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Stickiness of sound: An absolute lower limit on viscosity and the breakdown of second-order relativistic hydrodynamics

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Hydrodynamics predicts long-lived sound and shear waves. Thermal fluctuations in these waves can lead to the diffusion of momentum density, contributing to the shear viscosity and other transport coefficients. Within viscous hydrodynamics in 3 + 1 dimensions, this leads to a positive contribution to the shear viscosity, which is finite but inversely proportional to the microscopic shear viscosity. Therefore the effective infrared viscosity is bounded from below. The contribution to the second-order transport coefficient τ_π is divergent, which means that second-order relativistic viscous hydrodynamics is inconsistent below some frequency scale. We estimate the importance of each effect for the quark-gluon plasma, finding them to be minor if $\eta/s = 0.16$ but important if $\eta/s = 0.08$.

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I. INTRODUCTION

Heavy-ion collisions at the Relativistic Heavy Ion Collider [1–4] and the LHC [5,6] produce a medium whose evolution, at least at early times, is well described by hydrodynamics with a very small viscosity [7–15] (at least when normalized to the entropy density [16]). A major theoretical goal is now to determine this viscosity as accurately as possible, by modeling the development of a heavy-ion collision with viscous hydrodynamics. In order to do so it is necessary to go beyond the "first-order" (Navier-Stokes) formalism, because the first-order formalism leads to acausal and unstable evolution in a relativistic setting [17–21]. As shown by Israel and Stewart [22–24], this problem can be cured by working instead with hydrodynamics expanded to the second order in gradients, which should have the added advantage of being more accurate. Several groups have been involved in studying the hydrodynamics of heavy-ion collisions using such second-order formalisms [10,25–30]. However, within the community which studies hydrodynamics and kinetics of atomic gases, it has been known for almost 40 years that the gradient expansion in hydrodynamics fails beyond the first order [31]. Is this also true in the relativistic setting? If so, what implications does it have for the study of hydrodynamics via the second-order formalism?

II. SETUP AND INTUITIVE ARGUMENT

We start by reviewing relativistic hydrodynamics to second order. Hydrodynamics is the modeling of a fluid by solving the stress-energy conservation equations[1]

$$\partial_\mu T^{\mu\nu}(x) = 0 \tag{2.1}$$

assuming some functional form for the stress tensor. Ideal hydrodynamics assumes the equilibrium form

$$T^{\mu\nu}_{\rm eq}(x) = (\epsilon(x) + P)u^\mu(x)u^\nu(x) + Pg^{\mu\nu}, \qquad P = P(\epsilon), \tag{2.2}$$

where $g_{\mu\nu} = \mathrm{Diag}[-1, +1, +1, +1]$ is the metric, u^μ is the four-velocity determining the rest frame (normalized so $u_\mu u^\mu = -1$), $\epsilon = u_\mu u_\nu T^{\mu\nu}$ is the rest-frame energy density, and P is the pressure as determined by the equation of state $P = P(\epsilon)$. Viscous hydrodynamics assumes that the fluid is near equilibrium so that $T^{\mu\nu}$ is close to this form. Assuming that equilibration is a fast, local process, corrections to this form can be written in terms of an expansion in gradients. Israel and Stewart showed [23] that a slight reorganization of the second-order derivative expansion yields stable equations which are correct to second order, provided a particular additional term appears at second order:

$$\begin{aligned} T^{\mu\nu}(x) &= T^{\mu\nu}_{\rm eq} + \Pi^{\mu\nu},\\ \Pi^{\mu\nu} &= -2\eta_{\rm cl}\partial^{\langle\mu}u^{\nu\rangle} + \tau_\pi(u^\alpha\partial_\alpha\Pi^{\mu\nu} + \tfrac{1}{3}\Pi^{\mu\nu}\partial_\alpha u^\alpha)\\ &\qquad (+\text{other terms}), \end{aligned} \tag{2.3}$$

where $\eta_{\rm cl}$ is the "classical" viscosity coefficient that one would obtain from a microscopic calculation using the Kubo formula based on (2.3). The angular brackets in $\partial^{\langle\mu}u^{\nu\rangle}$ mean that the indices are to be symmetrized, projected to be spatial in the frame given by u^μ, and trace-subtracted.[2] The extra terms include bulk viscosity and

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[1] If there are conserved currents J^μ_a, one also considers current conservation $\partial_\mu J^\mu_a = 0$. In equilibrium $J^\mu_a = n_a u^\mu$; out of equilibrium there can be derivative corrections. However, in ultrarelativistic heavy-ion collisions the density of the conserved baryon number in the central rapidity region is small, so we will neglect it and will not discuss conserved currents further.

[2] That is, defining $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ which is a projector to local rest-frame spatial components, $2\partial^{\langle\mu}u^{\nu\rangle} = (P^{\mu\alpha}P^{\nu\beta} + P^{\nu\alpha}P^{\mu\beta} - \frac{2}{3}P^{\mu\nu}P^{\alpha\beta})\partial_\alpha u_\beta$.

nonlinear effects, and are catalogued in [32–34]. To simplify the discussion here we will consider a conformal fluid, in which case the bulk viscous term is absent. While this is not a very good approximation for QCD near the transition temperature, in practice the inclusion of bulk viscosity and a realistic equation of state would have only a small influence on our final results.[3]

Solving the hydrodynamic equations for small fluctuations in u^μ, ϵ about constant values, one finds two sorts of long-lived wave solutions, sound waves and shear waves. At lowest order in k, $\omega \ll \epsilon/\eta_{\rm cl}$, they obey dispersion relations of

$$\omega_{\rm shear} = -i\frac{\eta_{\rm cl}}{\epsilon + P}k^2, \qquad \omega_{\rm sound} = \frac{\pm k}{\sqrt{3}} - i\frac{2\eta_{\rm cl}}{3(\epsilon + P)}k^2. \tag{2.4}$$

Each sort of wave decays with time, but with a decay rate which vanishes quadratically in the small k limit, as well as becoming small for small $\eta_{\rm cl}$. Generically, out of equilibrium such waves will be present with large amplitudes. But equipartition of energy says that even in equilibrium such waves will be present, carrying energy which averages to $T/2$ per degree of freedom. The long decay times of these waves contradict the assumption that all degrees of freedom in a fluid equilibrate via rapid local processes. This imperils the assumption behind the gradient expansion in Eq. (2.3). If there are arbitrarily slowly equilibrating degrees of freedom, Eq. (2.3) can contain terms nonanalytic in gradients. In the nonrelativistic setting it is known that precisely this happens [31].

