PHYSICAL REVIEW D **84,** 024041 (2011)

Initial data for binary neutron stars with arbitrary spins

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(Received 13 May 2011; published 25 July 2011)

In general neutron stars in binaries are spinning. Because of the existence of millisecond pulsars we know that these spins can be substantial. We argue that spins with periods on the order of a few dozen milliseconds could influence the late inspiral and merger dynamics. Thus numerical simulations of the last few orbits and the merger should start from initial conditions that allow for arbitrary spins. We discuss quasiequilibrium approximations one can make in the construction of binary neutron star initial data with spins. Using these approximations we are able to derive two new matter equations. As in the case of irrotational neutron star binaries, one of these equations is algebraic and the other elliptic. If these new matter equations are solved together with the equations for the metric variables following the Wilson-Mathews or conformal thin sandwich approach, one can construct neutron star initial data. The spin of each star is described by a rotational velocity that can be chosen freely so that one can create stars in arbitrary rotation states. Our new matter equations reduce to the well-known limits of both corotating and irrotational neutron star binaries.

DOI: 10.1103/PhysRevD.84.024041 PACS numbers: 04.20.Ex, 04.30.Db, 97.60.Jd, 97.80.Fk

I. INTRODUCTION

Several gravitational wave detectors such as LIGO [1,2], Virgo [3,4] and GEO [5] have been operating over the last few years, while several others are in the planning or construction phase [6]. One of the most promising sources for these detectors are the inspirals and mergers of binary neutron stars. In order to make predictions about the last few orbits and the merger of such systems, fully nonlinear numerical simulations of the Einstein equations are required. To start such simulations we need initial data that describe the binary a few orbits before merger. The emission of gravitational waves tends to circularize the orbits [7,8]. Thus, during the inspiral, we expect the two neutron stars to be in quasicircular orbits around each other with a radius that shrinks on a time scale much larger than the orbital time scale. This means that the initial data should have an approximate helical Killing vector ξ^{μ}. To incorporate these ideas we will use the Wilson-Mathews approach [9,10], which is also known as conformal thin sandwich formalism [11], for the metric variables. The Wilson-Mathews approach has already been successfully used by several groups together with matter equations describing the neutron stars in either corotating [12–16] or irrotational [17–25] states. There have also been attempts to include intermediate rotation states [21,26]. However, as we will discuss in more detail later, these two attempts have certain drawbacks, because they do not correctly solve the Euler equation for the fluid. Thus, so far there is no canonical formalism to describe neutron star binaries with arbitrary spins. As pointed out by Bildsten and Cutler [27], the two neutron stars cannot be tidal locked, because the viscosity of neutron star matter is too low. Hence barring other effects like magnetic dipole radiation the spin of each star remains approximately constant. This means that initial data sequences of corotating configurations for different separations cannot be used to approximate the inspiral of two neutron stars. On the other hand, sequences of irrotational configurations can be used to approximate the inspiral of two neutron stars without spin. This fact explains why irrotational initial data are far more popular today. Nevertheless, astrophysical neutron stars will have a nonzero spin. Therefore a corotating configuration at some particular separation does have its place as a possible initial configuration with spin. It will just not remain corotating during the subsequent time evolution. Of course, real neutron stars will likely have spins that have periods different from the orbital period, and the spin direction may not be aligned with the orbital angular momentum. Thus it would be highly desirable to have a formalism that can be used to generate initial data for arbitrary initial spins.

In order to judge how important spins might be, let us discuss a few order of magnitude estimates. A typical neutron star has a mass of about 1.4 solar masses ($M_{\odot}$) and a radius on the order of 15 km. From Kepler's law the orbital period

$$P_o \sim \left(\frac{d}{50\ \mathrm{km}}\right)^{3/2}\left(\frac{M_{\odot}}{M}\right)^{1/2} 6\ \mathrm{ms} \tag{1}$$

is on the order of a few milliseconds during the last orbit before merger where the separation $d \sim 50$ km. Thus systems with spin periods that are much larger than P_o should be treatable as approximately irrotational, while systems with spin periods of a few milliseconds (such as millisecond pulsars) cannot be regarded as irrotational. Another way of judging how important spins could be during the evolution is to look at the dimensionless spin magnitude. If we assume that the spin S of a neutron star with mass m and radius R is related to its spin period P by $S = I(2\pi/P)$

with $I \sim mR^2$, we find that the dimensionless spin has a magnitude of

$$\frac{S}{m^2} \sim \left(\frac{R}{15\text{ km}}\right)^2 \frac{M_\odot}{m} \frac{3\text{ ms}}{P}. \tag{2}$$

