Relic gravitational waves from light primordial black holes

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The energy density of relic gravitational waves (GWs) emitted by primordial black holes (PBHs) is calculated. We estimate the intensity of GWs produced at quantum and classical scattering of PBHs, the classical graviton emission from the PBH binaries in the early Universe, and the graviton emission due to PBH evaporation. If nonrelativistic PBHs dominated the cosmological energy density prior to their evaporation, the probability of formation of dense clusters of PBHs and their binaries in such clusters would be significant and the energy density of the generated gravitational waves in the present-day universe could exceed that produced by other known mechanisms. The intensity of these gravitational waves would be maximal in the GHz frequency band of the spectrum or higher and makes their observation very difficult by present detectors but also gives a rather good possibility to investigate it by present and future high-frequency gravitational waves electromagnetic detectors. However, the low-frequency part of the spectrum in the range $f \sim 0.1$–10 Hz may be detectable by the planned space interferometers DECIGO/BBO. For sufficiently long duration of the PBH matter-dominated stage, the cosmological energy fraction of GWs from inflation would be noticeably diluted.

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I. INTRODUCTION

Since the prediction of gravitational waves (GW) by Albert Einstein in 1918 [1] on the basis of general relativity, they have been an object of intensive studies. Gravitational waves are thought to be fluctuations in the curvature of space-time, which propagate as waves, traveling outward from the source. Although gravitational radiation has not yet been directly detected, it has been indirectly shown to exist because it increases the pulsar orbital frequency [2] in good agreement with theoretical predictions.

Roughly speaking, there are two groups of possible sources of gravitational radiation which may be registered by gravitational wave detectors either on the Earth or by space missions. The first group includes energetic phenomena in the contemporary universe, such as emission of GWs by black hole or compact star binaries, supernova explosions, and possibly some other catastrophic phenomena. The second group contains gravitational radiation coming from the early Universe, which creates today an isotropic background, usually with rather low frequency. Such gravitational radiation could be produced at inflation, phase transitions in the primeval plasma, by the decay or interaction of topological defects, e.g., cosmic strings, *etc*.

The graviton (gravitational wave) production in the Friedmann-Robertson-Walker metric was first considered by Grishchuk [3], who noticed that the graviton wave equation is not conformal invariant and thus such quanta can be produced by conformal flat external gravitational field. Generation of gravitational waves at the De Sitter (inflationary) stage was studied by Starobinsky [4] (see also Ref. [5]). The stochastic homogeneous background of the low-frequency gravitational waves is now one of the very important predictions of inflationary cosmology, which may present a final proof of inflation.

In this work, we discuss one more source of gravitational wave (GW) radiation in the early Universe, namely, the interaction between primordial black holes (PBH). We consider relatively light PBH, such that they evaporated before the big bang nucleosynthesis (BBN) and so they are not constrained by the light element abundances. Cosmological scenario with early formed and evaporated primordial black holes producing gravitons was considered in Ref. [6]. Here we will remain in essentially the same framework and study in addition the GW emission in different processes with PBH.

According to Ref. [7,8] the lifetime of an evaporating black hole with initial mass M is equal to:

$$\tau_{\rm BH} = \frac{10240\pi}{N_{\rm eff}}\frac{M^3}{m_{\rm Pl}^4}, \qquad (1)$$

where the Planck mass is $m_{\rm Pl} = 2.176 \times 10^{-5}$ g and $N_{\rm eff}$ is the number of particle species with masses smaller than the black hole temperature:

$$T_{\rm BH} = \frac{m_{\rm Pl}^2}{8\pi M}. \qquad (2)$$

To avoid a conflict with BBN, the black holes should had been evaporated before cosmological time $t \approx 10^{-2}$ s [9] and thus their mass would be bounded from above by

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$$M < 1.75 \times 10^8 \left(\frac{N_{\rm eff}}{100}\right)^{1/3} \text{ g}. \tag{3}$$

The temperature of such PBHs should be higher than 3×10^4 GeV and correspondingly $N_{\rm eff} \geq 10^2$. On the other hand, as is discussed in what follows, the PBH mass is bounded from below, e.g., by Eq. (16). This is the mass range of PBHs considered in this work. Such PBH are not constrained by any astronomical data, which are applicable to heavier ones [9,10].

Primordial black holes should interact in the early Universe creating gravitational radiation. Below we estimate the efficiency of GW emission in several processes with PBH. In Sec. II some mechanisms of PBH production and PBH evolution in the early Universe are briefly described. We stress, in particular, a very important role played by the clumping of PBH due to gravitational instability at the matter-dominated stage. In Sec. III we consider the initial interaction between the PBHs when they started to "feel" each other and accelerate with respect to the background cosmological expansion. In Sec. IV the quantum bremsstrahlung of gravitons at PBH collisions is discussed, which is quite similar to the electromagnetic bremsstrahlung at Coulomb scattering of electrically charged particles. Next, in Sec. V we consider the classical emission of GW at accelerated motion of a pair of BHs in their mutual gravitational field. In Sec. VI we evaluate the energy loss of PBHs due to their mutual interaction. It may be relevant to the estimation of the probability of formation of PBH binaries. The gravitational radiation from PBH binaries in high-density clusters is discussed in Sec. VII. In Sec. VIII we calculate the present-day energy density of gravitons produced at PBH evaporation. In Sec. IX we review some mechanisms of the production of stochastic background of GWs. In Sec. X the status of existing and planned detectors of GWs is discussed. In Sec. XI we conclude.

II. PRODUCTION AND EVOLUTION OF PBH IN THE EARLY UNIVERSE

Formation of primordial black holes from the primordial density perturbations in the early Universe was first considered by Zeldovich and Novikov [11] and later by Hawking and Carr [12,13]. PBHs would be formed when the density contrast, $\delta\rho/\rho$, at horizon was of the order of unity or, in other words, when the Schwarzschild radius of the perturbation was of the order of the horizon scale. If PBH was formed at the radiation-dominated stage, when the cosmological energy density was $\rho(t) = 3m_{\rm Pl}^2/(32\pi t^2)$, and the horizon was $l_h = 2t$, the mass of PBH would be:

$$M(t) = m_{\rm Pl}^2 t \simeq 4 \times 10^{38} \left(\frac{t}{\rm sec}\right) \text{ g} \tag{4}$$

where t is the time elapsed since the big bang.

The fraction of the cosmological energy density of PBH produced by such mechanism depends upon the spectrum of the primordial density perturbations. We denote this fraction Ω_p and take it as a free parameter of the model. The data on the large-scale structure of the Universe and on the angular fluctuations of the cosmic microwave background radiation (CMB) show that the spectrum of the primordial density fluctuations is almost a flat Harrison-Zeldovich one. For such a spectrum, the probability of PBH production is quite low and $\Omega_p \ll 1$. However, the flatness of the spectrum is verified only for astronomically large scales, comparable with the galactic ones. The form of the spectrum for masses below 10^{10} g is not known. Inflation predicts that the spectrum remains flat for all the scales but there exist scenarios with strong deviation from flatness at small scales. In particular, in Ref. [14,15] a model of PBH formation has been proposed which leads to log-normal mass spectrum of the produced PBH:

$$\frac{dN}{dM} = C \exp\left[\frac{(M - M_0)^2}{M_1^2}\right], \tag{5}$$

where C, M_0, and M_1 are some model-dependent parameters. Quite naturally, the central value of PBH mass distribution may be in the desired range $M_0 < 10^9$ g. In this model, the value of Ω_p may be much larger than in the conventional model based on the flat spectrum of the primordial fluctuations. We will not further speculate on the value of Ω_p or on the form of the mass spectrum of PBH. In what follows, we assume for an order of magnitude estimate that the spectrum is well localized near some fixed mass value and that Ω_p is an arbitrary parameter. Different mechanisms of PBH production are reviewed, e.g., in Ref. [16,17].

We assume that PBHs were produced in radiation-dominated (RD) Universe, when the cosmological energy density was equal to

$$\rho_R = \frac{3m_{\rm Pl}^2}{32\pi t^2}. \tag{6}$$

If we neglect the PBH evaporation and possible coalescence, their number density would remain constant in the comoving volume, $n_{\rm BH}(t)a^3(t) = \text{const}$. In what follows, the instant decay approximation for evaporation is used. The cosmological evolution of PBHs with a more realistic account of their decay was studied in Ref. [18].

Since the black holes were nonrelativistic at production, their relative contribution to the cosmological energy density rose as the cosmological scale factor, $a(t)$:

$$\Omega_{\rm BH}(t) = \Omega_p \left(\frac{a(t)}{a_p}\right), \tag{7}$$

where a_p is the value of the scale factor at the PBH production and at RD stage $a(t)/a_p = (t/t_p)^{1/2}$. The moment t_p of the black hole production is connected with the PBH mass through Eq. (4). Hence

$$t_p = \frac{M}{m_{\rm Pl}^2}. \tag{8}$$

Thus, if PBHs lived long enough, they would dominate the cosmological energy density and the Universe would become matter-dominated at $t > t_{\rm eq}$, where

$$t_{\rm eq} = \frac{M}{m_{\rm Pl}^2 \Omega_p^2} = \frac{r_g}{2\Omega_p^2}, \tag{9}$$

and $r_g = 2M/m_{\rm Pl}^2$ is the gravitational (Schwartzschild) radius of a black hole.

In what follows we assume that all PBHs have the same mass M, but the results can be simply generalized by integration over the PBH mass spectrum.

Evidently at RD stage the number density of PBHs drops as:

$$n_{\rm BH}(t) = n_p\left(\frac{a_p}{a(t)}\right)^3 = n_p\left(\frac{t_p}{t}\right)^{3/2}, \tag{10}$$

while at matter-dominated (MD) stage

$$n_{\rm BH}(t) = n_p\left(\frac{t_p}{t_{\rm eq}}\right)^{3/2}\left(\frac{t_{\rm eq}}{t}\right)^2. \tag{11}$$

Cosmological mass fraction of a black hole (BH) as a function of time behaves as

$$\Omega_{\rm BH}(t) = \frac{n_{\rm BH}(t)M}{\rho_c} = \frac{16\pi}{3} r_g t^2 n_{\rm BH}(t), \tag{12}$$

i.e., $\Omega_{\rm BH} \sim t^{1/2}$ at RD stage. After the onset of the PBH dominance, $\Omega_{\rm BH}$ approached unity and remained constant until the PBH evaporation when $\Omega_{\rm BH}$ quickly dropped down to zero and the universe became dominated by relativistic particles produced by PBH evaporation. All relics from the earlier RD stage would be diluted by the redshift factor $(t_{\rm eq}/\tau_{\rm BH})^{2/3}$. In particular the energy density of GWs produced at inflation would be diminished by this factor with respect to the standard predictions. Such dilution may cause problems with baryogenesis. However, these problems may be resolved if baryogenesis took place at the process of PBH evaporation through the mechanism suggested by Zeldovich [19] and quantitatively studied in Ref. [20,21]. Somewhat similar model of baryogenesis by heavy particle decay (e.g., by bosons of GUT) created at PBH evaporation was considered in Ref. [22–25].

To survive until equilibration, the PBHs should live long enough so that their evaporation time $t_{\rm ev}$ would be larger than $t_{\rm eq}$ or $\tau_{\rm BH} > t_{\rm eq} - t_p$, which can be translated into the bound on the PBH mass:

$$M > \left(\frac{N_{\rm eff}}{3.2\times 10^4}\right)^{1/2} m_{\rm Pl}\left(\frac{1}{\Omega_p^2} - 1\right)^{1/2} \simeq 5.6\times 10^{-2}\left(\frac{N_{\rm eff}}{100}\right)^{1/2}\frac{m_{\rm Pl}}{\Omega_p}, \tag{13}$$

where $\Omega_p \ll 1$ and M is mass of PBHs at production.[1] Both constraints (3) and (13) would be satisfied if

$$\Omega_p > 0.7\times 10^{-14}\left(\frac{N_{\rm eff}}{100}\right)^{1/6}. \tag{14}$$

For example, if $\Omega_p = 10^{-10}$, the black holes should be heavier than 1.2×10^4 g.

When the Universe became dominated by nonrelativistic PBHs, primordial density perturbations, $\Delta = \delta\rho/\rho$, should rise as the cosmological scale factor. They could reach unity at cosmological time t_1 satisfying the condition:

$$\Delta_{\rm in}\left(\frac{t_1}{t_{\rm eq}}\right)^{2/3} \sim 1, \tag{15}$$

where $\Delta_{\rm in}$ is the initial magnitude of the primordial density perturbations. To be more accurate, the evolution of density perturbations depends upon the moment when they cross horizon, see below, Eq. (19). For the moment, we neglect this complication to make some simple estimates.