To see more intuitively how hydrodynamic waves can contribute to hydrodynamic coefficients, consider a shear wave, as illustrated in Fig. 1. Hydrodynamics says that this wave configuration will decay with time as the x momentum carried by the fluid diffuses in the y direction into neighboring regions, which are not flowing. The rate of this diffusion is controlled by shear viscosity, which is, by definition, the diffusion coefficient for the component of momentum transverse to the diffusion (the y diffusion of x momentum in this illustration). But one mechanism which can transmit x momentum is hydrodynamic waves, such as sound waves, with wave lengths shorter than the hydrodynamic structure considered. Sound waves leaving the x-moving fluid carry net x momentum away, diffusing away this component of momentum. The phase space of such waves scales as d^3k which is very UV dominated; but the distance propagated before dissipation scales as $1/k^2$ as we just saw, leading to a contribution to viscosity which scales as $\sim d^3k/k^2$. This is IR finite, though it would not be in two spatial dimensions [35]. The contribution is larger for smaller $\eta_{\rm cl}$, both because hydrodynamic waves propagate further and because the range of k where hydro is valid expands at smaller $\eta_{\rm cl}$, so the phase space of hydrodynamic waves is larger.

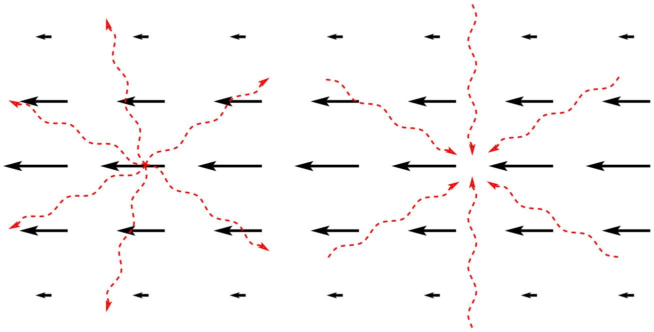

FIG. 1 (color online). A shear wave; fluid moves to the left in a band of fluid near the middle of the figure. Viscosity determines the loss (by diffusion) of the forward motion of this fluid. Sound waves (dotted red lines) leaving the left-moving fluid carry, on average, net left-moving momentum, which is not compensated by sound waves arriving in the left-moving fluid. Hence sound waves contribute to viscosity.

Similarly, τ_π can be interpreted as a relaxation time; if a fluid suddenly develops shear flow, τ_π is the time scale for momentum diffusion to be established. We already saw that hydrodynamic modes contribute to momentum diffusion, with a mode of wave number k contributing of order $1/k^2$. The time scale for this mode to leave equilibrium and establish its contribution to momentum diffusion also scales as $1/k^2$. This suggests that such a mode contributes to τ_π by an amount proportional to k^{-4}, leading to an $\int d^3k/k^4$ contribution to τ_π, which is small k divergent.

III. COMPUTATION OF THE CONTRIBUTION FROM HYDRODYNAMIC WAVES

Reference [32] derives the following Kubo relation for the shear viscosity $\eta_{\rm cl}$ and the relaxation time τ_π in terms of the retarded correlation function for two T^{xy} stress-tensor operators:

$$G_{\rm R}^{xy,xy}(\omega, k_z) = P - i\omega\eta_{\rm cl} + \left(\eta_{\rm cl}\tau_\pi - \frac{\kappa}{2}\right)\omega^2 - \frac{\kappa}{2}k_z^2 + \mathcal{O}(\omega^3, k^3). \tag{3.1}$$

Here κ is another transport coefficient discussed in [32]. We see from the above that κ can be extracted from the zero frequency behavior of the stress-stress correlation function, $G_{\rm R}^{xyxy}(\omega = 0, k_z)$. The zero frequency retarded function equals the Euclidean correlation function, so κ is a thermodynamic property [36], which is not sensitive to long-wavelength hydrodynamic waves. Since we are not interested in the value of κ, we will consider $G_{\rm R}^{xy,xy}$ at vanishing external spatial momentum $k = 0$ and small nonzero frequency ω.

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[3] In particular, our results are dominated by the effects of shear waves, which are not sensitive to bulk viscosity or the equation of state.

To compute the contribution of hydrodynamic waves to η and τ_π, we compute their contribution to the above correlation function, along the lines of what Kovtun and Yaffe did for the symmetrized correlator [37]. The operator T^{xy} to be used in the Green function above is the hydrodynamic one as described in Eqs. (2.2) and (2.3) above, allowing for thermally occupied fluctuations in u^μ and ϵ. In the absence of fluctuations $u^\mu = (1, 0, 0, 0)$, $\epsilon = \epsilon_0$, $P = \epsilon_0/3$, where ϵ_0 is the equilibrium energy density. Therefore T^{xy} arises at order (δu) due to the viscous terms, and at order $(\delta u)^2$ and higher from the equilibrium and viscous terms:

$$T^{xy} = [-\eta_{\rm cl}(\partial^x u^y + \partial^y u^x)] + \left[(\epsilon + P)u^x u^y - \eta_{\rm cl}(u^x \partial_0 u^y + u^y \partial_0 u^x) - \delta\epsilon \frac{d\eta_{\rm cl}}{d\epsilon}(\partial^x u^y + \partial^y u^x)\right] + \mathcal{O}(\delta^3). \qquad (3.2)$$