Thus millisecond pulsars have a dimensionless spin of order one. As in the case of binary black holes [28–33], spins of this magnitudes could have a significant influence on the merger dynamics. This means that neutron stars with spin periods of a few dozen milliseconds or less should not be considered irrotational. One could of course imagine that the neutron stars spin down before they enter the strongly relativistic regime of the last few orbits before merger that is usually considered in numerical relativity simulations. In order to address this question, let us look at the famous double pulsar PSR J0737-3039 which is the only neutron star binary where both spin periods and spin down rates are known [34]. Star A has mass $m_A = 1.34 M_\odot$ and spin period $P_A = 23$ ms, while star B has $m_B = 1.25 M_\odot$ and $P_B = 2.8$ s. The orbital period is $P_o = 2.4$ h. From these numbers one derives that the system will merge in about 85 My due to the emission of gravitational waves. Both stars are currently spinning down at a rate of $\dot{P}_A = 1.7 \times 10^{-18}$ and $\dot{P}_B = 8.8 \times 10^{-16}$ [34]. If one assumes that this spin down is due to magnetic dipole radiation and defines the characteristic ages given by $\tau_A = P_A/(2\dot{P}_A) = 210$ My and $\tau_B = P_B/(2\dot{P}_B) = 50$ My, one finds that the spin period of each star obeys [35]

$$P_{A/B}(t) = P_{A/B}(0)\sqrt{1 + \frac{t}{\tau_{A/B}}}, \tag{3}$$

where the time $t = 0$ is the time today. From this it is clear that the periods at merger (at $t = 85$ My) will be $P_A(t) = 27$ ms and $P_B(t) = 4.6$ s. Thus star A will not spin down enough to be well approximated by an irrotational configuration by this time. This example shows that neutron stars in binary systems can have appreciable spins a few orbits before merger. Of course since only about ten binary neutron stars have been observed so far [36], it is not clear yet how common binary neutron stars with high spins are. However, since there are numerous millisecond pulsars, it seems reasonable to expect that neutron stars in binaries can also have millisecond spin periods. Hence the widely held belief that only irrotational configurations are realistic is not necessarily correct. It is thus necessary to develop initial data for binary neutron stars with arbitrary spins. In the next sections we will describe what approximation one can make to derive a formalism that allows for this possibility. We will see that our new equations reduce to well-known accepted results in both the corotating and irrotational cases.

Throughout we will use units where $G = c = 1$. Latin indices such as i run from 1 to 3 and denote spatial indices, while Greek indices such as μ run from 0 to 3 and denote spacetime indices. The paper is organized as follows. Section II lists the general relativistic equations that govern binary neutron stars described by perfect fluids. We use three approximate quasiequilibrium conditions to simplify these equations. We find two new matter equations that allow us to set up binary neutron stars with arbitrary spins. In Sec. III we consider the Newtonian limit of our new equations. We conclude with a discussion of our method in Sec. IV. In the Appendix we discuss our quasiequilibrium conditions for a simple case.

II. BINARY NEUTRON STARS WITH ARBITRARY ROTATION STATES

In this section we describe the equations governing binary neutron stars in arbitrary rotation states in general relativity. The equations for the metric and matter variables discussed in Secs. II A, II B, II C, and II D, are well known. Our new results concerning quasiequilibrium conditions for neutron stars with arbitrary rotation states are presented in Secs. II E and II F.

A. ADM decomposition of Einstein's equations

We use the Arnowitt-Deser-Misner (ADM) decomposition of Einstein's equations (see e.g. [37]) and introduce the 3-metric

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{4}$$

Here $g_{\mu\nu}$ is the spacetime metric and n_μ is the unit normal to the $t =$ const hypersurface. The line element is then

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{5}$$

where the lapse α and shift β^i are related to n_μ via

$$n^\mu = (1/\alpha, -\beta^i/\alpha) \qquad n_\mu = (-\alpha, 0, 0, 0). \tag{6}$$

The extrinsic curvature is defined by

$$K_{ij} = -\frac{1}{2\alpha}(\partial_t \gamma_{ij} - \mathcal{L}_\beta \gamma_{ij}). \tag{7}$$

With these definitions Einstein's equations split into the evolution equations

$$\begin{aligned} \partial_t \gamma_{ij} &= -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}, \\ \partial_t K_{ij} &= \alpha(R_{ij} - 2K_{il}K^l_j + KK_{ij}) - D_i D_j \alpha \\ &\quad + \mathcal{L}_\beta K_{ij} - 8\pi S_{ij} + 4\pi \gamma_{ij}(S - \rho), \end{aligned} \tag{8}$$

and the Hamiltonian and momentum constraint equations

$$R - K_{ij}K^{ij} + K^2 = 16\pi\rho, \qquad D_j(K^{ij} - \gamma^{ij}K) = 8\pi j^i. \tag{9}$$

Here R_{ij} and R are the Ricci tensor and scalar computed from γ_{ij}, D_i is the derivative operator compatible with γ_{ij} and all indices here are raised and lowered with the 3-metric γ_{ij}. The source terms ρ, j^i, S_{ij} and $S = \gamma^{ij}S_{ij}$ are projections of the stress-energy tensor $T_{\mu\nu}$ given by

$$\rho = T_{\mu\nu}n^\mu n^\nu,$$
$$j^i = -T_{\mu\nu}n^\mu\gamma^{\nu i}, \tag{10}$$
$$S^{ij} = T_{\mu\nu}\gamma^{\mu i}\gamma^{\nu j},$$

and correspond to the energy density, flux and stress-tensor.