The initial density contrast is usually assumed to be of the order of $\Delta_{\rm in} \sim 10^{-5} - 10^{-4}$ which is not necessarily true at small scales and may be much larger, especially in the model of Ref. [14,15].

Evidently the BH lifetime, $\tau_{\rm BH}$, must be long enough so that the density fluctuations in BH matter would rise up to the values of the order of unity. The condition $t_{\rm ev} > t_1$ or equivalently $\tau_{\rm BH} > t_1 - t_p$ leads to the following restriction on the PBH mass:

$$M > M_{\rm low} = \left(\frac{N_{\rm eff}}{3.2\times 10^4}\right)^{1/2}\frac{m_{\rm Pl}}{\Omega_p \Delta_{\rm in}^{3/4}} \simeq 1.2\times 10^3\ {\rm g}\left(\frac{10^{-6}}{\Omega_p}\right)\left(\frac{10^{-4}}{\Delta_{\rm in}}\right)^{3/4}\left(\frac{N_{\rm eff}}{100}\right)^{1/2}. \tag{16}$$

We can see that Eq. (16) puts a stronger lower limit on PBHs mass than Eq. (13). The limits are comparable only if $\Delta_{\rm in} \approx 1$. Using Eqs. (16) and (3), we get a stronger than (14) restriction on Ω_p:

[1] In fact in Eq. (13) there must be the PBHs mass at the equilibrium time, $M(t_{\rm eq})$. Because of evaporation, the PBH mass as a function of time is given by $M(t) = M(t_p)\times(1 - t/\tau_{\rm BH})^{1/3}$ and it is easy to see that for $\tau_{\rm BH} > t_{\rm eq}$ it gives $M = M(t_p) \simeq M(t_{\rm eq})$, so hereafter we refer to M as the mass of PBH at production.

$$\Omega_p > 0.7 \times 10^{-11}\left(\frac{10^{-4}}{\Delta_{\rm in}}\right)^{3/4}\left(\frac{N_{\rm eff}}{100}\right)^{1/6}. \tag{17}$$

After Δ reached unity, the rapid structure formation would take place and high-density clusters of PBHs would be formed. As we see in what follows, generation of gravitational waves would be especially efficient from such high-density clusters of primordial black holes.

Let us assume that the spectrum of perturbations is the flat Harrison-Zeldovich one and that a perturbation with some wave length λ crossed horizon at moment $t_{\rm in}$. The mass inside the horizon at this moment was:

$$M_b(t_{\rm in}) = m_{\rm Pl}^2 t_{\rm in}. \tag{18}$$

It is the mass of the would-be high-density cluster of PBHs. This initial time is supposed to be larger than $t_{\rm eq}$ (9), i.e., the horizon crossing took place already at MD stage. For flat spectrum of perturbations density contrast, $\Delta = \delta\rho/\rho$, at horizon crossing is the same for all wave lengths. After horizon crossing the perturbations would continue to grow up as the scale factor, $\Delta(t) = \Delta_{\rm in}(t/t_{\rm in})^{2/3}$. The rise would continue until the moment $t_1(t_{\rm in})$ such that:

$$\Delta[t_1(t_{\rm in})] = \Delta_{\rm in}[t_1(t_{\rm in})/t_{\rm in}]^{2/3} = 1 \quad \text{or} \qquad t_1(t_{\rm in}) = t_{\rm in}\Delta_{\rm in}^{-3/2}. \tag{19}$$

The radius of the PBH cluster rose almost as the cosmological scale factor until $t = t_1(t_{\rm in})$. After the density contrast has reached unity, the cluster would decouple from the common cosmological expansion. In other words, the cluster stopped expanding together with the universe and, on the opposite, it would begin to shrink when gravity takes over the free streaming of PBHs. So the cluster size would drop down and both $n_{\rm BH}$ and ρ_b would rise. The density contrast would quickly rise from unity to $\Delta_b = \rho_b/\rho_c \gg 1$, where ρ_c and ρ_b are, respectively, the average cosmological energy density and the density of PBHs in the cluster (bunch). It looks reasonable that the density contrast of the evolved cluster could rise up to $\Delta = 10^5 - 10^6$, as in the contemporary galaxies. After the size of the cluster stabilized, the number density of PBH, $n_{\rm BH}$, as well as their mass density, $\rho_{\rm BH}$, would be constant too. But the density contrast, Δ_b would continue to rise as $(t/t_1)^2$ because ρ_c drops down as $1/t^2$. From time $t = t_1$ to $t = \tau_{\rm BH}$ the density contrast would additionally rise by the factor

$$\Delta(\tau_{\rm BH}) = \Delta(t_1)\left(\frac{\tau_{\rm BH}}{t_1}\right)^2 = \Delta(t_i)\left(\frac{M}{M_{\rm low}}\right)^4, \tag{20}$$

where t_1 and $M_{\rm low}$ are given by Eqs. (15) and (16) respectively.

The size of the high-density clusters of PBH would be

$$R_b = \Delta_b^{-1/3} t_1^{2/3} t_{\rm in}^{1/3} \tag{21}$$

and the average distance between the PBHs in the bunch can be estimated as:

$$d_b = (M/M_b)^{1/3} R_b = \Delta_b^{-1/3} t_1^{2/3} r_g^{1/3} = 2^{-2/3}\Delta_b^{-1/3}\Delta_{\rm in}^{-1}\Omega_p^{-4/3} r_g. \tag{22}$$

It does not depend upon $t_{\rm in}$. Here Eqs. (15) and (9) have been used.

The virial velocity inside the cluster would be

$$v = \sqrt{\frac{2M_b}{m_{\rm Pl}^2 R_b}} = 2^{1/2}\Delta_b^{1/6}\Delta_{\rm in}^{1/2} \approx 0.14\left(\frac{\Delta_b}{10^6}\right)^{1/6}\left(\frac{\Delta_{\rm in}}{10^{-4}}\right)^{1/2}. \tag{23}$$

So PBHs in the cluster can be moderately relativistic.

Later, when $t = \tau_{\rm BH}$, black holes would decay producing relativistic matter and the Universe would return to the normal RD regime. However, the previous history of the earlier RD stage would be forgotten.

For the future discussion it is convenient to introduce the average distance between the PBHs at arbitrary time, $d = n_{\rm BH}^{-1/3}$, where $n_{\rm BH} = \rho_{\rm BH}/M$ is the number density of PBHs. Since

$$\Omega_p = \frac{\rho_p}{\rho_c} = \frac{32\pi t_p^2 M n_p}{3 m_{\rm Pl}^2} = \frac{32\pi}{3}\left(\frac{t_p}{d_p}\right)^3, \tag{24}$$

the average distance between PBHs at the production moment is equal to

$$d_p = (4\pi/3)^{1/3} r_g \Omega_p^{-1/3}. \tag{25}$$

When the mutual gravitational attraction of PBH may be neglected, d rises as cosmological scale factor, $a(t)$.

Gravitational waves produced in the early universe will be hopefully registered in the present epoch. The sensitivity of GW detectors strongly depends upon the frequency of the signal. The frequency f_* of GW produced at time t_* during PBH evaporation, is redshifted down to the present-day value, f, according to:

$$f = f_*\left[\frac{a(t_*)}{a_0}\right] = 0.34 f_* \frac{T_0}{T_*}\left[\frac{100}{g_S(T_*)}\right]^{1/3}, \tag{26}$$

where $T_0 = 2.725$ K [26] is the temperature of the cosmic microwave background radiation at the present time, $T_* \equiv T(t_*)$ is the plasma temperature at the moment of radiation of the gravitational waves, and $g_S(T_*)$ is the number of species contributing to the entropy of the primeval plasma at temperature T_*. It is convenient to express T_0 in frequency units, $T_0 = 2.7\ {\rm K} = 5.4 \times 10^{10}$ Hz.

The temperature of the primeval plasma after the PBH evaporation can be approximately found from:

$$\rho = \frac{m_{\rm Pl}^2}{6\pi t^2} = \frac{\pi^2 g_*(T_*) T_*^4}{30}, \tag{27}$$

where $g_*(T_*) \approx 10^2$ is the contribution of different particle species to the energy density at temperature T_* and $t_1 < t < t_{ev}$. For relativistic plasma $g_*(T) = g_S(T)$. Since $t_{ev} = \tau_{BH} + t_p \simeq \tau_{BH}$, we obtain from Eq. (27) at time $t_* = \tau_{BH}$:

$$T_*(\tau_{BH}) = \left[\frac{30}{6\pi^3 g_S(T_*)}\right]^{1/4}\left(\frac{N_{eff}}{3.2\times 10^4}\right)^{1/2}\frac{m_{Pl}^{5/2}}{M^{3/2}}. \quad (28)$$

Substituting the numbers we find:

$$T_*(\tau_{BH}) \approx 0.011 m_{Pl}\left[\frac{100}{g_S(T_*)}\right]^{1/4}\left(\frac{N_{eff}}{100}\right)^{1/2}\left(\frac{m_{Pl}}{M}\right)^{3/2}. \quad (29)$$

For comparison, at the PBH production moment the temperature of the primeval plasma was:

$$T_p \approx 0.2 m_{Pl}\left(\frac{m_{Pl}}{M}\right)^{1/2}. \quad (30)$$

Using Eqs. (26) and (29), we find that the present-day frequency of the GWs, emitted at T_* (28) with frequency f_*, would be equal to:

$$f = 1.7\times 10^{12}\ \text{Hz}\left[\frac{100}{g_S(T_*)}\right]^{1/12}\left(\frac{100}{N_{eff}}\right)^{1/2}\left(\frac{f_*}{m_{Pl}}\right)\left(\frac{M}{m_{Pl}}\right)^{3/2}. \quad (31)$$

If we take the maximum frequency of the emitted gravitons $f_{max*} \approx r_g^{-1} = m_{Pl}^2/2M$, the GW maximum frequency today would be:

$$f_{max} \approx 8.6\times 10^{11}\ \text{Hz}\left(\frac{M}{m_{Pl}}\right)^{1/2} = 5.8\times 10^{16}\ \text{Hz}\left(\frac{M}{10^5\ \text{g}}\right)^{1/2}. \quad (32)$$

III. ONSET OF GW RADIATION

Once PBHs enter inside each other cosmological horizon[2] they start to interact and thus to radiate gravitational waves due to their mutual acceleration. The corresponding time moment t_h is determined by the condition $2t_h = d(t_h)$ and, remembering that it happened still at RD stage, we find

$$t_h = \frac{1}{2}\left(\frac{4\pi}{3}\right)^{2/3} r_g \Omega_p^{-2/3}. \quad (33)$$

For $t > t_h$, the curvature effects can be neglected and the PBH motion is completely determined by the Newtonian gravity:

$$\ddot{\mathbf{r}} = -\frac{M_{BH}}{m_{Pl}^2 r^2}\frac{\mathbf{r}}{r} \quad (34)$$

[2]The cosmological horizon is the distance which PBHs started interacting with each other, exchanging gravitons, and should not be confused with the black hole event horizon.

with the initial conditions $r_i \equiv |\mathbf{r}_i| = d(t_i)$ and $|\dot{\mathbf{r}}_i| = H(t_i)|\mathbf{r}_i|$, where $\mathbf{r}$ is the position vector of PBHs. For $t_i = t_h$, their relative initial velocity $|\dot{\mathbf{r}}_i| = v_i = 1$ and nonrelativistic approximation is invalid. To avoid that we should choose $t_i > t_h$ such that $v_i \ll 1$. The solution of the equation of motion demonstrates that the effects of mutual attraction at this stage and production of GW are weak.

After PBHs enter inside each other's horizon and Newtonian gravity can be applied, their acceleration toward each other becomes essential when their Hubble velocity drops below the capture velocity. The corresponding time moment, t_c, when it happened, is determined from the condition

$$\frac{1}{2}v^2(t_c) \equiv \frac{1}{2}[H(t_c)d(t_c)]^2 \lesssim \frac{M_{BH}}{m_{Pl}^2 d(t_c)}. \quad (35)$$

If it took place at the RD regime, the corresponding time moment would be equal to:

$$t_c = \frac{8\pi^2}{9}\frac{r_g}{\Omega_p^2}, \quad (36)$$

and the density parameter of PBHs at $t = t_c$ would be

$$\Omega_{BH}(t_c) = \Omega_p\left(\frac{t_c}{t_p}\right)^{1/2} = \frac{4\pi}{3} > 1. \quad (37)$$

Thus at $t = t_c$ the universe is already matter-dominated and we have to use the nonrelativistic expansion law, $a \sim t^{2/3}$, starting from the moment $t = t_{eq}$ (9). Accordingly the average distance between BHs, when $t > t_{eq}$, grows as

$$d(t) = d_p\left(\frac{t_{eq}}{t_p}\right)^{1/2}\left(\frac{t}{t_{eq}}\right)^{2/3}. \quad (38)$$

Now we find that the condition that the Hubble velocity, $v_H = (2/3t_c)d_c$, is smaller than the virial one, for average values, reads

$$\frac{4d_p^3}{9r_g t_p^{3/2} t_{eq}^{1/2}} < 1. \quad (39)$$

One can see that this condition is never fulfilled. However, this negative result does not mean that the acceleration of BHs and GW emission is suppressed, because of the effect, mentioned above, of rising density perturbations.

