Black hole enthalpy and an entropy inequality for the thermodynamic volume

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In a theory where the cosmological constant Λ or the gauge coupling constant g arises as the vacuum expectation value, its variation should be included in the first law of thermodynamics for black holes. This becomes $dE = TdS + \Omega_i dJ_i + \Phi_\alpha dQ_\alpha + \Theta d\Lambda$, where E is now the enthalpy of the spacetime, and Θ, the thermodynamic conjugate of Λ, is proportional to an effective volume $V = -\frac{16\pi\Theta}{D-2}$ "inside the event horizon." Here we calculate Θ and V for a wide variety of D-dimensional charged rotating asymptotically anti-de Sitter (AdS) black hole spacetimes, using the first law or the Smarr relation. We compare our expressions with those obtained by implementing a suggestion of Kastor, Ray, and Traschen, involving Komar integrals and Killing potentials, which we construct from conformal Killing-Yano tensors. We conjecture that the volume V and the horizon area A satisfy the inequality $R \equiv ((D-1)V/\mathcal{A}_{D-2})^{1/(D-1)}(\mathcal{A}_{D-2}/A)^{1/(D-2)} \geq 1$, where $\mathcal{A}_{D-2}$ is the volume of the unit $(D-2)$ sphere, and we show that this is obeyed for a wide variety of black holes, and saturated for Schwarzschild-AdS. Intriguingly, this inequality is the "inverse" of the isoperimetric inequality for a volume V in Euclidean $(D-1)$ space bounded by a surface of area A, for which $R \leq 1$. Our conjectured *reverse isoperimetric inequality* can be interpreted as the statement that the entropy inside a horizon of a given "volume" V is maximized for Schwarzschild-AdS. The thermodynamic definition of V requires a cosmological constant (or gauge coupling constant). However, except in seven dimensions, a smooth limit exists where Λ or g goes to zero, providing a definition of V even for asymptotically flat black holes.

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I. INTRODUCTION

In theories where physical constants such as Yukawa couplings, gauge coupling constants or Newton's constant G and the cosmological constant Λ are not fixed *a priori*, but arise as vacuum expectation values and hence can vary, their variation should be included in thermodynamic formulae such as the first law of black hole thermodynamics. In fact such "constants" are typically to be thought of as the values at infinity of scalar fields. In the case of modulus fields, the conjugate thermodynamic variables are scalar charges [1]. The cosmological constant Λ behaves like a pressure,

$$P = -\frac{D-2}{16\pi}\Lambda = \langle \mathcal{V} \rangle, \tag{1.1}$$

where $\mathcal{V}$ is the potential of any scalars, and the conjugate thermodynamic variable V is an effective volume inside the horizon, or alternatively a regularized version of the difference in the total volume of space with and without the black hole present [2–5].[1] Thus the first law of thermodynamics for black holes reads

$$dE = TdS + \sum_i \Omega_i dJ_i + \sum_\alpha \Phi_\alpha dQ_\alpha + VdP \tag{1.2}$$

and E should be thought of as the total *gravitational enthalpy*, which is the analogue of

$$H = U + PV, \tag{1.3}$$

where U is the total internal energy, so that

$$dU = TdS + \sum_i \Omega_i dJ_i + \sum_\alpha \Phi_\alpha dQ_\alpha - PdV. \tag{1.4}$$

(Some further discussion of varying the cosmological constant in the black hole thermodynamical context has recently been given in [6,7].)

Of course if the cosmological constant is not treated as a variable, then H, U and E coincide. However, even if the cosmological constant is not varied the quantities P and Θ enter the generalized *Smarr-Gibbs-Duhem relation*, since Λ affects the scaling properties of the thermodynamic variables. The Smarr-Gibbs-Duhem relation is a simple consequence of the first law (1.2), combined with dimensional analysis. In D spacetime dimensions it reads [4,5,8]

$$E = (D-2)\Big(TS + \sum_i \Omega_i J_i\Big) + \sum_\alpha \Phi_\alpha Q_\alpha - \frac{2}{D-3}VP. \tag{1.5}$$

Moreover, in the simplest case of the Schwarzschild anti-de Sitter metric, and in its single charged version, Reissner-Nordström anti-de Sitter, one finds that

[1]The term VdP is also written as $\Theta d\Lambda$.

$$V = \frac{\mathcal{A}_{D-2}}{D-1} r_H^{D-1}, \tag{1.6}$$

where $\mathcal{A}_{D-2}$ is the area of the unit $(D-2)$-sphere and r_H is the radius of the horizon expressed in terms of the *Schwarzschild radial coordinate*.

These general considerations become especially interesting in the case of gauged supergravity and string theories, where the cosmological constant and the gauge coupling constant g are related by

$$\Lambda = -(D-1)g^2, \qquad P = \frac{(D-2)(D-1)}{16\pi} g^2. \tag{1.7}$$

Such theories can be obtained by means of sphere reductions from the eleven- or ten-dimensional ungauged supergravity theories. The most interesting gauged supergravities arise in $D = 4$, obtained by an S^7 reduction from 11 dimensions; in $D = 5$, obtained by an S^5 reduction from ten dimensions; and in $D = 7$, obtained by an S^4 reduction from eleven dimensions. In these cases the cosmological constant and gauge coupling constant are related to the curvature of the compactifying sphere; that is, they are proportional to $(\text{radius})^{-2}$. Thus in these cases the term VdP in the first law incorporates the thermodynamics of the extra dimensional sphere, and its inclusion would be important if the size of the extra dimensions, i.e., the radius of the sphere, were to change with time.

If one is contemplating time-dependent extra dimensions, one should bear in mind that in descending from $(n + D)$ to D spacetime dimensions on a compact manifold K_n one has the relation

$$G_D = \frac{G_{D+n}}{\text{Vol}(K_n)} \tag{1.8}$$

between the Newton constants. Thus if G_{D+n} is regarded as fundamental and hence unchanging, then if $\text{Vol}(K^n)$ changes with time, so will G_D, *and its variation should also be contained in the first law.*

In the remainder of this paper, we shall use the cosmological constant Λ rather than the pressure P as the intensive thermodynamic variable, and the conjugate extensive variable will be taken to be Θ. Thus the first law will be

$$dE = TdS + \Omega_i dJ_i + \Phi_\alpha dQ_\alpha + \Theta d\Lambda, \tag{1.9}$$

where the metric in D dimensions is asymptotically AdS, with the Ricci tensor equal to (or, in the case of charged black holes, approaching) $R_{\mu\nu} = \Lambda g_{\mu\nu}$. We take $\Lambda = -(D-1)g^2$, where for solutions in gauged supergravities, g is the gauge coupling constant.

From dimensional scaling arguments, the generalized Smarr relation is

$$E = \frac{D-2}{D-3}(TS + \Omega_i J_i) + \Phi_\alpha Q_\alpha - \frac{2}{D-3}\Theta\Lambda. \tag{1.10}$$

The pressure and cosmological constant are related by (1.1), and so Θ is related to the volume by

$$\Theta = -\frac{(D-2)}{16\pi} V. \tag{1.11}$$

In this paper, we investigate the role of the volume term in the thermodynamics of asymptotically AdS black holes from various points of view. First of all, we note that since all the other quantities in the generalized Smarr relation are already known, we can simply use (1.10) to furnish a *definition* of Θ in all the known black hole examples. This will necessarily also be consistent with the generalized first law (1.9). It then becomes of interest to see whether V calculated via (1.11) admits a natural physical interpretation as a "volume" of the black hole.

In simple cases such as a Schwarzschild-AdS or Reissner-Nordström-AdS, it turns out that the volume calculated from (1.11) coincides with a "naive" integration

$$\int_{r_0}^{r_+} dr \int d\Omega \sqrt{-g} \tag{1.12}$$

over the interior of the black hole, where the radial coordinate ranges from the singularity at $r = r_0$ to the outer horizon at $r = r_+$. In fact, in such cases the volume V turns out to be expressible as

$$V = \frac{r_+ A}{D-1}, \tag{1.13}$$

where A is the area of the outer horizon. With an appropriate modification in the case that there are running scalar fields, a naive volume integration again allows the potential Θ to be calculated for static charged asymptotically AdS black holes.

We find, however, that the situation becomes more complicated in the case of rotating black holes. If, for example, we consider the Kerr-AdS black hole in D dimensions, then a natural integration over the volume interior to the horizon, of the form (1.12), again, remarkably, gives rise to the expression on the right-hand side of (1.13) (if one uses the standard radial coordinate that appears in the metrics given in [9,10]).[2] However, this volume, which we shall now call V', is *not* the one that gives rise to the correct thermodynamic potential Θ. Rather, it gives

$$V' \equiv -\frac{16\pi}{(D-2)}\Theta' = \frac{r_+ A}{(D-1)}, \tag{1.14}$$

where A is the area of the outer horizon. Θ' is related to the true thermodynamic potential [defined via (1.9) or (1.10)] by

$$\Theta' = \Theta + \frac{1}{2(D-1)} \sum_i a_i J_i, \tag{1.15}$$

[2]In four dimensions, the notion of a "black hole volume," obtained by integrating $\int d^4x\sqrt{-g}$, was discussed previously in [11,12].

where a_i are the rotation parameters and J_i the angular momenta of the black hole. We may refer to the associated volumes V and V' as the "thermodynamic volume" and the "geometric volume," respectively.

Since in general we now have two different candidate definitions, it becomes of interest to investigate the possible physical interpretations of each of the volumes V and V'. In view of the fact that the geometric volume for Kerr-AdS has the remarkable feature that $V' = r_H A/(D-1)$, as if it were just the volume inside a sphere in Euclidean space, it is interesting to test whether V' and A satisfy the isoperimetric inequality of Euclidean bounded volumes. Indeed, we find that

$$\left(\frac{(D-1)V'}{\mathcal{A}_{D-2}}\right)^{1/(D-1)} \leq \left(\frac{A}{\mathcal{A}_{D-2}}\right)^{1/(D-2)} \tag{1.16}$$

for all Kerr-AdS black holes, with equality attained when the rotation vanishes. However, we find that for electrically charged black holes, even without rotation (and hence V and V' are the same), the isoperimetric inequality is violated.

