf{P}_V e^{-\hat{P}^0/T_0 + \gamma \omega \cdot \hat{\mathbf{J}}/T_0 + \gamma \mu \hat{Q}/T_0} \hat{\Lambda}: \hat{V}^\nu(x): \hat{\Lambda}^{-1}]. \end{aligned}$$

As a consequence, general vector and tensor fields will be more complicated than in the unphysical infinite radius case and get additional components. The most general expressions of mean value of fields in the cases of interest for the stress-energy and spin tensor will be systematically determined in the next section. The second consequence is that boundary conditions for the quantum fields must be specified at a finite radius value R , but we will see that those conditions alone cannot ensure the validity of Eqs. (11) or (12), which are local conditions.

V. TENSOR FIELDS IN AN AXISYMMETRIC SYSTEM

In this section we will write down the most general forms of vector and tensor fields in an axisymmetric system, i.e., a system with the same symmetry features of the thermodynamical rotating system at equilibrium studied in the previous section. The goal of this section is to establish the conditions, if any, to be fulfilled by the superpotential to generate a good transformation of the stress-energy and spin tensors.

A. Vector field

The decomposition of a vector field will serve as a paradigm for more complicated cases. The idea is to take a suitable tetrad of space-time-dependent orthonormal four-vectors and decompose the vector field onto this basis. The tetrad we choose is dictated by the cylindrical symmetry

$$u = (\gamma, \gamma \mathbf{v}) \quad \tau = (\gamma v, \gamma \hat{\mathbf{v}}) \quad n = (0, \hat{\mathbf{r}}) \quad k = (0, \hat{\mathbf{k}}), \quad (32)$$

where $\hat{\mathbf{r}}$ is the radial versor in cylindrical coordinates, while $\hat{\mathbf{k}}$ is the versor of the z axis, that is, the axis of the cylinder (see Fig. 1).

Because of symmetry for reflections with respect to $z = \text{const}$ planes, the most general vector field V has vanishing component on k , and therefore

$$V = A(r)u + B(r)\tau + C(r)n, \quad (33)$$

where A, B, C are scalar functions which can only depend on the radial coordinate r , owing to the cylindrical symmetry. Note the presence of two additional components with respect to the infinitely extended cylinder case in Eq. (30). For symmetry reasons the only surviving component of the field at the axis is the time component, so $B(0) = C(0) = 0$.

If the field is divergence-free, then $C(r) \equiv 0$.

B. Rank-2 antisymmetric tensor field

Any antisymmetric tensor field of rank-2 can be decomposed first as

$$A^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} X_\rho u_\sigma + Y^\mu u^\nu - Y^\nu u^\mu,$$

where

$$X^\rho = -\frac{1}{2} \epsilon^{\rho\alpha\beta\gamma} A_{\alpha\beta} u_\gamma \quad Y^\rho = A^{\rho\alpha} u_\alpha$$

and, thus, X and Y are two spacelike vector fields such that $X \cdot u = Y \cdot u = 0$. Because of the reflection symmetry with respect to $z = \text{const}$ planes, one has $A_{xz} = A_{yz} = A_{0z} = 0$ and this in turn entails that, being $u_z = 0$, the only nonvanishing component of the pseudovector X is along k . Conversely, Y is a polar vector and it has components along τ and n which must vanish in $r = 0$. Altogether

$$\begin{aligned} A^{\mu\nu} &= D(r) \epsilon^{\mu\nu\rho\sigma} k_\rho u_\sigma + E(r) (\tau^\mu u^\nu - \tau^\nu u^\mu) \\ &+ F(r) (n^\mu u^\nu - n^\nu u^\mu) \end{aligned} \quad (34)$$

with $E(0) = F(0) = 0$. Since

$$\epsilon^{\mu\nu\rho\sigma} k_\rho u_\sigma = n^\mu \tau^\nu - n^\nu \tau^\mu$$

(which can be easily checked), the expression (34) can be rewritten as:

$$\begin{aligned} A^{\mu\nu} &= D(r) (n^\mu \tau^\nu - n^\nu \tau^\mu) + E(r) (\tau^\mu u^\nu - \tau^\nu u^\mu) \\ &+ F(r) (n^\mu u^\nu - n^\nu u^\mu). \end{aligned} \quad (35)$$

C. Rank-2 symmetric tensor field

For the symmetric tensor $S^{\mu\nu}$ we will employ an iteration method in order to write down the most general decomposition. First, we project the tensor onto the u field

$$S^{\mu\nu} = G(r) u^\mu u^\nu + q^\mu u^\nu + q^\nu u^\mu + \Theta^{\mu\nu},$$

where $q \cdot u = 0$ and $\Theta^{\mu\nu} u_\nu = 0$. Then, we decompose the spacelike polar vector field q according to (33)

$$q = H(r)\tau + I(r)n,$$

with $H(0) = I(0) = 0$, and we project the tensor Θ in turn onto the vector field τ

$$\begin{aligned} S^{\mu\nu} &= G(r)u^\mu u^\nu + H(r)(\tau^\mu u^\nu + \tau^\nu u^\mu) \\ &+ I(r)(n^\mu u^\nu + n^\nu u^\mu) + J(r)\tau^\mu \tau^\nu + h^\mu \tau^\nu \\ &+ h^\nu \tau^\mu + \Xi^{\mu\nu}, \end{aligned}$$

being $h \cdot u = h \cdot \tau = 0$ (whence $h = K(r)n$ with $K(0) = 0$) and $\Xi^{\mu\nu}\tau_\nu = \Xi^{\mu\nu}u_\nu = 0$. This procedure can be iterated projecting Ξ onto n and the thus-obtained new symmetric tensor onto k . Thereby, we get

$$\begin{aligned} S^{\mu\nu} &= G(r)u^\mu u^\nu + H(r)(\tau^\mu u^\nu + \tau^\nu u^\mu) \\ &+ I(r)(n^\mu u^\nu + n^\nu u^\mu) + J(r)\tau^\mu \tau^\nu \\ &+ K(r)(n^\mu \tau^\nu + n^\nu \tau^\mu) + L(r)n^\mu n^\nu + M(r)k^\mu k^\nu. \end{aligned} \quad (36)$$

However, since $u^\mu u^\nu - \tau^\mu \tau^\nu - n^\mu n^\nu - k^\mu k^\nu = g^{\mu\nu}$ the last term can be replaced with a linear combination of all other diagonal terms plus a term in $g^{\mu\nu}$ and the most general symmetric tensor can be rewritten, after a suitable redefinition of the scalar coefficients, as

$$\begin{aligned} S^{\mu\nu} &= G(r)u^\mu u^\nu + H(r)(\tau^\mu u^\nu + \tau^\nu u^\mu) \\ &+ I(r)(n^\mu u^\nu + n^\nu u^\mu) + J(r)\tau^\mu \tau^\nu \\ &+ K(r)(n^\mu \tau^\nu + n^\nu \tau^\mu) + L(r)n^\mu n^\nu - M(r)g^{\mu\nu} \end{aligned}$$

where $H(0) = I(0) = K(0) = 0$.

D. Rank-3 spinlike tensor field

The decomposition of a rank3 tensor is carried out in an iterative way, similarly to what we have just done for the rank2 symmetric tensor. First, we project the tensor onto the vector u and, taking the antisymmetry of $\mu\nu$ indices into account, one obtains

$$\begin{aligned} \Phi^{\lambda,\mu\nu} &= u^\lambda(f^\mu u^\nu - f^\nu u^\mu) + u^\lambda \Gamma^{\mu\nu} + \Sigma^{\lambda\mu} u^\nu \\ &- \Sigma^{\lambda\nu} u^\mu + \Upsilon^{\lambda,\mu\nu}, \end{aligned} \quad (37)$$

$$\begin{aligned} \Phi^{\lambda,\mu\nu} &= D(r)(n^\mu \tau^\nu - n^\nu \tau^\mu)u^\lambda + E(r)(\tau^\mu u^\nu - \tau^\nu u^\mu)u^\lambda + F(r)(n^\mu u^\nu - n^\nu u^\mu)u^\lambda + N(r)(n^\lambda \tau^\mu - n^\mu \tau^\lambda)u^\nu \\ &- N(r)(n^\lambda \tau^\nu - n^\nu \tau^\lambda)u^\mu + P(r)\tau^\lambda(\tau^\mu u^\nu - \tau^\nu u^\mu) + Q(r)(n^\lambda \tau^\mu + n^\mu \tau^\lambda)u^\nu - Q(r)(n^\lambda \tau^\nu + n^\nu \tau^\lambda)u^\mu \\ &+ R(r)n^\lambda(n^\mu u^\nu - n^\nu u^\mu) + S(r)k^\lambda(k^\mu u^\nu - k^\nu u^\mu) + T(r)(\tau^\mu n^\nu - \tau^\nu n^\mu)n^\lambda + U(r)\tau^\lambda(\tau^\mu n^\nu - \tau^\nu n^\mu) \\ &+ V(r)k^\lambda(k^\mu n^\nu - k^\nu n^\mu) + W(r)k^\lambda(k^\mu \tau^\nu - k^\nu \tau^\mu), \end{aligned} \quad (43)$$

with $E(0) = F(0) = Q(0) = T(0) = 0$.