IV. BREMSSTRAHLUNG OF GRAVITONS

PBH scattering in the early Universe should be accompanied by the graviton emission almost exactly as the scattering of charged particles is accompanied by the emission of photons. The cross section of the graviton bremsstrahlung in particle collisions was calculated in Ref. [27] for the case of two spineless particles (here black holes) with masses m and M under assumption that $m \ll M$. In nonrelativistic approximation, $\mathbf{p}^2 \ll m^2$, the differential cross section reads

$$d\sigma = \frac{64M^2m^2}{15m_{\rm Pl}^6}\frac{d\xi}{\xi}\left[5\sqrt{1-\xi}+\frac{3}{2}(2-\xi)\ln\frac{1+\sqrt{1-\xi}}{1-\sqrt{1-\xi}}\right], \tag{40}$$

where ξ is the ratio of the emitted graviton frequency, $\omega = 2\pi f$, to the kinetic energy of the incident black hole, i.e., $\xi = 2m\omega/\mathbf{p}^2$. We will use expression (40) for an order of magnitude estimate, assuming that it is approximately valid for arbitrary m and M, in particular, for $m \sim M$.

The energy density of gravitational waves emitted at the time interval t and $t + dt$ in the frequency range ω and $\omega + d\omega$ is given by

$$\frac{d\rho_{\rm GW}}{d\omega} = v_{\rm rel} n_{\rm BH}^2 \omega\left(\frac{d\sigma}{d\omega}\right)dt, \tag{41}$$

where $n_{\rm BH}$ is the number density of PBH and $v_{\rm rel}$ is their relative velocity.

The energy emitted in the frequency interval $\omega \in [0, \omega_{\max}]$ per unit time is proportional to the integral

$$I(\omega_{\max}) = \frac{\mathbf{p}^2}{2m}\int_0^{\xi\,\max} d\xi\left[5\sqrt{1-\xi}+\frac{3}{2}(2-\xi)\times \ln\frac{1+\sqrt{1-\xi}}{1-\sqrt{1-\xi}}\right]. \tag{42}$$

The maximum value of the frequency of the emitted gravitons should be smaller than either the kinetic energy of the colliding BHs, $E_{\rm kin} = p^2/(2M)$ or the BH inverse gravitational radius, $1/r_g = m_{\rm Pl}^2/2M$, depending on which of the two is smaller. Their ratio is $E_{\rm kin} r_g = M^2 v^2/m_{\rm Pl}^2$, so for $M < m_{\rm Pl} v^{-1}$ the maximum frequency would be the PBH kinetic energy and in this case $\xi_{\max} = 1$. It corresponds to the situation when PBH is nearly captured. It loses practically all its kinetic energy, which goes to the graviton. For PBHs in the high-density clusters, when $v \sim 0.1$, the maximum frequency would be $\omega_{\max} \sim 1/r_g$ for all PBHs heavier than $10 m_{\rm Pl}$. In this case $\xi_{\max} = (m_{\rm Pl}/Mv)^2$.

The first, rather exotic case, when $M < m_{\rm Pl}/v$ can be realized only if $\Omega_p \geq 0.01$, see Eq. (13). If $\xi_{\max} = 1$, then $\omega_{\max} \sim \mathbf{p}^2/2m$ and the integral can be taken analytically:

$$I(\omega_{\max} = p^2/2M) = \frac{25}{3}\frac{\mathbf{p}^2}{2m} = \frac{25}{3}\omega_{\max}. \tag{43}$$

In this case, the energy taken by GWs is of the order of the kinetic energy of PBH and correspondingly $\Omega_{\rm GW} \sim M n_{\rm bh} v^2/\rho_{\rm BH} = v^2$.

Below we will consider more natural situation when $M > m_{\rm Pl} v^{-1}$. Integral (42) in the limit of small $\xi_{\max}$ is

$$I(\omega_{\max} = 1/r_g) = \frac{p^2}{2M}\xi_{\max}[8 + 3\ln(4/\xi_{\max})] \tag{44}$$

This expression is accurate within 30% up to $\xi_{\max} = 1$. So, in what follows, we will use this result as $I(\omega_{\max}) \approx 25\omega_{\max}/3$, keeping in mind that normally $\omega_{\max} = 1/r_g \ll p^2/2M$.

The fraction of the cosmological energy density of the emitted gravitational waves which has been produced during time interval t and $t + dt$, which is smaller than or comparable to the cosmological time $t_1 \lesssim t \lesssim t_{\rm ev} \simeq \tau_{\rm BH}$, can be obtained by the integration of Eq. (41) over ω from 0 to $\omega_{\max}$ taking into account that the energy density of GWs goes with the redshift as $(1 + z)^{-4}$, and the integration over cosmological time, t, which is connected with the redshift by the relation[3]

$$dt = -\frac{dz}{H_*(1+z)[\Omega_{\rm BH*}(1+z)^3 + \Omega_{r*}(1+z)^4]^{1/2}}, \tag{45}$$

where H_*, $\Omega_{\rm BH*}$, and Ω_{r*} are, respectively, the Hubble parameter, the matter density parameter, and the radiation density parameter evaluated at cosmological time $t_* = \tau_{\rm BH}$, just before the PBH decay. Recall that we use the instant decay approximation, so the Universe at $t = \tau_{\rm BH}$ was still at MD stage. In this case, all quantities such as H_* and ρ_c are taken at this stage: $H_* = 2/3t_*$, $\rho_c = m_{\rm Pl}^2/6\pi\tau_{\rm BH}^2$, $\Omega_{\rm BH*} = 1$, and $\Omega_{r*} = 0$.

We need to calculate the energy density of GWs at the moment of the PBH evaporation. The rate of GW production is given by Eq. (41). To take into account the redshift of the energy density of the gravitational waves we have to divide $d\rho_{\rm GW}/d\omega$ by $(1 + z)^4$, to substitute $\omega = (1 + z)\omega_*$, where ω_* is the GW frequency at $t = \tau_{\rm BH}$, and to express time through the redshift as $dt = (3/2)\tau_{\rm BH}(1 + z)^{-5/2}dz$. As a result we obtain at $t_* = \tau_{\rm BH}$:

$$d\rho_{\rm GW}(\tau_{\rm BH}) = \frac{32M^2 v_{\rm rel}}{5m_{\rm Pl}^6}[\rho_{\rm BH}^{(\rm cluster)}]^2\tau_{\rm BH}(1+z)^{-13/2} \times f[\omega_*(z+1)]d[(1+z)\omega_*]dz. \tag{46}$$

Here $\rho_{\rm BH}^{(\rm cluster)}$ is the energy density of the PBHs in the cluster (which is denoted above as ρ_b). Note that $\rho_{\rm BH}^{(\rm cluster)} = \text{const}$ before the PBH decay. We parametrize this quantity as $\rho_{\rm BH}^{(\rm cluster)} = \rho_{\rm BH}^{(c)}(\tau_{\rm BH})\Delta(\tau_{\rm BH})$, where $\rho_{\rm BH}^{(c)}(\tau_{\rm BH}) = m_{\rm Pl}^2/(6\pi\tau_{\rm BH}^2)$ is the average cosmological energy density of PBH and $\Delta(\tau_{\rm BH})$ is given by Eq. (20); see also the discussion above this equation. Function $f(\omega)$ is the function of $\xi = 2m\omega/p^2$ in the square brackets of Eq. (40).

To find the cosmological energy fraction of GWs at $t = \tau_{\rm BH}$, we need to integrate the expression above over frequency, using Eq. (43), and over redshift and to divide it by the total average cosmological energy density

[3]In this paper we consider flat space with curvature $k = 0$ and neglect cosmological constant, $\Lambda = 0$.

$\rho_{\rm BH}^{(c)}(\tau_{\rm BH}) = m_{\rm Pl}^2/(6\pi\tau_{\rm BH}^2)$. Since we have to average over the whole cosmological volume, one factor Δ disappears and we remain with the first power of Δ. So the cosmological energy fraction of GWs would be

$$\Omega_{\rm GW}(\omega_{\max}, \tau_{\rm BH}) \approx 16Q\left(\frac{v_{\rm rel}}{0.1}\right)\left(\frac{\Delta}{10^5}\right)\left(\frac{N_{\rm eff}}{100}\right)\left(\frac{\omega_{\max}}{M}\right). \qquad (47)$$

Here coefficient Q reflects the uncertainty in the cross section due to the unaccounted-for Sommerfeld enhancement [28,29]. Note that Δ may be considerably larger than 10^5.

With $v_{\rm rel} = 0.1$, $\Delta = 10^5$, $Q = 100$, and $f_{\max} = r_g^{-1}$, the fraction of the cosmological energy density of the GWs emitted by the bremsstrahlung of gravitons from the PBHs collisions, when the Universe age was equal to the lifetime of the PBH, could reach

$$\Omega_{\rm GW}(\tau_{\rm BH}) \sim 3.8 \times 10^{-17}\left(\frac{10^5\ {\rm g}}{M}\right)^2. \qquad (48)$$

It looks as if, for very light PBH, $M < 50m_{\rm Pl}$, the fraction of GW might exceed unity, which is evidently a senseless result. However, one should remember the lower bound on the PBH mass (16) and that $m_{\rm Pl}/M < \Omega_p/20$ and $m_{\rm Pl}/M < 10^{-7}(\Omega_p/10^{-6})$.

It may be interesting to calculate the contribution to $\Omega_{\rm GW}(\tau_{\rm BH})$ from the earlier period before the cluster formation. The mass density of PBHs at that stage was equal to the cosmological energy density, but since it was quite high and the effect is proportional to the density squared, the contribution from this period might be non-negligible. The result can be obtained from Eq. (46), where $\rho_{\rm BH}$ is taken equal to the average cosmological energy density. Since ρ_c evolves with time, we need to insert into the integral over dz the factor $(1+z)^6$ where the redshift is taken from some initial time, presumably $t_i = t_{\rm eq}$, down to the moment of the cluster formation, t_1. So the energy density of gravitational waves produced by bremsstrahlung from $t = t_{\rm eq}$ (9) until $t = t_1$ (15) would be

$$d\rho_{\rm GW}^{(1)} = \frac{32M^2 v_{\rm rel}}{5m_{\rm Pl}^6}[\rho_{\rm BH}^{(c)}(t_1)]^2 t_1(1+z)^{-1/2} \times f[\omega_*(z+1)]d[(1+z)\omega_*]dz, \qquad (49)$$

where $\rho_{\rm BH}^{(c)} = m_{\rm Pl}^2/(6\pi t_1^2)$ and $(1+z)$ runs from 1 up to $(t_1/t_{\rm eq})^{2/3}$. We have introduced an upper index (1) to indicate that this is the energy density of GWs generated before the cluster formation time $t = t_1$. The integration over z gives the enhancement factor $(1+z_{\max})^{1/2} = (t_1/t_{\rm eq})^{1/3}$. According to Eqs. (9) and (15), this ratio is $\Delta_{\rm in}^{-1/2} \sim 10^2$. Another enhancement factor comes from a larger cosmological energy density $\rho^{(c)}(t_1) = \rho^{(c)}(\tau_{\rm BH}) \times (\tau_{\rm BH}/t_1)^2$. The other factor $\rho_{\rm BH}^{(c)}(t_1)$ disappears in the ratio $\Omega_{\rm GW} = \rho_{\rm GW}/\rho^{(c)}$. On the other hand, $\Omega_{\rm GW}$ is redshifted by $(\tau_{\rm BH}/t_1)^{2/3}$. Correspondingly

$$\frac{\Omega_{\rm GW}^{(1)}(\tau_{\rm BH})}{\Omega_{\rm GW}(\tau_{\rm BH})} = \frac{11\Delta_{\rm in}^{-1/2}}{\Delta(\tau_{\rm BH})}\frac{v_{\rm rel}^{(1)}}{v_{\rm rel}}\left(\frac{\tau_{\rm BH}}{t_1}\right)^{1/3}, \qquad (50)$$

where the coefficient 11 came from the ratio of the integrals over z of Eqs. (46) and (49) and

$$\left(\frac{\tau_{\rm BH}}{t_1}\right)^{1/3} = \left(\frac{32170}{N_{\rm eff}}\right)^{1/3}\Omega_p^{2/3}\left(\frac{M}{m_{\rm Pl}}\right)^{2/3}. \qquad (51)$$

The ratio of relative velocities of PBHs before and after the cluster formation, $v_{\rm rel}^{(1)}/v_{\rm rel}$, is tiny, according to the estimates of Sec. III, and this introduces another strong suppression factor to the production of GWs at an earlier stage. In accordance with Eq. (20), the density contrast rises as $\Delta = \Delta(t_1)(\tau_{\rm BH}/t_1)^2$, where $\Delta(t_1)$ is supposed to be large, say, 10^4–10^5 due to the fast rise of density perturbations at MD stage after they reached unity. Thus the generation of GWs in high-density PBH clusters is much more efficient than at the earlier stage.