If we instead use the thermodynamic volume V, then we find that the isoperimetric inequality is *always* violated by rotating Kerr-AdS black holes. Furthermore, we find strong indications that using V, the isoperimetric inequality is violated for all black holes, with or without rotation and/or charge. This leads us to conjecture that all black holes satisfy the *reverse isoperimetric inequality*, which asserts that

$$\left(\frac{(D-1)V}{\mathcal{A}_{D-2}}\right)^{1/(D-1)} \geq \left(\frac{A}{\mathcal{A}_{D-2}}\right)^{1/(D-2)}, \tag{1.17}$$

where V is the thermodynamic volume of the black hole and A is the area of the outer horizon. Equality is attained for Schwarzschild-AdS.

The reverse isoperimetric inequality may be rephrased as the statement that *for a black hole of given thermodynamic volume V, the entropy is maximized for Schwarzschild-AdS*.

The Smarr relation for black hole solutions of the vacuum Einstein equations can be derived by the Komar procedure, based on the integration of the identity $d * d\xi = 0$ over a spacelike hypersurface intersecting the horizon, where $\xi = \xi_\mu dx^\mu$ and ξ^μ is a Killing vector that is timelike at infinity. A generalization to the case with a cosmological constant Λ has been discussed in [5,13,14]. One writes ξ in terms of a 2-form Killing potential ω as $\xi = *d * \omega$, and then integrates the identity $d * d\xi + 2\Lambda d * \omega = 0$ over the spacelike hypersurface. After using Stokes' theorem the integration of $*\omega$ contributes a term on the sphere at infinity that removes a divergent contribution from $*d\xi$ to give a finite expression for the mass E, and a term on the horizon that furnishes an expression for Θ. One might hope that this could provide a further insight into the question of whether the "thermodynamic" or the "geometric" Θ is to be preferred. Unfortunately, however, there is an ambiguity in the definition of the Killing potential (the freedom to add a co-closed but not co-exact 2-form to ω), and this allows the expressions for E and for Θ to be adjusted in tandem. As we discuss later, the best that one can do is to choose a gauge for ω such that the mass E comes out to be the correct value, as already determined by other means. Necessarily, the integral yielding Θ then produces the "thermodynamic" expression rather than the geometric one.

We shall see that although the concept of the thermodynamic volume V requires that one consider an asymptotically AdS black hole in a theory with a nonvanishing cosmological constant, it is possible (except in $D = 7$) to take a smooth limit in the expression for V in which the cosmological constant is set to zero. Since the thermodynamic volume still, in general, differs from the geometric volume in this limit, one may define, for an asymptotically flat black hole, the thermodynamic volume by first obtaining its expression in the more general asymptotically AdS case, and then taking the limit where the cosmological constant goes to zero. We find that this limit exists for all the known asymptotically AdS black holes except for those in seven-dimensional gauged supergravity. This case is exceptional because of the existence of an odd-dimensional self-duality constraint in the seven-dimensional theory. It has the consequence that the volume diverges in this case if the three rotation parameters and the electric charge are all nonvanishing.

The organization of the paper is as follows. In Sec. II, we use the Smarr relation, or, equivalently, the first law of thermodynamics, to calculate Θ for the various static multi-charged black holes in four, five and seven-dimensional gauged supergravities, and we show how Θ is related to a volume integral of the scalar potential. In Sec. III, we use the same methods to calculate the thermodynamic expressions for Θ for the rotating Kerr-AdS black holes in arbitrary dimensions. We also show how these expressions are related to the geometric quantities Θ' that are directly given by volume integrations. We also perform similar calculations for some examples of charged rotating black holes in four and five-dimensional gauged supergravities. In Sec. IV we examine the isoperimetric inequality, and we show, in particular, that the reverse isoperimetric inequality holds for all the black hole examples we have considered. In Sec. V we review the derivation for the Smarr relation using the generalization of the Komar procedure, and then we give a detailed construction of the required Killing potentials ω for Kerr-AdS, making use of the conformal Killing-Yano tensors that exist in these backgrounds. The paper ends with conclusions in Sec. VI. In an appendix, we present some explicit results for the Killing potentials in four and five-dimensional Kerr-AdS.

II. STATIC CHARGED BLACK HOLES

In this section, we consider charged static black hole solutions in gauged supergravities in $D = 4$, 5, and 7 dimensions. We shall work in conventions where Newton's constant is set to 1, and the action takes the form

$$I=\int\sqrt{-g}\left[\frac{1}{16\pi}R-\frac{1}{16\pi}f(\phi)F^{\mu\nu}F_{\mu\nu}-\mathcal{V}(\phi)+\cdots\right], \tag{2.1}$$

where $F_{\mu\nu}$ is a $U(1)$ field strength (there may be just one, or several), $f(\phi)$ represents the coupling of scalar fields, and $\mathcal{V}(\phi)$ is the potential term for the scalar fields. In the solutions we shall consider, the scalar fields go to zero at infinity, and then $f(\phi)$ approaches 1, and the potential approaches

$$\mathcal{V}\rightarrow-\frac{(D-1)(D-2)}{16\pi}g^2, \tag{2.2}$$

where g is the gauge coupling constant. Thus the black hole solutions are asymptotic to AdS_D with $R_{\mu\nu}\rightarrow-(D-1)g^2g_{\mu\nu}$. Details of the black hole solutions can be found in [15–17], where they were constructed, and further discussion of their thermodynamics can be found in [18]. In what follows, we summarize the pertinent properties of the black holes for each of the dimensions 4, 5, and 7, and we calculate the quantity Θ in each case.

A. Charged AdS black holes in $D = 4$

The metric, electromagnetic potentials, and scalars fields are given by [16]

$$ds_4^2=-\prod_{i=1}^4H_i^{-1/2}fdt^2+\prod_{i=1}^4H_i^{1/2}(f^{-1}dr^2+r^2d\Omega_2^2),$$
$$A^i=\frac{\sqrt{q_i(q_i+\mu)}}{2(r+q_i)}dt,\qquad X_i=H_i^{-1}\prod_{j=1}^4H_j^{1/4}, \tag{2.3}$$

where

$$f=1-\frac{2m}{r}+g^2r^2\prod_{i=1}^4H_i,\qquad H_i=1+\frac{q_i}{r}. \tag{2.4}$$

The four scalar fields X_i, subject to the constraint $\prod_{i=1}^4X_i=1$, have the potential

$$\mathcal{V}=-\frac{g^2}{16\pi}\sum_{i<j}X_iX_j. \tag{2.5}$$

The relevant thermodynamic quantities are given by

$$E=m+\frac{1}{4}\sum_iq_i,\qquad Q_i=\frac{1}{2}\sqrt{q_i(q_i+2m)},$$
$$S=\pi\prod_i(r_++q_i)^{1/2},\qquad T=\frac{f'(r_+)}{4\pi}\prod_iH_i^{-1/2}(r_+),$$
$$\Phi_i=\frac{\sqrt{q_i(q_i+2m)}}{2(r_++q_i)}, \tag{2.6}$$

where the outer horizon is located at $r=r_+$, the largest root of $f(r_+)=0$.

Substituting into the first law (1.9) or the Smarr relation (1.10), we find that Θ is given by

$$\Theta=\Theta(r_+)=-\frac{r_+^3}{24}\prod_iH_i(r_+)\sum_j\frac{1}{H_j(r_+)}. \tag{2.7}$$

To interpret our result we note the nontrivial relation

$$-\Lambda\frac{d\Theta(r)}{dr}=\int d\Omega_2\mathcal{V}\sqrt{-g}. \tag{2.8}$$

Thus we may introduce a quantity r_0 so that

$$\Lambda\Theta=-W, \tag{2.9}$$

where W is the integral of the scalar potential

$$W=\int_{r_0}^{r_+}dr\int d\Omega_2\mathcal{V}\sqrt{-g}. \tag{2.10}$$

It is easily seen that r_0 is the largest root of

$$4r_0^3+3r_0^2\sum_iq_i+2r_0\sum_{i<j}q_iq_j+\sum_{i<j<k}q_iq_jq_k=0. \tag{2.11}$$

The appearance of W in (2.9) is not unexpected since it appears in the classical action but we do not, as yet, have an independent definition of r_0 other than via (2.10). The situation is similar in all of the examples which follow and we shall give the analogues of (2.9), (2.10), and (2.11) without further detailed comment.

B. Charged AdS black holes in $D = 5$

The metric, electromagnetic potentials, and scalar fields are given by [15]

$$ds_5^2=-\prod_{i=1}^3H_i^{-2/3}fdt^2+\prod_{i=1}^3H_i^{1/3}(f^{-1}dr^2+r^2d\Omega_3^2),$$
$$A^i=\frac{\sqrt{q_i(q_i+2m)}}{(r^2+q_i)}dt,\qquad X_i=H_i^{-1}\prod_{j=1}^3H_j^{1/3}, \tag{2.12}$$

where

$$f=1-\frac{2m}{r^2}+g^2r^2\prod_{i=1}^3H_i,\qquad H_i=1+\frac{q_i}{r^2}. \tag{2.13}$$

The three scalar fields X_i, subject to the constraint $\prod_{i=1}^3X_i=1$, have the potential

$$\mathcal{V}=-\frac{g^2}{4\pi}\sum_i\frac{1}{X_i}. \tag{2.14}$$

The relevant thermodynamic quantities are given by

$$E=\frac{1}{4}\pi[3m+q_1+q_2+q_3], \quad Q_i = \tfrac{1}{4}\pi\sqrt{q_i(q_i+2m)},$$
$$S=\frac{1}{2}\pi^2\prod_i(r_+^2+q_i)^{1/2}, \quad T=\frac{f'(r_+)}{4\pi}\prod_i H_i^{-1/2}(r_+),$$
$$\Phi_i=\frac{\sqrt{q_i(q_i+2m)}}{(r_+^2+q_i)}, \tag{2.15}$$

where the outer horizon is located at $r = r_+$, the largest root of $f(r_+) = 0$.