The contribution at first order in fluctuations arises only from the viscous term $-\eta_{\rm cl}(\partial^x u^y + \partial^y u^x)$. The *symmetrized* correlator is

$$G_{\rm S}^{xyxy}[\text{1-order}] = \int d^3x dt e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \left\langle \frac{1}{2}\{(-\eta_{\rm cl}[\partial^x u^y + \partial^y u^x](\mathbf{x}, t)), (-\eta_{\rm cl}[\partial^x u^y + \partial^y u^x](0, 0))\}\right\rangle, \qquad (3.3)$$

which is related to the retarded correlator via $G_{\rm S}(\omega) = -[1 + 2n_b(\omega)]{\rm Im}G_{\rm R}(\omega)$. The expression above is automatically $\mathcal{O}(k^2)$ and so it does not contribute in the $k \to 0$ limit. But there can also be local contributions, that is, contributions proportional to $\delta^4(x)$ or its derivatives (contact terms). We will present a direct calculation of such contact terms within hydrodynamic theory in a future publication [38]. For our current purposes we will instead extract them by using stress conservation,

$$\partial_\mu T^{\mu y} = 0 \Rightarrow k_x^2 G^{xyxy}(k_x, \omega) = \omega^2 G^{0y0y}(k_x, \omega), \qquad (3.4)$$

together with the known expression for $G_{\rm S}^{0y0y}$ [see Eq. (33d) of [37]]:

$$G_{\rm S}^{0y0y}(k_x, \omega) = \frac{2k_x^2 \eta_{\rm cl} T}{(k_x^2 \eta_{\rm cl}/(\epsilon + P))^2 + \omega^2} \quad \text{and hence } G_{\rm S}^{xyxy}(k_x, \omega) = 2\eta_{\rm cl} T\left(1 - \frac{(k_x^2 \eta_{\rm cl}/(\epsilon + P))^2}{(k_x^2 \eta_{\rm cl}/(\epsilon + P))^2 + \omega^2}\right). \qquad (3.5)$$

The first term in (3.5) is the contact term, which gives rise to the $-i\omega\eta_{\rm cl}$ term in Eq. (3.1). The second term is the contribution from the correlator of the $-\eta_{\rm cl}\partial^x u^y$ part of T^{xy}, which vanishes at small k.

Now we extend this calculation to second order in fluctuations. At this order there are higher order corrections to the terms involving a single power of δu in T^{xy}, and the lowest order contribution from terms involving two powers of u^x, u^y, $\delta\epsilon$ in T^{xy}. But since the terms involving a single power of δu in T^{xy} always involve spatial derivatives, these terms all vanish in the $k \to 0$ limit, like the second term in Eq. (3.5) above. So we skip their calculation and concentrate on contributions from second order in fluctuation terms in T^{xy}. There are several such terms in Eq. (3.2), but we will concentrate on the term $(\epsilon + P)u^x u^y$ and explain why the other terms can be neglected at the end. Since $u^x u^y$ is already quadratic in fluctuations, we can neglect fluctuations in ϵ and replace $(\epsilon + P)$ with $\epsilon_0 + P_0 = \frac{4}{3}\epsilon_0$. Since fluctuations in ϵ will play no further role in the discussion, we will henceforth write ϵ, P for ϵ_0, $P(\epsilon_0)$.

Define the correlation function of the fluid velocity u^i to be

$$\Delta_{\rm S,R}^{ij}(\omega', \mathbf{p}) \equiv \int dt d^3\mathbf{x} e^{-i\mathbf{p}\cdot\mathbf{x} + i\omega t}\langle u^i(t, \mathbf{x}) u^j(0, 0)\rangle_{\rm S,R}, \qquad (3.6)$$

where $\langle\rangle_{\rm S,R}$ indicate whether the operators are to be symmetrized (S) or if the retarded correlator is to be used (R). Using the expression we have for the stress tensor above, the symmetrized correlation function of two stress tensors at vanishing external spatial momentum is

$$G_{\rm S}^{xyxy}[\text{2-order}](\omega, k = 0) = (\epsilon + P)^2 \int \frac{d\omega'}{2\pi} \int \frac{d^3\mathbf{p}}{(2\pi)^3}[\Delta_{\rm S}^{xx}(\omega', \mathbf{p})\Delta_{\rm S}^{yy}(\omega - \omega', -\mathbf{p}) + \Delta_{\rm S}^{xy}(\omega', \mathbf{p})\Delta_{\rm S}^{yx}(\omega - \omega', -\mathbf{p})], \qquad (3.7)$$

where we assumed small, nearly linear, hydrodynamic fluctuations and small frequencies (details will be given in Ref. [38]).

We could use the Kubo-Martin-Schwinger (KMS) relation $G_{\rm S}(\omega) = -[1 + 2n_b(\omega)]{\rm Im}G_{\rm R}(\omega)$ to extract the shear viscosity from this correlation function, but since τ_π depends on the real part of the retarded function, we would need to invert the KMS condition through a Kramers-Kronig relation to get the real part of $G_{\rm R}$. It is more economical to compute $G_{\rm R}$ directly:

$$
\begin{aligned}
G_{\mathrm{R}}^{xyxy}[\text{2-order}](\omega, k=0) = (\epsilon + P)^2 \int \frac{d\omega'}{2\pi} \int \frac{d^3\mathbf{p}}{(2\pi)^3} [&\Delta_{\mathrm{S}}^{xx}(\omega', \mathbf{p}) \Delta_{\mathrm{R}}^{yy}(\omega - \omega', -\mathbf{p}) + \Delta_{\mathrm{S}}^{xy}(\omega', \mathbf{p}) \Delta_{\mathrm{R}}^{yx}(\omega - \omega', -\mathbf{p}) \\
&+ \Delta_{\mathrm{R}}^{xx}(\omega', \mathbf{p}) \Delta_{\mathrm{S}}^{yy}(\omega - \omega', -\mathbf{p}) + \Delta_{\mathrm{R}}^{xy}(\omega', \mathbf{p}) \Delta_{\mathrm{S}}^{yx}(\omega - \omega', -\mathbf{p})],
\end{aligned} \tag{3.8}
$$

which in the small frequency limit reproduces G_{S}^{xyxy} when using the KMS condition. This contribution is to be added to the first-order contribution, which, as we discussed, reproduces the terms present in Eq. (3.1).

Now $\Delta_{\mathrm{S}}^{ij}(\omega', \mathbf{p})$ was determined by Kovtun and Yaffe [37]. They included the effect of the viscous η_{cl} term (first-order gradients) but dropped the τ_π term (second-order gradients), a procedure we will follow. One finds

$$
\begin{aligned}
\Delta_{\mathrm{S}}^{ij}(\omega', \mathbf{p}) &= \frac{2T}{\epsilon + P} \left[\frac{p^i p^j}{p^2} \frac{\tilde{\gamma}_\eta p^2 \omega^2}{(\omega^2 - p^2/3)^2 + (\tilde{\gamma}_\eta p^2 \omega)^2} + \left(\delta^{ij} - \frac{p^i p^j}{p^2} \right) \frac{\gamma_\eta p^2}{(\gamma_\eta p^2)^2 + \omega^2} \right] \quad \text{with} \quad \gamma_\eta \equiv \frac{\eta_{\mathrm{cl}}}{\epsilon + P}, \\
\tilde{\gamma}_\eta &\equiv \frac{4\eta_{\mathrm{cl}}}{3(\epsilon + P)}.
\end{aligned} \tag{3.9}
$$

The first and second terms in Δ^{ij} represent sound and shear waves, respectively. This expression differs slightly from the one found by Kovtun and Yaffe [37] because they assume $p\gamma_\eta \ll 1$ which allows them to split the sound mode contribution into separate terms for $\omega \simeq p/\sqrt{3}$ and $\omega \simeq -p/\sqrt{3}$ propagation.