The density parameter of the gravitational waves at the present time is related to cosmological time t_* as

$$\Omega_{\rm GW}(t_0) = \Omega_{\rm GW}(t_*)\left(\frac{a(t_*)}{a(t_0)}\right)^4\left(\frac{H_*}{H_0}\right)^2, \qquad (52)$$

where $H_0 = 100h_0$ km/s/Mpc is the Hubble parameter and $h_0 = 0.74 \pm 0.04$ [30,31].

Using expression for redshift (26) and taking the emission time $t_* = \tau_{\rm BH}$, we obtain:

$$\Omega_{\rm GW}(t_0) = 1.67 \times 10^{-5}h_0^{-2}\left(\frac{100}{g_S(T(\tau_{\rm BH}))}\right)^{1/3}\Omega_{\rm GW}(\tau_{\rm BH}). \qquad (53)$$

Now using both Eqs. (48) and (53) we find that the total density parameter of gravitational waves integrated up to the maximum frequency is:

$$h_0^2\Omega_{\rm GW}(t_0) \approx 0.6 \times 10^{-21}K\left(\frac{10^5\ {\rm g}}{M}\right)^2, \qquad (54)$$

where K is the numerical coefficient:

$$K = \left(\frac{v_{\rm rel}}{0.1}\right)\left(\frac{\Delta}{10^5}\right)\left(\frac{N_{\rm eff}}{100}\right)\left(\frac{Q}{100}\right)\left(\frac{100}{g_S(T(\tau_{\rm BH}))}\right)^{1/3}. \qquad (55)$$

Presumably K is of the order of unity but since Δ may be much larger than 1, see Eq. (20), K may also be large.

V. GW FROM PBH SCATTERING. CLASSICAL TREATMENT

Classical radiation of gravitational waves by nonrelativistic masses is well described in quadrupole approximation, see, e.g., [32–34]. However, as we have seen, in high-density clusters of PBH, their relative velocity could be high, see Eq. (23), and relativistic corrections may be non-negligible. This problem was studied by Peters [35], who considered emission of the GWs by two bodies with masses M and m, where the former is supposed to be heavy and at rest and the latter, lighter one, moves with velocity v. For nonrelativistic motion, when $v \ll 1$, and the minimal distance between the bodies is larger than their gravitational radii, the energy of gravitational waves emitted in a single scattering process is equal to

$$\delta E_{\rm GW} = \frac{37\pi}{15}\frac{M^2 m^2 v}{b^3 m_{\rm Pl}^6}, \qquad v \ll 1, \tag{56}$$

where b is the impact parameter.

For the relativistic motion, $1 - v^2 < 1$, the emitted energy is:

$$\delta E_{\rm GW} = \frac{M^2 m^2}{b^3 m_{\rm Pl}^6 (1 - v^2)^{3/2}}. \tag{57}$$

The frequency of the emitted gravitational waves in this process is peaked near $\omega \sim 2\pi/\delta t$, where δt is the transition time which, for nonrelativistic motion is $\delta t = b/v$, according to Ref. [35], while for the relativistic one it is equal to $\delta t \sim b(1 - v^2)^{1/2}$. For an order of magnitude estimate let us take $M \sim m$; then the radiated energy, as a function of frequency, would be

$$\delta E_{\rm GW}(\omega) \approx \frac{M^4}{m_{\rm Pl}^6}\omega^3. \tag{58}$$

This and the previous equations are true for sufficiently large impact parameter, $b \gg r_g$, for which the space-time between the scattered PBHs may be considered as flat and their gravitational mass defect can be neglected. The energy loss in a single scattering event cannot be larger than

$$\delta E_{\rm max} = \frac{pq}{M}, \tag{59}$$

where $p = M v_{\rm rel}$ is the relative momentum of two scattered PBHs and q is the momentum transfer which by an order of magnitude is $q = 1/b$. Here and in what follows, we use nonrelativistic approximation. So Eqs. (56) and (57) can be true only for

$$b > b_{\rm min} = \sqrt{\frac{37\pi}{15}\frac{M^2}{m_{\rm Pl}^3}}. \tag{60}$$

For smaller impact parameters, the radiation of gravitational waves would be considerably stronger but the approximation used becomes invalid. For the (near) "head-on" collision of black holes, a bound state of two BH (a binary) or a larger black hole could be formed and the energy loss might be comparable to the BH mass due to gravitational mass defect. However, we are interested in gravitational waves at the low-frequency part of the spectrum, such that they could be registered by existing or not-so-distant-future GW detectors. For such low-frequency gravitational waves the approximation used here is an adequate one.

The differential cross section of the gravitational scattering of two PBHs in nonrelativistic regime, $q^2 \ll 2M^2$, can be taken as:

$$d\sigma = \frac{M^2}{m_{\rm Pl}^2}\frac{dq^2}{q^4} = \frac{2M^2}{m_{\rm Pl}^2} b db. \tag{61}$$

The differential energy density of GWs emitted at time and frequency intervals $[t, t + dt]$ and $[\omega, \omega + d\omega]$ respectively can be calculated as follows. The rate of the energy emission by GWs is

$$d\dot\rho_{\rm GW} = d\sigma n_{\rm BH}^2 v_{\rm rel} \delta E_{\rm GW}, \tag{62}$$

where we take for δE nonrelativistic expression (56). We assume that the impact parameter is related to the radiated frequency as $\omega = 2\pi v_{\rm rel}/b$, as is discussed below Eq. (57). So $bdb = b^3 d\omega/(2\pi v_{\rm rel})$. So we find

$$d\rho_{\rm GW} = \frac{74\pi v_{\rm rel}}{15}\rho_{\rm BH}^2 \frac{M^4}{m_{\rm Pl}^8}\frac{d\omega}{2\pi} dt. \tag{63}$$

The energy density parameter of GW at the moment of BH evaporation can be obtained by integrating this expression over time and frequency. Thus we obtain

$$\Omega_{\rm GW}(\tau_{\rm BH}) = 2 \times 10^{-10}\left(\frac{v_{\rm rel}}{0.1}\right)^2\left(\frac{\Delta_b}{10^5}\right)\left(\frac{N_{\rm eff}}{100}\right)\left(\frac{10^5\ {\rm g}}{M}\right). \tag{64}$$

If we do not confine ourselves to the impact parameter bounded by condition (60) and allow for $b \sim r_g$, the energy density of GWs at the moment of PBHs evaporation might be comparable to unity.

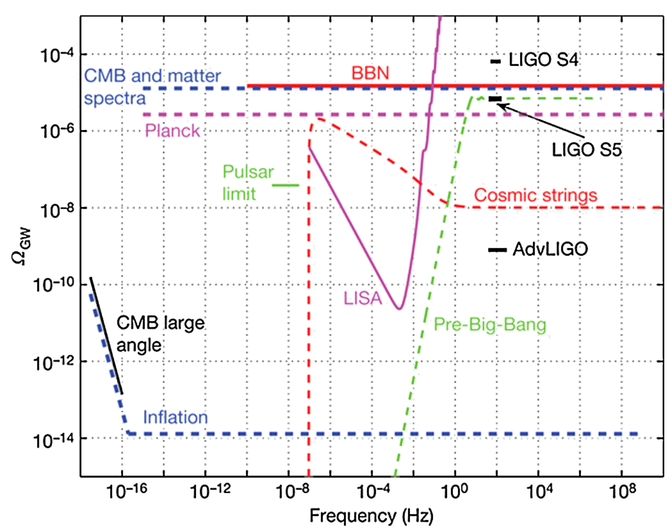

FIG. 1 (color online). $\log[h_0^2\Omega_{\rm GW}(f)]$ vs $\log(f[{\rm Hz}])$ for different models of production of stochastic background of GWs as given in Ref. [45].

Let us now take into account the redshift of GWs emitted at different moments during the lifetime of the high-density clusters. The energy density of GWs emitted at some time t is redshifted to the moment of BH decay as $1/(z+1)^4$. The frequency of GW is redshifted as $\omega = (z+1)\omega_*$, where ω_* is the frequency of GWs at $t = \tau_{BH}$. Integration over time or redshift is trivial and we find from Eq. (63) that the energy density parameter of gravitational waves per logarithmic interval of frequency or the spectral density parameter, which is defined according to Ref. [36] as

$$\Omega_{GW}(f;t) \equiv \frac{1}{\rho_c}\frac{d\rho_{GW}}{d\ln f}, \qquad (65)$$

at time $t = \tau_{BH}$ is equal to:

$$\Omega_{GW}(f_*;\tau_{BH}) \approx 8.5\left(\frac{v_{rel}}{0.1}\right)\left(\frac{\Delta_b}{10^5}\right)\left(\frac{N_{eff}}{100}\right)\left(\frac{M}{m_{Pl}^2}\right)f_*. \qquad (66)$$

Now using Eqs. (31) and (53) we can calculate the relative energy density of GWs per logarithmic frequency at the present time

$$h_0^2\Omega_{GW}(f;t_0) \approx 1.23\times 10^{-12}\alpha'\left(\frac{f}{\mathrm{GHz}}\right)\left(\frac{10^5\ \mathrm{g}}{M}\right)^{1/2}, \qquad (67)$$

where α' is the coefficient at least of the order of unity

$$\alpha' = \left(\frac{v_{rel}}{0.1}\right)\left(\frac{\Delta_b}{10^5}\right)\left(\frac{N_{eff}}{100}\right)^{3/2}\left(\frac{100}{g_S(T(\tau_{BH}))}\right)^{1/4}. \qquad (68)$$

It may be much larger if $\Delta_b \gg 10^5$.

As we mentioned above, the classical approximation is valid if the impact parameter is bounded from below by Eq. (60). Since the frequency of the radiated GWs is of the order of v/b, the maximum present-day frequency of GWs, produced at cosmological time $t = \tau_{BH}$, for which the classical nonrelativistic approximation is still valid, would be:

$$f_{max} \sim 9\times 10^5\ \mathrm{Hz}\left(\frac{v_{rel}}{0.1}\right)\left(\frac{100}{g_S(T(\tau_{BH}))}\right)^{1/12} \times\left(\frac{100}{N_{eff}}\right)^{1/2}\left(\frac{10^5\ \mathrm{g}}{M}\right)^{1/2}. \qquad (69)$$

For $M = 10^5$ g the minimum impact parameter is $b_{min} \approx 10^{-13}$ cm. The frequency of the order of 1 Hz today corresponds to the impact parameter six orders of magnitude larger. If we demand that the impact parameter should be smaller than the average distance between PBHs in the clusters, then using Eqs. (22) and (60) we find that it can be true if the following condition is fulfilled:

$$\Omega_p < 1.8\times 10^{-6}\left(\frac{10^5\ \mathrm{g}}{M}\right)^{3/4}\left(\frac{10^5}{\Delta_b}\right)^{1/4}\left(\frac{10^{-4}}{\Delta_{in}}\right)^{3/4}. \qquad (70)$$

VI. ENERGY LOSS OF PBHS

We calculate here the total energy loss of PBHs in the high-density clusters, in order to understand how probable could be the formation of the PBH binaries. First, let us estimate the total energy loss of PBHs due to the graviton bremsstrahlung. The loss of the kinetic energy per unit time due to the graviton emission is

$$-\left(\frac{dE_{kin}}{dt}\right)_{brem} = n_{BH}v_{rel}\int_0^{\omega_{max}} d\omega\,\omega\left(\frac{d\sigma}{d\omega}\right)_{brem}, \qquad (71)$$

where ω_{max} is defined in Sec. IV. The total loss of kinetic energy of a single PBH during the time interval equal to the PBH lifetime, $\delta E_{kin} = -\dot{E}_{kin}\tau_{BH}$, normalized to the original kinetic energy of the PBH can be estimated as

$$\frac{\delta E_{kin}}{E_{kin}} = 6\times 10^4\kappa_2\left(\frac{m_{Pl}}{M}\right)^2, \qquad (72)$$

where

$$\kappa_2 = \left(\frac{0.1}{v_{rel}}\right)\left(\frac{\Delta_b}{10^5}\right)\left(\frac{N_{eff}}{100}\right)\left(\frac{Q}{10}\right). \qquad (73)$$

Clearly the energy loss is essential for very light PBHs which could form dense clusters only if Ω_p is sufficiently high, see Eq. (16).