B. Matter equations

We assume that the matter in both stars is a perfect fluid with a stress-energy tensor

$$T^{\mu\nu} = [\rho_0(1+\epsilon) + P]u^\mu u^\nu + Pg^{\mu\nu}. \tag{11}$$

Here ρ_0 is the mass density (which is proportional to the number density of baryons), P is the pressure, ϵ is the internal energy density divided by ρ_0 and u^μ is the 4-velocity of the fluid. The matter variables in Eq. (10) are then

$$\rho = \alpha^2[\rho_0(1+\epsilon) + P]u^0u^0 - P,$$
$$j^i = \alpha[\rho_0(1+\epsilon) + P]u^0u^0(u^i/u^0 + \beta^i),$$
$$S^{ij} = [\rho_0(1+\epsilon) + P]u^0u^0(u^i/u^0 + \beta^i)(u^j/u^0 + \beta^j) + P\gamma^{ij}. \tag{12}$$

From $\nabla_\nu T^{\mu\nu} = 0$ we obtain the relativistic Euler equation

$$[\rho_0(1+\epsilon) + P]u^\nu\nabla_\nu u^\mu = -(g^{\mu\nu} + u^\mu u^\nu)\nabla_\nu P, \tag{13}$$

which together with the continuity equation

$$\nabla_\nu(\rho_0 u^\nu) = 0 \tag{14}$$

governs the fluid.

In order to simplify the problem we assume that internal energy ϵ is a function of ρ_0 alone (which implies a temperature of zero), and use a polytropic equation of state

$$P = \kappa\rho_0^{1+1/n}. \tag{15}$$

We also introduce the specific enthalpy

$$h = 1 + \epsilon + P/\rho_0. \tag{16}$$

Changes in h at zero temperature obey

$$dh = dP/\rho_0. \tag{17}$$

Using Eqs. (16) and (17) we can rewrite the Euler equation (13) as

$$u^\mu\nabla_\mu\tilde{u}_\nu + \nabla_\nu h = 0, \tag{18}$$

where

$$\tilde{u}^\nu = hu^\nu. \tag{19}$$

It is often convenient to introduce the dimensionless ratio

$$q = P/\rho_0, \tag{20}$$

which we can use to write

$$h = (n+1)q + 1, \qquad \rho_0 = \kappa^{-n}q^n,$$
$$P = \kappa^{-n}q^{n+1}, \qquad \epsilon = nq. \tag{21}$$

C. Decomposition of 3-metric and extrinsic curvature

As in [9,10], the 3-metric γ_{ij} is decomposed into a conformal factor ψ and a conformal metric $\bar{\gamma}_{ij}$ such that

$$\gamma_{ij} = \psi^4\bar{\gamma}_{ij}. \tag{22}$$

The extrinsic curvature is split into its trace K and its tracefree part A_{ij} by writing it as

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K. \tag{23}$$

D. Quasiequilibrium assumptions for the metric variables

We now make some additional simplifying assumptions. First we assume that the binary is in an approximately circular orbit and that the spins of each star remain approximately constant. As in the case of binary black holes (see e.g. [38,39]), this implies the existence of an approximate helical Killing vector ξ^μ with $\mathcal{L}_\xi g_{\mu\nu} \approx 0$. In order to clarify the meaning of the approximate sign, we now briefly discuss two cases.

If both spins are parallel to the orbital angular momentum, we have $\mathcal{L}_\xi g_{\mu\nu} = O(P_o/T_{\rm ins})$, where we assume the inspiral time scale $T_{\rm ins}$ to be much longer than the orbital time scale P_o. That is, in a corotating coordinate system all metric time derivatives are of order $O(P_o/T_{\rm ins})$ and thus small. For arbitrary spins the situation becomes more complicated. We can again use corotating coordinates, but in this coordinate system the spin vectors will be precessing on an orbital time scale P_o. This means there are matter currents that change on a time scale P_o, while the matter distribution itself only changes on the inspiral time scale $T_{\rm ins}$. In this case it is useful to consider gravity to be made up of gravitoelectric and gravitomagnetic fields [37,40]. The gravitoelectric parts of the metric are sourced by the matter distribution and thus change only on the time scale $T_{\rm ins}$, while the gravitomagnetic parts of the metric are sourced by matter currents and thus change on the shorter time scale P_o. However, the gravitomagnetic parts are smaller than the gravitoelectric parts by $O(v/c)$ [37,40]. Thus we now have $\mathcal{L}_\xi g_{\mu\nu} = O(v/c) \approx 0$, where we assume that the orbital velocity v is smaller than the speed of light.