We are now in a position to find out the conditions to be fulfilled by the superpotential Φ to be a good transformation of the stress-energy and spin tensors in a thermodynamically equilibrated system with angular momentum, as derived at the end of Sec. II.

where all vector and tensor fields have vanishing contractions with u for any index. Particularly, using the general expressions (33) and (35), the vector field f and the antisymmetric tensor Γ read

$$f = E(r)\tau^\mu + F(r)n^\mu \quad \Gamma^{\mu\nu} = D(r)(n^\mu \tau^\nu - n^\nu \tau^\mu), \quad (38)$$

with $E(0) = F(0) = 0$. The tensor Σ can be decomposed as the sum of a symmetric and an antisymmetric part; having vanishing contractions with u , according to Eqs. (35) and (36), it can be written as

$$\begin{aligned} \Sigma^{\lambda\mu} &= N(r)(n^\lambda \tau^\mu - n^\mu \tau^\lambda) + P(r)\tau^\lambda \tau^\mu \\ &+ Q(r)(n^\lambda \tau^\mu + n^\mu \tau^\lambda) + R(r)n^\lambda n^\mu + S(r)k^\lambda k^\mu, \end{aligned} \quad (39)$$

with $Q(0) = 0$. The tensor Υ is projected in turn onto n and the above procedure is iterated. Then, similarly to Eq. (37)

$$\begin{aligned} \Upsilon^{\lambda,\mu\nu} &= \chi^{\mu\nu} n^\lambda + (h^\mu n^\nu - h^\nu n^\mu)n^\lambda + \Theta^{\lambda\mu} n^\nu \\ &- \Theta^{\lambda\nu} n^\mu + \Lambda^{\lambda,\mu\nu}, \end{aligned} \quad (40)$$

where all tensors have vanishing contractions with u and n . The antisymmetric tensor χ must be orthogonal to u and n and, therefore, according to Eq. (35), vanishes. On the other hand, the vector field h can only have nonvanishing component on τ and so $h = T(r)\tau$ with $T(0) = 0$. Finally, the tensor Θ must be orthogonal to n , besides u , hence, using Eqs. (35) and (36), can only be of the form

$$\Theta^{\lambda\mu} = U(r)\tau^\lambda \tau^\mu + V(r)k^\lambda k^\mu. \quad (41)$$

Likewise, the tensor Λ can be decomposed onto τ and, because of vanishing contractions with u and n , it can be written as

$$\Lambda^{\lambda,\mu\nu} = W(r)k^\lambda(k^\mu \tau^\nu - k^\nu \tau^\mu). \quad (42)$$

Putting together Eqs. (37)–(42), the general decomposition of a rank-3 tensor with antisymmetric $\mu\nu$ indices is obtained

Let us start from Eq. (12), which is the most constraining. Since

$$u^\lambda u^\mu - \tau^\lambda \tau^\mu - n^\lambda n^\mu - k^\lambda k^\mu = g^{\lambda\mu},$$

we can write a rank3 tensor (43) in the form of Eq. (12) as long as

$$\begin{aligned}
V(r) &= U(r) = F(r) & P(r) &= R(r) = S(r) \\
E(r) &= W(r) = -T(r) & D(r) &= N(r) = Q(r) = 0,
\end{aligned} \quad (44)$$

which are definitely nontrivial conditions. If these are fulfilled, then the superpotential (43) reduces to

$$\begin{aligned}
\Phi^{\lambda, \mu\nu} &= (F(r)n^\mu + E(r)\tau^\mu + P(r)u^\mu)g^{\lambda\nu} \\
&\quad - (F(r)n^\nu + E(r)\tau^\nu + P(r)u^\nu)g^{\lambda\mu} \\
&\equiv K^\mu g^{\lambda\nu} - K^\nu g^{\lambda\mu}.
\end{aligned}$$

Now, the field $K^\mu = F(r)n^\mu + E(r)\tau^\mu + P(r)u^\mu$ ought to be a constant one, according to Eq. (13). Since its divergence vanishes, then $F(r) = 0$ and, by using the definitions (32), we readily obtain the conditions

$$F(r) = 0 \quad P(r)/\gamma = \text{const} \quad E(r) = -P(r)\omega r. \quad (45)$$

In conclusion, only if a quantum superpotential is such that its mean value, calculated with the density operator (31), fulfills conditions (44) and (45), is the corresponding transformation (3) possible. Otherwise, the original and transformed stress-energy and spin tensors are inequivalent because they imply different values of mean energy, momentum or angular momentum densities. Since the most general form of the mean value of the superpotential, i.e., Eq. (43) is highly nontrivial, the inequivalence will occur far more often than equivalence. To demonstrate this, we will consider a specific instance involving the most familiar quantum field endowed with a spin tensor.

VI. AN EXAMPLE: THE FREE DIRAC FIELD

We now come to the possibly most significant result obtained in this work: the proof of a concrete instance of inequivalence, involving the simplest quantum field theory endowed with a spin tensor, namely, the free Dirac field.

It is well known that the from the Lagrangian density

$$\mathcal{L} = \frac{i}{2} \bar{\Psi} \gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - m \bar{\Psi} \Psi \quad (46)$$

one obtains, by means of the Noether theorem, the canonical stress-energy and spin tensors [10]

$$\begin{aligned}
\hat{T}^{\mu\nu} &= \frac{i}{2} \bar{\Psi} \gamma^\mu \overleftrightarrow{\partial}^\nu \Psi \\
\hat{S}^{\lambda, \mu\nu} &= \frac{1}{2} \bar{\Psi} \{ \gamma^\lambda, \Sigma^{\mu\nu} \} \Psi = \frac{i}{8} \bar{\Psi} \{ \gamma^\lambda, [\gamma^\mu, \gamma^\nu] \} \Psi,
\end{aligned} \quad (47)$$

where

$$\Sigma_{ij} = \epsilon_{ijk} \begin{pmatrix} \sigma_k/2 & 0 \\ 0 & \sigma_k/2 \end{pmatrix}$$

and σ_k are Pauli matrices. The spin tensor obeys the Eq. (2)

$$\partial_\lambda \hat{S}^{\lambda, \mu\nu} = \hat{T}^{\nu\mu} - \hat{T}^{\mu\nu} = \frac{i}{2} \bar{\Psi} \gamma^\nu \overleftrightarrow{\partial}^\mu \Psi - \frac{i}{2} \bar{\Psi} \gamma^\mu \overleftrightarrow{\partial}^\nu \Psi.$$

The couple of quantum tensors in (47) can be changed through the psuedo-gauge transformation in Eq. (3). Accordingly, if we take $\hat{\Phi} = \hat{S}$, namely, the superpotential as the original spin tensor itself, a symmetrized stress-energy tensor and a vanishing spin tensor are obtained

$$\hat{T}'^{\mu\nu} = \frac{i}{4} [\bar{\Psi} \gamma^\mu \overleftrightarrow{\partial}^\nu \Psi + \bar{\Psi} \gamma^\nu \overleftrightarrow{\partial}^\mu \Psi] \quad \hat{S}'^{\lambda, \mu\nu} = 0. \quad (48)$$

This transformation is well known as Belinfante's symmetrization procedure.

One may wonder whether these tensors, fulfilling continuity equations, still exist in a bounded region breaking the global translational and Lorentz symmetry, such as our cylinder with finite radius; or if, because of the boundary, they get additional terms with respect to the usual form. The problem of Dirac field with boundary has been tackled and solved by the authors of the MIT bag model [11]. First of all, it should be pointed out that the continuity Eqs. (41) certainly apply to tensors (47) and (48) on shell, i.e., for fields obeying the free Dirac equation within the cylinder. Furthermore, it is possible to find suitable boundary conditions, discussed in the next subsection, such that the fluxes (6) vanish, as needed, without introducing an *ad hoc* discontinuity in the Dirac field. Thereby, the stress-energy and spin tensors retain the same form as in the usual no-boundary case and the integrals over the bounded region of the time components have the same physical meaning of conserved generators. It is also possible to derive field equations and canonical tensors (47) from an action, which is shown in Appendix A.

The spin tensor in Eq. (47) has a remarkable feature which makes it easier to check the equivalence of the two couples in Eqs. (47) and (48): because of the special properties of gamma matrices, the spin tensor is also antisymmetric in the first two indices

$$\hat{S}^{\lambda, \mu\nu} = -\hat{S}^{\mu, \lambda\nu} \quad (49)$$

and thus the mean value of this tensor is greatly simplified. The antisymmetry in the indices (λ, μ) dictates that all coefficients of symmetric $\lambda\mu$ terms of the general form of this kind of tensor found in Eq. (43) vanish

$$\begin{aligned}
E(r) &= F(r) = P(r) = Q(r) = R(r) = S(r) = T(r) \\
&= U(r) = V(r) = W(r) = 0
\end{aligned}$$

and that $D(r) = N(r)$, so that \mathcal{S} is simply given by:

$$\begin{aligned}
\mathcal{S}^{\lambda, \mu\nu} &= D(r) [(n^\mu \tau^\nu - n^\nu \tau^\mu) u^\lambda + (n^\lambda \tau^\mu - n^\mu \tau^\lambda) u^\nu \\
&\quad - (n^\lambda \tau^\nu - n^\nu \tau^\lambda) u^\mu]
\end{aligned} \quad (50)$$

and it is described by just one unknown radial function $D(r)$. Therefore, according to the conditions (44), the Belinfante tensors (48) are equivalent to the canonical ones (47) only if $D(r) = 0$, i.e., only if the spin tensor has a vanishing mean value.

For $\lambda = 0$, Eq. (50) reads:

$$\mathcal{S}^{0,\mu\nu} = D(r)[(n^\mu \tau^\nu - n^\nu \tau^\mu)u^0 - \tau^0(n^\mu u^\nu - n^\nu u^\mu)]$$

and, because of the antisymmetry, the only nonvanishing components are those with both μ and ν equal to 1, 2, 3, indices that we denote with i, j . We can then write, using (32)

$$\begin{aligned} \mathcal{S}^{0,ij} &= D(r)[(n^i \tau^j - n^j \tau^i)u^0 - \tau^0(n^i u^j - n^j u^i)] \\ &= D(r)[\gamma^2(n^i \hat{v}^j - n^j \hat{v}^i) - \gamma^2 v^2(n^i \hat{v}^j - n^j \hat{v}^i)] \\ &= D(r)(n^i \hat{v}^j - n^j \hat{v}^i) = D(r)\epsilon_{ijk} \hat{k}^k, \end{aligned} \quad (51)$$

Therefore, as expected, the time part of the spin tensor, contributing to the angular momentum density, is equivalent to a pseudovector field $\mathbf{D}(r)$ directed along z axis.