The energy loss due to classical GW emission might be somewhat more efficient. According to the previous section the energy loss by a single PBH per unit time is:

$$\Delta\dot{E}_{class} = n_{BH}v\int_{b_{min}}^{\infty} db\left(\frac{d\sigma}{db}\right)_{class}\delta E(b), \qquad (74)$$

where $\delta E(b)$ and b_{min} are given, respectively, by Eqs. (56) and (60).

Taking the integral over b and time we find for the fractional energy loss of PBH due to classical emission of the gravitational waves

$$\frac{\Delta E_{class}}{E_{kin}} = 0.9\times 10^3\frac{\Delta_b}{10^5}\frac{N_{eff}}{100}\frac{m_{Pl}}{M}. \qquad (75)$$

One should remember, however, that this energy loss comes from the PBHs scattering with rather large impact parameter $b > b_{min}$. For smaller b, when the simple approximation used in this work is inapplicable, the energy loss might be much larger. Moreover, according to Eqs. (9), (15), and (20) the density amplification factor Δ_b may be much larger than 10^5

$$\Delta_b(\tau_{BH}) = 10^4\Delta(t_1)\Delta_{in}^3\Omega_p^4\left(\frac{100}{N_{eff}}\right)^2\left(\frac{M}{m_{Pl}}\right)^4, \qquad (76)$$

where we may expect, e.g., that $\Delta(t_1) \sim 10^5$, $\Delta_{in} \sim 10^{-4}$, and $\Omega_p \sim 10^{-6}$.

PBHs in the high-density clouds could also lose their energy by dynamical friction, see, e.g., [37]. A particle moving in the cloud of other particles would transfer its energy to these particles due to their gravitational interaction. However, one should keep in mind that the case of dynamical friction is essentially different from the energy loss due to gravitational radiation. In the latter case the energy leaks out of the system, cooling it down, while dynamical friction does not change the total energy of the cluster. Nevertheless, a particular pair of black holes moving toward each other with acceleration may transmit their energy to the rest of the system and became gravitationally captured, forming a binary.

For an order of magnitude estimate we will use Chandrasekhar's formula, which is valid for a heavy particle moving in the gas of lighter particles having the Maxwellian velocity distribution with dispersion σ. The deceleration of a BH moving at velocity $v_{\rm BH}$ with respect to the rest frame of the gas is given by

$$\frac{d}{dt}\vec{v}_{\rm BH} = -4\pi G_N^2 M_{\rm BH}\rho_b \ln\Lambda \frac{\vec{v}_{\rm BH}}{v_{\rm BH}^3} \times \left[\mathrm{erf}(X) - \frac{2X\exp(-X^2)}{\sqrt{\pi}}\right], \tag{77}$$

where $X \equiv v_{\rm BH}/(\sqrt{2}\sigma)$, erf is the error function, ρ_b is the density of the background particles, and $\ln\Lambda \approx \ln(M_*/M_{\rm BH})$ is the Coulomb logarithm, which is defined as [37]:

$$\ln\Lambda = \ln\frac{b_{\max} m_{\rm Pl}^2 \sigma^2}{M_{\rm BH} + m}.$$

Here $b_{\max}$ is the maximum impact parameter, σ^2 is the mean square velocity of the gas and m is the mass of particles in the gas. Numerical simulations show that $b_{\max}$ can be assumed to be of the order the radius of the cloud, R_b, which is given by Eq. (21). Since $\sigma^2 \sim M_b/(m_{\rm Pl}^2 R_b)$, a reasonable estimate of Λ is $M_b/M_{\rm BH}$.

Equation (77) was solved in Ref. [38] in two limits $v > \sigma$ and $v < \sigma$. In both cases the characteristic dynamical friction time was of the order of

$$\tau_{\rm DF} = \frac{\sigma^3 m_{\rm Pl}^4}{4\pi M_{\rm BH}\rho_b \ln\Lambda} \approx \left(\frac{\sigma}{0.1}\right)^3 \left[\frac{25}{\ln(10^{-6}/\Omega_p)}\right]\left(\frac{100}{N_{\rm eff}}\right)\left(\frac{M}{1\ \mathrm{g}}\right)\left(\frac{10^6}{\Delta}\right)\tau_{\rm BH}. \tag{78}$$

For PBH masses below a few grams, dynamical friction would be an efficient mechanism of PBH cooling, leading to frequent binary formation. Moreover, dynamical friction could result in the collapse of small PBHs into much larger BH with the mass of the order of M_b (18). This process would be accompanied by a burst of GW emission.

VII. GRAVITATIONAL WAVES FROM PBH BINARIES

Binary systems of PBH could be formed with non-negligible probability in the high-density clusters. As we have seen in the previous section, PBHs could lose their energy due to emission of gravitational waves and due to dynamical friction [37]. As a result, they would be mutually captured. Determination of the capture probability is a complicated task, which could probably be solved by numerical simulation. Since it is outside of the scope of the present work, we simply assume that the mass or number fraction of PBH binaries in the high-density bunches of PBH is equal to ϵ, where ϵ is a dimensionless parameter which is hopefully not too small in comparison with unity.

Gravitationally bound systems of two massive bodies in circular orbit are known to emit gravitational waves with stationary rate and fixed frequency, which is twice the rotation frequency of the orbit. In this approximation orbital frequency, $\omega_{\rm orb}$, and orbit radius, R, are fixed. Luminosity of GW radiation from a single binary in the stationary approximation is well known, see, e.g., [32]

$$L_s \equiv \dot{E} = \frac{32 M_1^2 M_2^2 (M_1 + M_2)}{5 R^5 m_{\rm Pl}^8} = \frac{32}{5} m_{\rm Pl}^2 \left(\frac{M_c \omega_{\rm orb}}{m_{\rm Pl}^2}\right)^{10/3}, \tag{79}$$

where M_1, M_2 are the masses of two bodies in the binary system and M_c is the chirp mass which is defined as

$$M_c = \frac{(M_1 M_2)^{3/5}}{(M_1 + M_2)^{1/5}} \tag{80}$$

and

$$\omega_{\rm orb}^2 = \frac{M_1 + M_2}{m_{\rm Pl}^2 R^3}. \tag{81}$$

In the case of elliptic orbit with large semiaxis a and eccentricity e the luminosity is somewhat larger (if $R = a$):

$$L_e = \frac{32 M_1^2 M_2^2 (M_1 + M_2)}{5 a^5 m_{\rm Pl}^8 (1 - e^2)^{7/2}}\left(1 + \frac{73 e^2}{24} + \frac{37 e^4}{96}\right). \tag{82}$$

The emission of GWs costs energy which is provided by the sum of the kinetic and potential energy of the system. To compensate for the energy loss, the radius of the binary system decreases and the frequency rises making the stationary approximation invalid. As a result the system goes into the so called inspiral regime. Ultimately the two rotating bodies coalesce and produce a burst of gravitational waves. To reach this stage the characteristic time of the coalescence should be shorter than the lifetime of the system. In our case it is the lifetime of PBH with respect to the evaporation.

In the inspiral regime, the initially circular orbit may remain approximately circular if radial velocity of the orbit, $\dot{R}$, is much smaller than the tangential velocity, $\omega_{\rm orb} R$. This regime is called quasicircular motion and is valid as long as (see, e.g., [39])

$$\dot{\omega}_{\rm orb} \ll \omega_{\rm orb}^2. \tag{83}$$

Equation (83) can be translated into the lower bound on the radius of the orbit

$$R \gg r_g^{(\rm eff)} = \frac{M_1 + M_2}{m_{\rm Pl}^2}, \tag{84}$$

which is the condition of the validity of the Newtonian approximation. It was shown by Peters [40] that the orbits with initial $e_0 = 0$ would remain quasicircular as far as condition (83) is fulfilled, while for the orbits with $e_0 \neq 0$ the eccentricity rapidly approaches zero due to backreaction of the gravitational radiation.

Most probably, binaries are formed in elliptic orbits with high eccentricity. However in the calculation of the GW emission by binaries we assume for simplicity that all orbits are circular. The result would be a lower bound on GW emission, hopefully not too far from the real case.

In what follows we will consider both stationary and inspiral regimes, since they both might be realized for different values of the parameters. We will use the instant decay approximation, when the PBH mass is supposed to be constant until $t = \tau_{\rm BH}$ and then BH would instantly disappear. The case of the realistic decrease of PBH mass will be considered elsewhere.

The stationary orbit approximation would be valid if time of coalescence, $\tau_{\rm co}$, would be much larger than the BH lifetime, $\tau_{\rm co} > \tau_{\rm BH}$. The former can be found as follows (see, e.g., [32]). According to the virial theorem the total (kinetic plus potential) energy of the system is $\mathcal{E} = -M_1 M_2/(2R m_{\rm Pl}^2)$. Since luminosity (79) is $L_s = -d\mathcal{E}/dt$, the radius varies with time according to

$$\dot{R} = -\frac{64 M_1 M_2 (M_1 + M_2)}{5 R^3 m_{\rm Pl}^6}. \tag{85}$$

Correspondingly

$$R(t) = R_0 \left(\frac{t_0 + \tau_{\rm co} - t}{\tau_{\rm co}}\right)^{1/4}, \tag{86}$$

where R_0 is the initial value of the radius, t_0 is the initial time, and the coalescence time is given by

$$\tau_{\rm co} = \frac{5 R_0^4 m_{\rm Pl}^6}{256 M_1 M_2 (M_1 + M_2)}. \tag{87}$$

The condition $\tau_{\rm co} > \tau_{\rm BH}$ can be translated into the lower bound on R (for $M_1 = M_2$)

$$R > R_{\rm min} = 4.6 \times 10^5 \left(\frac{100}{N_{\rm eff}}\right)^{1/4} \left(\frac{M}{10^5~\rm g}\right)^{1/2} r_g. \tag{88}$$

Keeping in mind that the frequency of GWs emitted at circular motion of the binary is twice the orbital frequency, $f_s = \omega_{\rm orb}/\pi$, we find from Eq. (81) that lower bound (88) leads to the following upper bound on the GW frequency:

$$f_s < \omega_{\rm max}/\pi \approx 2 \times 10^{24}~{\rm Hz} \left(\frac{N_{\rm eff}}{100}\right)^{3/8} \left(\frac{10^5~\rm g}{M}\right)^{7/4}. \tag{89}$$

On the other hand, the radius of the binary orbit should be smaller than the average distance between PBHs in the cluster (22) and probably quite close to it. Using Eqs. (22) and (88) we find

$$\frac{R_{\rm min}}{d_b} = 1.3 \times 10^{-5} \left(\frac{\Delta_b}{10^5}\right)^{1/3} \left(\frac{\Delta_{\rm in}}{10^{-4}}\right) \left(\frac{\Omega_p}{10^{-6}}\right)^{4/3} \left(\frac{M}{10^5 g}\right)^{1/2}. \tag{90}$$

So it seems natural that $R_{\rm min} \ll d_b$ and the PBH binaries should be mostly in the quasistationary regime. $R_{\rm min}$ would be equal to d_b roughly speaking for quite large mass fraction of the produced PBHs, $\Omega_p > 10^{-3}$.

The condition $R_{\rm min} = d_b$ gives a lower bound on orbital frequency, $\omega_{\rm orb}$,

$$\omega_{\rm orb} > \omega_{\rm min} \approx 9.4 \times 10^{17}~{\rm sec}^{-1} \left(\frac{\Delta_b}{10^5}\right)^{1/2} \times \left(\frac{\Delta_{\rm in}}{10^{-4}}\right)^{3/2} \left(\frac{\Omega_p}{10^{-6}}\right)^2 \left(\frac{10^5~\rm g}{M}\right). \tag{91}$$

During the inspiral phase, for which $\tau_{\rm co} < \tau_{\rm BH}$, we expect that binaries emit GWs in the frequency range

$$2 \times 10^{24}~{\rm Hz} \left(\frac{N_{\rm eff}}{100}\right)^{3/8} \left(\frac{10^5~\rm g}{M}\right)^{7/4} < f < 0.6 \times 10^{33}~{\rm Hz} \left(\frac{10^5~\rm g}{M}\right). \tag{92}$$

The upper bound corresponds to $\omega \sim 1/r_g$.