An approximate helical Killing vector with $\mathcal{L}_\xi g_{\mu\nu} \approx 0$ implies that

$$\mathcal{L}_\xi \bar{\gamma}_{ij} \approx \mathcal{L}_\xi K \approx 0, \tag{24}$$

which is what we will need to assume here for the metric variables. In a corotating coordinate system where the time evolution vector lies along ξ^μ, the time derivatives of these metric variables are then equal to zero. From $\partial_t \bar{\gamma}_{ij} = 0$ it follows that

$$A^{ij} = \frac{1}{2\psi^4 \alpha} (\bar{L}\beta)^{ij}, \tag{25}$$

where

$$(\bar{L}\beta)^{ij} = \bar{D}^i \beta^j + \bar{D}^j \beta^i - \frac{2}{3} \bar{D}_k \beta^k, \tag{26}$$

and $\bar{D}_k$ is the derivative operator compatible with $\bar{\gamma}_{ij}$. The assumption $\partial_t K = 0$ together with the evolution equation of K [derived from Eq. (8)] implies

$$\psi^{-5}[\bar{D}_k \bar{D}^k(\alpha\psi) - \alpha \bar{D}_k \bar{D}^k \psi] = \alpha(R + K)^2 + \beta^i \bar{D}_i K + 4\pi\alpha(S - 3\rho). \tag{27}$$

E. Quasiequilibrium assumptions for the matter variables

In an inertial frame (i.e. a frame with $\lim_{r\to\infty} \beta^i = 0$), the approximate helical Killing vector has the components

$$\xi^\mu = (1, -\Omega[x^2 - x^2_{\rm CM}], \Omega[x^1 - x^1_{\rm CM}], 0). \tag{28}$$

Here $x^i_{\rm CM}$ denotes the center of mass position of the system (which can be obtained from surface integrals at infinity, e.g. Eq. (20.11) in [37]), and Ω is the orbital angular velocity, which we have chosen to lie along the x^3-direction. Following Shibata [41], we decompose the fluid velocity u^μ into a piece along ξ^μ and a spatial vector V^μ and write

$$u^\mu = u^0(\xi^\mu + V^\mu), \tag{29}$$

where $u^0 = -u^\mu n_\mu / \alpha$.

In terms of ξ^μ and V^μ the fluid equations (14) and (18) can be recast as

$$D_i(\rho_0 \alpha u^0 V^i) + \alpha[\mathcal{L}_\xi(\rho_0 u^0) + \rho_0 u^0 g^{\mu\nu} \mathcal{L}_\xi g_{\mu\nu}] = 0 \tag{30}$$

and

$$D_i\left(\frac{h}{u^0} + {}^{(3)}\tilde{u}_k V^k\right) + V^k(D_k {}^{(3)}\tilde{u}_i - D_i {}^{(3)}\tilde{u}_k) + \gamma_i^\nu \mathcal{L}_\xi \tilde{u}_\nu = 0, \tag{31}$$

where

$${}^{(3)}\tilde{u}^i = \gamma^i_\nu \tilde{u}^\nu. \tag{32}$$

When one constructs neutron star initial data for corotating or irrotational configurations one usually assumes that the Lie derivatives of all matter variables with respect to ξ^μ vanish [13,41,42]. However, for arbitrary spins this may not be the best approximation, since the portion of the fluid velocity responsible for the star's spin is not constant along ξ^μ if the spin remains constant while the stars orbit around each other. So we should not assume that $\mathcal{L}_\xi \tilde{u}^\mu$ vanishes. Rather we will split $\tilde{u}^\mu$ into an irrotational and a rotational part and assume that only the Lie derivative of the irrotational part vanishes. In the irrotational (zero spin) case, we have $D_i {}^{(3)}\tilde{u}_j - D_j {}^{(3)}\tilde{u}_i = 0$ and thus ${}^{(3)}\tilde{u}_i$ is derivable from a potential. For general rotation states we write

$${}^{(3)}\tilde{u}^i = D^i\phi + w^i, \tag{33}$$

so that $D^i\phi$ and w^i denote the irrotational and rotational pieces of the velocity. In order to assure that w^i is purely rotational, one usually requires that

$$D_i w^i = 0. \tag{34}$$

In Sec. II F we will show how one can choose w^i such that Eq. (34) is satisfied. However, Eq. (34) is not explicitly used in any of the derivations in this subsection.

Note that, once ${}^{(3)}\tilde{u}_i$ is known, $\tilde{u}^0 = -\tilde{u}^\mu n_\mu / \alpha$ can be obtained from $\tilde{u}^\mu \tilde{u}_\mu = -h^2$. If we choose $w^\mu n_\mu = 0$, the split of ${}^{(3)}\tilde{u}_i$ in Eq. (33) can be extended to

$$\tilde{u}^\mu = \nabla^\mu \phi + w^\mu, \tag{35}$$

where the time dependence of ϕ is now chosen such that it satisfies $\nabla^0 \phi = \tilde{u}^0$.