In the small frequency limit, the factor $2T$ in the numerator of Eq. (3.9) should really be interpreted as $\omega(1 + 2n_b(\omega))$, with $n_b(\omega) = (e^{\omega/T} - 1)^{-1}$ as usual. Then the KMS relation $\Delta_S^{ij}(\mathbf{p}, \omega) = -(1 + 2n_b(\omega)) \operatorname{Im} \Delta_{\mathrm{R}}^{ij}(\mathbf{p}, \omega)$ and the analytic properties of Δ_{R} (no poles in the upper half-plane) uniquely establish

$$
\begin{aligned}
\Delta_{\mathrm{R}}^{ij}(\mathbf{p}, \omega) = \frac{1}{\epsilon + P} \bigg[&\frac{p^i p^j}{p^2} \frac{\omega^2}{i\tilde{\gamma}_\eta p^2 \omega + (\omega^2 - p^2/3)} \\
&+ \left(\delta^{ij} - \frac{p^i p^j}{p^2} \right) \frac{-\gamma_\eta p^2}{-i\omega + \gamma_\eta p^2} \bigg].
\end{aligned} \tag{3.10}
$$

We are now ready to compute Eq. (3.8). Since our calculation is done entirely within hydrodynamics, or equivalently assuming momenta to be small, our result will not be applicable at large momenta, neither in the argument of G_{R}^{xyxy}, nor for the hydrodynamic propagators Δ^{ij} inside the integral of Eq. (3.8). Hence, we need to restrict the calculation to the highest wave number $p_{\max}$, or the inverse of the shortest length scale, where the hydrodynamic description is valid. Cutting off the p integration at $p_{\max}$ and considering first the shear-shear contribution, we can perform the index contractions and angular integrals, finding

$$
\begin{aligned}
G_{\mathrm{R,shear\text{-}shear}}^{xyxy}(\omega) &= \frac{14}{15} \frac{1}{2\pi^2} \int_0^{p_{\max}} p^2 dp \int \frac{d\omega'}{2\pi} \frac{2\gamma_\eta p^2 T}{(\gamma_\eta p^2 - i\omega')(\gamma_\eta p^2 + i\omega')} \frac{-\gamma_\eta p^2}{\gamma_\eta p^2 - i\omega + i\omega'} \\
&= \frac{7T}{30\pi^2} \int_0^{p_{\max}} dp \frac{-p^4}{p^2 - i\omega/(2\gamma_\eta)} = \frac{7T}{30\pi^2} \int_0^{p_{\max}} dp \left[-p^2 - \frac{i\omega}{2\gamma_\eta} + \frac{\omega^2/(4\gamma_\eta^2)}{p^2 - i\omega/(2\gamma_\eta)} \right].
\end{aligned} \tag{3.11}
$$

The p^2 term is an uninteresting contribution of hydrodynamic waves to the pressure.[4] The two "interesting" terms are

$$
G_{\mathrm{R,shear\text{-}shear}}^{xyxy}(\omega) \simeq -i\omega \frac{7T p_{\max}}{60\pi^2 \gamma_\eta} + (i + 1)\omega^{3/2} \frac{7T}{240\pi \gamma_\eta^{3/2}}. \tag{3.12}
$$

Comparing with Eq. (3.1), we see that the first term is a positive contribution to the shear viscosity arising from shear waves. The imaginary part of the second term is a frequency-dependent reduction in the shear viscosity, which vanishes as $\omega \to 0$. Therefore, there is no problem defining the shear viscosity in terms of the zero frequency limit of $\partial G_{\mathrm{R}}^{xyxy}/\partial\omega$. The real part of the $\omega^{3/2}$ term has the same sign as the τ_π term in Eq. (3.1), but the wrong ω dependence. This term can be interpreted as a frequency-dependent correction to τ_π which diverges at small frequency; or it can be interpreted as a breakdown of the validity of the hydrodynamic expansion beyond one-derivative order.

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[4] The pressure contribution has an unexpected sign. In a weakly coupled theory, the contributions of ordinary particles in loops also have the wrong sign, which is over-canceled by a contact term; see [39].

We should also include the sound $(p^i p^j/p^2)$ terms in Δ^{ij} shown in Eqs. (3.9) and (3.10). If we assume $p_{\max}\gamma_\eta \ll 1$ then the sound part in (3.10) may be approximated as

$$\frac{\omega^2}{i\tilde{\gamma}_\eta p^2\omega + (\omega^2 - p^2/3)} \to \frac{\omega}{2}\bigg(\frac{1}{\omega + i\tilde{\gamma}_\eta/2p^2 - p/\sqrt{3}} + \frac{1}{\omega + i\tilde{\gamma}_\eta/2p^2 + p/\sqrt{3}}\bigg)$$

which has the advantage that now the ω' poles in (3.8) are simple. The integration is then straightforward, and for the mixed shear-sound term, one has

$$\frac{-T}{5\pi^2}\int_{-p_{\max}}^{p_{\max}} dp\,\frac{p^4}{p^2 - 3i(\omega - p/\sqrt{3})/5\gamma_\eta} = -\frac{2T}{3\pi^2}\gamma_\eta^2 p_{\max}^5 + i\frac{2\gamma_\eta T}{3\pi^2}\omega p_{\max}^3 + \ldots.$$

One thus finds that the mixed shear-sound term is suppressed with respect to the result (3.12) by extra powers of $p_{\max}\gamma_\eta$. The sound-sound term contains a part that has the same structure as the shear-shear contribution, as well as other parts that are again suppressed by powers of $p_{\max}\gamma_\eta$. Evaluation of the interesting contribution to leading order in $p_{\max}\gamma_\eta$ gives

$$G_{\mathrm{R}}^{xyxy}(\omega \ll p_{\max} \ll \gamma_\eta^{-1}) \simeq -i\omega\frac{17Tp_{\max}}{120\pi^2\gamma_\eta} + (i+1)\omega^{3/2}\frac{(7 + (\frac{3}{2})^{3/2})T}{240\pi\gamma_\eta^{3/2}} + \mathcal{O}(p_{\max}^2\gamma_\eta^2, \omega^2). \tag{3.13}$$

This is our main result.