Substituting into the first law (1.9) or the Smarr relation (1.10), we find that Θ is given by

$$\Theta = -\frac{\pi r_+^4}{32}\prod_i H_i(r_+)\sum_j\frac{1}{H_j(r_+)}. \tag{2.16}$$

With the integral of the scalar potential defined by

$$W = \int_{r_0}^{r_+} dr \int d\Omega_3 \mathcal{V}\sqrt{-g}, \tag{2.17}$$

where r_0 is taken to be the largest root of

$$3r_0^4 + 2r_0^2\sum_i q_i + \sum_{i<j} q_i q_j = 0, \tag{2.18}$$

we find that Θ can again be written as

$$\Theta = -\frac{1}{\Lambda}W. \tag{2.19}$$

C. Charged AdS black holes in $D = 7$

The metric, electromagnetic potentials, and scalars fields are given by [17]

$$ds_7^2 = -(H_1H_2)^{-4/3}fdt^2 + (H_1H_2)^{1/5}(f^{-1}dr^2 + r^2d\Omega_5^2),$$
$$A^i = \frac{\sqrt{q_i(q_i+2m)}}{(r^4+q_i)}dt, \qquad X_i = H_i^{-1}(H_1H_2)^{2/5}, \tag{2.20}$$

where

$$f = 1 - \frac{2m}{r^4} + g^2r^2H_1H_2, \qquad H_i = 1 + \frac{q_i}{r^4}. \tag{2.21}$$

The two scalar fields X_i have the potential

$$\mathcal{V} = -\frac{g^2}{4\pi}\Big(4X_1X_2 + 2X_1^{-1}X_2^{-2} + 2X_2^{-1}X_1^{-2} - \frac{1}{2}(X_1X_2)^{-4}\Big). \tag{2.22}$$

The relevant thermodynamic quantities are

$$E=\frac{1}{8}\pi^2[5m+2(q_1+q_2)], \quad Q_i=\frac{1}{4}\pi^2\sqrt{q_i(q_i+2m)},$$
$$S=\frac{1}{4}\pi^3r_+\prod_i(r_+^4+q_i)^{1/2}, \quad T=\frac{f'(r_+)}{4\pi}(H_1(r_+)H_2(r_+))^{-1/2},$$
$$\Phi_i=\frac{\sqrt{q_i(q_i+2m)}}{(r_+^4+q_i)}, \tag{2.23}$$

where the outer horizon is located at $r = r_+$, the largest root of $f(r_+) = 0$.

Substituting into the first law (1.9) or the Smarr relation (1.10), we find that Θ is given by

$$\Theta = -\frac{\pi^2 r_+^6}{96}[H_1(r_+)H_2(r_+) + 2H_1(r_+) + 2H_2(r_+)]. \tag{2.24}$$

With the integral of the scalar potential defined by

$$W = \int_{r_0}^{r_+} dr \int d\Omega_5 \mathcal{V}\sqrt{-g}, \tag{2.25}$$

where r_0 is taken to be the largest root of

$$5r_0^8 + 3(q_1+q_2)r_0^4 + q_1q_2 = 0, \tag{2.26}$$

we find that Θ can again be written as

$$\Theta = -\frac{1}{\Lambda}W. \tag{2.27}$$

III. ROTATING BLACK HOLES

A. Kerr-AdS black holes in all dimensions

The Kerr-(A)dS solution in all dimensions, which generalizes the asymptotically flat rotating black hole solutions of [19], was obtained in [9,10]. The metric obeys the vacuum Einstein equations $R_{\mu\nu} = -(D-1)g^2g_{\mu\nu}$. In the ‘‘generalized'' Boyer-Lindquist coordinates it takes the form

$$ds^2 = -W(1+g^2r^2)dt^2 + \frac{2m}{U}\Big(Wdt - \sum_{i=1}^N\frac{a_i\mu_i^2d\phi_i}{\Xi_i}\Big)^2 + \sum_{i=1}^N\frac{r^2+a_i^2}{\Xi_i}(\mu_i^2d\phi_i^2 + d\mu_i^2) + \frac{Udr^2}{V-2m} - \frac{g^2}{W(1+g^2r^2)}\Big(\sum_{i=1}^N\frac{r^2+a_i^2}{\Xi_i}\mu_id\mu_i + \epsilon r^2\nu d\nu\Big)^2 + \epsilon r^2d\nu^2, \tag{3.1}$$

where

$$W \equiv \sum_{i=1}^{N} \frac{\mu_i^2}{\Xi_i} + \epsilon\nu^2, \quad V \equiv r^{\epsilon-2}(1+g^2r^2)\prod_{i=1}^{N}(r^2+a_i^2),$$

$$U \equiv \frac{V}{1+g^2r^2}\left(1-\sum_{i=1}^{N}\frac{a_i^2\mu_i^2}{r^2+a_i^2}\right), \quad \Xi_i = 1-g^2a_i^2. \tag{3.2}$$

Here $N \equiv [(D-1)/2]$, where $[A]$ means the integer part of A and we have defined ϵ to be 1 for D even and 0 for odd. The coordinates μ_i are not independent, but obey the constraint

$$\sum_{i=1}^{N} \mu_i^2 + \epsilon\nu^2 = 1. \tag{3.3}$$

In the remainder of the paper, we shall not in general indicate the range of the i index in summations or products; it will always be understood to be for $1 \le i \le N$, with $N = (D-1)/2$ in odd dimensions, and $N = (D-2)/2$ in even dimensions.

The calculation of Θ is slightly different in the two cases that the dimension D is odd or even. We discuss these cases in the following two subsections.

1. Odd-dimensional Kerr-AdS black holes

Here, we take $D = 2N+1$. As discussed in [8], the various thermodynamic quantities are given by

$$\begin{aligned}
E &= \frac{m\mathcal{A}_{D-2}}{4\pi\prod_j \Xi_j}\left(\sum_i \frac{1}{\Xi_i} - \frac{1}{2}\right),\\
J_i &= \frac{ma_i\mathcal{A}_{D-2}}{4\pi\Xi_i\prod_j\Xi_j},\\
S &= \frac{\mathcal{A}_{D-2}}{4r_+}\prod_i \frac{r_+^2+a_i^2}{\Xi_i},\\
T &= \frac{r_+(1+g^2r_+^2)}{2\pi}\sum_i \frac{1}{r_+^2+a_i^2} - \frac{1}{2\pi r_+},\\
\Omega_i &= \frac{(1+g^2r_+^2)a_i}{r_+^2+a_i^2},
\end{aligned} \tag{3.4}$$

where m and a_i are the "mass" and the N rotation parameters appearing in the Kerr-AdS metrics, the summations and products are taken over $1 \le i \le N$, the horizon radius is determined by the relation

$$2m = \frac{1}{r_+^2}(1+g^2r_+^2)\prod_i(r_+^2+a_i^2), \tag{3.5}$$

and the Ξ_i are given by $\Xi_i = 1-g^2a_i^2$. The quantity $\mathcal{A}_{D-2}$ is the volume of the unit-radius $(D-2)$ sphere, and is given by

$$\mathcal{A}_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma[(D-1)/2]}. \tag{3.6}$$

After substituting into (1.9) or (1.10), we find that Θ is given by

$$\Theta\Lambda = \frac{m\mathcal{A}_{D-2}}{8\pi\prod_j\Xi_j}\left(\sum_i\frac{1}{\Xi_i} + \frac{D-3}{2} - \frac{D-2}{1+g^2r_+^2}\right) \tag{3.7}$$

$$= \frac{1}{2}E - \frac{m(D-2)\mathcal{A}_{D-2}}{16\pi\prod_i\Xi_i}\frac{1-g^2r_+^2}{1+g^2r_+^2}. \tag{3.8}$$

This may in fact be written more simply if we introduce another quantity Θ', such that

$$\Theta = \Theta' - \frac{1}{2(D-1)}\sum_i a_iJ_i, \tag{3.9}$$

with Θ' being given by

$$\Theta' = -\frac{(D-2)m\mathcal{A}_{D-2}}{8\pi(D-1)\prod_i\Xi_i}\frac{r_+^2}{1+g^2r_+^2} = -\frac{(D-2)}{(D-1)}\frac{r_+A}{16\pi}, \tag{3.10}$$

where $A = 4S$ is the area of the horizon. Remarkably, r_+A is related to the spatial integral of $\sqrt{-g}$ up to the horizon radius. Specifically, we define

$$V(r_+) = \int_{r_0}^{r_+} dr \int d\Omega\sqrt{-g}, \tag{3.11}$$

where $d\Omega$ denotes the integration over the coordinates parameterizing the $(D-2)$-sphere surfaces, and r_0 is given by $r_0^2 = -a_{\min}^2$, where $a_{\min}^2$ is the smallest amongst the values of the a_i^2. (The $(D-2)$ spheres are not *round* spheres, of course.) Using the expression for $\sqrt{-g}$ obtained in the appendix of [8], we then find after some algebra that

$$V(r_+) = \frac{r_+A}{D-1}. \tag{3.12}$$

This therefore implies that

$$\Theta' = -\frac{(D-2)}{16\pi}V(r_+). \tag{3.13}$$

(Note that in performing the integration in Eq. (3.11), it is really more appropriate to use $x = r^2$ as the radial variable, since in odd dimensions r^2 can be negative.)