In order to simplify Eqs. (30) and (31) we now assume that

$$\mathcal{L}_\xi(\rho_0 u^0) \approx \mathcal{L}_\xi g_{\mu\nu} \approx 0, \tag{36}$$

but we will not assume that $\mathcal{L}_\xi \tilde{u}_\nu$ vanishes as well. Instead we assume that

$$\gamma_i^\nu \mathcal{L}_\xi(\nabla_\nu \phi) \approx 0, \tag{37}$$

so that the time derivative of the irrotational piece of the fluid velocity vanishes in corotating coordinates. Furthermore we also assume that

$$\gamma_i^\nu \mathcal{L}_{\bar{\xi}} w_\nu \approx 0, \tag{38}$$

where we have defined

$$\bar{\xi}^\mu = \frac{\nabla^\mu \phi}{\tilde{u}^0}. \tag{39}$$

The assumption in Eq. (38) describes the fact that the rotational piece of the fluid velocity (which gives rise to the spin) is constant along $\bar{\xi}^\mu$ which is parallel to the worldline of the star center. Defining

$$\Delta\xi^\mu = \xi^\mu - \bar{\xi}^\mu = (0, \Delta k^i) \tag{40}$$

and using Eqs. (37) and (38), the Lie derivative term in Eq. (31) can be written as

$$\gamma_i^\nu \mathcal{L}_\xi \tilde{u}_\nu \approx \gamma_i^\nu \mathcal{L}_\xi w_\nu = \gamma_i^\nu \mathcal{L}_{\bar{\xi}+\Delta\xi} w_\nu \approx \gamma_i^\nu \mathcal{L}_{\Delta\xi} w_\nu = {}^{(3)}\mathcal{L}_{\Delta k} w_i. \tag{41}$$

Here ${}^{(3)}\mathcal{L}$ is the Lie derivative in three dimensions. Thus Eqs. (30) and (31) simplify and can be rewritten as

$$D_i(\rho_0 \alpha u^0 V^i) = 0 \tag{42}$$

and

$$D_i\left(\frac{h}{u^0} + V^k D_k \phi\right) + {}^{(3)}\mathcal{L}_{V+\Delta k} w_i = 0. \tag{43}$$

In order to further simplify Eq. (43), note that

$$V^i + \Delta k^i = \frac{u^i}{u^0} - \xi^i + \Delta\xi^i = \frac{\tilde{u}^i}{\tilde{u}^0} - \bar{\xi}^i = \frac{w^i}{\tilde{u}^0}, \tag{44}$$

which follows from Eqs. (29), (33), (39), and (40). Hence

$${}^{(3)}\mathcal{L}_{V+\Delta k} w_i = \frac{w_i}{\tilde{u}^0} \mathcal{L}_{w/\tilde{u}^0} \tilde{u}^0 + w^{k(3)} \mathcal{L}_{w/\tilde{u}^0} \gamma_{ik} \approx 0, \tag{45}$$

where we have assumed that both $\tilde{u}^0$ and γ_{ik} are approximately constant along the 3-vector $w^i/\tilde{u}^0$, which lies along the direction of the fluid's rotational velocity piece w^i. Note that ${}^{(3)}\mathcal{L}_{V+\Delta k} w_i$ is of order $O(w)^2$, while assumptions (37) and (38) are $O(1)$ and $O(w)$ in w^i. Thus alternatively we can view Eq. (45) as an assumption that will hold if w^i is small compared to $D^i \phi$. All three assumptions (37), (38), and (45) are discussed in the Appendix for a simple case.

With the last assumption in Eq. (45), the Euler equation (43) yields

$$\frac{h}{u^0} + V^k D_k \phi = -C, \tag{46}$$

where C is a constant of integration that is in general different for each star.

In the corotating case where $V^\mu = 0$, Eq. (42) is identically satisfied and Eq. (46) reduces to

$$h = -C u^0, \tag{47}$$

The u^0 here can be computed from $u_\mu u^\mu = -1$ and reduces to

$$u^0 = 1 \Big/ \sqrt{\alpha^2 - (\beta_i + \xi_i)(\beta^i + \xi^i)} \tag{48}$$

for $V^\mu = 0$.

If the stars are not corotating, V^i is given by

$$V^i = \frac{D^i \phi + w^i}{h u^0} - (\beta^i + \xi^i). \tag{49}$$

In this case the continuity equation (42) becomes

$$D_i\left[\frac{\rho_0 \alpha}{h}(D^i \phi + w^i) - \rho_0 \alpha u^0 (\beta^i + \xi^i)\right] = 0. \tag{50}$$

Note that $u_\mu u^\mu = -1$ yields

$$u^0 = \frac{\sqrt{h^2 + (D_i \phi + w_i)(D^i \phi + w^i)}}{\alpha h}, \tag{51}$$

so that Eq. (50) is a nonlinear elliptic equation for ϕ. Using u^0 from Eq. (51), the integrated Euler equation (46) can then be solved for h with the result

$$h = \sqrt{L^2 - (D_i \phi + w_i)(D^i \phi + w^i)}, \tag{52}$$

where we use the abbreviations

$$L^2 = \frac{b + \sqrt{b^2 - 4\alpha^4 [(D_i \phi + w_i) w^i]^2}}{2\alpha^2} \tag{53}$$

and

$$b = [(\xi^i + \beta^i) D_i \phi - C]^2 + 2\alpha^2 (D_i \phi + w_i) w^i. \tag{54}$$

Note that the rotational piece of the fluid velocity w^i can be freely chosen, and that the fluid equations (50) and (52) reduce to the well-known result for irrotational stars [41,42] if $w^i = 0$.