Let us finally comment about whether or not we need to consider other contributions to G_{R}^{xyxy} coming, for instance, from the $-\eta_{\mathrm{cl}}u^x\partial_0 u^y$ term in Eq. (3.2). Since hydrodynamics is predicated on the convergence of the derivative expansion introduced in Eqs. (2.2) and (2.3), $p_{\max}$ should be chosen as the largest momentum scale where successive terms in the series are successively smaller, which requires $p_{\max} \ll (\epsilon + P)/\eta_{\mathrm{cl}}$ (comparing the zero- and one-derivative terms) and $p_{\max} \ll \eta_{\mathrm{cl}}/(\eta_{\mathrm{cl}}\tau_\pi) = \tau_\pi^{-1}$ (comparing the one-derivative and two-derivative terms). This also ensures that hydrodynamic waves with $p < p_{\max}$ will have $\mathrm{Im}\,\omega \ll p$, so that hydrodynamic waves are well-defined, long-lived excitations in the plasma. And it ensures that the real parts of the propagating frequencies of the two sound waves and the shear wave are more widely separated than their imaginary parts. These conditions ensure that contributions to G_{R}^{xyxy} from higher-derivative terms in Eq. (2.3), and contributions arising from interference between sound and shear waves in Eq. (3.8), are small compared to the terms we have computed. In particular, terms arising from $T^{xy} \supset -\eta_{\mathrm{cl}}u^x\partial_0 u^y$ in Eq. (2.3) will give rise to an integrand similar to Eq. (3.11) but with an extra power of $(p\gamma_\eta)^2$. The resulting term will be analytic in the frequency and will give corrections suppressed by $p_{\max}^2\gamma_\eta^2$ relative to the terms we have computed. We will not attempt to compute such suppressed corrections here.

IV. INTERPRETATION AND DISCUSSION

A. Viscosity

We found that, besides the classical viscosity η_{cl}, there is an additional contribution to viscosity as measured on very long distance and time scales, generated by relatively short-wavelength hydrodynamic waves. It is given by the coefficient of $-i\omega$ in Eq. (3.13):

$$\eta_{\mathrm{new}} = \frac{17p_{\max}T(\epsilon + P)}{120\pi^2\eta_{\mathrm{cl}}}(1 + \mathcal{O}(p_{\max}^2\gamma_\eta^2)), \tag{4.1}$$

where higher power corrections in $p_{\max}$ can be neglected as long as $p_{\max}\gamma_\eta \ll 1$. The new contribution from Eq. (3.13) scales as an inverse power of η_{cl}. Therefore, $\eta \sim \eta_{\mathrm{cl}} + \mathcal{O}(1/\eta_{\mathrm{cl}})$ has a positive minimum at a nonzero value of η_{cl}. This places a lower bound on the total (infrared) value of η in any given theory.

To estimate the size of the new term and the minimum possible viscosity, we need to estimate $p_{\max}$. As discussed, $p_{\max}$ should be the largest wave number such that the gradient expansion converges and a hydrodynamic description is self-consistent. Following the discussion after Eq. (3.13), we will estimate $p_{\max} \simeq \tau_\pi^{-1}/2$.

The total viscosity η is then the classical plus the new contribution,

$$\eta = \eta_{\mathrm{cl}} + \frac{17p_{\max}\gamma_\eta T(\epsilon + P)^2}{120\pi^2\eta_{\mathrm{cl}}^2}, \tag{4.2}$$

which leads to an absolute lower bound on the total viscosity:

$$\eta > \bigg(\frac{153}{160\pi^2}T(\epsilon + P)^2 p_{\max}\gamma_\eta\bigg)^{1/3}. \tag{4.3}$$

Note that an actual measurement of correlators in a fluid would give information about η, whereas η_{cl} turns out to be only a calculational quantity.

In a theory with many colors and a weak coupling so that parametrically [40–42]

$$\epsilon \sim N_{\mathrm{c}}^2 T^4, \qquad s \sim N_{\mathrm{c}}^2 T^3, \qquad \eta \sim N_{\mathrm{c}}^2\alpha^{-2}T^3, \qquad \tau_\pi \sim N_{\mathrm{c}}^0\alpha^{-2}T^{-1}, \tag{4.4}$$

the new contribution is parametrically

$$\eta_{\mathrm{new}} \sim \frac{\tau_\pi^{-1}\epsilon T}{\eta} \sim \alpha^4 T^3 \sim \alpha^6 N_{\mathrm{c}}^{-2}\eta, \tag{4.5}$$

showing that the new contribution is safely subdominant in any theory which is either weakly coupled or has many fields.

However, in the real world QCD has three colors and the coupling at scales of current interest is not small. How large is this new contribution to η in this case? To address this, we need estimates for $\eta_{\rm cl}$, τ_π, and $\epsilon + P = sT$ for real-world QCD. The easiest is sT, which can be determined on the lattice. According to Borsányi *et al.* [43] (Fig. 12), between $T = 200$ MeV and $T = 300$ MeV s/T^3 rises from about 10 to about 14. Bazavov *et al.* [44] (Fig. 8) give comparable numbers, with s/T^3 rising from 11 to 16 in the same range. τ_π naturally scales as $\tau_\pi \sim \eta_{\rm cl}/sT = \gamma_\eta$, and estimates in QCD vary from $2.6\gamma_\eta$, the value in strongly coupled $\mathcal{N} = 4$ super Yang-Mills theory [32], to $\sim 5\gamma_\eta$, the value at weak coupling [42]. The trend is that, while τ_π/γ_η is a pure number of order unity, it is smaller in more strongly coupled contexts and larger in weakly coupled contexts. So in the context of QCD with a small shear viscosity, it probably makes sense to assume τ_π/γ_η is at the low end of this estimated range. Regarding $\eta_{\rm cl}$, the lowest estimates for $\eta_{\rm cl}/s$ are around 0.08, while $\eta_{\rm cl}/s = 0.16$ may be on the high side in terms of fitting elliptic flow data [10].