According to Eq. (8), the variation of energy-momentum density reads

$$\begin{aligned} \frac{1}{2} \partial_\alpha (\Phi^{\alpha,0\nu} - \Phi^{0,\alpha\nu} - \Phi^{\nu,\alpha 0}) &= \frac{1}{2} \partial_\alpha (\mathcal{S}^{\alpha,0\nu} - \mathcal{S}^{0,\alpha\nu} - \mathcal{S}^{\nu,\alpha 0}) \\ &= -\frac{1}{2} \partial_\alpha \mathcal{S}^{0,\alpha\nu}, \end{aligned} \quad (52)$$

which implies at once that the energy density is unchanged because $\mathcal{S}^{0,\alpha 0} = 0$ in view of (49), whereas the momentum density varies by a derivative. Using (51) and recalling the expression of curl in cylindrical coordinates

$$\begin{aligned} T_{\text{Belinfante}}^{0i} &= T_{\text{canonical}}^{0i} - \frac{1}{2} \partial_\alpha \mathcal{S}^{0,\alpha i} \\ &= T_{\text{canonical}}^{0i} - \frac{1}{2} \partial_\alpha \epsilon_{\alpha ik} D(r) \hat{k}^k \\ &= T_{\text{canonical}}^{0i} + \frac{1}{2} (\text{rot} \mathbf{D})^i \\ &= T_{\text{canonical}}^{0i} - \frac{1}{2} \frac{dD(r)}{dr} \hat{v}^i. \end{aligned} \quad (53)$$

Note that this last equation implies that the mean value of the canonical stress-energy tensor of the Dirac field has a nontrivial antisymmetric part if $D'(r) \neq 0$ as, according to (2)

$$\partial_\alpha \mathcal{S}^{0,\alpha i} = -\partial_\alpha \mathcal{S}^{\alpha,0i} = T^{0i} - T^{i0}.$$

Now we can write the angular momentum density variation by thermal-averaging Eq. (4) with $\hat{\Phi} = \hat{\mathcal{S}}$

$$\begin{aligned} \mathcal{J}_{\text{Belinfante}}^{0,\mu\nu} &= \mathcal{J}_{\text{canonical}}^{0,\mu\nu} + \frac{1}{2} \partial_\alpha [x^\mu (\Phi^{\alpha,0\nu} - \Phi^{0,\alpha\nu} - \Phi^{\nu,\alpha 0}) \\ &\quad - x^\nu (\Phi^{\alpha,0\mu} - \Phi^{0,\alpha\mu} - \Phi^{\mu,\alpha 0})] \\ &= \frac{1}{2} [x^\mu \partial_\alpha (\Phi^{\alpha,0\nu} - \Phi^{0,\alpha\nu} - \Phi^{\nu,\alpha 0}) \\ &\quad + \Phi^{\mu,0\nu} - \Phi^{0,\mu\nu} - \Phi^{\nu,\mu 0} - (\mu \leftrightarrow \nu)]. \end{aligned}$$

The sum of all terms linear in the superpotential returns a $-\Phi^{0,\mu\nu}$ (see Eq. (12)) while for the derivative terms we can use Eq. (52)

$$\begin{aligned} \mathcal{J}_{\text{Belinfante}}^{0,\mu\nu} &= \mathcal{J}_{\text{canonical}}^{0,\mu\nu} + \frac{1}{2} [x^\mu \partial_\alpha (\Phi^{\alpha,0\nu} - \Phi^{0,\alpha\nu} - \Phi^{\nu,\alpha 0}) \\ &\quad - (\mu \leftrightarrow \nu)] - \Phi^{0,\mu\nu} \\ &= \mathcal{J}_{\text{canonical}}^{0,\mu\nu} - \frac{1}{2} [x^\mu \partial_\alpha \mathcal{S}^{0,\alpha\nu} - x^\nu \partial_\alpha \mathcal{S}^{0,\alpha\mu}] - \mathcal{S}^{0,\mu\nu}. \end{aligned} \quad (54)$$

Therefore, by plugging the expression of the mean value of the spin tensor in Eq. (51), the angular momentum pseudovector corresponding to the angular momentum density in (54) can be finally written

$$\begin{aligned} \mathcal{J}_{\text{Belinfante}} &= \mathcal{J}_{\text{canonical}} - \frac{1}{2} \left(\mathbf{x} \times \frac{dD(r)}{dr} \hat{\mathbf{v}} \right) - \mathbf{D}(r) \\ &= \mathcal{J}_{\text{canonical}} - \left(\frac{1}{2} r \frac{dD(r)}{dr} + D(r) \right) \hat{\mathbf{k}}. \end{aligned} \quad (55)$$

In order for the canonical and Belinfante tensors to be equivalent, as has been mentioned and as it is apparent from Eqs. (53) and (55), the function $D(r)$ ought to vanish everywhere. If $D'(r) \neq 0$, the two stress-energy tensors give two different momentum densities and are thus inequivalent; if, on top of that, $D'(r) \neq -2D(r)/r$, then the angular momentum densities are inequivalent as well. In the rest of this section we will prove that this is exactly the case, i.e., neither of these conditions is fulfilled. In order to show that this is not a problem arising from peculiar values of the field at the boundary, we will conservatively enforce boundary conditions such that the *total* energy, momentum and angular momentum operators obtained by integrating the fields within the cylinder are invariant by transformation (3). Note that for this to be true, in the case under consideration, it is necessary that the function $D(r)$ vanishes at the boundary, i.e., $D(R) = 0$, because the difference between total angular momenta is

$$\begin{aligned} &\int_V d^3x (\mathcal{J}_{\text{Belinfante}} - \mathcal{J}_{\text{canonical}}) \\ &= - \int_V dz d\varphi dr \left(\frac{1}{2} r \frac{dD(r)}{dr} + D(r) \right) \hat{\mathbf{k}} \\ &= -2\pi \int_{-\infty}^{+\infty} dz \int_0^R dr \frac{d}{dr} \left(\frac{r^2}{2} D(r) \right) \hat{\mathbf{k}}. \end{aligned}$$

Thereby, we will demonstrate that, although the stress-energy and spin tensors in (47) and (48) lead to the same quantum generators, their respective mean densities are inconsistent. The problem we are facing is then to solve the Dirac equation within a cylinder with finite radius and second-quantize the field.

A. The Dirac field in a cylinder

The problem of the Dirac field within a cylinder with finite radius has been tackled by several authors in the

context of the MIT bag model [12]. We first stress that, as we have done thus far, we take the viewpoint of an external inertial observer in Minkowski space-time, seeing the spinning cylinder globally at rest. This observer can use either Cartesian coordinates or cylindrical coordinates to describe the system, the former being certainly more convenient to express tensor fields components while the latter are fit to solve the Dirac equation, as we will see. The most important issue in searching for a solution of this problem is the choice of appropriate boundary conditions, not an easy task because the Dirac equation is a first-order partial differential equation. The authors of the bag model [11] have shown that the following condition³

$$i\hat{n}\Psi(R) = in^\mu\gamma_\mu\Psi(R) = -\Psi(R) \quad (56)$$

ensures the vanishing of the fluxes in Eq. (6) through the border and allows nontrivial solutions of the Dirac equation within the cylinder (see Fig. 1) which, however, extend to the whole space without any discontinuity in the field. These boundary conditions, whence the whole solution, are not affected by the rotation of a possible material support defining the outer surface of the cylinder, as, for the inertial observer, the motion transforms the boundary into itself. The above equation, in the nonrelativistic limit, entails the vanishing of the “large” components of the Dirac field at the boundary, that is, one is left with the Schrödinger equation with Dirichlet boundary conditions. The Eq. (56) implies that $\bar{\Psi}(R)\Psi(R) = 0$ at the boundary [11], whence the vanishing of the outward current flux $j^\mu(R)n_\mu$. Moreover, since $\bar{\Psi}\Psi(R) = 0$, for any value of φ , z , t , the outer surface of the cylinder must be such that $\partial^\mu\bar{\Psi}\Psi|_R = \hat{\Xi}(R)n^\mu$ or

$$n^\mu\partial_\mu(\bar{\Psi}\Psi)(R) = \frac{\partial}{\partial r}\bar{\Psi}\Psi|_{r=R} = -\hat{\Xi}(R). \quad (57)$$

Thus, the flux of energy-momentum of the canonical tensor at the boundary is vanishing because, using Eq. (47), (56), and (57)

$$\begin{aligned} \int_{\partial V} dS\hat{T}^{\mu\nu}n_\mu &= \frac{i}{2} \int_{\partial V} dS\bar{\Psi}\hat{n}\partial^\nu\Psi - \partial^\nu\bar{\Psi}\hat{n}\Psi \\ &= \frac{1}{2} \int_{\partial V} dS\bar{\Psi}\partial^\nu\Psi + \partial^\nu\bar{\Psi}\Psi \\ &= \frac{1}{2} \int_{\partial V} dS\partial^\nu(\bar{\Psi}\Psi) \\ &= -\frac{\hat{\Xi}(R)}{2} \int_{\partial V} dSn^\nu = 0. \end{aligned}$$

³Actually, in the paper [11], the boundary condition chosen is $i\hat{n}\Psi(R) = \Psi(R)$, but the change of sign is indeed immaterial.

Likewise, for the orbital part of the angular momentum flux

$$\begin{aligned} &\int_{\partial V} dSx^\mu\hat{T}^{\lambda\nu}n_\lambda - (\mu \leftrightarrow \nu) \\ &= \frac{1}{2} \int_{\partial V} dSx^\mu\partial^\nu(\bar{\Psi}\Psi) - (\mu \leftrightarrow \nu) \\ &= \frac{\hat{\Xi}(R)}{2} \int_{\partial V} dS(x^\mu n^\nu - x^\nu n^\mu) = 0, \end{aligned}$$

where the last integral vanishes because of the geometrical symmetry $z \rightarrow -z$. Finally, the flux of the spin tensor also vanishes at the boundary because, using (47) and (56)

$$\begin{aligned} n_\lambda\hat{S}^{\lambda,\mu\nu}(R) &= \frac{1}{2}(\bar{\Psi}\hat{n}\Sigma^{\mu\nu}\Psi + \bar{\Psi}\Sigma^{\mu\nu}\hat{n}\Psi) \\ &= -\frac{i}{2}(\bar{\Psi}\Sigma^{\mu\nu}\Psi - \bar{\Psi}\Sigma^{\mu\nu}\Psi) = 0. \end{aligned} \quad (58)$$

Therefore, the Eq. (6) applies and the integrals

$$\hat{P}^\nu = \int_V d^3x\hat{T}^{0\nu} \quad \hat{J}^{\mu\nu} = \int_V d^3x\hat{\mathcal{J}}^{0,\mu\nu} \quad (59)$$

are conserved. Since, we also have, from the Lagrangian, the usual anticommutation relations at equal times

$$\begin{aligned} \{\Psi_a(t, \mathbf{x}), \Psi_b^\dagger(t, \mathbf{x}')\} &= \delta_{ab}\delta^3(\mathbf{x} - \mathbf{x}') \\ \{\Psi_a(t, \mathbf{x}), \Psi_b(t, \mathbf{x}')\} &= \{\Psi_a^\dagger(t, \mathbf{x}), \Psi_b^\dagger(t, \mathbf{x}')\} = 0, \end{aligned}$$

it is easy to check that the conserved Hamiltonian $i/2 \int d^3x\Psi^\dagger\vec{\partial}_t\Psi$ is indeed, as expected, the generator of time translations, i.e.,

$$[\hat{H}, \Psi] = -i\frac{\partial}{\partial t}\Psi \quad [\hat{H}, \Psi^\dagger] = -i\frac{\partial}{\partial t}\Psi^\dagger$$

and, therefore, putting together the above equation with Eqs. (59) and (47) we conclude that

$$[\hat{H}, \hat{J}_i] = 0 \quad (60)$$

for the case under examination.