The frequency spectrum of the gravitational waves in inspiral but quasicircular motion can be found in the adiabatic approximation as follows. Since the gravitational waves are emitted in a narrow band near twice the orbital frequency, the spectrum of the luminosity (79) can be approximated as

$$d\dot{E} = \frac{32 M_1^2 M_2^2 (M_1 + M_2)}{5 R^5(t) m_{\rm Pl}^8} \delta(\omega - 2\omega_{\rm orb}(R)) d\omega \tag{93}$$

To find the energy spectrum we have to integrate this expression over time from initial time, $t_{\rm min} = t_0$, to maximum time $t_{\rm max} = \min[\tau_{\rm BH} + t_p, \tau_{\rm co} + t_0]$, where t_0 and t_p are, respectively, the time of the binary formation (it may be different for different binaries but here we neglect this possible spread) and the time of PBH formation (it is different for PBH with different masses). Note that the coalescence time, $\tau_{\rm co}$ is also different for binaries with different initial radius R_0.

Using Eqs. (81) and (85) and the expression $dt = (dR/dt)^{-1}(dR/d\omega_{\rm orb}) d\omega_{\rm orb}$, we find

$$\frac{dE}{d\ln\omega} = \frac{2^{1/3}\omega^{2/3}}{3} \frac{M_1 M_2}{m_{\rm Pl}^{4/3}(M_1 + M_2)^{1/3}} \tag{94}$$

in agreement with Refs. [39,41]. This expression is valid for the frequencies in the interval determined by Eq. (81) with $R_{\max} = R_0$ and $R_{\min} = R(t_{\max})$.

In expression (94), we have not taken into account the redshift, which is different for different frequencies and thus this leads to spectrum distortion. According to Eqs. (81) and (86) frequency ω is emitted at the time moment

$$t(\omega) = t_0 + \tau_{\rm co}\left[1 - \left(\frac{\omega_{\min}}{\omega}\right)^{8/3}\right], \qquad (95)$$

where

$$\omega_{\min} = 2\left(\frac{M_1 + M_2}{m_{\rm Pl}^2}\right)^{1/2} R_0^{-3/2} \qquad (96)$$

is the minimal frequency emitted at initial moment $t = t_0$. To the moment of the PBH evaporation the frequency of the GWs emitted at $t = t(\omega)$ is redshifted by the frequency dependent factor

$$\omega_* = \frac{\omega}{1 + z(\omega)} = \left[\frac{t(\omega)}{t_p + \tau_{\rm BH}}\right]^{2/3}\omega, \qquad (97)$$

where ω_* is the frequency of GWs at $t = t_p + \tau_{\rm BH}$. This equation implicitly determines ω as a function of ω_*.

The spectrum of the gravitational waves at PBH evaporation can be obtained from Eq. (94) dividing it by $(1 + z)$ (the redshift of the graviton energy, E) and with substitution $\omega = (z + 1)\omega_*$. Correspondingly

$$d\omega = \frac{z + 1}{1 - \omega_*(dz/d\omega)} d\omega_* \qquad (98)$$

As a result we find

$$\frac{dE_*}{d\ln\omega_*} = \frac{2^{1/3}\omega_*^{2/3}}{3}\frac{M_1 M_2}{m_{\rm Pl}^{4/3}(M_1 + M_2)^{1/3}} \times \frac{[1 - \omega_*(dz/d\omega)]^{-1}}{(1 + z)^{1/3}}. \qquad (99)$$

Here $z(\omega)$ should be taken as a function of ω_* according to Eq. (97) and ω_* varies between $\omega_{\min}$ and $\omega_{\max}$ divided by the corresponding redshift factor. In particular, $\omega_{*(\min)} = \omega_{\min}[t_0/(t_p + \tau_{\rm BH})]^{2/3}$. Note that R_0 enters explicitly into Eq. (99), while in Eq. (94) it enters only through the limits in which ω varies. Because of that the frequency spectrum depends upon the distribution of binaries over their initial radius, R_0. As is shown below, it is especially profound in the case of long coalescence time when the frequency spectrum of a single binary with fixed R is close to delta-function.

In the stationary approximation, when the change of the orbit radius can be neglected, we expect that a single binary emits GWs in a narrow band of frequencies close to twice the orbital frequency. However the distribution of binaries over their initial radius, $dn_{\rm BIN} = F(R_0)dR_0$ spreads up the spectrum. Here $dn_{\rm BIN}$ is the number density of binaries with the radius in the interval $[R_0, R_0 + dR_0]$. Since in this approximation the radius is approximately constant, we do not distinguish between R and R_0. The cosmological energy density of the gravitational waves emitted per unit time is equal to:

$$d\dot{\rho}_{\rm GW}^{\rm (stat)} = \frac{2F(R)R}{3}\frac{n_{\rm BH}^c}{n_{\rm BH}^b}\frac{d\omega}{\omega}L_s, \qquad (100)$$

where $n_{\rm BH}^b$ is the number density of PBH in the high-density bunch (cluster), $n_{\rm BH}^c$ is the average cosmological number density of PBH, $R = R(\omega_{\rm orb})$ according to Eq. (81), and we used the relation $dR = -2(R/3) \times (d\omega/\omega)$. Distribution, $F(R)$, is normalized as:

$$\int dRF(R) = n_{\rm BIN} = \epsilon n_{\rm BH}^b. \qquad (101)$$

We assume for simplicity that $F(R)$ does not depend upon R in some interval $[R_1, R_2]$ and vanishes outside it. So $F(R) = \epsilon n_{\rm BH}^b/(R_1 - R_2)$.

A more realistic fit to the PBH distribution over radius could be a Gaussian one

$$F(R) = \frac{1}{\sqrt{2\pi}\sigma}\epsilon n_{\rm BH}\exp[-(R - \langle R\rangle)^2/2\sigma^2], \qquad (102)$$

where σ is the mean-square deviation of R from the average value $\langle R\rangle$.

The small factor $n_{\rm BH}^c/n_{\rm BH}^b$ enters Eq. (100) because we are interested in the cosmological energy density of GWs averaged over the whole universe volume. The cosmological number density of PBH is expressed through their energy density as $n_{\rm BH} = \rho_{\rm BH}/M = \rho_c(t)/M$. The number density of binaries in the cluster is parametrized according to

$$n_{\rm BIN}(t) = \epsilon(t)n_{\rm BH}^b(t) = \epsilon(t)\rho_c(t)\Delta(t)/M, \qquad (103)$$

where, we recall, $\rho_c(t)$ is the total cosmological energy density and $\Delta(t) = \rho_b/\rho_c \gg 1$ is the density contrast of the cluster. The time dependence of $n_{\rm BH}^b$ disappears when the cluster reaches the stationary state (see discussion in Sec. II) and $\Delta(t)$ evolves according to Eq. (20). When the stationary orbit approximation is valid, ϵ remains constant.

Collecting all the factors and integrating Eq. (100) over time with an account of the frequency redshift, $\omega = \omega_*(1 + z)$ and the total redshift of the energy density of GWs, $\rho_{\rm GW}(t_*) = \rho_{\rm GW}(t)/(1 + z)^4$, we find

$$d\rho_{\rm GW}^{\rm (stat)}(\omega_*; \tau_{\rm BH}) = \frac{2^{7/3}}{5}\left[\frac{n_{\rm BH}^c(\tau_{\rm BH})}{n_{\rm BH}^b}\right]\frac{(M_1^2 M_2^2)(\tau_{\rm BH} + t_p)}{(M_1 + M_2)^{1/3} m_{\rm Pl}^{16/3}} \times F(R)\omega_*^{5/3} d\omega_* \int_{x_{\min}}^1 x^{11/6}dx, \qquad (104)$$

where $x = a(t)/a(t_*) = 1/(1 + z)$, $x_{\min} = a(t_0)/a(t_*)$, t_0 is the time moment of binary formation and we make use of Eq. (45). Dividing this result by the critical energy density just before complete evaporation of the PBHs, $n_{BH}(\tau_{BH}) \approx \rho_c(\tau_{BH})/M$, we find the cosmological fraction of the energy density of GWs at $t = \tau_{BH}$ per logarithmic interval of frequency $f = \omega/(2\pi)$ (below we assume that all BHs have equal masses, M):

$$\Omega_{GW}^{(stat)}(f_*; \tau_{BH}) = \frac{3 \times 2^{17/3}}{85} \frac{\epsilon \times (t_p + \tau_{BH})}{R_1 - R_2} \times \left(\frac{\pi f_* M}{m_{Pl}^2}\right)^{8/3} [1 - x_{min}^{17/6}], \quad (105)$$

where for the sake of a simple estimate we assumed that $F(R) = $ const. We assume also that all the binaries are formed at the same time, $t_0 \ll \tau_{BH}$ and so $x_{min} \ll 1$. Note that the frequency of GWs coming from the binaries with radii between R_1 and R_2 is confined, according to Eq. (81).

To make an order of magnitude estimate of the fraction of the energy density of GWs at the moment of PBH evaporation, we take $(R_1 - R_2) \sim R_1 \sim R(\omega)$, where $R(\omega)$ is determined by Eq. (81) and take into account that the stationary approximation is valid if the radii of the binaries are bounded from below by Eq. (88). Hence, if the stationary regime is realized, the spectral density parameter today would be:

$$h_0^2 \Omega_{GW}^{(stat)}(f; t_0) \approx 10^{-8} \epsilon \left[\frac{N_{eff}}{100}\right]^{2/3} \left[\frac{100}{g_S(T(\tau_{BH}))}\right]^{1/18} \times \left[\frac{M}{10^5 \text{ g}}\right]^{1/3} \left[\frac{f}{\text{GHz}}\right]^{10/3}. \quad (106)$$

The expected range of the present-day frequencies of the GWs from the binaries in the stationary approximation is given by Eqs. (91) and (89). The emitted frequency is determined by the binary radius, so a single binary emits GWs with a very narrow spectrum. However, the distribution of binaries over their radius could lead to a significant spread of the spectrum. In principle, the frequencies emitted may have any value in the specified above range. The minimal present-day frequency of such GWs today can be found by plugging Eq. (91) into Eq. (31):

$$f \geq 4.3 \text{ Hz} \beta \left(\frac{10^5 \text{ g}}{M}\right)^{1/2}, \quad (107)$$

where β is given by

$$\beta = \left(\frac{\Delta_b}{10^5}\right)^{1/2} \left(\frac{\Delta_{in}}{10^{-4}}\right)^{3/2} \left(\frac{\Omega_p}{10^{-6}}\right)^2 \times \left(\frac{100}{g_S(T(\tau_{BH}))}\right)^{1/12} \left(\frac{100}{N_{eff}}\right)^{1/2}. \quad (108)$$

For binaries formed with $R > R_{min}$, see Eqs. (31), (88), and (89), the frequency of emitted GWs today is bounded from above by

$$f \leq 5.7 \times 10^7 \text{ Hz} \left(\frac{100}{g_S(T(\tau_{BH}))}\right)^{1/12} \left(\frac{100}{N_{eff}}\right)^{1/8} \left(\frac{10^5 \text{ g}}{M}\right)^{1/4}. \quad (109)$$

Let us estimate now the energy density of GWs in the inspiral case, when $\tau_{co} < \tau_{BH}$ and the GW emission from a single binary proceeds in a wide range of frequencies due to shrinking of the binary radius. The radiation frequency spans from $f_{s,min}$, which is the GW frequency at the initial PBH separation, to $f_{s,max}$, which corresponds to GWs emitted at $R \sim r_g$. The energy spectrum of GWs is given by Eq. (94) where, in what follows, we change to cyclic frequency, $f = \omega/2\pi$.

After the cluster evolution was over, the number density of PBHs in high-density clusters remained approximately constant until the PBH evaporation, but in the inspiral phase the fraction of binaries, $\epsilon(t)$, decreased due to their coalescence. So the tail of the distribution function at small initial R_0 is eaten up, and the average value of R drops down. In distribution function, $F(R_0)$, we have to substitute for R_0 its expression through R and time according to

$$R_0 \to \left[R^4 + \left(\frac{256 M_1 M_2 (M_1 + M_2)}{5 m_{Pl}^6}\right)(t - t_0)\right]^{1/4} \quad (110)$$

with the corresponding change of $R_0^3 dR_0 \to R^3 dR$.