If we estimate that $\tau_\pi = 3\eta_{\rm cl}/sT$, $\eta_{\rm cl}/s = 0.08$, and $s = 10T^3$, we find $p_{\rm max} \sim 2T$ and $\eta_{\rm new} \sim 0.36T^3 \sim 0.036s$. The true value of η/s would then be $0.08 + 0.036 = 0.116$. Varying the value of $\eta_{\rm cl}/s$ while holding the other estimates fixed, this is close to the minimum value of the *total* viscosity. On the other hand, if $\eta_{\rm cl} = 0.16s$ but the other estimates are the same, then $p_{\rm max} \sim 1T$ and $\eta_{\rm new} \sim 0.09T^3 \sim 0.01s$, which is a negligible correction to the total viscosity. Therefore, the importance of the "new" contribution to viscosity is quite sensitive to the value of $\eta_{\rm cl}/s$; for real-world values of other parameters $\eta/s = 0.08$ appears to be impossible, but $\eta/s = 0.16$ is not.

Finally, we point out that these viscosity values depend on the choice for the parameter $p_{\rm max}$, the UV scale up to which hydrodynamics can be used. We estimated $p_{\rm max} \simeq 1/(2\tau_\pi)$, which should be reliable up to a factor of 2. Note that this uncertainty in $p_{\rm max}$ translates into an uncertainty of $\sim 25\%$ in the viscosity bound (4.3), not enough to jeopardize the above estimate for the real-world value of η/s.

B. Relaxation time τ_π

The presence of a nonanalytic term in the frequency expansion for $G_{\rm R}^{xyxy}(\omega)$ implies that hydrodynamics at second order in gradients does not, strictly speaking, work. However, in practice we are usually interested in applying hydrodynamics over some range of time, with some limited time resolution, and with a finite accuracy tolerance. If the nonanalytic term is sufficiently small compared to the ω^2 term for the frequency scales which are of actual importance in a particular problem, then there may be no issue, in practice, with using second-order hydrodynamics. To see whether this is the case, we should estimate the frequency $\omega_{\rm min}$ where the $\omega^{3/2}$ term is larger than the $\eta_{\rm cl}\tau_\pi\omega^2$ term. Clearly, for $\omega < \omega_{\rm min}$, the second-order treatment becomes invalid.

Again using the parametric estimates of Eq. (4.4), the frequency expansion of $G_{\rm R}^{xyxy}$, Eq. (3.1) plus Eq. (3.13), is parametrically of the form

$$\begin{aligned} G_{\rm R}^{xyxy}(\omega) &\sim P - i(\eta_{\rm cl} + \eta_{\rm new})\omega + \eta_{\rm cl}\tau_\pi\omega^2 \\ &\sim N_{\rm c}^2 T^4\omega^0 - iN_{\rm c}^2\alpha^{-2}T^3\omega^1 + (1+i)N_{\rm c}^0\alpha^3T^{5/2}\omega^{3/2} \\ &\quad + N_{\rm c}^2\alpha^{-4}T^2\omega^2. \end{aligned} \tag{4.6}$$

The frequency scale where the $\omega^{3/2}$ term first dominates the ω^2 term is parametrically

$$N_{\rm c}^0\alpha^3T^{5/2}\omega^{3/2} \sim N_{\rm c}^2\alpha^{-4}T^2\omega^2 \Rightarrow \omega_{\rm min} \sim N_{\rm c}^{-4}\alpha^{14}T. \tag{4.7}$$

Therefore, in a theory which *either* has a large number of colors *or* has weak coupling, the frequency scale where the new term becomes important is parametrically tiny. Then there is no obstacle to using second-order hydrodynamics for $\omega > \omega_{\rm min}$; the second-order treatment will only become invalid at a frequency scale where it is, in any case, almost irrelevant compared to the ω^1 viscous term. Note that, in particular, the large N_c limit is the reason why one does not encounter the new term e.g. in the context of AdS/CFT duality calculations. However, it has been shown that it reemerges when $1/N_c$ corrections are taken into account, cf. [45].

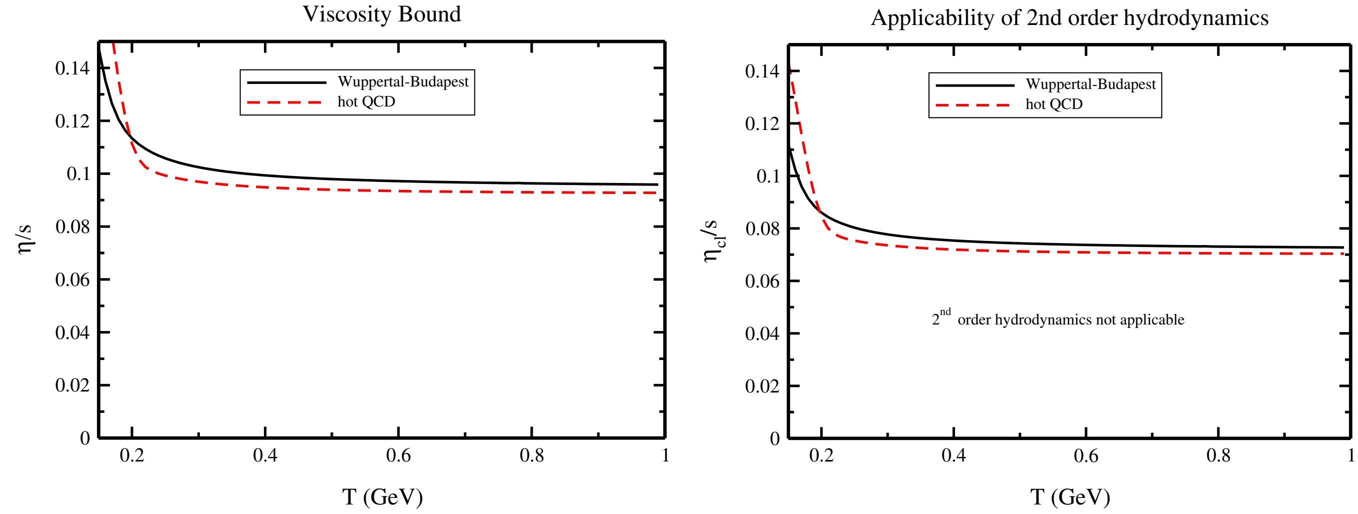
Now we turn to real-world QCD. To determine the breakdown scale of second-order hydrodynamics, we again compute the scale where the real part of the $\omega^{3/2}$ term in Eq. (3.13) equals the $\eta_{\rm cl}\tau_\pi\omega^2$ term in Eq. (3.1):

$$\omega_{\rm min} = \frac{(7 + (3/2)^{3/2})^2 T}{(240\pi)^2}\left(\frac{\eta_{\rm cl}}{s}\right)^{-7}\left(\frac{\tau_\pi}{\gamma_\eta}\right)^{-2}\left(\frac{s}{T^3}\right)^{-2}. \tag{4.8}$$