It is interesting also that Θ' can be obtained from a Smarr relation if one works in a certain frame that is rotating at infinity. Specifically, we have

$$E' = \frac{D-2}{D-3}(TS + \Omega_i'J_i) + \Phi_\alpha Q_\alpha - \frac{2}{D-3}\Theta'\Lambda, \tag{3.14}$$

where E' and Ω'_i are the energy and the angular velocities of the horizon measured with respect to a frame defined by sending the azimuthal coordinates ϕ_i in the black hole metrics to $\phi_i + a_i g^2 t$. This implies that[3]

$$E' = E - g^2 \sum_i a_i J_i = \frac{(D-2)m\mathcal{A}_{D-2}}{8\pi \prod_i \Xi_i}, \tag{3.15}$$

$$\Omega'_i = \Omega_i - a_i g^2 = \frac{a_i \Xi_i}{r_+^2 + a_i^2}. \tag{3.16}$$

The Einstein action in D dimensions is (with $G = 1$)

$$I_D = \frac{1}{16\pi}\int \sqrt{-g}[R - (D-2)\Lambda] d^D x. \tag{3.17}$$

Thus if we define the potential W to be

$$W \equiv \frac{(D-2)\Lambda}{16\pi}\int_{r_0}^{r_+} dr \int d\Omega \sqrt{-g}, \tag{3.18}$$

then we have

$$\Theta' = -\frac{1}{\Lambda} W. \tag{3.19}$$

2. Even-dimensional Kerr-AdS black holes

Here, we take $D = 2N + 2$. As discussed in [8], the various thermodynamic quantities are now given by

$$\begin{aligned}
E &= \frac{m\mathcal{A}_{D-2}}{4\pi \prod_j \Xi_j}\sum_i \frac{1}{\Xi_i},\\
J_i &= \frac{m a_i \mathcal{A}_{D-2}}{4\pi \Xi_i \prod_j \Xi_j},\\
S &= \frac{1}{4}\mathcal{A}_{D-2}\prod_i \frac{r_+^2 + a_i^2}{\Xi_i},\\
T &= \frac{r_+(1+g^2 r_+^2)}{2\pi}\sum_i \frac{1}{r_+^2 + a_i^2} - \frac{1 - g^2 r_+}{4\pi r_+},\\
\Omega_i &= \frac{(1+g^2 r_+^2)a_i}{r_+^2 + a_i^2},
\end{aligned} \tag{3.20}$$

and the location of the horizon is determined by the equation

$$2m = \frac{1}{r_+}(1 + g^2 r_+^2)\prod_i (r_+^2 + a_i^2), \tag{3.21}$$

where the summations and products are over $1 \le i \le N$. We find from (1.9) or from (1.10) that

$$\Theta\Lambda = \frac{m\mathcal{A}_{D-2}}{8\pi \prod_j \Xi_j}\left(\sum_i \frac{1}{\Xi_i} + \frac{D-2}{2} - \frac{D-2}{1+g^2 r_+^2}\right) \tag{3.22}$$

$$= \frac{1}{2}E - \frac{m(D-2)\mathcal{A}_{D-2}}{16\pi \prod_i \Xi_i}\frac{1 - g^2 r_+^2}{1 + g^2 r_+^2}. \tag{3.23}$$

Again we find that Θ can be expressed more simply in the form (3.9), with Θ' given by (3.10). As in the odd-dimensional case, we again find that if we define a volume "inside the horizon" as in (3.11), then the relation (3.12) again holds, and hence Θ' is again related to the potential W by Eq. (3.19). The only difference from the odd-dimensional case is that in the volume integral (3.11), the lower limit for the radial integration should now be $r_0 = 0$. (This is really the same rule as is used in odd dimensions, since in even dimensions there is effectively a "missing" rotation parameter that is equal to zero.)

B. Rotating pairwise-equal 4-charge black hole in $D = 4$ gauged supergravity

The metric for this black hole is obtained in [20]. The various thermodynamic quantities are given by [21]

$$\begin{aligned}
E &= \frac{m + q_1 + q_2}{\Xi^2}, \qquad & S &= \frac{\pi(r_1 r_2 + a^2)}{\Xi},\\
J &= \frac{a(m + q_1 + q_2)}{\Xi^2}, \qquad & Q_1 = Q_2 &= \frac{\sqrt{q_1(q_1+m)}}{2\Xi},\\
Q_3 = Q_4 &= \frac{\sqrt{q_2(q_2+m)}}{2\Xi}, \qquad & T &= \frac{\Delta'_r}{4\pi(r_1 r_2 + a^2)},\\
\Omega &= \frac{a(1 + g^2 r_1 r_2)}{r_1 r_2 + a^2}, \qquad & \Phi_1 = \Phi_2 &= \frac{2r_1\sqrt{q_1(q_1+m)}}{r_1 r_2 + a^2},\\
\Phi_3 = \Phi_4 &= \frac{2r_2\sqrt{q_2(q_2+m)}}{r_1 r_2 + a^2},
\end{aligned} \tag{3.24}$$

where $r_1 = r + 2q_1$, $r_2 = r + 2q_2$,

$$\Delta_r = r^2 + a^2 - 2mr + g^2 r_1 r_2 (r_1 r_2 + a^2), \tag{3.25}$$

and all r-dependent quantities in (3.24) are evaluated at the horizon radius r_+, determined as the largest root of $\Delta_r(r_+) = 0$.

Substituting into either (1.9) or (1.10), we can determine Θ. As usual in rotating black holes, the expression is quite complicated, and it is most elegantly expressed, *via* (3.9), in terms of Θ' defined in the rotating frame:

$$\Theta = \Theta' - \tfrac{1}{6}aJ, \tag{3.26}$$

where

[3]It should be noted, however, that the thermodynamic variables E' and Ω'_i do *not* satisfy the first law of thermodynamics. Thus, for example, if we hold Λ fixed then dE' is not equal to $TdS + \Omega'_i dJ_i$, and indeed, the latter is not even an exact differential. (See [8] for a detailed discussion.)

$$\Theta' = -\frac{r + q_1 + q_2}{6\Xi}(r_1 r_2 + a^2), \tag{3.27}$$

(evaluated at $r = r_+$).

There is a scalar potential in the four-dimensional gauged supergravity, given by

$$\mathcal{V} = -\frac{g^2}{16\pi}(4 + 2\cosh\varphi + e^{\varphi}\chi^2), \tag{3.28}$$

and in the black hole solution we have [20]

$$e^{\varphi} = \frac{r_1^2 + a^2\cos^2\theta}{r_1 r_2 + a^2\cos^2\theta}, \qquad \chi = \frac{a(r_2 - r_1)\cos\theta}{r_1^2 + a^2\cos^2\theta}. \tag{3.29}$$

If we define

$$U(r) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \mathcal{V}\sqrt{-g}, \tag{3.30}$$

then we find that

$$\frac{d\Theta'}{dr_+} = -\frac{1}{\Lambda}U(r_+). \tag{3.31}$$

In integral form, if we define the potential term

$$W = \int_{r_0}^{r_+} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \mathcal{V}\sqrt{-g}, \tag{3.32}$$

then

$$\Theta' = -\frac{1}{\Lambda}W, \tag{3.33}$$

where the lower limit of integration is taken to be

$$r_0 = -q_1 - q_2 + \sqrt{(q_1 - q_2)^2 - a^2}. \tag{3.34}$$

C. Charged rotating black hole in minimal $D = 5$ gauged supergravity

The metric for this black hole is obtained in [22]. It has the thermodynamic quantities

$$\begin{aligned}
E &= \frac{m\pi(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b) + 2\pi qabg^2(\Xi_a + \Xi_b)}{4\Xi_a^2\Xi_b^2},\\
S &= \frac{\pi^2[(r_+^2 + a^2)(r_+^2 + b^2) + abq]}{2\Xi_a\Xi_b r_+},\\
J_a &= \frac{\pi(2am + qb(1 + a^2g^2)]}{4\Xi_a^2\Xi_b},\\
J_b &= \frac{\pi(2bm + qa(1 + b^2g^2)]}{4\Xi_b^2\Xi_a},\\
Q &= \frac{\sqrt{3}\pi q}{4\Xi_a\Xi_b},\\
T &= \frac{r_+^4[1 + g^2(r_+^2 + a^2 + b^2)] - (ab + q)^2}{2\pi r_+[(r_+^2 + a^2)(r_+^2 + b^2) + abq]},\\
\Phi &= \frac{\sqrt{3}qr_+^2}{(r_+^2 + a^2)(r_+^2 + b^2) + abq},\\
\Omega_a &= \frac{a(r_+^2 + b^2)(1 + g^2r_+^2) + bq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq},\\
\Omega_b &= \frac{b(r_+^2 + a^2)(1 + g^2r_+^2) + aq}{(r_+^2 + a^2)(r_+^2 + b^2) + abq},
\end{aligned} \tag{3.35}$$

where the location of the horizon is determined by the equation

$$2m = \frac{(r_+^2 + a^2)(r_+^2 + b^2)(1 + g^2r_+^2) + q^2 + 2abq}{r_+^2}. \tag{3.36}$$

From (1.9) or from (1.10) we find that

$$\begin{aligned}
\Theta = \Theta_0 &- \frac{abq\pi}{16\Xi_a^2\Xi_b^2r_+^2}\\
&\times [2r_+^2 + a^2 + b^2 - g^2(r_+^2(a^2 + b^2) + 2a^2b^2)]\\
&- \frac{\pi q^2(a^2 + b^2 - 2a^2b^2g^2)}{32\Xi_a^2\Xi_b^2r_+^2},
\end{aligned} \tag{3.37}$$

where Θ_0 is the value for five-dimensional Kerr-AdS, as given in (3.7) for $D = 5$. As in the Kerr-AdS examples, the quantity Θ' evaluated in the asymptotically rotating frame, and defined by (3.9), is much simpler, and is given in this case by

$$\Theta' = -\frac{\pi}{32\Xi_a\Xi_b}[3(r_+^2 + a^2)(r_+^2 + b^2) + 2abq]. \tag{3.38}$$

The metric in [22] has

$$\sqrt{-g} = \frac{r\sin\theta\cos\theta(r^2 + a^2\cos^2\theta + b^2\sin^2\theta)}{\Xi_a\Xi_b}, \tag{3.39}$$

and hence if we define

$$U(r) \equiv -\frac{3g^2}{4\pi}\int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \int_0^{(1/2)\pi} d\theta \sqrt{-g} = \frac{3g^2 \pi r(2r^2 + a^2 + b^2)}{4\Xi_a \Xi_b}, \tag{3.40}$$

(where $-3g^2/(4\pi)$ is the coefficient of the cosmological term in the Lagrangian), then we see that

$$\frac{d\Theta'}{dr_+} = -\frac{1}{\Lambda} U(r_+). \tag{3.41}$$

To integrate this we introduce the radial variable $x = r^2$, and integrate from $x = x_0$ to $x = r_+^2$, where x_0 is the less negative of the two possibilities

$$x_0 = -\frac{1}{2}(a^2 + b^2) \pm \frac{1}{2}\sqrt{(a^2 - b^2)^2 - \frac{8}{3}abq}. \tag{3.42}$$