F. Further simplifications and boundary conditions

Next we also choose a maximal slice with $K = 0$ and assume that the conformal 3-metric is flat and given by [9,10]

$$\bar{\gamma}_{ij} = \delta_{ij}. \tag{55}$$

This latter assumption merely simplifies our equations and could in principle be improved by e.g. choosing a post-Newtonian expression for $\bar{\gamma}_{ij}$ or by matching a post-Newtonian metric with a single neutron star solution similar to [43–48]. Using Eq. (55), the Hamiltonian and momentum constraints in Eqs. (9) and (27) simplify, and we obtain

$$\begin{aligned}
\bar{D}^2 \psi &= -\frac{\psi^5}{32\alpha^2} (\bar{L}B)^{ij} (\bar{L}B)_{ij} - 2\pi \psi^5 \rho, \\
\bar{D}_j (\bar{L}B)^{ij} &= (\bar{L}B)^{ij} \bar{D}_j \ln(\alpha \psi^{-6}) + 16\pi \alpha \psi^4 j^i, \\
\bar{D}^2 (\alpha \psi) &= \alpha \psi \left[\frac{7\psi^4}{32\alpha^2} (\bar{L}B)^{ij} (\bar{L}B)_{ij} + 2\pi \psi^4 (\rho + 2S)\right],
\end{aligned} \tag{56}$$

where $(\bar{L}B)^{ij} = \bar{D}^i B^j + \bar{D}^j B^i - \frac{2}{3}\delta^{ij} \bar{D}_k B^k$, $\bar{D}_i = \partial_i$, and

$$B^i = \beta^i + \xi^i + \Omega \epsilon^{ij3} (x^j - x^j_{\rm CM}). \tag{57}$$

The elliptic equations (56) have to be solved subject to the boundary conditions

$$\lim_{r\to\infty} \psi = 1, \qquad \lim_{r\to\infty} B^i = 0, \qquad \lim_{r\to\infty} \alpha \psi = 1 \tag{58}$$

at spatial infinity.

Equations (56) need to be solved together with the fluid equations (50) and (52). These fluid equations simplify in corotating coordinates where $\xi^i = 0$. Furthermore they can be expressed in terms of the derivative operator $\bar{D}_i$ by noting that

$$D_i\phi = \bar{D}_i\phi, \qquad D^i\phi = \psi^{-4}\bar{D}^i\phi. \tag{59}$$

In addition, w^i can be replaced by

$$w^i = \psi^{-6}\bar{w}^i. \tag{60}$$

The latter scaling is useful since

$$D_i w^i = \psi^{-6}\bar{D}_i\bar{w}^i, \tag{61}$$

so that, if we choose $\bar{D}_i\bar{w}^i = 0$, we automatically obtain $D_i w^i = 0$. One obvious choice for the conformal rotational velocity could be

$$\bar{w}^{\,i} = \epsilon^{ijk}\omega^j(x^k - x^k_{C*}), \tag{62}$$

where x^k_{C*} is the location of the star center, which could be defined as the point with the highest rest mass density ρ_0 or as the center of mass of the star. However, it is also possible to choose

$$\bar{w}^{\,i} = f(|x^n - x^n_{C*}|)\epsilon^{ijk}\omega^j(x^k - x^k_{C*}), \tag{63}$$

where $f(|x^n - x^n_{C*}|)$ is any function that depends only on the conformal distance from the star's center. Thus the method described here is capable of giving an arbitrary rotational velocity to each star.

Also note that we need a boundary condition at the star surface to solve Eq. (50). This boundary condition can be obtained from Eq. (50) itself by evaluating Eq. (50) on the boundary where $\rho_0 \to 0$ but $\bar{D}_i\rho_0 \neq 0$. Taking this limit, we obtain

$$(D^i\phi)D_i\rho_0 + w^i D_i\rho_0 = hu^0(\beta^i + \xi^i)D_i\rho_0 \tag{64}$$

at the star surface. In applications it may be a good idea to choose $\bar{w}^i$ such that $\bar{w}^i\bar{D}_i\rho_0$ vanishes; otherwise the rotational velocity has a component perpendicular to the star's surface. Also notice that Eq. (50) together with its boundary condition in Eq. (64) do not uniquely specify the solution. If ϕ solves both Eqs. (50) and (64), then $\phi + \text{const}$ will be a solution as well. In numerical codes this kind of ambiguity is usually removed by adding e.g. the volume integral of ϕ over the star to the boundary condition.