We illustrate the consequences for real-world QCD in Table I, using some of the estimates for s/T^3, τ_π, and $\eta_{\rm cl}$ discussed in the last subsection to evaluate the frequency scale $\omega_{\rm min}$. Below this frequency second-order hydrodynamics is certainly not consistent. What we see is that, for reasonable values of s/T^3 and of τ_π, this frequency scale is very low if $\eta_{\rm cl}/s = 0.16$, but it is very high for $\eta_{\rm cl}/s = 0.08$. Therefore, second-order hydrodynamics can be applied to QCD if $\eta_{\rm cl}/s \sim 0.16$ above the transition

TABLE I. Frequency scale $\omega_{\rm min}$, where the $\omega^{3/2}$ term equals the $\eta_{\rm cl}\tau_\pi\omega^2$ term in $G_{\rm R}^{xyxy}(\omega)$. For $\omega < \omega_{\rm min}$, second-order hydrodynamics is inconsistent.

$\epsilon + P$	τ_π	$\eta_{\rm cl}/s$	$\omega_{\rm min}$
$10T^4$	$3\,\frac{\eta_{\rm cl}}{sT}$	0.08	$7T$
$10T^4$	$5\,\frac{\eta_{\rm cl}}{sT}$	0.08	$2.6T$
$16T^4$	$3\,\frac{\eta_{\rm cl}}{sT}$	0.08	$2.8T$
$10T^4$	$3\,\frac{\eta_{\rm cl}}{sT}$	0.16	$T/18$
$16T^4$	$5\,\frac{\eta}{sT}$	0.16	$T/125$


Viscosity Bound

Applicability of 2nd order hydrodynamics

FIG. 2 (color online). Examples for the viscosity over entropy density bound (4.3) (left panel) and applicability of second-order viscous hydrodynamics (4.9) (right panel). The unknown parameters τ_π and $p_{\max}$ were assumed to be $\tau_\pi/\gamma_\eta = 3$ and $p_{\max} = 1/(2\tau_\pi)$, and s/T^3 was evaluated using lattice QCD equations of state from the hotQCD [44] and Wuppertal-Budapest [43] collaborations.

temperature; it breaks down below frequency scales of order $T/20$, which corresponds to time scales above 20 Fermi, safely above the microphysical scale it is intended to model in heavy-ion collisions (5–10 fm). But if $\eta_{\rm cl}/s \sim 0.08$, there is a problem. Second-order hydrodynamics is then already inconsistent on frequency scales of order $2.5T$, corresponding to time scales above 0.4 Fermi. This is a shorter time scale than our estimate $p_{\max}^{-1} \sim 1/2T$ for the shortest scale on which hydrodynamics is to be reliable. So in this case there would be *no* range of scales on which second-order hydrodynamics is applicable.

C. Viscosity—again

Let us accentuate the conclusions drawn in the previous subsection by again considering the correction to the viscosity in (3.13). As discussed above, Eq. (3.13) contains a constant, $p_{\max}$-dependent correction to the viscosity, but its actual value depends on the estimate for $p_{\max}$.

A different way to bound what values of viscosity are compatible with second-order hydrodynamics can be obtained by considering the imaginary part for the $\omega^{3/2}$ term, which can be interpreted as a frequency-dependent correction to the viscosity,

$$\eta(\omega) = \eta_{\rm cl} - \omega^{1/2}\frac{(7+(3/2)^{3/2})T}{(240\pi)\gamma_\eta^{3/2}}.$$

Note that the first and second terms become of the same order at $\omega = \omega_{\max}$, where

$$\omega_{\max} = \gamma_\eta^5 s^2 \frac{(240\pi)^2}{(7+(3/2)^{3/2})^2}.$$

Second-order hydrodynamics is certainly no longer applicable at $\omega > \omega_{\max}$, because the frequency-dependent contribution to the viscosity becomes large. On the other hand, we found above that second-order hydrodynamics also breaks down for $\omega < \omega_{\min}$, because there the frequency-dependent contribution to the relaxation time becomes dominant.

So for a second-order hydrodynamic description to be applicable at all, there has to be a frequency window $\omega_{\min} < \omega < \omega_{\max}$ which can only exist if

$$\eta_{\rm cl} > \frac{s^{2/3}T(7+(3/2)^{3/2})^{1/3}}{(240\pi)^{1/3}(\tau_\pi/\gamma_\eta)^{1/3}}. \tag{4.9}$$

This is a lower bound on viscosity if second-order hydrodynamics is to have any range of validity. If we require the range of allowed frequencies ω to be nonzero, the viscosity will have to be even higher than this bound.

The temperature dependence of the minimal viscosity can be evaluated when using lattice QCD data [43,44] for the entropy s and using $\tau_\pi/\gamma_\eta = 3$, and it is shown in Fig. 2. We find that the minimal viscosity as defined in Eq. (4.9) varies from about $\eta/s = 0.09$ at 200 MeV to about $\eta/s = 0.07$ in the Stefan-Boltzmann limit. These values are comparable to what we found to be the lowest possible value of η/s in Sec. IVA.

V. CONCLUSIONS

We have shown that, just as in nonrelativistic hydrodynamics, in a relativistic setting it is not self-consistent to consider the hydrodynamic gradient expansion to second order. The effects of thermal fluctuations in the hydrodynamic variables themselves contribute to the hydrodynamic evolution of the longest wavelength modes. These effects are suppressed both in weakly coupled theories and in theories with many degrees of freedom such as QCD with many colors, but they can be important in real-world QCD.

The correction to the shear viscosity is positive and finite, so shear viscosity is a well-defined quantity in $3+1$ dimensions. However, in real-world QCD, if $\eta_{\rm cl}/s$ is very small then the hydrodynamic fluctuations can make a significant additional contribution; estimating their size from our calculation, it does not appear to be possible for the shear viscosity to entropy ratio of real-world QCD at $T = 200$ MeV to be smaller than $\eta/s \simeq 0.1$.