To calculate the cosmological energy fraction of GWs at the PBH evaporation moment we can proceed along the same lines as we have done deriving Eq. (99), introducing additional factor $F(R_0)dR_0$ which depends upon time according to Eq. (110). However, at the level of calculations in the present model with many unknown parameters, it can be sufficient to neglect such subtleties and to use a simplified estimate

$$\frac{d\rho_{GW}}{d(\log f_s)} = \epsilon_{co} n_{BH}^c(t) \frac{dE_{GW}}{d(\log f_s)}, \quad (111)$$

where ϵ_{co} is the fraction of binaries with coalescence time shorter or equal to PBH lifetime. For an estimate by an order of magnitude we assume also that the number of binaries is independent on the redshift. To some extent the decrease of the binary number may be compensated by their continuous formation. We neglect possible difference of binary masses and take $M_1 = M_2$. We approximately take the redshift into account from the moment of the coalescence to the PBH decay, $(z_{co} + 1) \approx (\tau_{BH}/\tau_{co})^{2/3}$. This corresponds to the assumption that the binaries radiated all GWs only at the moment of τ_{co}. So the $f_* = f(1 + z_{co})$. Thus we obtain as an order of magnitude estimate

$$\Omega_{GW}(f_*, \tau_{BH}) = \frac{\epsilon_{co}}{3} \left(\frac{\pi f_* M}{m_{Pl}^2}\right)^{2/3} (z_{co} + 1)^{-1/3}. \quad (112)$$

Using Eqs. (31) and (53), we find that the energy density parameter of gravitational waves today is equal to:

$$h_0^2\Omega_{\rm GW}(f) \approx 5\times 10^{-9}\epsilon_{\rm co}\left(\frac{100}{g_S(T(\tau_{\rm BH}))}\right)^{5/18}\left(\frac{N_{\rm eff}}{100}\right)^{1/3} \times\left(\frac{f}{10^{12}\ {\rm Hz}}\right)^{2/3}\left(\frac{10^5\ {\rm g}}{M}\right)^{1/3}, \quad (113)$$

where we neglected possibly weak redshift dilution of GWs by the factor $(\tau_{\rm co}/\tau_{\rm BH})^{2/9}$.

If the system goes to the inspiral phase, then according to Eq. (92) we would expect today a continuous spectrum in the range from $f_{\rm min}\sim 0.9\times 10^7$ Hz to $f_{\rm max}\sim 3\times 10^{14}$ Hz. However if we take into account the redshift of the early formed binaries from the moment of their formation to the PBH decay, the lower value of the frequency may move to about 1 Hz.

VII. GRAVITONS FROM PBH EVAPORATION

In the previous sections, we have considered only gravitational waves emitted through mutual acceleration of PBHs in the high-density clusters. On the other hand, PBHs could directly produce gravitons by evaporation. This process in connection with creation of cosmological background of relic GWs was considered in Ref. [6] and later in Ref. [42]. In the last reference a possible clumping of PBHs at the matter-dominated stage was also considered. Though such clumping does not influence the probability of the GW emission by PBHs, it may change the mass spectrum of PBHs due to their merging.

The PBHs reduce their mass according to the equation

$$M(t) = M_0\left(1-\frac{t-t_p}{\tau_{\rm BH}}\right)^{1/3}, \quad (114)$$

where M_0 is the initial mass of an evaporating BH and t_p is the time of BH production after the big bang. Equation (114) shows that the BH mass can be approximately considered as constant until the moment of the evaporation and may be approximated as $\theta(t-t_p-\tau_{\rm BH})$. Because of evaporation, a BH emits all kind of particles with masses $m < T_{\rm BH}$ and, in particular, gravitons. The total energy emitted by BH per unit time and frequency ω (energy) of the emitted particles, is approximately given by the equation (see, e. g. [43])

$$\left(\frac{dE}{dtd\omega}\right) = \frac{2N_{\rm eff}}{\pi}\frac{M^2}{m_{\rm Pl}^4}\frac{\omega^3}{e^{\omega/T_{\rm BH}}-1}, \quad (115)$$

where T is the BH temperature (2). Because of the impact of the gravitational field of BH on the propagation of the evaporated particles, their spectrum is distorted [8] by the so called grey factor $g(\omega)$, but we disregard it in what follows.

Let us now estimate the amount of the gravitational radiation from the graviton evaporation. After their production, PBHs started to emit thermal gravitons independently on the PBH clustering. Hence the thermal graviton emission depends only on PBH number density, $n_{\rm BH}$. The energy density of gravitons in logarithmic frequency band emitted in the time interval t and $t+dt$ is

$$\frac{d\rho_{\rm GW}(\omega;t)}{d\omega} = 10^{-2}n_{\rm BH}(t)\left(\frac{dE}{dtd\omega}\right)dt, \quad (116)$$

where factor 10^{-2} takes into account that about 1% of the emitted energy goes into gravitons. The density parameter of GWs per logarithmic frequency interval at cosmological time $t_* = \tau_{\rm BH}$ can be obtained by integrating expression (116) over redshift with an account of the dropoff of the graviton energy density by $(1+z)^{-4}$ and the redshift of the emitted frequency so that at $t_*=\tau_{\rm BH}$, $\omega=\omega_*(1+z)$. Note that, in the instant decay approximation, the BH temperature remains constant. One has also to take into account that the number density of PBH behaves as $n_{\rm BH}(t) = n_p(t_p)(1+z)^3$, so if we normalize our result to $n_{\rm BH}(\tau_{\rm BH})$, the integrand should be multiplied by $(1+z)^3$. Finally we obtain

$$\frac{d\rho_{\rm GW}(\omega_*,\tau_{\rm BH})}{d\ln\omega_*} = \frac{0.03N_{\rm eff}M\omega_*^4}{\pi m_{\rm Pl}^4}(3\tau_{\rm BH})\rho_{\rm BH}(\tau_{\rm BH})I\left(\frac{\omega_*}{T_{\rm BH}}\right), \quad (117)$$

where

$$I\left(\frac{\omega_*}{T_{\rm BH}}\right) \equiv \int_0^{z_{\max}}\frac{dz(1+z)^{1/2}}{\exp[(z+1)\omega_*/T_{\rm BH}]-1}, \quad (118)$$

and

$$1+z_{\max} = \left(\frac{\tau_{\rm BH}}{t_{\rm eq}}\right)^{2/3}\left(\frac{t_{\rm eq}}{t_p}\right)^{1/2} = \left(\frac{32170}{N_{\rm eff}}\right)^{2/3}\left(\frac{M}{m_{\rm Pl}}\right)^{4/3}\Omega_p^{1/3}, \quad (119)$$

where the effective time of integration is equal to $3\tau_{\rm BH}$ because of the instant decay approximation. One can check that in this case the total evaporated energy would be equal to the PBH mass.

The spectral density parameter of GWs at $t=\tau_{\rm BH}$ is equal to:

$$\Omega_{\rm GW}(\omega_*;\tau_{\rm BH}) \approx \frac{2.9\times 10^3 M^4\omega_*^4}{\pi m_{\rm Pl}^8}I\left(\frac{\omega_*}{T_{\rm BH}}\right). \quad (120)$$

The spectrum is not a thermal one, though rather similar to it. It has more power at small frequencies due to redshift of higher frequencies into lower band and less power at high ω_*. The spectral density parameter reaches maximum at $\omega_*^{\rm peak}/T_{\rm BH} = 2.8$. Accordingly the maximum value of the spectral density parameter when PBHs completely evaporated is equal to

$$\Omega_{\rm GW}^{\rm peak}(\omega_*^{\rm peak};\tau_{\rm BH}) \approx 3.8\times 10^{-3}. \quad (121)$$

Integrating Eq. (120) first over ω_* and then over redshift, we find that the total fraction of energy of GWs is 0.006 which is reasonably (in view of the used approximations) close to the expected 0.01. At BBN, the energy fraction of such GWs would be about 0.005. So the total number of additional effective neutrino species would be close to 0.045, where 0.03 comes from neutrino heating by e^+e^- annihilation and 0.01 comes from the plasma corrections (see, e.g., the review in [44]). Of course, the GWs produced by the considered mechanism are safely below the BBN bound [45]. Using Eq. (53) and taking into account the redshift from $t = \tau_{\rm BH}$ to the present time, we find that the total density parameter of GWs today due to PBH evaporation would be about 10^{-7}.

The total energy density of GWs from the PBH evaporation is quite large but it is concentrated at high frequencies. According to Eq. (31) the redshifted peak frequency emitted at time $t_* = \tau_{\rm BH}$ becomes today

$$f^{(\rm peak)} = 2\times 10^{15}\ {\rm Hz}\left(\frac{g_S(T(\tau_{\rm BH}))}{100}\right)^{1/12} \times\left(\frac{100}{N_{\rm eff}}\right)^{1/2}\left(\frac{M}{10^5\ {\rm g}}\right)^{1/2}. \tag{122}$$

The energy density of GWs at small f drops down in accordance with Eq. (121). The spectral density today can be calculated from Eq. (120) with an account of the redshift to the present day

$$h_0^2\Omega_{\rm GW}(f;t_0) = 1.36\times 10^{-27}\left(\frac{N_{\rm eff}}{100}\right)^2\left(\frac{10^5\ {\rm g}}{M}\right)^2 \times\left(\frac{f}{10^{10}\ {\rm Hz}}\right)^4\times I\left(\frac{2\pi\times f}{T_0}\right), \tag{123}$$

where we used $\omega = 2\pi f$ and T_0 is the BH temperature redshifted to the present time

$$T_0 = \left[\frac{a(\tau_{\rm BH})}{a(t_0)}\right]T_{\rm BH} = 4.53\times 10^{15}\ {\rm Hz}\left(\frac{100}{g_S(T(\tau_{\rm BH}))}\right)^{1/12} \times\left(\frac{100}{N_{\rm eff}}\right)^{1/2}\left(\frac{M}{10^5\ {\rm g}}\right)^{1/2}. \tag{124}$$

IX. STOCHASTIC BACKGROUND OF GRAVITATIONAL WAVES. AN OVERVIEW

Stochastic background of relic gravitational waves can be produced by several mechanisms. The theoretical predictions are model-dependent due to the uncertainties in the cosmological framework and on the values of the redshift from the production epoch. Below we briefly describe some of the production scenarios. For a more detailed review on stochastic background of GWs production mechanisms and their spectra the reader can consult more specific Ref. [46–48].

Inflationary models. It was established long ago that gravitational waves could be produced in cosmology due to an amplification of vacuum fluctuations by external gravitational field (quantum particle production). It was first studied by Grishchuck [3] and first applied to an inflationary model by Starobinsky [4]. The gravitational waves could be quite efficiently produced at inflation. Their spectrum at large wavelengths is independent of the details of inflationary models. The frequency band of these gravitons today is quite wide and the associated density parameter is very low. The predicted density parameter of gravitational waves in the frequency range from 3×10^{-18} Hz $< f < 10^{-16}$ Hz is

$$h_0^2\Omega_{\rm GW}(f) \simeq 6.71\times 10^{-10}\left(\frac{10^{-18}\ {\rm Hz}}{f}\right)^2\left(\frac{H}{10^{15}\ {\rm Gev}}\right)^2, \tag{125}$$

while in the frequency range 2×10^{-15} Hz $< f < f_{\rm max} \simeq 10^9$ Hz the spectrum is flat and the density parameter is

$$h_0^2\Omega_{\rm GW}(f) \simeq 6.71\times 10^{-14}\left(\frac{H}{10^{15}\ {\rm Gev}}\right)^2, \tag{126}$$

where H is the Hubble parameter at inflation.

A near scale-invariant spectrum over a wide range of frequencies is a key prediction of the standard inflationary model [49,50]. The relative amplitude of GWs spectrum to density perturbations spectrum is usually expressed in terms of the ratio, r, of tensor to scalar perturbations. From observations of WMAP, the current limit on B-mode of the CMB polarization demands $r \lesssim 0.22$, which rules out some models of inflation [51,52]. The spectrum of GWs can be expressed in terms of the tensorial spectral index, n_t, and is almost flat in the frequency range 2×10^{-15} Hz $< f < f_{\rm max} \simeq 10^{10}$ Hz. The density parameter is proportional to a power of the frequency

$$h_0^2\Omega_{\rm GW}(f) \propto f^{n_t}. \tag{127}$$

Since the tensorial spectral index is negative, $n_t < 0$, the spectrum is decreasing rather than flat. Depending on inflationary model, the value of the tensorial spectral index changes; there are some models that predict $r \sim 10^{-3}$.

Preheating phase at the end of inflation. At this stage the energy of scalar field ϕ is spent to generate new particles and heat the Universe. The first estimate of the density parameter of GWs during the preheating phase was done by Klebnihkov and Tkachev [53], who found the density parameter of the order of $h_0^2\Omega_{\rm GW} \sim 10^{-11}$ for the gravitational waves with the present-day frequency $f \sim 10^6$ Hz, in the models with quartic potential, $\lambda\phi^4$. Later, this mechanism was reconsidered by Easther and Lim [54,55], who studied the models with the potentials of the form $\lambda\phi^4$ and $m^2\phi^2$. The authors have found numerically that $h_0^2\Omega_{\rm GW} \sim 10^{-10}$ in the frequency range $f \sim 10^8$–10^9 Hz.