III. THE NEWTONIAN LIMIT

We now investigate the Newtonian limit of the approximate matter equations derived above. If φ is the Newtonian potential satisfying $\partial_i\partial^i\varphi = 4\pi\rho_0$ and $v^i = u^i/u^0$ is the Newtonian fluid velocity (in inertial coordinates), we can express the Newtonian limit as

$$\begin{aligned}
g_{00} &\to -1 - 2\varphi,\\
\alpha &\to 1 + \varphi,\\
g_{0i} = \beta_i &\to 0,\\
g_{ij} = \gamma_{ij} &\to \delta_{ij},\\
\xi^i &\to [\Omega \times x]^i,\\
u^i &\to u_i \to \tilde{u}_i \to {}^{(3)}\tilde{u}_i \to v_i,\\
v_i &\to \partial_i\phi + w_i,\\
V^i &\to \partial^i\phi + w^i - \xi^i,\\
u^0 &\to 1 + \frac{v^2}{2} - \varphi,\\
u_0 = g_{0\mu}u^\mu &\to -1 - \frac{v^2}{2} - \varphi,\\
h &= 1 + h_N = 1 + \epsilon + \frac{P}{\rho_0},
\end{aligned} \tag{65}$$

where $v = \sqrt{v^i v_i}$ and V^i is the fluid velocity in corotating coordinates.

Using Eqs. (65) and (50) reduces to

$$\partial_i(\rho_0 V^i) = 0, \tag{66}$$

which is the Newtonian continuity equation in corotating coordinates where $\partial_{t'}\rho_0 = 0$.

In order to examine the limit of Eq. (46), we first note that

$$\frac{h}{u^0} + {}^{(3)}\tilde{u}_k V^k = -hu_\mu\xi^\mu \tag{67}$$

and

$$-D_i(V^k w_k) = V^k(D_k{}^{(3)}\tilde{u}_i - D_i{}^{(3)}\tilde{u}_k) - {}^{(3)}\mathcal{L}_V w_i. \tag{68}$$

Using Eqs. (67) and (68) together with the limits in Eqs. (65), the gradient of Eq. (46) yields

$$\partial_i\left(h_N + \frac{v^2}{2} + \varphi + v_k\xi^k\right) + V^k(\partial_k v_i - \partial_i v_k) = {}^{(3)}\mathcal{L}_V w_i. \tag{69}$$

In order to show that this is the Euler equation of Newtonian physics, we first note that the time derivative $\partial_{t'}$ in corotating coordinates is related to the time derivative ∂_t in inertial coordinates by $\partial_{t'} = \partial_t + {}^{(3)}\mathcal{L}_\xi$. Then

$$\begin{aligned}
\partial_{t'}V_i &= \partial_{t'}(\partial_i\phi + w_i - \xi_i)\\
&= \partial_{t'}(\partial_i\phi) + \partial_t w_i + {}^{(3)}\mathcal{L}_\xi w_i - \partial_{t'}\xi_i\\
&= \partial_{t'}(\partial_i\phi) - \partial_{t'}\xi_i + (\partial_t w_i + {}^{(3)}\mathcal{L}_\xi w_i)\\
&\quad + {}^{(3)}\mathcal{L}_{V+\Delta k}w_i - {}^{(3)}\mathcal{L}_V w_i.
\end{aligned} \tag{70}$$

In the last equality, all terms but the last vanish if we make the same assumptions as in Eqs. (37), (38), and (45). Hence Eq. (69) can be rewritten as

$$\partial_{t'} V_i + V^k \partial_k V_i + 2[\Omega \times V]_i + [\Omega \times (\Omega \times x)]_i = -\frac{\partial_i P}{\rho_0} - \partial_i \varphi, \qquad (71)$$

which is simply the well-known Euler equation of Newtonian physics expressed in corotating coordinates. Thus we see that our new matter equations reduce to the correct result in the Newtonian limit.

IV. DISCUSSION

Realistic neutron stars in binaries will be spinning. From observations of millisecond pulsars we know that these spins can be substantial enough to influence the late inspiral and merger dynamics of the binary.

There have been prior attempts to construct initial data for spinning neutron stars. In [21] (hereafter MS), the Euler equation is not solved directly. Rather it is replaced by an equation equivalent to $\frac{h}{u^0} + {}^{(3)}\tilde{u}_k V^k = -C$. However, as already pointed out by MS, this equation agrees with the integrated Euler equation (46) only for the corotating and the irrotational case. Thus in general the Euler equation is violated in the MS approach. Furthermore, MS split u^i/u^0 and not ${}^{(3)}\tilde{u}^i$ into an irrotational and a rotational part [see Eq. (33)]. This has two consequences. First, their equations do not have the correct limit in the irrotational case. And second, since u^μ/u^0 is not a purely spatial vector, it is inconsistent to set u^i/u^0 equal to something like $D^i\phi$, which is a purely spatial vector. This explains why the continuity equation of MS has no shift terms, unlike in Eq. (50) and in [41,42]. When MS compare their results for a particular corotating case with [20], they find that their approach introduces errors of about 2% in the angular momentum.

Another approach to include spin that is aligned with the orbital angular momentum was proposed in [26] (hereafter BS). This approach does not seek to analytically integrate the Euler equation as we have done here. Instead the divergence of Eq. (31) is set to zero, which leads to another elliptic equation. However, as pointed out first by Gourgoulhon [49], in general, the Euler equation itself is not satisfied if we only enforce its divergence to be zero. Hence the BS approach can lead to initial data that do not obey the Euler equation. Furthermore, the boundary condition given by BS for their new elliptic equation seems to imply that the star surface is always at the same location. If we consider the usual numerical treatment where we start from an initial guess for the stars which is iteratively refined, it is unclear how the star surface can change during the iterations.