The issue becomes more severe for the second-order coefficient τ_π. Strictly speaking, this coefficient cannot be defined; its definition assumes an analytic structure for the stress-stress correlation function which is violated due to long-wavelength hydrodynamic fluctuations. In practice, this is only really an issue at long time scales (low frequencies), where the inclusion of second-order effects may not be important anyway. However, how long "long" is depends on the value of the viscosity and the entropy density. For real-world QCD, if $\eta_{\rm cl}/s \sim 0.16$ then there is a wide range of frequencies where the second-order theory is applicable; it only fails at such low frequencies that the difference between first-order and second-order hydro is insignificant at the affected scales. In this case there would be no real problem, in practice, with using the second-order theory to e.g. model heavy-ion collisions. But if $\eta_{\rm cl}/s \sim 0.08$, then there is *no* range of scales where the application of second-order hydrodynamics is consistent in real-world QCD.

While our calculation focused on the case of relativistic hydrodynamics, a similar calculation should be possible in the nonrelativistic setting, presumably with qualitatively similar results. We leave this interesting project for future work.

The effects we describe should not be present in dissipative hydrodynamic simulations of heavy-ion collisions as they are currently conducted. That is because, currently, such simulations include dissipative viscous effects, but they do not include the fluctuations required by the fluctuation-dissipation theorem to ensure that hydrodynamic modes equilibrate with mean thermal excitation amplitudes. So the short-wavelength hydrodynamic waves responsible for the effects we are discussing get quenched in existing hydrodynamic simulations. It would be very interesting to try to include thermal fluctuations in hydrodynamic variables consistently in hydrodynamic studies of heavy-ion collisions.

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[1] K. Adcox *et al.* (PHENIX Collaboration), Nucl. Phys. **A757**, 184 (2005).
[2] B. B. Back *et al.*, Nucl. Phys. **A757**, 28 (2005).
[3] I. Arsene *et al.* (BRAHMS Collaboration), Nucl. Phys. **A757**, 1 (2005).
[4] J. Adams *et al.* (STAR Collaboration), Nucl. Phys. **A757**, 102 (2005).
[5] K. Aamodt *et al.* (ALICE Collaboration), arXiv:1011.3914.
[6] K. Aamodt *et al.* (ALICE Collaboration), Phys. Rev. Lett. **105**, 252301 (2010).
[7] P. F. Kolb and U. W. Heinz, in *Quark Gluon Plasma 3*, edited by R. C. Hwa and X. N. Wang (World Scientific, Singapore, 2003), pp. 634–714.
[8] T. Hirano, U. W. Heinz, D. Kharzeev, R. Lacey, and Y. Nara, Phys. Lett. B **636**, 299 (2006).
[9] P. Huovinen and P. V. Ruuskanen, Annu. Rev. Nucl. Part. Sci. **56**, 163 (2006).
[10] M. Luzum and P. Romatschke, Phys. Rev. C **78**, 034915 (2008); **79**, 039903(E) (2009).
[11] D. A. Teaney, arXiv:0905.2433.
[12] M. Luzum and P. Romatschke, Phys. Rev. Lett. **103**, 262302 (2009).
[13] M. Luzum, Phys. Rev. C **83**, 044911 (2011).
[14] H. Song, S. A. Bass, U. W. Heinz, T. Hirano, and C. Shen, Phys. Rev. C **83**, 054910 (2011).
[15] H. Song, S. A. Bass, and U. W. Heinz, Phys. Rev. C **83**, 054912 (2011).
[16] P. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005).
[17] P. Romatschke, Int. J. Mod. Phys. E **19**, 1 (2010).
[18] T. Schafer and D. Teaney, Rep. Prog. Phys. **72**, 126001 (2009).
[19] U. W. Heinz, arXiv:0901.4355.
[20] A. Monnai and T. Hirano, Nucl. Phys. **A847**, 283 (2010).
[21] E. Calzetta and J. Peralta-Ramos, Phys. Rev. D **82**, 106003 (2010).
[22] I. Müller, Z. Phys. **198**, 329 (1967).
[23] W. Israel, Ann. Phys. (N.Y.) **100**, 310 (1976); W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).
[24] W. A. Hiscock and L. Lindblom, Ann. Phys. (N.Y.) **151**, 466 (1983); Phys. Rev. D **31**, 725 (1985); **35**, 3723 (1987); Phys. Lett. A **131**, 509 (1988).
[25] K. Dusling and D. Teaney, Phys. Rev. C **77**, 034905 (2008).

[26] H. Song and U. W. Heinz, Phys. Rev. C **78**, 024902 (2008).
[27] J. Peralta-Ramos and E. Calzetta, Phys. Rev. C **82**, 054905 (2010).
[28] H. Niemi, G. S. Denicol, P. Huovinen, E. Molnar, and D. H. Rischke, Phys. Rev. Lett. **106**, 212302 (2011).
[29] B. Schenke, S. Jeon, and C. Gale, Phys. Rev. C **82**, 014903 (2010).
[30] A. K. Chaudhuri, J. Phys. G **37**, 075011 (2010).
[31] I. M. De Schepper, H. Van Beyeren, and M. H. Ernst, Physica (Amsterdam) **75**, 1 (1974).
[32] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, J. High Energy Phys. 04 (2008) 100.
[33] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, J. High Energy Phys. 02 (2008) 045.
[34] P. Romatschke, Classical Quantum Gravity **27**, 025006 (2010).
[35] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).
[36] G. D. Moore and K. A. Sohrabi, Phys. Rev. Lett. **106**, 122302 (2011).
[37] P. Kovtun and L. G. Yaffe, Phys. Rev. D **68**, 025007 (2003).
[38] P. Kovtun, G. D. Moore, and P. Romatschke (unpublished).
[39] P. Romatschke and D. T. Son, Phys. Rev. D **80**, 065021 (2009).
[40] G. Policastro, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001).
[41] P. B. Arnold, G. D. Moore, and L. G. Yaffe, J. High Energy Phys. 11 (2000) 001.
[42] M. A. York and G. D. Moore, Phys. Rev. D **79**, 054011 (2009).
[43] S. Borsányi *et al.*, J. High Energy Phys. 11 (2010) 077.
[44] A. Bazavov *et al.*, Phys. Rev. D **80**, 014504 (2009).
[45] S. Caron-Huot and O. Saremi, J. High Energy Phys. 11 (2010) 013.