The complete solution of the free Dirac equation for a massive particle in a longitudinally unlimited cylinder with finite transverse radius, with boundary conditions of the kind (56) has been obtained by Bezerra de Mello *et al.* in Ref. [13] and we summarize it here. In a longitudinally unlimited cylinder, but with finite transverse radius R , the field is expanded in terms of eigenfunctions of the longitudinal momentum, third component of angular momentum, transverse momentum and an additional “spin” quantum number [13]. The relevant quantum numbers $\mathbf{n} = (p_z, M, \zeta_{(M,\xi,l)}, \xi)$ take on continuous (p_z) and discrete values ($M, \zeta_{(M,\xi,l)}, \xi$). The third component of the angular momentum M takes on all semi-integer values $\pm 1/2, \pm 3/2, \dots$; the spin quantum number ξ can be ± 1 and the transverse momentum quantum number

$$\zeta_{(M,\xi,l)} = p_{Tl}R \quad (61)$$

takes on discrete values which are zeroes, sorted in ascending order with the label $l = 1, 2, \dots$ and depending on M and ξ , of the equation

$$J_{|M-(1/2)|}(p_T R) + \text{sgn}(M) b_\xi^{(+)} J_{|M+(1/2)|}(p_T R) = 0, \quad (62)$$

where J are Bessel functions and

$$b_\xi^{(\pm)} = \frac{\pm m + \xi m_T}{p_T}, \quad (63)$$

m being the mass and

$$m_T = \sqrt{p_T^2 + m^2}$$

the transverse mass⁴; we note in passing that $b_\xi^{(+)} = 1/b_\xi^{(-)}$. The Dirac field itself can be written as an expansion

$$\Psi(x) = \sum_{\mathbf{n}} U_{\mathbf{n}}(x) a_{\mathbf{n}} + V_{\mathbf{n}}(x) b_{\mathbf{n}}^\dagger, \quad (64)$$

where $a_{\mathbf{n}}$ and $b_{\mathbf{n}}$ are destruction operators of quanta \mathbf{n} while

$$\sum_{\mathbf{n}} \equiv \sum_M \sum_{\xi=\pm 1} \sum_{\zeta(M,\xi,l)} \int_{-\infty}^{+\infty} dp_z = \sum_M \sum_{\xi=-1,1} \sum_{l=1}^{\infty} \int_{-\infty}^{+\infty} dp_z.$$

The eigenspinors $U_{\mathbf{n}}$ and $V_{\mathbf{n}}$ read, in the Dirac representation of the γ matrices and in cylindrical coordinates (t, r, φ, z)

$$U_{\mathbf{n}}(x) = C_{\mathbf{n}} \begin{pmatrix} J_{|M(1/2)|}(p_T r) \\ i \text{sgn}(M) \kappa_\xi b_\xi^{(+)} J_{|M+(1/2)|}(p_T r) e^{i\varphi} \\ \kappa_\xi J_{|M-(1/2)|}(p_T r) \\ -i \text{sgn}(M) b_\xi^{(+)} J_{|M+(1/2)|}(p_T r) e^{i\varphi} \end{pmatrix} \\ \times \frac{1}{\sqrt{2\pi}} e^{i[(M-1/2)\varphi + p_z z - \varepsilon t]} \\ V_{\mathbf{n}}(x) = \frac{C_{\mathbf{n}}}{b_\xi^{(-)}} \begin{pmatrix} J_{|M+(1/2)|}(p_T r) \\ i \text{sgn}(M) \kappa_\xi b_\xi^{(-)} J_{|M-(1/2)|}(p_T r) e^{i\varphi} \\ \kappa_\xi J_{|M+(1/2)|}(p_T r) \\ -i \text{sgn}(M) b_\xi^{(-)} J_{|M-(1/2)|}(p_T r) e^{i\varphi} \end{pmatrix} \\ \times \frac{1}{\sqrt{2\pi}} e^{-i[(M+1/2)\varphi + p_z z - \varepsilon t]} \quad (65)$$

with

$$\kappa_\xi = \frac{\varepsilon + \xi \sqrt{\varepsilon^2 - p_z^2}}{p_z}$$

and $\varepsilon = \sqrt{p_z^2 + p_{Tl}^2 + m^2}$ being the energy. The eigenspinors (65) are normalized so as to

$$\int_V d^3x \Psi^\dagger \Psi = \sum_{\mathbf{n}} a_{\mathbf{n}}^\dagger a_{\mathbf{n}} + b_{\mathbf{n}} b_{\mathbf{n}}^\dagger,$$

⁴In the rest of this section the symbol p_{Tl} stands for a discrete variable taking on (M, ξ, l) -dependent values given by the Eq. (61) or, later on, by Eq. (79).

that is, with

$$\int_V d^3x U_{\mathbf{n}}^\dagger(x) U_{\mathbf{n}'}(x) = \int_V d^3x V_{\mathbf{n}}^\dagger(x) V_{\mathbf{n}'}(x) = \delta_{\mathbf{n}\mathbf{n}'} \quad (66) \\ \int_V d^3x U_{\mathbf{n}}^\dagger(x) V_{\mathbf{n}'}(x) = 0,$$

being $\delta_{\mathbf{n}\mathbf{n}'} = \delta_{MM'} \delta_{\xi\xi'} \delta_{ll'} \delta(p_z - p_z')$ and the anticommutation relations of creation and destruction operators

$$\{a_{\mathbf{n}}, a_{\mathbf{n}'}^\dagger\} = \{b_{\mathbf{n}}, b_{\mathbf{n}'}^\dagger\} = \delta_{\mathbf{n}\mathbf{n}'} \quad \{a_{\mathbf{n}}, b_{\mathbf{n}'}\} = \{a_{\mathbf{n}}^\dagger, b_{\mathbf{n}'}\} = 0. \quad (67)$$

The normalization coefficient in (65) obtained from the condition (66) reads [13]

$$(C_{\mathbf{n}})^{-2} = 2\pi R^2 J_{|M-(1/2)|}^2(p_{Tl} R) \frac{\kappa_\xi^2 + 1}{p_{Tl}^2 R^2} \\ \times (2R^2 m_{Tl}^2 + 2\xi M R m_{Tl} + mR). \quad (68)$$

B. Proving the inequivalence

For what we have seen so far, from a purely quantum field theoretical point of view, the Belinfante tensors (48) for the Dirac field in the cylinder could be regarded as equivalent to the canonical ones in Eq. (47) because they give, once integrated, the same generators (59). This happens because the condition (7) is met for $\hat{\Phi} = \hat{S}$ (what follows from Eq. (58)) and this implies, taking Eq. (49) into account, that all the integrands of (7) vanish at the boundary. Yet, these two set of tensors are thermodynamically inequivalent because, as it will be shown hereafter, it turns out that, using Eq. (51),

$$S^{0,ij} = \frac{1}{2} \text{tr}_V(\hat{\rho}: \bar{\Psi}\{\gamma^0, \Sigma^{ij}\}\Psi:) \\ = D(r) \epsilon_{ijk} k^k \neq 0 \Rightarrow D(r) \neq 0 \quad (69)$$

at some $r \neq R$ (we have used the Eq. (51)), with $\hat{\rho}$ written in Eq. (18). This will be enough to conclude that either the energy-momentum or the angular momentum densities or both have different values for different sets of quantum tensors, as previously discussed. Note that the boundary condition (58) together with the general expression of the mean value of the spin tensor (50) implies that $D(R) = 0$, i.e., its vanishing at the boundary.

We can rewrite the inequality (69) by taking advantage of the commutation relation

$$[\gamma^\lambda, \Sigma^{\mu\nu}] = i g^{\lambda\mu} \gamma^\nu - i g^{\lambda\nu} \gamma^\mu,$$

implying

$$S^{0,ij} = \text{tr}_V[\hat{\rho}: \Psi^\dagger \Sigma^{ij} \Psi:] - i g^{0i} \text{tr}[\hat{\rho}: \bar{\Psi} \gamma^j \Psi:] \\ + i g^{0i} \text{tr}[\hat{\rho}: \bar{\Psi} \gamma^\mu \Psi:] \\ = \text{tr}_V[\hat{\rho}: \Psi^\dagger \Sigma^{ij} \Psi:] \neq 0$$

or, equivalently

$$D(r) = \frac{1}{2} \epsilon_{ij3} \mathcal{S}^{0,ij} = \frac{1}{2} \text{tr}[\hat{\rho}:\Psi^\dagger \epsilon_{ij3} \Sigma^{ij} \Psi:]$$

$$\equiv \text{tr}[\hat{\rho}:\Psi^\dagger \Sigma_3 \Psi:] \neq 0, \quad (70)$$

where the indices i, j can only take on the value 1 or 2. In the above equation and henceforth, we can take the Heisenberg field operators at some fixed time $t = 0$ because of the stationarity of density operator $\hat{\rho}$. Hence we just need to show that

$$\text{tr}_V[\hat{\rho}:\Psi^\dagger(0, \mathbf{x}) \Sigma_z \Psi(0, \mathbf{x}):] \neq 0, \quad (71)$$

with

$$\Sigma_z = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (72)$$

for some point \mathbf{x} within the cylinder and our goal is achieved.

To calculate the mean value of the spin density in Eq. (71), we start by observing that (see Appendix B for the proof)

$$\text{tr}[\hat{\rho} a_{\mathbf{n}}^\dagger a_{\mathbf{n}'}] = \frac{\delta_{\mathbf{n}\mathbf{n}'}}{e^{(\varepsilon - M\omega + \mu)/T} + 1}$$

$$\text{tr}[\hat{\rho} b_{\mathbf{n}}^\dagger b_{\mathbf{n}'}] = \frac{\delta_{\mathbf{n}\mathbf{n}'}}{e^{(\varepsilon - M\omega - \mu)/T} + 1} \quad (73)$$

$$\text{tr}[\hat{\rho} a_{\mathbf{n}}^\dagger b_{\mathbf{n}'}] = \text{tr}[\hat{\rho} a_{\mathbf{n}} b_{\mathbf{n}'}] = 0,$$

which allows us to work it out by plugging in there the field expansion (64)

$$\text{tr}_V[\hat{\rho}:\Psi^\dagger(0, \mathbf{x}) \Sigma_z \Psi(0, \mathbf{x}):]$$

$$= \sum_{\mathbf{n}} \frac{1}{e^{(\varepsilon - M\omega + \mu)/T} + 1} [U_{\mathbf{n}}^\dagger(x) \Sigma_z U_{\mathbf{n}}(x)]$$

$$- \frac{1}{e^{(\varepsilon - M\omega - \mu)/T} + 1} [V_{\mathbf{n}}^\dagger(x) \Sigma_z V_{\mathbf{n}}(x)]. \quad (74)$$

where we have taken into account that the normal ordering of fermions is such that $:b_{\mathbf{n}} b_{\mathbf{n}'}^\dagger := -b_{\mathbf{n}'}^\dagger b_{\mathbf{n}}$. By using Eq. (65) and (72),