IV. REVERSE ISOPERIMETRIC INEQUALITY

The isoperimetric inequality for the volume V of a connected domain in Euclidean space $\mathbb{E}^{D-1}$ whose area is A states that

$$\left(\frac{(D-1)V}{\mathcal{A}_{D-2}}\right)^{D-2} \le \left(\frac{A}{\mathcal{A}_{D-2}}\right)^{D-1} \tag{4.1}$$

with equality if and only if the domain is a standard round ball. Thus we may restate the inequality as $R \le 1$, where we define

$$R \equiv \left(\frac{(D-1)V}{\mathcal{A}_{D-2}}\right)^{1/(D-1)} \left(\frac{\mathcal{A}_{D-2}}{A}\right)^{1/(D-2)}. \tag{4.2}$$

It is interesting to examine whether or not the area of the black hole horizon and the "volume" defined via either Θ or Θ' satisfy the isoperimetric inequality. Let us first consider the case of electrically neutral black holes; i.e., the rotating Kerr-AdS black holes in arbitrary dimensions. Intriguingly, we find that if we use the quantity Θ' to define the volume of the black hole, then the isoperimetric inequality is always satisfied in Kerr-AdS, with equality being attained for the nonrotating Schwarzschild-AdS limit. If, on the other hand, we use the quantity Θ, which arises naturally from thermodynamic considerations, to define the volume, then the opposite is true, and the isoperimetric inequality is always violated, except in the nonrotating limit.

A. Isoperimetric inequality for the Θ' volume

For the Kerr-AdS metrics, if A is the area of the event horizon, then in all cases

$$V' = -\frac{16\pi}{D-2}\Theta' = \frac{r_+ A}{D-1}, \tag{4.3}$$

and if D is odd

$$A = \frac{\mathcal{A}_{D-2}}{r_+}\prod_i \frac{r_+^2 + a_i^2}{\Xi_i}, \tag{4.4}$$

while if D is even

$$A = \mathcal{A}_{D-2}\prod_i \frac{r_+^2 + a_i^2}{\Xi_i}. \tag{4.5}$$

A simple calculation shows that in both odd and even dimensions, R' defined by (4.2), and using the volume V', given by

$$R' = \prod_i \left(\frac{1 + a_i^2/r_+^2}{\Xi_i}\right)^{-1/((D-1)(D-2))}. \tag{4.6}$$

Since $\Xi_i = 1 - g^2 a_i^2 \le 1$ for each i, it is evident that $R' \le 1$, with equality when all a_i vanish.

Thus remarkably, the geometrical V' and the surface area A of the black hole satisfy the standard isoperimetric inequality for a ball in flat Euclidean space $\mathbb{E}^{D-1}$. There is an obvious analogy here with the liquid drop model, which regards a nucleus as a ball of incompressible fluid, whose volume is thus fixed. If the energy is solely due to positive surface tension, then the configuration which minimizes the energy is spherical.

B. Reverse isoperimetric inequality for the Θ volume

We saw in Eq. (3.9) that the thermodynamic quantity Θ in Kerr-AdS is more negative than Θ', and hence it follows that the associated volume V is larger than V'. In fact, from (4.3) and (3.9) we find that

$$V = \frac{r_+ A}{(D-1)}\left[1 + \frac{(1 + g^2 r_+^2)}{(D-2) r_+^2}\sum_i \frac{a_i^2}{\Xi_i}\right]. \tag{4.7}$$

This suggests the possibility that although V' and A satisfy the isoperimetric inequality, as we saw above, it might be that the volume V and the area A could violate it in Kerr-AdS black holes. This is indeed exactly what we find. Since, as it turns out, this violation seems to be a universal property, for all rotating and/or charged black holes, we may elevate this to the status of a conjecture in its own right. Thus we make the conjecture that the ratio R defined in (4.2) actually satisfies the *reverse isoperimetric inequality*

$$R \ge 1 \tag{4.8}$$

for all black holes, if one uses the "thermodynamic" definition of the volume V. We now demonstrate the validity of the conjecture for a variety of black hole solutions.

1. Kerr-AdS

Defining the (necessarily non-negative) dimensionless quantity

$$z = \frac{(1+g^2r_+^2)}{r_+^2}\sum_i \frac{a_i^2}{\Xi_i}, \tag{4.9}$$

we consider R^{D-1}, where R is given by (4.2), and observe that in odd dimensions

$$\begin{aligned} R^{D-1} &= r_+\Big[1+\frac{z}{D-2}\Big]\Big[\frac{1}{r_+}\prod_i \frac{(r_+^2+a_i^2)}{\Xi_i}\Big]^{-1/(D-2)} \\ &= \Big[1+\frac{z}{D-2}\Big]\Big[\prod_i \frac{(r_+^2+a_i^2)}{r_+^2\Xi_i}\Big]^{-1/(D-2)} \\ &\geq \Big[1+\frac{z}{D-2}\Big] \\ &\quad\times\Big[\frac{2}{D-1}\Big(\sum_i \frac{1}{\Xi_i}+\sum_i\frac{a_i^2}{r_+^2\Xi_i}\Big)\Big]^{-(D-1)/(2(D-2))} \\ &= \Big[1+\frac{z}{D-2}\Big]\Big[1+\frac{2z}{D-1}\Big]^{-(D-1)/(2(D-2))} \\ &\equiv F(z), \end{aligned} \tag{4.10}$$

where the inequality follows from $(\prod_i x_i)^{1/N} \leq (1/N)\sum_i x_i$ for non-negative quantities x_i.

Noting that $F(0) = 1$, and that

$$\frac{d\log F(z)}{dz} = \frac{(D-3)z}{(D-2)(D-2+z)(D-1+2z)}, \tag{4.11}$$

which is positive for non-negative z in $D > 3$ dimensions, it follows that $F(z) \geq 1$, and hence the reverse isoperimetric inequality (4.8) is satisfied by all odd-dimensional Kerr-AdS black holes.

In even dimensions the calculation is rather similar, since now we have

$$\begin{aligned} R^{D-1} &= r_+\Big[1+\frac{z}{D-2}\Big]\Big[\prod_i \frac{(r_+^2+a_i^2)}{\Xi_i}\Big]^{-1/(D-2)} \\ &= \Big[1+\frac{z}{D-2}\Big]\Big[\prod_i \frac{(r_+^2+a_i^2)}{r_+^2\Xi_i}\Big]^{-1/(D-2)} \\ &\geq \Big[1+\frac{z}{D-2}\Big]\Big[\frac{2}{D-2}\Big(\sum_i \frac{1}{\Xi_i}+\sum_i\frac{a_i^2}{r_+^2\Xi_i}\Big)\Big]^{-(1/2)} \\ &= \Big[1+\frac{z}{D-2}\Big]\Big[1+\frac{2z}{D-2}\Big]^{-(1/2)} \\ &\equiv G(z). \end{aligned} \tag{4.12}$$

Thus $G(0) = 1$ and $d\log G(z)/dz \geq 0$, and so again we conclude that $R \geq 1$. Thus the reverse isoperimetric inequality holds for even-dimensional Kerr-AdS black holes also.

2. Charged static black holes

All of the charged static black hole solutions in gauged supergravity satisfy the reverse isoperimetric inequality also. There is no distinction between the V and V' volumes in this case, since there is no rotation. Consider, for example, the 4-charge solution given in Sec. II A. The volume and area are given by

$$\begin{aligned} V &= \frac{1}{3}\pi\sum_i \frac{1}{r_++q_i}\prod_j (r_++q_j), \\ A &= 4\pi\prod_i (r_++q_i), \end{aligned} \tag{4.13}$$

and so from (4.2) we have

$$R^3 = \frac{1}{4}\sum_i \frac{1}{r_++q_i}\prod_j (r_++q_j)^{1/4}, \tag{4.14}$$

and so using the inequality

$$\prod_i (r_++q_i)^{-(1/4)} \leq \frac{1}{4}\sum_i \frac{1}{r_++q_i}, \tag{4.15}$$

we see that $R \geq 1$.

Very similar calculations show that the inequality $R \geq 1$ holds for the static charged black holes in $D = 5$ and $D = 7$ also.

3. Charged rotating black holes

We have verified explicitly that $R \geq 1$ for the rotating black hole in four-dimensional gauged supergravity with pairwise equal charges (described in Sec. III B), and also for the charged rotating black hole in five-dimensional ungauged minimal supergravity (i.e., setting $g = 0$ in the solution described in Sec. III C). In each case, the calculations are quite complicated, and we shall not present them here.

In the case of the rotating black hole in five-dimensional *gauged* minimal supergravity, we have constructed an analytical proof that $R \geq 1$ in the case that the product abq is non-negative. Numerical investigations indicate that $R \geq 1$ also if abq is negative.

It is worth remarking that while we can obtain an expression for the volume V of an asymptotically flat black hole in ungauged supergravity (or with zero cosmological constant) by sending $g \to 0$ or $\Lambda \to 0$ in the expressions obtained for an asymptotically AdS black hole, we do not have an intrinsic way in general of defining V for an asymptotically flat black hole if the more general asymptotically AdS solution is not itself known.

The dependence of volume on g is smooth; there are no discontinuities for $g \to 0$ or in the large to small black hole transition. To illustrate this point we display the $V = V(g)$ dependence for a Kerr-AdS black hole of fixed mass in Fig. 1.