The purpose of this paper is thus to introduce a new method for the computation of binary neutron star initial data with arbitrary rotation states. Our method is derived from the standard matter equations of perfect fluids together with certain quasiequilibrium assumptions. We assume that there is an approximate helical Killing vector ξ^μ and that Lie derivatives of the metric variables with respect to ξ^μ vanish. We also assume that scalar matter variables such as h or ρ_0 have Lie derivatives that vanish with respect to ξ^μ. However, as discussed in the Appendix, the Lie derivative of the fluid velocity u^μ is expected to be nonzero for arbitrary spins. We split the fluid velocity u^μ into an irrotational piece (derived from a potential ϕ) and a rotational piece w^i, and assume that only the irrotational piece has a vanishing Lie derivative [see Eq. (37)] with respect to ξ^μ. This can be interpreted as the natural generalization of the irrotational case where one commonly assumes $\mathcal{L}_\xi h u^\mu = 0$. Furthermore we know that the spin of each star remains approximately constant since the viscosity of the stars is insufficient for tidal coupling [27]. To incorporate this fact, we use Eq. (38) which is based on the assumption that w^i is constant along the star's motion described by the irrotational velocity piece $\nabla^\mu\phi$. Since $\nabla^\mu\phi$ is equivalent to the velocity of the star center, this latter assumption captures the fact that the spin or rotational velocity w^i of each star remains approximately constant. With these two assumptions, the Euler equation simplifies to Eq. (43). In order to analytically integrate Eq. (43), we use the additional assumption (45) that $\tilde{u}^0$ and γ_{ij} are constant along the field lines of the rotational velocity piece. We then arrive at the two matter equations (50) and (52). These equations reduce to well-known equations [41,42] for the irrotational case of $w_i = 0$. They also reduce to the corotating limit (where $V^i = 0$) as is evident from Eqs. (42) and (46) which are written in terms of V^i. Furthermore, our equations reduce to the correct Newtonian limit.

The elliptic equation in Eq. (50) can be solved (for ϕ) together with the Eqs. (56) for the metric variables once the enthalpy h is known. However, the enthalpy given by Eq. (52) depends on the metric variables, ϕ and w^i. Apart from their dependence on w^i, this set of equations has a similar structure as for the case of irrotational neutron stars (where w^i vanishes). The standard way (see e.g. [16]) to solve such a mixture of elliptic and algebraic equations is by iteration, where at each step we first solve the elliptic equations for a given h and then use the algebraic Eq. (52) to update h. At each step we also need to specify w^i. One way to do this would be by choosing a constant $\bar{w}^i$ as in Eq. (62). Note however, that other choices for w^i are possible. We plan to investigate these possibilities in future numerical studies of our new method. For such studies it might useful to use a numerical code like LORENE [18,50–52] or SGRID [16,53,54], where the star surface is always at a domain boundary so that the boundary condition in Eq. (64) can be easily implemented.

ACKNOWLEDGMENTS

It is a pleasure to thank Pedro Marronetti for helpful discussions. This work was supported by NSF Grant No. PHY-0855315.

APPENDIX A: THE MATTER QUASIEQUILIBRIUM ASSUMPTIONS IN A SIMPLIFIED CASE

In the Newtonian limit, ${}^{(3)}\tilde{u}_i$ is equal to the fluid velocity in the inertial frame. When the two stars are well separated, it is clear that each star is well approximated by an orbiting and spinning sphere. In this case the fluid velocity inside a star is given by

$$ {}^{(3)}\tilde{u}_i \approx [\Omega \times x_{C*}]_i + [\omega \times (x - x_{C*})]_i, \qquad \text{(A1)} $$

where x_{C*i} is the (time-dependent) location of the star center, and Ω_i and ω_i are the angular velocities of the orbital and spinning motion of the star. Within this approximation we then have

$$ \phi \approx [\Omega \times x_{C*}]_k x^k, \qquad w_i \approx [\omega \times (x - x_{C*})]_i. \qquad \text{(A2)} $$

It is then easy to verify that the assumptions in Eqs. (37) and (38) are identically satisfied. Furthermore for approximate spherical symmetry we see that the assumptions in Eq. (45) hold as well. In addition, we find that

$$ \Delta k_i = [\Omega \times x]_i - D_i\phi = [\Omega \times (x - x_{C*})]_i. \qquad \text{(A3)} $$

From this it follows that

$$ \gamma_i^\nu \mathcal{L}_\xi \tilde{u}_\nu = {}^{(3)}\mathcal{L}_{\Delta k} \tilde{w}_i = (\Omega_i \omega_j - \omega_i \Omega_j)(x^j - x^j_{C*}), \qquad \text{(A4)} $$

which illustrates that $\mathcal{L}_\xi \tilde{u}_\nu$ does not vanish even in this simplified case. The only case when $\mathcal{L}_\xi \tilde{u}_\nu$ can vanish is if the spin is aligned with the orbital angular momentum, i.e. if $\omega_i = a\Omega_i$ for some constant a.

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