$$U_{\mathbf{n}}^\dagger(x) \Sigma_z U_{\mathbf{n}}(x) = \frac{C_{\mathbf{n}}^2}{4\pi} [J_{|M-(1/2)|}^2(p_{Tl}r) - \kappa_\xi^2 b_\xi^{(+2)} J_{|M+(1/2)|}^2(p_{Tl}r) + \kappa_\xi^2 J_{|M-(1/2)|}^2(p_{Tl}r) - b_\xi^{(+2)} J_{|M+(1/2)|}^2(p_{Tl}r)]$$

$$= \frac{C_{\mathbf{n}}^2}{4\pi} [J_{|M-(1/2)|}^2(p_{Tl}r) - b_\xi^{(+2)} J_{|M+(1/2)|}^2(p_{Tl}r)] (1 + \kappa_\xi^2)$$

$$V_{\mathbf{n}}^\dagger(x) \Sigma_z V_{\mathbf{n}}(x) = \frac{C_{\mathbf{n}}^2}{4\pi} b_\xi^{(-2)} [J_{|M+(1/2)|}^2(p_{Tl}r) - b_\xi^{(-2)} J_{|M-(1/2)|}^2(p_{Tl}r)] (1 + \kappa_\xi^2)$$

$$= \frac{C_{\mathbf{n}}^2}{4\pi} [-J_{|M-(1/2)|}^2(p_{Tl}r) + b_\xi^{(+2)} J_{|M+(1/2)|}^2(p_{Tl}r)] (1 + \kappa_\xi^2) = -U_{\mathbf{n}}^\dagger(x) \Sigma_z U_{\mathbf{n}}(x),$$

hence, by using Eqs. (68) and (70), we can rewrite Eq. (74) as

$$\text{tr}_V[\hat{\rho}:\Psi^\dagger(0, \mathbf{x}) \Sigma_z \Psi(0, \mathbf{x}):] = D(r)$$

$$= \sum_M \sum_{\xi=\pm 1} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} dp_z \left[\frac{1}{e^{(\varepsilon - M\omega + \mu)/T} + 1} + \frac{1}{e^{(\varepsilon - M\omega - \mu)/T} + 1} \right]$$

$$\times \frac{p_{Tl}^2 [J_{|M-(1/2)|}^2(p_{Tl}r) - b_\xi^{(+2)} J_{|M+(1/2)|}^2(p_{Tl}r)]}{8\pi^2 R J_{|M-(1/2)|}^2(p_{Tl}r) (2Rm_{Tl}^2 + 2\xi M m_{Tl} + m)}. \quad (75)$$

The mean value of the spin tensor is therefore given by the sum of a particle and an antiparticle term which are equal only for $\mu = 0$. As expected, it vanishes for $r = R$ in view of the Eq. (62), yet our goal is to show that it is non-vanishing at some point \mathbf{x} not belonging to the boundary. It is worth pointing out that, if this is the case, the spin tensor has a *macroscopic* value because, as is apparent from (75), it is proportional to the number density (in phase space) of quanta $1/\exp[(\varepsilon - M\omega \pm \mu)/T + 1]$.

It is most convenient to consider a point belonging to the rotation axis, i.e., with radial coordinate $r = 0$ because Bessel functions of all orders but 0 vanish therein. By working out the Eq. (75), it can then be shown that $D(0) = 0$ for $\omega = 0$ and that it is an increasing function of ω/T thereafter (see Appendix C). It then follows that

$$D(0) \neq 0 \quad \text{for } \omega/T > 0$$

and therefore the inequality (71) must be true for small, yet finite, values of ω/T around the rotation axis. We note in passing that for $\omega = 0$ the whole function $D(r)$ must be vanishing because of symmetry reasons. In fact, if $\omega = 0$, the density operator (31) enjoys an additional symmetry, that is, the rotation of an angle π around any axis orthogonal to the cylinder axis, say $R_2(\pi)$. This transformation corresponds to flipping over the cylinder, which leaves the system invariant provided that $\omega = 0$, and has the consequence that any pseudovector field directed along the axis must vanish.

Moreover, it is easy to show, again by using Eq. (75), that the derivative of the function $D(r)$ vanishes in $r = 0$ for it is proportional to terms, with $N \geq 0$,

$$2J_N(0)J'_N(0) = J_N(0)(J_{N-1}(0) - J_{N+1}(0)),$$

which all vanish because of the known properties of Bessel functions. Hence, the mean angular momentum density in $r = 0$ differs between canonical and Belinfante tensors, i.e., rewriting the Eq. (55) for $r = 0$,

$$\mathcal{J}_{\text{Belinfante}}(0) = \mathcal{J}_{\text{canonical}}(0) - D(0)\hat{\mathbf{k}},$$

where $D(0)$ is finite for finite ω and positive. Thus, the Belinfante angular momentum density is lower than the canonical one by some finite and macroscopic amount.

We point out that, had we used the definition of mean values (1) without normal ordering, this conclusion would be unaffected. Indeed, the spin tensor (47) is a bilinear in the fields and therefore the difference between the two definitions is a vacuum expectation value of the spin tensor

$$\text{tr}(\hat{\rho}:\hat{\mathcal{S}}:) = \text{tr}(\hat{\rho}\hat{\mathcal{S}}) - \langle 0|\hat{\mathcal{S}}|0\rangle. \quad (76)$$

Vectorial irreducible parts of the spin tensor, such as $\hat{\mathcal{S}}^{0,ij}$, have a vanishing vacuum expectation value if the vacuum is invariant under general rotations. The vacuum of the free

Dirac field in the cylinder—defined by $a_{\mathbf{n}}|0\rangle = 0$ —is indeed rotationally invariant. If degenerate vacua existed, the commutation of the Hamiltonian with angular momentum operators that was shown before (see Eq. (60)) would ensure that they belong to some irreducible representation of the SU(2) group. However, for the free Dirac field in the cylinder, the angular momentum operator along the z axis turns out to be

$$\hat{J}_z = \sum_{\mathbf{n}} M(a_{\mathbf{n}}^\dagger a_{\mathbf{n}} + b_{\mathbf{n}}^\dagger b_{\mathbf{n}})$$

so that $\hat{J}_z|0\rangle = 0$ on all possible degenerate vacua. This means that the only possible multiplet is one-dimensional and, thereby, the vacuum is nondegenerate and the second term in the Eq. (76) vanishes.

C. The nonrelativistic limit

It would be very interesting to calculate the function $D(r)$ numerically to “see” the difference between the Belinfante and the canonical tensors and to make sure that this difference is not a rapidly oscillating function on a microscopic scale, which would render the macroscopic observation of the difference impossible. This is, though, very hard in the fully relativistic case but relatively easy in the nonrelativistic limit $m/T \gg 1$ because in this case the Eq. (62) yielding the quantized transverse momenta reduces to the vanishing of one single Bessel function. This happens because, in the nonrelativistic limit,

$$b_\xi^{(+)} = \frac{\xi m_T + m}{p_T} = \begin{cases} \frac{m_T + m}{p_T} \simeq \frac{2m + p_T^2/2m}{p_T} \simeq \frac{2m}{p_T} \gg 1 & \text{for } \xi = 1 \\ \frac{m - m_T}{p_T} \simeq \frac{p_T^2/2m}{p_T} \simeq \frac{p_T}{2m} \ll 1 & \text{for } \xi = -1 \end{cases} \quad (77)$$

so that the Eq. (62) in fact reduces to

$$\begin{cases} J_{|M+(1/2)|}(p_T R) = \text{sgn}(M) \frac{p_T}{2m} J_{|M-(1/2)|}(p_T R) \simeq 0 & \text{for } \xi = 1 \\ J_{|M-(1/2)|}(p_T R) = \text{sgn}(M) \frac{p_T}{2m} J_{|M+(1/2)|}(p_T R) \simeq 0 & \text{for } \xi = -1 \end{cases} \quad (78)$$

Altogether, we can solve the equation $J_L(p_T R) = 0$ for all integers $L = M + \xi/2$ and take the quantized transverse momenta

$$p_{Tl} = \frac{\zeta_{L,l}}{R}, \quad (79)$$

where $\zeta_{L,l} = 1, 2, \dots$ are now the familiar zeroes of the Bessel function of integer order L .

We can now separate the particle and antiparticle terms in the Eq. (75)

$$D(r)^\pm = \sum_M \sum_{\xi=\pm 1} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} dp_z \frac{1}{e^{(\varepsilon - M\omega \pm \mu)/T} + 1} \frac{p_{Tl}^2 [J_{|M-(1/2)|}^2(p_{Tl} r) - b_\xi^{(+)} J_{|M+(1/2)|}^2(p_{Tl} r)]}{8\pi^2 R J_{|M-(1/2)|}^2(p_{Tl} R) (2Rm_{Tl}^2 + 2\xi M m_{Tl} + m)},$$

with $D(r) = D(r)^+ + D(r)^-$. In the nonrelativistic limit one has

$$2Rm_T^2 + 2\xi Mm_T + m \simeq 2Rm^2 + 2\xi Mm + m \simeq 2Rm^2, \quad (80)$$

where the last approximation is due to the obvious assumption $Rm \gg 1$ and that the term $|\xi Mm|$ can be comparable to Rm^2 only if $|M|$ is very large. However, terms with large $|M|$ are either suppressed by the exponential $\exp[\omega M/T]$ or by the Bessel functions, which effectively implements the semiclassical equality $M \approx Rp_T$; since nonrelativistically $Rm^2 \gg Rp_T m \approx |M|m$, the approximation (80) is justified. We then calculate the terms with $\xi = 1$ and

$\xi = -1$ in the sum in Eq. (78) separately. For $\xi = 1$ one sets $M + 1/2 = L$ and writes the integrand of Eq. (75), including approximation (80) and taking into account (77),

$$\frac{1}{e^{(\varepsilon-L\omega+\omega/2\pm\mu)/T} + 1} \frac{p_T^2}{16\pi^2 R^2 m^2} \frac{J_{|L-1|}^2(p_{Tl}r) - \frac{4m^2}{p_T^2} J_{|L|}^2(p_{Tl}r)}{J_{|L-1|}^2(p_{Tl}R)} \simeq -\frac{1}{e^{(\varepsilon-L\omega+\omega/2\pm\mu)/T} + 1} \frac{1}{4\pi^2 R^2} \frac{J_{|L|}^2(p_{Tl}r)}{J_{|L-1|}^2(p_{Tl}R)}, \quad (81)$$

where p_{Tl} is a solution of the first equation in (78). Similarly, for $\xi = -1$ one sets $M - 1/2 = L$ and obtains, by using the second of the Eqs. (78),

$$\frac{1}{e^{(\varepsilon-L\omega+\omega/2\pm\mu)/T} + 1} \frac{p_T^2}{16\pi^2 R^2 m^2} \frac{J_{|L|}^2(p_{Tl}r) - \frac{p_{Tl}^2}{4m^2} J_{|L+1|}^2(p_{Tl}r)}{J_{|L|}^2(p_{Tl}R)} \simeq \frac{1}{e^{(\varepsilon-L\omega-\omega/2\pm\mu)/T} + 1} \frac{p_T^2}{16\pi^2 R^2 m^2} \frac{J_{|L|}^2(p_{Tl}r)}{\frac{p_{Tl}^2}{4m^2} J_{|L+1|}^2(p_{Tl}R)} = \frac{1}{e^{(\varepsilon-L\omega-\omega/2\pm\mu)/T} + 1} \frac{1}{4\pi^2 R^2} \frac{J_{|L|}^2(p_{Tl}r)}{J_{|L+1|}^2(p_{Tl}R)}. \quad (82)$$