We have not checked our Reverse Isoperimetric Conjecture for all the known examples of charged rotating black holes in gauged supergravities. We have, however, examined the recent construction in [23] of the rotating black hole in four-dimensional maximal gauged supergravity with two zero charges and the other two freely specifiable. With nonzero gauge coupling the complexity of the

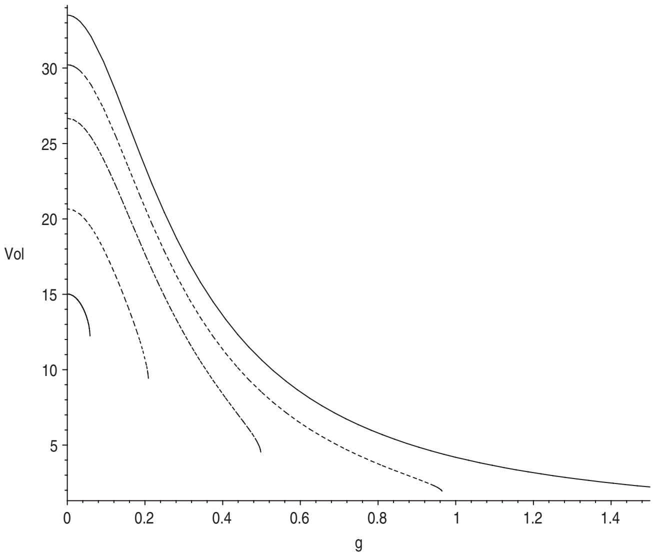

FIG. 1. *Thermodynamic volume of the Kerr-AdS black hole*. The graph displays the dependence of V on gauge coupling g, $\Lambda = -3g^2$, for various rotation parameters, while we keep the total gravitational enthalpy fixed, $E = 1$ ($J = a$). The upper curve represents Schwarzschild-AdS ($a = 0$), the lower curves, in descending order, correspond to Kerr-AdS with $a = 0.5$, $a = 0.7$, $a = 0.9$ and $a = 0.99$, respectively. Obviously, the smooth limit exists for $g \to 0$, the volume is smooth also in the transition between large and small black holes.

metric has so far prevented us from obtaining an analytic proof, but the indications from numerical analysis are that the conjecture is satisfied. The expression for the thermodynamic volume V is much simpler in the limit that $g = 0$, and in this case we have been able to show analytically that the reverse isoperimetric conjecture is satisfied.

We have evaluated the volume for all solutions known to us in all dimensions $D \leq 7$. When $g \neq 0$, the volume may be obtained from the Smarr formula (1.10) by dividing by $\Lambda \sim -g^2$. If $D \leq 6$, the numerator is always found to be proportional to g^2, and hence a smooth limit exists as g tends to zero. In $D = 7$, however, the numerator contains in addition terms proportional to g (times the product of the three rotation parameters a_i), and hence the $g \to 0$ limit diverges if all the a_i are nonvanishing. The fact that the $D = 7$ solutions [24] are not invariant under $g \to -g$ may be traced back to the self-duality constraint for the 3-form gauge potential in the seven-dimensional gauged supergravity theory (see for example, Eq. (3.8) in [24]), since this equation contains a term linear in g.

V. KOMAR INTEGRATION, SMARR FORMULA AND KILLING POTENTIALS FOR KERR-ADS BLACK HOLES

A. Komar derivation of the Smarr relation

In a D-dimensional stationary, axisymmetric black hole spacetime, let Σ denote a spacelike hypersurface that intersects $\mathcal{H}^+$, a Killing horizon of $\xi = k + \Omega_i m_i$, in a $(D-2)$ sphere H. Here, k is a Killing vector that is timelike at infinity, m_i are $U(1)$ Killing fields that generate rotations in the orthogonal spatial 2-planes, and Ω_i are the corresponding angular velocities of the horizon. Since any Killing vector satisfies $\nabla_\mu K^\mu = 0$ and $\Box K_\mu + R_{\mu\nu}K^\nu = 0$ it follows that if the metric is Ricci flat, corresponding to the case of an asymptotically flat black hole, then $d * d\xi = 0$, and hence

$$0 = \int_\Sigma d * d\xi = \int_{\partial\Sigma} *d\xi = \int_{S_\infty} *d\xi - \int_H *d\xi, \quad (5.1)$$

where S_∞ denotes the sphere at infinity. One can show that the Komar integrals constructed using the Killing vectors k and m_i give the energy and angular momenta of the asymptotically flat black hole

$$E = -\frac{(D-2)}{16\pi(D-3)}\int_{S_\infty} *dk, \qquad J_i = \frac{1}{16\pi}\int_{S_\infty} *dm_i, \quad (5.2)$$

while the integral of $*d\xi$ over the horizon gives

$$\frac{1}{16\pi}\int_H *d\xi = \frac{\kappa A}{8\pi} = TS, \quad (5.3)$$

and so from (5.1) one obtains the Smarr relation

$$E = \frac{(D-2)}{(D-3)}\left(\frac{\kappa A}{8\pi} + \Omega_i J_i\right) = \frac{(D-2)}{(D-3)}(TS + \Omega_i J_i) \quad (5.4)$$

for an asymptotically flat black hole.

If the cosmological constant is negative rather than zero, then the above Komar derivation of the Smarr relation requires modification. Following the arguments in [5,13,14], one may note that since any Killing vector satisfies $d * K = 0$, there must always exist, locally, a 2-form *Killing potential* ω_K such that K may be written as $K = *d * \omega_K$. In view of the fact that with $R_{\mu\nu} = \Lambda g_{\mu\nu}$ we now have $d * d\xi + 2\Lambda * \xi = 0$, and it follows that

$$d * d\xi + 2\Lambda d * \omega_\xi = 0. \quad (5.5)$$

By integrating this over Σ, one thereby obtains

$$\begin{aligned} 0 &= \int_\Sigma (d * d\xi + 2\Lambda d * \omega_\xi) = \int_{\partial\Sigma}(*d\xi + 2\Lambda * \omega_\xi) \\ &= \int_{S_\infty}(*dk + 2\Lambda * \omega_\xi) + \Omega_i \int_{S_\infty} *dm_i \\ &\quad - \int_H *d\xi - 2\Lambda\int_H *\omega_\xi. \end{aligned} \quad (5.6)$$

Of course the Killing potential ω_ξ is not unique; one may add any co-closed 2-form ν to ω_ξ. If ν is co-exact, $\nu = *d * \eta$ for any 3-form η, then the integrals of $*\omega_\xi$ in (5.6) will be unaltered, since

$$\int_{S_\infty} d*\eta = \int_{\partial S_\infty} *\eta = 0, \qquad \int_H d*\eta = \int_{\partial H} *\eta = 0. \tag{5.7}$$

However, if ν is co-closed but not co-exact, each of the integrals $\int_{S_\infty} *\omega_\xi$ and $\int_H *\omega_\xi$ will be separately changed by the addition of ν, although their difference will be unaltered, since

$$\int_{S_\infty} *\nu - \int_H *\nu = \int_\Sigma d*\nu = 0. \tag{5.8}$$

By analogy with the asymptotically flat case we discussed above, one would like to interpret the integrals over S_∞ in (5.6) as being proportional, respectively, to the energy and the angular momenta of the black hole. Indeed, one again finds that the integrals of $*dm_i$ give the angular momenta, as in (5.2). The integral $\int_{S_\infty} *dk$ by itself now diverges, as does $\int_{S_\infty} *\omega_\xi$, but remarkably, the combination $\int_{S_\infty}(*dk + 2\Lambda * \omega_\xi)$ turns out to be finite. Since, however, as we remarked above, its value is altered if one exploits the gauge freedom to add a co-closed 2-form ν to ω_ξ, one cannot use $\int_{S_\infty}(*dk + 2\Lambda * \omega_\xi)$ to provide an unambiguous definition of the energy of the black hole. The best that can be done is to make a gauge choice for ω_ξ such that

$$E = -\frac{(D-2)}{16\pi(D-3)} \int_{S_\infty} (*dk + 2\Lambda * \omega_\xi) \tag{5.9}$$

yields the true mass E of the black hole, which itself is determined by other means.

The easiest and most reliable way of calculating the mass of an asymptotically AdS black hole is by means of the conformal definition of Ashtekar, Magnon, and Das (AMD) [25,26]. This has the great advantage over other methods, such as that of Abbott and Deser [27], that it involves an integration at infinity of a finite quantity, computed from the Weyl tensor, that does not require any infinite subtraction of a pure AdS background. The AMD mass for the Kerr-AdS black hole in arbitrary dimension was calculated in [8], and it was shown to be consistent with the first law of thermodynamics.

Having chosen a gauge for ω_ξ for which the integration in (5.9) yields the AMD mass E, the remaining integrals in (5.6) can be evaluated. Defining

$$\Theta = \frac{(D-2)}{16\pi} \int_H *\omega_\xi, \tag{5.10}$$

we recover precisely the Smarr relation (1.10) for the uncharged case,

$$E = \frac{D-2}{D-3}(TS + \Omega_i J_i) - \frac{2}{D-3}\Theta\Lambda. \tag{5.11}$$

B. Killing potentials from the conformal Killing-Yano tensor

In this subsection we review the work of [28], which shows how one may construct the towers of hidden and explicit symmetries of a spacetime that admits a *principal conformal Killing-Yano (PCKY) tensor*. In this discussion we closely follow [29], and then we present a new method for constructing the Killing potentials for the Killing vectors.

The PCKY tensor h is a nondegenerate closed conformal Killing-Yano 2-form [28]. This means that there exists a 1-form η such that

$$\nabla_\mu h_{\nu\rho} = 2g_{\mu[\nu}\eta_{\rho]}. \tag{5.12}$$

The condition of nondegeneracy means that at a generic point of the manifold, the skew-symmetric matrix $h_{\mu\nu}$ has the maximum possible (matrix) rank, and that the eigenvalues of $h_{\mu\nu}$ are functionally independent in some spacetime domain. The Eq. (5.12) implies

$$dh = 0, \qquad \eta = \frac{1}{D-1} * d * h. \tag{5.13}$$

This means that there exists a 1-form PCKY potential b, such that

$$h = db. \tag{5.14}$$

The 1-form η associated with h is called *primary*, and turns out to be a Killing 1-form.