Now, by using approximations (80)–(82) we can write the nonrelativistic limit of $D(r)^\pm$ as

$$D(r)^\pm = \frac{1}{4\pi^2 R^2} \sum_{L=-\infty}^{\infty} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} dp_z \frac{1}{e^{(\varepsilon-L\omega-\omega/2\pm\mu)/T} + 1} \frac{J_{|L|}^2(p_{Tl}r)}{J_{|L+1|}^2(p_{Tl}R)} - \frac{1}{e^{(\varepsilon-L\omega+\omega/2\pm\mu)/T} + 1} \frac{J_{|L|}^2(p_{Tl}r)}{J_{|L-1|}^2(p_{Tl}R)}, \quad (83)$$

where the first term is to be associated to particles with spin projection $+1/2$ along the z axis and the second term to those with projection $-1/2$. Finally, the integral over p_z in Eq. (83) can be worked out by first introducing the nonrelativistic approximation $\varepsilon = m + p_T^2/2m + p_z^2/2m$ and then expanding the Fermi distribution. The final result is

$$D(r)^\pm = \frac{1}{4\pi^2 R^2} \sum_{L=-\infty}^{\infty} \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \sqrt{\frac{2\pi mKT}{n}} e^{-n(mc^2 \pm \mu + p_T^2/2m - L\hbar\omega)/KT} \times \left\{ e^{n\hbar\omega/2KT} \frac{J_{|L|}^2(p_{Tl}r/\hbar)}{J_{|L+1|}^2(p_{Tl}R/\hbar)} - e^{-n\hbar\omega/2KT} \frac{J_{|L|}^2(p_{Tl}r/\hbar)}{J_{|L-1|}^2(p_{Tl}R/\hbar)} \right\},$$

where we have purposely restored, for reasons to become clear shortly, the natural constants.

It is very interesting to observe that the functions $D(r)^\pm$, hence $D(r)$, are nonvanishing in the exact nonrelativistic limit $c \rightarrow \infty$. Indeed, it can be seen from Eq. (84) that no factor \hbar or c or powers thereof appear as proportionality constants in front of it, because the $D(r)^\pm$ dimension is already—in natural units—that of an angular momentum; the only c^2 needed is in the exponent, which is compensated by a shift of the chemical potential, and the only \hbar 's needed are those multiplying ω and in the argument of Bessel functions. Since \hbar multiplies ω everywhere and $D(r)$ vanishes for $\omega = 0$, we also see that the difference between canonical and Belinfante densities is essentially a quantum effect, as it vanishes in the limit $\hbar \rightarrow 0$; this is expected as the spin tensor exists only for quantum fields.

For very small values of $\hbar\omega/KT$ these two functions are proportional to $\hbar\omega/KT$ itself since $D(r)|_{\omega=0} = 0$, as discussed in the previous subsection. Retaining only the $n = 1$ term of the series, corresponding to the Boltzmann limit of Fermi-Dirac statistics, and expanding the exponentials $\exp(\pm n\hbar\omega/2KT)$ at first order, one obtains the noteworthy equality

$$D(r)^\pm = \hbar \operatorname{tr}[\hat{\rho}(:\Psi^\dagger \Sigma_z \Psi:)^{\pm}] \simeq \frac{1}{2} \frac{\hbar\omega}{KT} \hbar \operatorname{tr}[\hat{\rho}(:\Psi^\dagger \Psi:)^{\pm}] = \hbar \frac{1}{2} \frac{\hbar\omega}{KT} \left(\frac{dn}{d^3x} \right)^\pm, \quad (84)$$

where the superscript \pm implies that one retains either the particle or the antiparticle term in the expansion of the free field and $(dn/d^3x)^\pm$ is, apparently, the particle or antiparticle density. The Eq. (84) can be shown by retracing all the

steps of the calculations carried out for the spin tensor just replacing Σ_z with the identity matrix.

The function $D(r)$ can be computed with available numerical routines finding a sufficient number of zeroes of Bessel functions, according to Eq. (78). For the numerical computation to be accurate enough one has to make the series in L , l and n quickly convergent at any r . For the series in L , two requirements should be met: first (in natural units) $\omega/T \ll 1$ in order to keep the exponential $\exp[L\omega/T]$ relatively small and, secondly, the radius R should be such that $R\sqrt{mT}$ is not too large; this condition stems from the fact that, as the Bessel functions effectively implement the semiclassical approximation $|L| \simeq p_T R$ and $p_T \approx \sqrt{mT}$, the effective maximal value of L is of the order of $R\sqrt{mT}$. For the series in l , one has to set $m/T \gg 1$, so that large p_T 's are strongly suppressed; this is also the nonrelativistic limit condition. For the series in n , one has to choose μ so as to keep far from the degenerate Fermi gas case. The function $D(r)$ as a function of r is shown in Fig. 2 for $\mu = 0$, $R = 300$, $T = 0.01$, $m = 1$ and two different values of ω , 10^{-4} and 2×10^{-4} ; the function $(r/2)D'(r) - D(r)$, which is the difference between angular momentum densities for the canonical and Belinfante tensors, is shown in Fig. 3.

The plots in Figs. 2 and 3 show that the angular momentum density is larger in the canonical than in the Belinfante case almost everywhere, except for a narrow space near the boundary, whose thickness is plausibly determined by the microscopic scales of the problem (thermal wavelength or Compton wavelength). Thereby, the observable macroscopic value of the differences between angular momentum densities, for a rotating system

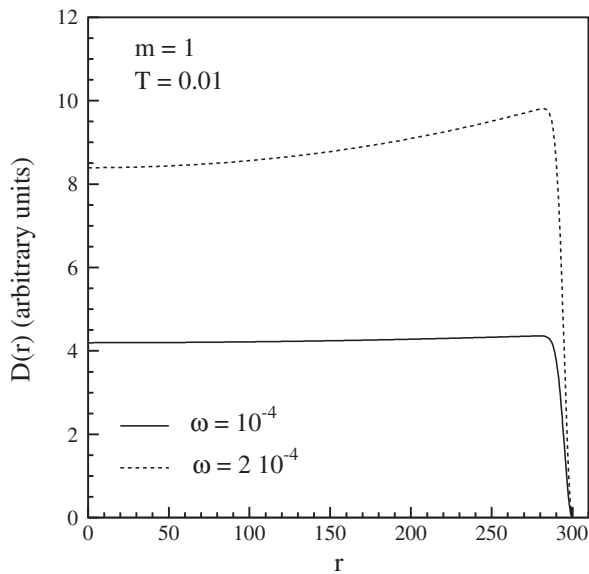


FIG. 2. Function $D(r)$, corresponding to the mean value of the canonical spin tensor for the free Dirac field, in a rotating cylinder at thermodynamical equilibrium as a function of radius r , in the nonrelativistic limit.

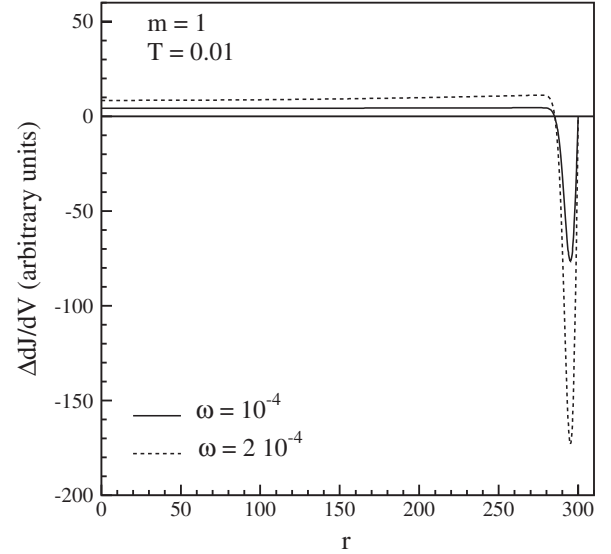


FIG. 3. Difference between the mean value of the canonical angular momentum density and the Belinfante angular momentum density for the free Dirac field, in a rotating cylinder at thermodynamical equilibrium as a function of radius r , in the nonrelativistic limit.

of free fermions, is the slowly varying positive one in the bulk. While the boundary conditions are needed to ensure the invariance of the total angular momentum, the rapid drop to zero within a microscopic distance from the cylinder surface tells us that the chosen boundary conditions at a macroscopic scale of observation correspond to a discontinuity or a surface effect. Any *macroscopic* coaxial subcylinder of the full cylinder with a radius $r < R$ will therefore have different total angular momenta whether one chooses the canonical or the Belinfante tensors in Eqs. (47) and (48), respectively. Such an ambiguity is physically unacceptable and can be solved only by admitting that these tensors are in fact inequivalent.

VII. CONCLUSIONS AND OUTLOOK

In conclusion, we have shown that, in general, couples of stress-energy and spin tensors related by a pseudogauge transformation (3) and allegedly equivalent in quantum field theory, are in fact thermodynamically inequivalent. The inequivalence shows up only for thermodynamical rotating systems and not for the systems—familiar in thermal field theory—locally at rest in an inertial frame. We have worked out exhaustively an instance of such inequivalence involving the free Dirac field and have shown that, surprisingly, the canonical and Belinfante tensors imply the same mean energy density but different mean densities of momentum and angular momentum. Particularly, the latter is almost everywhere larger in the canonical than in the Belinfante case for a small, yet macroscopic, amount. We would like to stress that this result does not depend on an inappropriate treatment of

the quantum field problem in a region with finite transverse size. First of all, field boundary conditions have been chosen so as to guarantee the invariance of the integrated quantities—i.e., the generators—and, secondly, the final spin tensor value (84) is proportional to particle density through fixed parameters ω and T . Hence, its nonvanishing does not apparently depend on spurious factors depending on the radius R which would probably be there if that was the result of an inaccurate calculation of finite size effects.

What is the right couple of tensors? Needless to say, this is a very important issue; for instance, if it was found that the quantum spin tensor is not the trivial Belinfante one (i.e., vanishing) this would have major consequences in hydrodynamics and gravity, even more if its associated stress-energy tensor had a nonsymmetric part, because this could imply a torsion of the space-time (for a recent discussion see, e.g., Ref. [14]). Thus far in this work, no method has been discussed to answer the above question. Indeed, the problem of determining the right tensors can be approached both theoretically and experimentally.

From an experimental viewpoint, in principle we could decide if a specific stress-energy or a spin tensor is *wrong* by measuring with sufficient accuracy the angular momentum density of a rotating system at full thermodynamical equilibrium kept at fixed temperature T and angular velocity ω . This measurement would, for instance, be able to reject the canonical or Belinfante tensor without even the need of resorting to relativistic systems as their difference has a nonvanishing, nonrelativistic limit, as has been discussed at the end of last section. In practice, at a glance, this measurement would not seem an easy one. According to Eq. (84), in the nonrelativistic limit the difference between these two tensors is of the order of $\hbar\omega/KT$ times \hbar times the particle density, that is, particles have a polarization of the order of $\hbar\omega/KT$. This ratio is extremely small for ordinary macroscopic systems; assuming a large angular velocity ω , say 100 Hz, at room temperature $T = 300^\circ\text{K}$ it turns out to be of the order of 10^{-12} . Notwithstanding, this is precisely the polarization responsible for the observed magneto-mechanical phenomena, the Barnett [15] (magnetization induced by a rotation) and Einstein-De Haas (rotation induced by magnetization) effects. It is therefore possible that with some suitable experiment of this sort one can discriminate between spin tensors; this will be the subject of further investigation. We would like to point out that the effect could be enhanced by lowering the temperature so much to increase the ratio $\hbar\omega/KT$, e.g., with cold atom techniques.

From a theoretical viewpoint, we cannot, for the present, determine a thermodynamically best couple of stress energy and spin tensor. Yet, we can argue, on the basis of a thermodynamical argument—which can in principle be used to assess any other couple of tensors—that the canonical spin tensor is favored over the Belinfante one; the argument also elucidates why the source of the inequivalence is ultimately

the second law of thermodynamics. Let us first write down the entropy of a system with cylindrical symmetry and large but finite radius R , using Eq. (18),

$$S = -\text{tr}_V[\hat{\rho} \log \hat{\rho}] = \log Z_\omega + \frac{\langle \hat{H} \rangle}{T} - \frac{\mu}{T} \langle \hat{Q} \rangle - \frac{\boldsymbol{\omega}}{T} \cdot \langle \hat{\mathbf{J}} \rangle,$$

where $\langle \rangle$ stands for $\text{tr}[\hat{\rho}]$. All average quantities in the above equation, namely, energy, charge and angular momentum, have a density, meaning that they are given by a volume integral of a supposedly objective function. Now let us assume that entropy also has a physically objective density, which is expected to happen for a macroscopic system; in this case, according to the above equation, the potential $\log Z_\omega$ has an objective density as well. Furthermore, if the system is properly thermodynamical, the derivatives of this density $d \log Z_\omega / dV$ with respect to the intensive parameters $1/T, \mu/T$ and ω/T should give the energy density, the charge density and the angular momentum density, similarly to what happens for the integrated quantities.

Suppose that the system is initially nonrotating, i.e., with $\omega = 0$. We have seen in Sec. VI that in this case the function $D(r)$ vanishes, hence there is no difference between the mean values of canonical and Belinfante tensors. Let us then focus on a coaxial subcylinder with radius $r \ll R$, yet large enough for it to be macroscopic. Keeping the temperature T and chemical potential μ fixed, let us turn on very slowly a small angular velocity $\Delta\omega$. Thermodynamically, we can think of the subcylinder as “the system” and the rest as an angular momentum reservoir at full thermodynamical equilibrium with it as both have the same angular velocity. It is well known that a thermodynamical system at fixed T, V and μ maximizes the value of the thermodynamical potential $\log Z$. Likewise, a thermodynamical system with fixed T, V, μ and ω maximizes the value of the potential $\log Z_\omega$. For the subcylinder with radius r , according to our previous thermodynamical assumptions, this potential is given by the integral over the region V_r of the density $d \log Z_\omega / dV$ and its variation after switching on the rotational motion is

$$\Delta \log Z_{\omega,r} \simeq \left. \frac{\partial \log Z_{\omega,r}}{\partial (\omega/T)} \right|_{V_r, T, \mu} \Delta \left(\frac{\omega}{T} \right) = \langle J \rangle_r \Delta \left(\frac{\omega}{T} \right),$$

where $\langle J \rangle_r$ is the angular momentum of the subcylinder. This variation should be maximal and since, for what we have seen in Sec. VI, in any properly macroscopic coaxial subcylinder with radius $r < R$, the canonical value of the angular momentum is larger than the Belinfante one, the final value of the thermodynamical potential of the subcylinder will be larger in the canonical than in the Belinfante case. Hence, if the system could choose between canonical and Belinfante tensors, the former would be certainly favored. Of course, this method allows to discriminate between two couples of tensors but, for the present, does not permit to single out the best couple of tensors for a

given quantum field theory. For instance, for the free Dirac field, it may happen that neither the canonical nor the Belinfante tensors maximize the thermodynamic potential of such a subsystem and another couple does.

ACKNOWLEDGMENTS

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Note added in proof—After the acceptance of this paper, F. W. Hehl brought to our attention his paper [17], in which he wrote the most general transformations of the stress-energy and spin tensor. The transformations (3) that we have used are a special case of his with a tensor field of rank-4 (called z in his paper) set to zero or constant. We note that this choice does not affect any argument of this work.

APPENDIX A: ACTION AND CANONICAL STRESS-ENERGY TENSOR FOR THE DIRAC FIELD IN A BOUNDED REGION

Consider an action A of general fields ψ^a in a cylindrical region in Fig. 1 and its variations

$$\begin{aligned} \delta A &= \int_V d^4x \left(\frac{\partial \mathcal{L}}{\partial \psi^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^a} \right) \delta \psi^a \\ &\quad + \int_V d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^a} \delta \psi^a \right) \\ &= \int_V d^4x \left(\frac{\partial \mathcal{L}}{\partial \psi^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^a} \right) \delta \psi^a \\ &\quad + \int_{\partial V} dt dS n^\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^a} \delta \psi^a. \end{aligned} \quad (\text{A1})$$

Particularly, if we take the Dirac action

$$A = \int_V d^4x \left(\frac{i}{2} \bar{\Psi} \gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - m \bar{\Psi} \Psi \right) \quad (\text{A2})$$

and require it to be stationary with respect to variation of the fields Ψ , $\bar{\Psi}$ with the boundary conditions (56), we obtain the free Dirac equation. This can be shown by working out the derivatives in the boundary integral with \mathcal{L} as in Eq. (46)

$$\begin{aligned} n^\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^a} \delta \psi^a &= n^\mu \frac{i}{2} \bar{\Psi} \gamma_\mu \delta \Psi - n^\mu \frac{i}{2} \delta \bar{\Psi} \gamma_\mu \Psi \\ &= \frac{i}{2} (\bar{\Psi} \hat{n} \delta \Psi - \delta \bar{\Psi} \hat{n} \Psi). \end{aligned}$$

As the fields meet boundary condition (56)

$$\begin{aligned} \frac{i}{2} (\bar{\Psi} \hat{n} \delta \Psi - \delta \bar{\Psi} \hat{n} \Psi) &= -\frac{1}{2} (\bar{\Psi} \delta \Psi + \delta \bar{\Psi} \Psi) \\ &= -\frac{1}{2} \delta (\bar{\Psi} \Psi) \end{aligned}$$

as well as $\bar{\Psi} \Psi(R) = 0$ as a consequence of (56), then $\delta \bar{\Psi} \Psi$ vanishes at the boundary and the second integral in Eq. (A1) vanishes. We are thus left with a bulk integral which has to vanish for general variations of the field, leading to free Dirac equations, as usual.

For the calculation of canonical tensors from the action (A2), we follow Ref. [16] and use a space-dependent variation of the fields

$$\delta \Psi(x) = \Psi(x + \epsilon(x)) - \Psi(x) \simeq \epsilon^\mu(x) \partial_\mu \Psi(x)$$

with small $\epsilon(x)$. This is a particular variation of the field which fulfills the condition (56) if $\epsilon(x) = 0$ at the boundary and this is what we set. If Ψ is the solution of the equation of motion, then the variation of the action should vanish. After some easy calculations (see Ref. [16])

$$\begin{aligned} 0 &= \delta A \\ &= \int_V d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^a} \partial^\nu \psi^a - g^{\mu\nu} \mathcal{L} \right) \epsilon_\nu \\ &\quad + \int_V d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^a} \partial^\nu \psi^a \epsilon_\nu \right). \end{aligned}$$

The second term can be turned into a boundary integral which vanishes because $\epsilon = 0$ there, as has been mentioned. The first term should also vanish and since $\epsilon(x)$ is an arbitrary function, the divergence of the what can be easily recognized as the canonical stress-energy tensor must vanish.

A similar reasoning leads to the conclusion that the angular momentum tensor is conserved.