The PCKY tensor generates a tower of closed conformal Killing-Yano (CKY) tensors [28]

$$h^{(j)} \equiv h^{\wedge j} = \underbrace{h \wedge \cdots \wedge h}_{\text{total of } j \text{ factors}}. \tag{5.15}$$

The CKY tensor $h^{(j)}$ is a $(2j)$ form, and in particular $h^{(1)} = h$. Since h is nondegenerate, one has a set of $N + \varepsilon$ nonvanishing closed CKY tensors in dimension $D = 2N + 1 + \varepsilon$, where $\varepsilon = 0$ in odd dimensions and $\varepsilon = 1$ in even dimensions. In an even-dimensional spacetime, $h^{(N+1)}$ is proportional to the totally antisymmetric tensor, whereas it is dual to a Killing vector in odd dimensions. In both cases such a CKY tensor is trivial, and can be excluded from the tower of hidden symmetries. Therefore we take $j = 1, \ldots, N - 1 + \varepsilon$.

The CKY tensors (5.15) can be generated from the potentials $b^{(j)}$,

$$b^{(j)} = b \wedge h^{\wedge(j-1)}, \qquad h^{(j)} = db^{(j)}. \tag{5.16}$$

For each $(2j)$ form $h^{(j)}$, its Hodge dual is a $(D-2j)$ form, denoted by

$$f^{(j)} = *h^{(j)}. \tag{5.17}$$

In their turn, these tensors give rise to the Killing tensors $K^{(j)}$,

$$K^{(j)}_{\mu\nu} \equiv \frac{1}{(D-2j-1)!(j!)^2} f^{(j)}{}_{\mu\rho_1\ldots\rho_{D-2j-1}} f^{(j)}{}_{\nu}{}^{\rho_1\ldots\rho_{D-2j-1}}. \tag{5.18}$$

(The coefficient in this definition (5.18) is a convenient choice in the canonical basis, see [29].) The metric itself trivially satisfies the conditions for a Killing tensor, and it is convenient to define $K^{(0)}_{\mu\nu} = -g_{\mu\nu}$, extending the range of the j index so that $j = 0, \ldots, N-1+\varepsilon$.

The PCKY tensor also naturally generates $(N+1)$ vectors $\eta^{(k)}$ $(k = 0, \ldots, N)$ which turn out to be the independent commuting Killing vector fields. These are given as

$$\eta^{(j)\mu} = K^{(j)\mu}{}_{\nu}\eta^{\nu}, \qquad j = 0, \ldots, N-1+\varepsilon, \tag{5.19}$$

where η^{μ} is the Killing vector given by (5.13). In odd dimensions the last Killing vector is in the tower is given by the N-th Killing-Yano tensor

$$\eta^{(N)} = -\frac{1}{N!} f^{(N)}. \tag{5.20}$$

The *canonical spacetimes* with all these symmetries were constructed in [30,31]. When the Einstein equation is imposed, they are the general Kerr-NUT-AdS spacetimes constructed in [32].

1. Killing potentials

We shall now show how the PCKY tensor may be used in order to construct the Killing potentials for the Killing vectors. We define the following 2-forms, for $j = 0, \ldots N-1+\varepsilon$:

$$\omega^{(j)}_{\mu\nu} = \frac{1}{D-2j-1} K^{(j)}_{\mu\rho} h^{\rho}{}_{\nu}, \qquad \omega^{(N)} = \frac{\sqrt{-c}}{N!} * b^{(N)}, \tag{5.21}$$

where the second expression, for $\omega^{(N)}$, applies only in odd dimensions. $\sqrt{-c}$ is some appropriately chosen constant (see [29]). It is easy to verify (for example in the canonical basis) that these are Killing potentials for the previously constructed Killing fields, i.e., we have

$$\eta^{(i)} = *d * \omega^{(i)}. \tag{5.22}$$

Note that although in odd dimensions the gauge freedom $b \to b + d\lambda$ affects $\omega^{(N)}$,

$$\omega^{(N)} \to \omega^{(N)} + \frac{\sqrt{-c}}{N!} * d(\lambda h^{(N-1)}), \tag{5.23}$$

its divergence $*d * \omega^{(N)}$ remains unchanged. Since any Killing vector ξ in a canonical spacetime is a linear combination of the $\eta^{(i)}$, of the form $\xi = \sum_{i=0}^{N} c_i \eta^{(i)}$, the problem of finding its Killing potential reduces to the algebraic problem of finding the constant coefficients $c_i(\xi)$ of this expansion:

$$\xi = *d * \omega_{\xi}, \qquad \omega_{\xi} = \sum_{i=0}^{N} c_i(\xi)\omega^{(i)}, \tag{5.24}$$

where $\omega^{(i)}$ are given by (5.21).

C. Kerr-AdS black holes

The Kerr-AdS black hole metrics (3.1) possess a closed conformal Killing-Yano 2-form h [33] which can be derived from the potential b, $h = db$, given by

$$b = \frac{1}{2}\Big\{\Big[r^2 + \sum_{i=1}^{N} a_i^2\mu_i^2\Big(1 + g^2\frac{r^2 + a_i^2}{\Xi_i}\Big)\Big]dt - \sum_{i=1}^{N} a_i\mu_i^2\frac{r^2 + a_i^2}{\Xi_i}d\phi_i\Big\}. \tag{5.25}$$

The 2-form h is nondegenerate, i.e., it is a PCKY tensor when all rotations a_i are nonzero and distinct. In that case any Killing vector of the spacetime is a linear combination of the (independent) Killing fields $\eta^{(i)}$, and its Killing potential is given by (5.24), where in odd dimensions we identify the constant $\sqrt{-c} = \prod_{i=1}^{N} a_i$.[4]

The outer Killing horizon of the Kerr-AdS metric (3.1) is located at $r = r_+$, the largest root of $V(r_+) - 2m = 0$. It is a Killing horizon for the Killing field

$$\xi = \partial_t + \Omega_i\partial_{\phi_i}, \qquad \Omega_i = \frac{a_i(1 + g^2 r_+^2)}{r_+^2 + a_i^2}. \tag{5.26}$$

The Killing potential ω_{ξ}, (5.24), now reads

$$\omega_{\xi} = \frac{r_+^{2N}}{\prod_{i=1}^{N}(r_+^2 + a_i^2)} \sum_{j=0}^{N} \frac{1}{r_+^{2j}}\omega^{(j)}. \tag{5.27}$$

Before using ω_{ξ} in (5.6) to derive the Smarr relation, we must first consider the gauge freedom to add to it a nontrivial co-closed 2-form ν. Since co-closure, or divergence freedom, can be written as

$$\partial_{\mu}(\sqrt{-g}\nu^{\mu\nu}) = 0, \tag{5.28}$$

it is clear that a co-closed ν is obtained if we take all its contravariant components to vanish except for $\nu^{tr} = \text{constant}/\sqrt{-g}$. This is equivalent to the statement that

$$*\,\nu = \alpha\Omega_{D-2}, \tag{5.29}$$

where α is a constant and Ω_{D-2} is the volume element of the unit $(D-2)$ sphere. Evaluating (5.9) with ω_{ξ} given by (5.27) plus ν,

$$\omega_{\xi} \to \tilde{\omega}_{\xi} = \omega_{\xi} - \alpha * \Omega_{D-2}, \tag{5.30}$$

[4] If the a_i are not distinct or if some of them vanish, then h is degenerate. In such a case one does not recover all the Killing fields of the spacetime by the construction (5.19) and (5.20). However, the formula for the Killing potential ω_{ξ} obtained in the next section, Eq. (5.27), still applies.

we find that in order for (5.9) to produce the correct AMD mass for the Kerr-AdS black holes, we must choose

$$\alpha = -\frac{2m}{(D-1)(D-2)(\prod_j \Xi_j)} \sum_i \frac{a_i^2}{\Xi_i}. \tag{5.31}$$

Using $\tilde{\omega}_\xi$ in this gauge in (5.10), we find that it indeed reproduces the expressions for Θ that we obtained in Sec. III from the thermodynamic calculations.

The construction of the Killing potential (5.27) by means of Killing-Yano tensors that we have described is essentially unique. It is interesting, therefore, to observe that if we choose not to add the "gauge correction" term ν to the Killing potential given in (5.27), then the integral (5.10) over the horizon produces precisely the modified quantity Θ' that we discussed in Sec. III, which can be written in terms of the geometric volume $V' = r_+ A/(D-1)$ as in (3.10). It is not clear whether there is some simple geometrical explanation for this.

VI. CONCLUSIONS

In this paper, we have investigated some of the consequences of treating the cosmological constant, or the gauge coupling constant in a gauged supergravity, to become a dynamical variable. In particular, this means that it should then be treated as a thermodynamic variable in the first law of thermodynamics for black holes. Since the cosmological constant can be thought of as a pressure, this means that its conjugate variable in the first law is proportional to a volume. Using the first law, we have calculated this "thermodynamic volume" V for a wide variety of black holes, including static multicharge solutions in four, five and seven-dimensional gauged supergravities; rotating Kerr-AdS black holes in arbitrary dimensions; and certain charged rotating black holes in four and five-dimensional gauged supergravities.

When there is no rotation, the thermodynamic volume V can be interpreted as an integral of the scalar potential over the volume "inside the event horizon" of the black hole. In cases without scalar fields, this corresponds precisely to a naive geometrical notion of the "volume" inside the horizon. When there is rotation, however, the thermodynamic volume V differs from the notion of the "geometric volume" V' by a shift related to the angular momenta of the black hole. We showed that although in some examples the geometric volume has certain intriguing characteristics suggestive of a volume in Euclidean space that is "excluded" by the black hole, it appears that the thermodynamic volume has a more universal character. In particular, we have found that it and the horizon area obey the "reverse isoperimetric inequality" (1.17), which can be restated as the property that at fixed geometric volume V, the black hole with the largest entropy is Schwarzschild-AdS.

Although the concept of the thermodynamic volume V requires that one consider an asymptotically AdS black hole in a theory with a nonvanishing cosmological constant, interestingly it is nevertheless possible (except in $D = 7$) to take a smooth limit in the expression for V in which the cosmological constant is set to zero. Since the thermodynamic volume still, in general, differs from the geometric volume in this limit, it appears that to give a definition of V for an asymptotically flat black hole, one needs first to obtain the expression in the more general asymptotically AdS case. For example, for the Myers-Perry asymptotically flat rotating black holes, the thermodynamic volume is given by setting $g = 0$ in (4.7). As we discussed in Sec. IV, this limiting procedure also works for the known rotating black holes in all gauged supergravities except in $D = 7$. The case $D = 7$ is exceptional because of the g dependence of the odd-dimensional self-duality constraint in the seven-dimensional gauged supergravity. As a consequence, the volume diverges in the $g \to 0$ limit if the electric charges and all three rotation parameters are nonzero.