APPENDIX B: CALCULATION OF THE MEAN VALUE OF PRODUCTS OF CREATION AND DESTRUCTION OPERATORS

We follow the argument used in [1]. The aim is to calculate

$$\text{tr}_V [\hat{\rho} a_n^\dagger a_n]$$

with $\hat{\rho}$ given by Eq. (18). For this purpose we define, with $\beta = 1/T$,

$$a_n^\dagger(\beta) = e^{-\beta(\hat{H} - \omega \hat{J} - \mu \hat{Q})} a_n^\dagger e^{\beta(\hat{H} - \omega \hat{J} - \mu \hat{Q})} \quad (\text{B1})$$

and similarly for a_n , b_n and b_n^\dagger . From the above equation it ensues:

$$\frac{\partial a_n^\dagger(\beta)}{\partial \beta} = [a_n^\dagger(\beta), \hat{H} - \omega \hat{J} - \mu \hat{Q}] \quad (\text{B2})$$

and, since

$$[\hat{H}, a_n^\dagger] = \varepsilon a_n^\dagger \quad [\hat{J}, a_n^\dagger] = M a_n^\dagger \quad [\hat{Q}, a_n^\dagger] = q a_n^\dagger,$$

which is solved by, being $a_n^\dagger(0) = a_n^\dagger$,

one readily obtains that Eq. (B2) is equivalent to:

$$a_n^\dagger(\beta) = a_n^\dagger e^{-\beta(\varepsilon - M\omega - \mu q)}. \quad (\text{B3})$$

$$\frac{\partial a_n^\dagger(\beta)}{\partial \beta} = (-\varepsilon + M\omega + \mu q) a_n^\dagger(\beta),$$

We can now write

$$\begin{aligned} \text{tr}_V[\hat{\rho} a_n^\dagger a_{n'}] &= \text{tr}_V[\hat{\rho} a_n^\dagger e^{\beta(\hat{H} - \omega \hat{J} - \mu \hat{Q})} e^{-\beta(\hat{H} - \omega \hat{J} - \mu \hat{Q})} a_{n'}] = \text{tr}_V[e^{-\beta(\hat{H} + \omega \hat{J} - \mu \hat{Q})} a_{n'} \hat{\rho} a_n^\dagger e^{\beta(\hat{H} - \omega \hat{J} - \mu \hat{Q})}] \\ &= \frac{1}{Z_\omega} \text{tr}_V[e^{-\beta(\hat{H} - \omega \hat{J} - \mu \hat{Q})} a_{n'} e^{-\beta(\hat{H} - \omega \hat{J} - \mu \hat{Q})} a_n^\dagger e^{\beta(\hat{H} - \omega \hat{J} - \mu \hat{Q})}] = \frac{1}{Z_\omega} \text{tr}_V[e^{-\beta(\hat{H} - \omega \hat{J} - \mu \hat{Q})} a_{n'} a_n^\dagger(\beta)] \\ &= \text{tr}_V[\hat{\rho} a_{n'} a_n^\dagger(\beta)], \end{aligned}$$

where we have used the cyclicity of the trace, the definition of $\hat{\rho}$ in Eq. (18) and the Eq. (B1). It should be pointed out that the cyclicity of the trace can be used safely because a complete set of states for the cylinder with finite radius can be constructed with eigenvectors of the operators \hat{H} , \hat{J}_z and \hat{Q} ; we could have also used the full trace and insert the operator P_V discussed in Sec. IV but this would not have changed the final result as this operator commutes with $\hat{\rho}$. By using Eq. (B3) and the anticommutation relation (67), the above equation can also be written as

$$\begin{aligned} \text{tr}_V[\hat{\rho} a_n^\dagger a_{n'}] &= \text{tr}_V[\hat{\rho} a_{n'} a_n^\dagger(\beta)] \\ &= \text{tr}_V[\hat{\rho} a_{n'} a_n^\dagger] e^{-\beta(\varepsilon - M\omega - \mu q)} \\ &= (-\text{tr}_V[\hat{\rho} a_n^\dagger a_{n'}] + \delta_{nn'}) e^{-\beta(\varepsilon - M\omega - \mu q)}, \end{aligned}$$

whence

$$\text{tr}_V[\hat{\rho} a_n^\dagger a_{n'}] = \frac{\delta_{nn'}}{e^{\beta(\varepsilon - M\omega - \mu q)} + 1}.$$

The above method can be used for the calculation of other bilinear combinations of creation and destruction operators, leading to the equalities reported in Eq. (73).

APPENDIX C: CALCULATION OF THE FUNCTION $D(r)$ ON THE ROTATION AXIS

We calculate $D(0)|_{\omega=0}$ to show that it is vanishing as well as its derivative with respect to ω/T to show that is strictly positive. The function $D(r)$ in Eq. (75) is the sum of a particle $D(r)^+$ and an antiparticle $D(r)^-$ term: we focus on the particle term as the calculation for antiparticle is a trivial extension. Since Bessel functions of all orders but zero vanish ($J_0(0) = 1$), in $r = 0$ in the sum of Eq. (75) only terms with $M = -1/2$ and $M = 1/2$ survive

$$\begin{aligned} D(0)^+ &= \frac{1}{8\pi^2 R} \sum_{\xi=\pm 1} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} dp_z \left\{ \frac{p_{+, \xi}^2}{(e^{(\varepsilon - (1/2)\omega + \mu)/T} + 1) J_0(p_{+, \xi} R)^2 [2R(p_{+, \xi}^2 + m^2) + \xi \sqrt{p_{+, \xi}^2 + m^2} + m]} \right. \\ &\quad \left. - \frac{p_{-, \xi}^2 b_{\xi}^{(+2)}}{(e^{(\varepsilon + (1/2)\omega + \mu)/T} + 1) J_1(p_{-, \xi} R)^2 [2R(p_{-, \xi}^2 + m^2) - \xi \sqrt{p_{-, \xi}^2 + m^2} + m]} \right\}, \end{aligned} \quad (\text{C1})$$

where we have defined $p_{\pm, \xi} = \frac{\xi(\pm \frac{1}{2}, \xi, l)}{R}$ (see Eq. (61)). We can rearrange the above sum by noting that the Eq. (62), depending on indices (M, ξ) is the same for $(-M, -\xi)$. In fact,

$$J_{|-M-(1/2)|}(\xi) + \text{sgn}(-M) b_{-\xi}^{(+)} J_{|-M+(1/2)|}(\xi) = J_{|M+(1/2)|}(\xi) - \text{sgn}(M) b_{-\xi}^{(+)} J_{|M-(1/2)|}(\xi),$$

However, because of (63), $-b_{-\xi}^{(+)} = b_{\xi}^{(-)} = 1/b_{\xi}^{(+)}$, and so multiplying the right-hand side of above equation by $\text{sgn}(M) b_{\xi}^{(+)}$ one gets the left hand side of Eq. (62). Hence, the zeroes of Eq. (62) and the one with ‘‘reflected’’ indices $(-M, -\xi)$ are the same

$$\zeta_{(-M, -\xi, l)} = \zeta_{(M, \xi, l)} \quad (\text{C2})$$

for any $l = 1, 2, \dots$. Now we can redefine the indices in the second term of the sum in Eq. (C1) by turning ξ into $-\xi$, which changes nothing as $\xi = -1, +1$ and write

$$D(0)^+ = \frac{1}{8\pi^2 R} \sum_{\xi=\pm 1} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} dp_z \left\{ \frac{p_{+, \xi}^2}{[e^{(\varepsilon-(1/2)\omega+\mu)/T} + 1] J_0(p_{+, \xi} R)^2 [2R(p_{+, \xi}^2 + m^2) + \xi \sqrt{p_{+, \xi}^2 + m^2} + m]} - \frac{p_{-, -\xi}^2 b_{-\xi}^{(+2)}}{[e^{(\varepsilon+(1/2)\omega+\mu)/T} + 1] J_1(p_{-, -\xi} R)^2 [2R(p_{-, -\xi}^2 + m^2) + \xi \sqrt{p_{-, -\xi}^2 + m^2} + m]} \right\}.$$

We can replace $p_{-, -\xi}$ with $p_{+, \xi}$ because of (C2) and therefore:

$$D(0)^+ = \frac{1}{8\pi^2 R} \sum_{\xi=\pm 1} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} dp_z \left\{ \frac{p_{+, \xi}^2}{[e^{(\varepsilon-(1/2)\omega+\mu)/T} + 1] J_0(p_{+, \xi} R)^2 [2R(p_{+, \xi}^2 + m^2) + \xi \sqrt{p_{+, \xi}^2 + m^2} + m]} - \frac{p_{+, \xi}^2 b_{-\xi}^{(+2)}}{[e^{(\varepsilon+(1/2)\omega+\mu)/T} + 1] J_1(p_{+, \xi} R)^2 [2R(p_{+, \xi}^2 + m^2) + \xi \sqrt{p_{+, \xi}^2 + m^2} + m]} \right\}. \quad (C3)$$

We are now going to prove that this latter expression is nonvanishing when $\omega \neq 0$. First, we note that it does vanish when $\omega = 0$. In this case Eq. (C3) yields

$$D(0)^+|_{\omega=0} = \frac{1}{8\pi^2 R} \sum_{l, \xi} \int_{-\infty}^{\infty} dp_z \frac{p_{+, \xi}^2 [J_1(p_{+, \xi} R)^2 - b_{-\xi}^{(+2)} J_0(p_{+, \xi} R)^2]}{[e^{(\varepsilon+\mu)/T} + 1] J_1(p_{+, \xi} R)^2 J_0(p_{+, \xi} R)^2 [2R(p_{+, \xi}^2 + m^2) + \xi \sqrt{p_{+, \xi}^2 + m^2} + m]}.$$

By using again (C2) to replace $p_{+, \xi}$ with $p_{-, -\xi}$ it is easy to show that the numerator of the integrand vanishes as

$$J_1(p_{+, \xi} R)^2 - b_{-\xi}^{(+2)} J_0(p_{+, \xi} R)^2 = J_1(p_{-, -\xi} R)^2 - b_{-\xi}^{(+2)} J_0(p_{-, -\xi} R)^2 = J_1(\zeta_{(-1/2, -\xi, l)})^2 - b_{-\xi}^{(+2)} J_0(\zeta_{(-1/2, -\xi, l)})^2 = 0$$

in view of the Eq. (62). Therefore, the spin tensor density in $r = 0$ vanishes for a nonrotating system, as expected. To show that it is no longer zero for $\omega \neq 0$ we just need to show that the derivative with respect to ω/T in $\omega = 0$ is not zero. One has

$$\frac{\partial}{\partial(\omega/T)} D(0)^+|_{\omega=0} = \frac{1}{16\pi^2 R} \sum_{l, \xi} \int_{-\infty}^{\infty} dp_z \frac{e^{(\varepsilon+\mu)/T} p_{+, \xi}^2 [J_1(p_{+, \xi} R)^2 + b_{-\xi}^{(+2)} J_0(p_{+, \xi} R)^2]}{[e^{(\varepsilon+\mu)/T} + 1]^2 J_1(p_{+, \xi} R)^2 J_0(p_{+, \xi} R)^2 [2R(p_{+, \xi}^2 + m^2) + \xi \sqrt{p_{+, \xi}^2 + m^2} + m]}.$$

All terms are manifestly positive except $[2R(p_{Tl}^2 + m^2) + \xi \sqrt{p_{Tl}^2 + m^2} + m]$ in the denominator when $\xi = -1$. However, in this case,

$$2R(p_{Tl}^2 + m^2) - \sqrt{p_{Tl}^2 + m^2} + m = \sqrt{p_{Tl}^2 + m^2} (2R\sqrt{p_{Tl}^2 + m^2} - 1) > \sqrt{p_{Tl}^2 + m^2} (2Rm - 1),$$

which is positive for a radius greater than half the Compton wavelength of the particle, that is, positive for any actually macroscopic value of the radius R . The very same argument applies to the antiparticle term $D(0)^-$ of the $D(r)$ function in Eq. (75) with the immaterial replacement $\mu \rightarrow -\mu$, hence,

$$D(0)|_{\omega=0} = 0 \quad \frac{\partial}{\partial(\omega/T)} D(0)|_{\omega=0} > 0.$$

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