We also studied the derivation of the Smarr relation when the cosmological constant is allowed to become a thermodynamic variable. This procedure, which is a generalization of the Komar method for asymptotically flat black holes, involves the introduction of a Killing potential 2-form ω whose divergence gives the asymptotic timelike Killing vector. Because of the gauge freedom to add a co-closed 2-form to ω, the procedure does not provide an unambiguous computation of the conjugate variable Θ unless one first fixes the gauge ambiguity by requiring that the integration at infinity yield the correct expression for the mass of the black hole. Having made this gauge choice, we showed that one then recovers the thermodynamic result for Θ.

We also presented a method for constructing the Killing potentials for the Killing vectors in the Kerr-AdS black holes, based on the existence of conformal Killing-Yano tensors in these metrics. They occur because of certain "hidden symmetries" in the Kerr-AdS metrics, associated with the separability of equations such as the Dirac equation in these backgrounds. The procedure for constructing the Killing potential from the Killing-Yano tensors is an essentially unique one, and it yields the result in a very specific gauge. Interestingly, it is the gauge in which the integral $\int_H *\omega$ generates the "geometric volume" V'. This suggests that the other remarkable properties of the geometric volume, such as the fact that it is given by the Euclidean space formula (3.12), might be related to the existence of the hidden symmetries of the Kerr-AdS metrics.

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APPENDIX A: KILLING POTENTIALS IN $D = 4$ AND $D = 5$ KERR-ADS

In this appendix, for illustrative purposes, we present explicit results for the Killing potentials in the four-dimensional and five-dimensional Kerr-AdS metrics.

1. $D = 4$ Kerr-AdS

In the frame that is nonrotating at infinity, the four-dimensional Kerr-AdS metric, satisfying $R_{\mu\nu} = -3g^2 g_{\mu\nu}$, can be written as

$$ds_4^2 = -\frac{(1+g^2r^2)\Delta_\theta dt^2}{\Xi} + \frac{(r^2+a^2)\sin^2\theta d\phi^2}{\Xi} + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{2mr}{\Xi^2\rho^2}(\Delta_\theta dt - a\sin^2\theta d\phi)^2, \tag{A1}$$

where

$$\Delta_r = (r^2+a^2)(1+g^2r^2) - 2mr, \quad \Delta_\theta = 1 - a^2g^2\cos^2\theta,$$
$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Xi = 1 - a^2g^2. \tag{A2}$$

The 1-form potential b given by (5.25) is

$$b = \frac{1}{2}(r^2 + a^2\sin^2\theta)dt - \frac{a(r^2+a^2)\sin^2\theta}{2\Xi}(d\phi - ag^2dt). \tag{A3}$$

Following the steps described in Sec. V B for constructing the Killing potentials $\omega^{(0)}$ and $\omega^{(1)}$ in this case, we find that their contravariant components are given by

$$\omega^{(0)tr} = -\frac{r(r^2+a^2)}{3\rho^2}, \qquad \omega^{(0)t\theta} = -\frac{a^2\sin\theta\cos\theta}{3\rho^2},$$
$$\omega^{(0)r\phi} = \frac{ar(1+g^2r^2)}{3\rho^2}, \qquad \omega^{(0)\theta\phi} = \frac{a\Delta_\theta\cot\theta}{3\rho^2}, \tag{A4}$$

$$\omega^{(1)tr} = -\frac{a^2r(r^2+a^2)\cos^2\theta}{\rho^2}, \quad \omega^{(1)t\theta} = \frac{a^2r^2\sin\theta\cos\theta}{\rho^2},$$
$$\omega^{(1)r\phi} = \frac{a^3r(1+g^2r^2)\cos^2\theta}{\rho^2}, \quad \omega^{(1)\theta\phi} = -\frac{ar^2\Delta_\theta\cot\theta}{\rho^2}. \tag{A5}$$

These Killing potentials give rise to the corresponding Killing vectors

$$\nabla_\mu\omega^{(0)\mu\nu}\partial_\nu = \frac{\partial}{\partial t} + ag^2\frac{\partial}{\partial\phi},$$
$$\nabla_\mu\omega^{(1)\mu\nu}\partial_\nu = a^2\frac{\partial}{\partial t} + a\frac{\partial}{\partial\phi}. \tag{A6}$$

Thus the Killing potential for the Killing vector

$$\xi = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi} \tag{A7}$$

that is null on the horizon is

$$\omega_\xi = \frac{r_+^2}{(r_+^2+a^2)}\bigg(\omega^{(0)} + \frac{1}{r_+^2}\omega^{(1)}\bigg). \tag{A8}$$

2. $D = 5$ Kerr-AdS

In the frame that is nonrotating at infinity, the five-dimensional Kerr-AdS metric, satisfying $R_{\mu\nu} = -4g^2 g_{\mu\nu}$, can be written as

$$ds_5^2 = -\frac{(1+g^2r^2)\Delta_\theta dt^2}{\Xi_1\Xi_2} + \frac{(r^2+a_1^2)\sin^2\theta d\phi_1^2}{\Xi_1} + \frac{(r^2+a_2^2)\cos^2\theta d\phi_2^2}{\Xi_2} + \frac{\rho^2dr^2}{\Delta_r} + \frac{\rho^2d\theta^2}{\Delta_\theta} + \frac{2m}{\rho^2}\bigg[\frac{\Delta_\theta dt}{\Xi_1\Xi_2} - \frac{a_1\sin^2\theta d\phi_1}{\Xi_1} - \frac{a_2\cos^2\theta d\phi_2}{\Xi_2}\bigg]^2, \tag{A9}$$

where

$$\Delta_r = \frac{(r^2+a_1^2)(r^2+a_2^2)(1+g^2r^2)}{r^2} - 2m,$$
$$\Delta_\theta = 1 - a_1^2g^2\cos^2\theta - a_2^2g^2\sin^2\theta,$$
$$\rho^2 = r^2 + a_1^2\cos^2\theta + a_2^2\sin^2\theta,$$
$$\Xi_1 = 1 - a_1^2g^2,$$
$$\Xi_2 = 1 - a_2^2g^2. \tag{A10}$$

The 1-form potential b given by (5.25) is

$$b = \frac{1}{2}(r^2 + a_1^2\sin^2\theta + a_2^2\cos^2\theta)dt - \frac{a_1(r^2+a_1^2)\sin^2\theta}{2\Xi_1}(d\phi_1 - a_1g^2dt) - \frac{a_2(r^2+a_2^2)\cos^2\theta}{2\Xi_2}(d\phi_2 - a_2g^2dt). \tag{A11}$$

Following the steps described in Sec. V B for constructing the Killing potentials $\omega^{(0)}$, $\omega^{(1)}$ and $\omega^{(2)}$ in this case, we find that their contravariant components are given by

$$\omega^{(a)tr} = -\frac{(r^2+a_1^2)(r^2+a_2^2)}{4r\rho^2}\Big[1, \frac{2(1-\Delta_\theta)}{g^2}, a_1^2a_2^2\Big],$$
$$\omega^{(a)t\theta} = \frac{(a_1^2-a_2^2)\sin\theta\cos\theta}{4\rho^2}[-1, 2r^2, a^2b^2],$$
$$\omega^{(a)r\phi_1} = \frac{a_1(r^2+a_2^2)}{4r\rho^2}\Big[(1+g^2r^2), \frac{2(1+g^2r^2)(1-\Delta_\theta)}{g^2}, a_2^2(r^2+a_1^2)\Big],$$
$$\omega^{(a)r\phi_2} = \frac{a_2(r^2+a_1^2)}{4r\rho^2}\Big[(1+g^2r^2), \frac{2(1+g^2r^2)(1-\Delta_\theta)}{g^2}, a_1^2(r^2+a_2^2)\Big],$$
$$\omega^{(a)\theta\phi_1} = \frac{a_1\cot\theta}{4\rho^2}[\Delta_\theta, -2r^2\Delta_\theta, a_2^2(a_1^2-a_2^2)\sin^2\theta],$$
$$\omega^{(a)\theta\phi_2} = -\frac{a_2\tan\theta}{4\rho^2}[\Delta_\theta, -2r^2\Delta_\theta, -a_1^2(a_1^2-a_2^2)\cos^2\theta], \tag{A12}$$

where the components for $\omega^{(a)}$ with $a = 0$, 1 and 2 correspond to the first, second and third entries of the square bracketed factors, respectively.

The three Killing potentials give rise to the following Killing vectors:

$$\nabla_\mu\omega^{(0)\mu\nu}\partial_\nu = \frac{\partial}{\partial t} + a_1g^2\frac{\partial}{\partial\phi_1} + a_2g^2\frac{\partial}{\partial\phi_2},$$
$$\nabla_\mu\omega^{(1)\mu\nu}\partial_\nu = (a_1^2+a_2^2)\frac{\partial}{\partial t} + a_1(1+a_2^2g^2)\frac{\partial}{\partial\phi_1} + a_2(1+a_1^2g^2)\frac{\partial}{\partial\phi_2},$$
$$\nabla_\mu\omega^{(2)\mu\nu}\partial_\nu = a_1^2a_2^2\Big(\frac{\partial}{\partial t} + \frac{1}{a_1}\frac{\partial}{\partial\phi_1} + \frac{1}{a_2}\frac{\partial}{\partial\phi_2}\Big). \tag{A13}$$

Thus the Killing potential for the Killing vector

$$\xi = \frac{\partial}{\partial t} + \Omega_1\frac{\partial}{\partial\phi_1} + \Omega_2\frac{\partial}{\partial\phi_2} \tag{A14}$$

that is null on the horizon is

$$\omega_\xi = \frac{r_+^4}{(r_+^2+a_1^2)(r_+^2+a_2^2)}\Big(\omega^{(0)} + \frac{1}{r_+^2}\omega^{(1)} + \frac{1}{r_+^4}\omega^{(2)}\Big). \tag{A15}$$

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