First-order phase transitions. At the end of inflation, first-order phase transitions could have generated a large amount of gravitational waves. At such transitions, the bubble nucleation of true vacuum states and percolation can occur, accompanied by the bubble collisions. In a series of papers [56–59], the energy of gravitational waves generated from bubble collisions at strongly first-order phase transitions was estimated and the results were later extended to the electroweak first-order phase transitions. The amount of GWs from strongly first-order phase transition at its end is of the order $1.3 \times 10^{-3}(\tau/H)$, where τ is the duration of the phase transition, H is the Hubble constant, and the peak frequency is $\omega_*^{\rm peak} = 3.8/\tau$. The present-day density parameter of GWs produced at the electroweak first-order phase transition was found to be of the order $\Omega_{\rm GW} \sim 10^{-22}$, with characteristic frequency $f \sim 4 \times 10^{-3}$. Since it was later found that there is no first-order electroweak phase transition in the standard model [60], the mechanism was reconsidered by Grojean and Servant [61]. The authors estimated the GW production in the temperature range 100 GeV $-$ 10^7 GeV. The spectrum of the GWs today in this temperature range extends from 10^{-3} Hz to 10^2 Hz. The associated density parameter was found to be quite large, $h_0^2\Omega_{\rm GW}(f_{\rm peak}) \sim 10^{-9}$ depending on the parameters of the model.

Topological defects and cosmic strings. In practically all inflationary models, the gravitational wave spectrum is almost flat in the frequency range from 10^{-15} Hz $< f < f_{\rm max} \simeq 10^{10}$ Hz, with some variations coming from preheating and reheating phases, for which the frequency peaks near GHz region. There are other mechanisms of GWs production, e.g., by cosmic strings, which predict an almost flat spectrum in a wide range of frequencies. Many of the proposed observational tests for the existence of cosmic strings are based on their gravitational interactions [62,63]. Particularly interesting are GWs produced by closed string loops which oscillate in relativistic regime. The spectrum of the gravitational waves produced by such relativistic oscillations is almost flat in the region 10^{-8} Hz $< f < f_{\rm max} \simeq 10^{10}$ Hz, with a peak at low frequency near $f \sim 10^{-12}$ Hz. The density parameter in the frequency range $f \gg 10^{-4}$ Hz, according to Ref. [64], is equal to

$$h_0^2\Omega_{\rm GW}(f) \simeq 10^{-8}\left(\frac{G\mu}{10^{-8}}\right)^{1/2}\left(\frac{\gamma}{50}\right)^{1/2}\left(\frac{\alpha}{0.1}\right)^{1/2}, \quad (128)$$

where $G\mu$, α, and γ are, respectively, the string tension, the initial loop size as a fraction of the Hubble radius, and the radiation efficiency. From the pulsar timing data, the authors of Ref. [65] constrained the density parameter of GWs from the cosmic strings in the frequency range $f \gg 10^{-6}$ Hz and put the limit

$$h_0^2\Omega_{\rm GW}(f) \lesssim 10^{-8}. \quad (129)$$

It is generally assumed that at the end of inflation the inflaton oscillates and eventually decays. If nontopological solitons, the so called Q-balls, are produced at the inflaton decay, such Q-balls could be a source of GWs. According to the calculations of Ref. [66] the density parameter of such GWs would be of the order of $h_0^2\Omega_{\rm GW} \sim 10^{-9}$ with a peak frequency $f \sim 10^{10}$ Hz.

X. GRAVITATIONAL WAVES DETECTORS. PRESENT STATUS

For most of the models mentioned above, the stochastic background of GWs is beyond the sensitivity of the current and planned interferometers (see 1). We have seen that inflationary models predict an almost flat spectrum of GWs in a wide range of frequencies. There is a narrow band of frequencies of this background that falls into the range of the present detectors such as LIGO and VIRGO. Unfortunately, the density parameter predicted by inflationary models is too low to be detected by the present detectors. Almost all the models mentioned above predict the density parameter of the order of$h_0^2\Omega_{\rm GW} \lesssim 10^{-5}$ and actually LIGO and VIRGO are not able to detect such a quantity because of the frequency dependence of the density parameter. This can be seen from the relation between the expected amplitude of stochastic gravitational waves $h_c(f)$ with the density parameter as presented in the Ref. [47]

$$h_c(f) = 1.3 \times 10^{-18}\sqrt{h_0^2\Omega_{\rm GW}(f)}\left(\frac{1\ {\rm Hz}}{f}\right). \quad (130)$$

Present detectors such as LIGO and VIRGO with enhanced technologies operate in the frequency range 1 Hz $-$ 10^4 Hz and can reach, respectively, the strain sensitivity $h_{\rm rms} \sim 10^{-23}$ Hz$^{-1/2}$ and $h_{\rm rms} \sim 10^{-22}$ Hz$^{-1/2}$ in the frequency band $f \sim 10^2$–10^3. The planned detectors such as Advanced LIGO, Advanced VIRGO, and LISA have better chances to detect this stochastic background. In fact, LISA can reach the density parameter of the order of $h_0^2\Omega_{\rm GW} \lesssim 10^{-11}$ at frequency $f \sim 10^{-3}$ Hz and Advanced LIGO can reach a $h_0^2\Omega_{\rm GW} \lesssim 10^{-9}$ at frequency $f \sim 10^2$ Hz. These planned detectors can register the stochastic background of GWs coming from cosmic strings and the pre-big-bang stage. The gap between LISA and the ground-based detectors will be covered by DECIGO/BBO detectors, which will operate in the frequency range from 0.1 Hz to 10 Hz and have 10^3 better sensitivity than LISA from 0.1 Hz to 1 Hz [67]. DECIGO will be able to observe the stochastic background of GWs produced at inflation and can reach $h_0^2\Omega_{\rm GW} \sim 10^{-20}$ at $f \sim 1$ Hz after three years of observation [68,69]. All the above mentioned GW detectors cover a frequency range 10^{-7} Hz $-$ 10^3 Hz and the high-frequency range will hopefully be explored by future high-frequency GWs detectors. The principle of a high-frequency detector is based on the electromagnetic-gravitational resonance first proposed by Braginsky and Mensky [70–73]. Actually there is a renewed interest on these new detectors, for which a prototype has been constructed at Birmingham University [74–76] and which reaches a strain sensitivity of the order of $h_{\rm rms} \sim 10^{-14}$ Hz $-$ $1/2$ at $f \sim 10^8$ Hz. The main goal of this detector is detection of high-frequency stochastic background of GWs from the early Universe and black hole interactions in higher dimensional gravitational theories.

XI. SUMMARY AND RESULTS

We have analyzed the formation and evolution of light primordial black holes in the early Universe, which created a transient matterdominated regime, in contrast to the present standard cosmology, where the early Universe after inflation was normally radiation-dominated. PBHs with masses less than $M \sim 10^8$ g evaporated before primordial nucleosynthesis, leaving no trace. Thus the fraction of the energy density of such PBHs, Ω_p, in this case is a free parameter of the model, not constrained by any existing observations.

At MD stage, the PBHs could form high-density clusters which would be efficient sources of the primordial GWs. PBHs could have dominated the Universe for a short time of the order of their lifetime, $\tau_{\rm BH}$, generating relic gravitational waves by various mechanisms of their mutual interactions as well as due to their evaporation. In the former case we have shown that production of GWs is most efficient after BH density started to dominate over radiation. After that moment, high-density clusters of PBHs could have been formed, leading to an efficient production of GWs. To survive until cluster formation, the PBH mass at production must be bounded from below by $M \sim 4 \times 10^{-5}$ g$\Omega_p^{-1}\Delta_{\rm in}^{-3/4}N_{\rm eff}^{1/2}$, which leads to a lower bound $\Omega_p > 10^{-14}\Delta_{\rm in}^{-3/4}N_{\rm eff}^{1/2}$. According to the standard cosmology the amplitude of primordial density perturbations is of the order of $\sim 10^{-4}$, which in our case leads to a lower bound on the density parameter of PBHs, $\Omega_p \gtrsim 10^{-11}$.

In this context, we have calculated the density parameter of GWs today from scattering of PBHs in both classical (Fig.) 2 and quantum regime, GWs emission from binaries, and from black hole evaporation. We have shown that a substantial amount of gravitational waves has been emitted by all mechanisms considered here. In the case of scattering of PBHs, we considered *only* scattering between them neglecting the possibility of PBH mergers, which results in an underestimate on $h_0^2\Omega_{\rm GW}$. Even in this case, the density parameter is substantial at high frequencies, reaching values of the order of $h_0^2\Omega_{\rm GW} \sim 10^{-9}$ for classical scattering and the total density parameter $h_0^2\Omega_{\rm GW} \sim 10^{-10}$ at $f \sim$ GHz for very light primordial black holes. In the low-frequency limit, the density parameter in the classical case is of the order of $h_0^2\Omega_{\rm GW} \sim 10^{-17}$–$10^{-20}$ in the frequency range $f \sim 10^{-1}$–10^2 Hz, which falls into the detection band of DECIGO/BBO.

The number of PBHs that form binaries after cluster formation is subject to uncertainties and in this paper we parametrized it through factor ϵ. The exact value of this parameter could be calculated elsewhere by numerical calculations. Since the density in such clusters is very high, we expect that ϵ is not very small in comparison with unity. In Fig. 3 the expected value of the density parameter today is presented. We can see that a large amount of gravitational waves has been emitted in the high-frequency regime with $h_0^2\Omega_{\rm GW} \sim 10^{-14}$–$10^{-12}$ at frequency $f \sim 10^{10}$ Hz depending on the BH initial mass. In the low-frequency part of the spectrum the spectral density parameter is utterly negligible, making it impossible to detect GWs produced by this mechanism at present and probably in the near future. In our derivation, we have

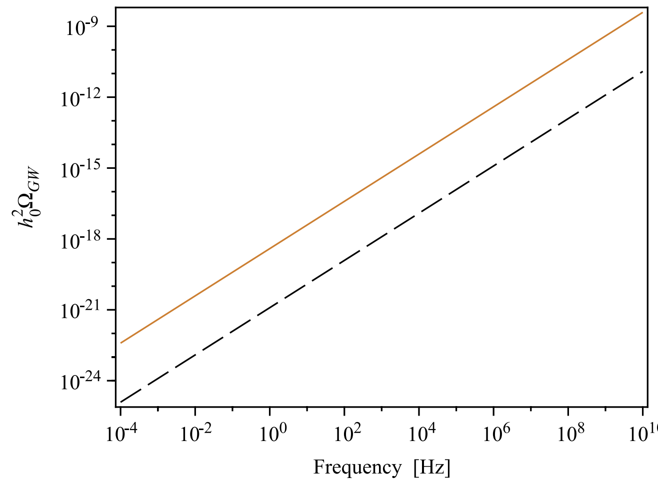

FIG. 2 (color online). Log-log plot of density parameter today, $h_0^2\Omega_{\rm GW}$, as a function of expected frequency today in classical approximation for $N_{\rm eff} \sim 100$, $g_S \sim 100$, $\Delta_b \sim 10^5$, and $v_{\rm rel} \sim 0.1$ for different values of PBH mass $M \sim 1$ g (solid line) and $M \sim 10^5$ g (dashed line).

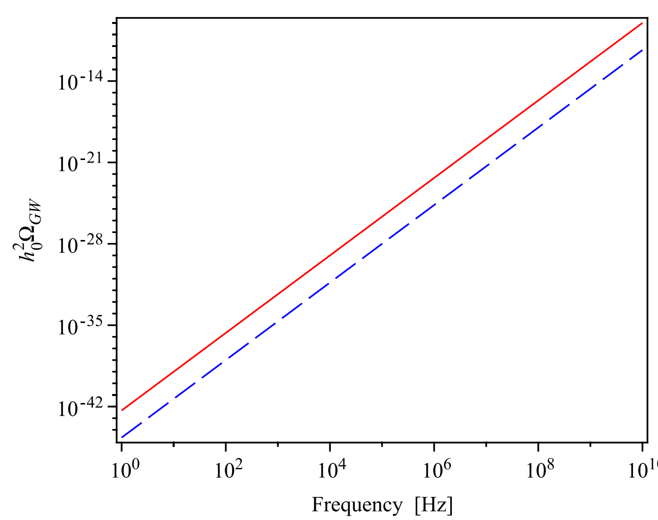

FIG. 3 (color online). Log-log plot of density parameter today, $h_0^2\Omega_{\rm GW}$, as a function of expected frequency today for PBH binaries in the stationary approximation for $\beta \sim 1$, $\epsilon \sim 10^{-5}$, $N_{\rm eff} \sim 100$, $g_S \sim 100$, PBH mass $M \sim 10^7$ g (solid line) and $M \sim 1$ g (dashed line).

considered both stationary and inspiral phases of binaries leading to a wide range of the frequencies emitted. We have considered only binaries in circular orbits; the problem with elliptical orbits will be treated later. If elliptical orbits were frequent, the amount of GWs will be presumably higher over a wide range of frequencies. We assumed that all binaries are formed with initial radius less than the average distance between PBHs and greater than the gravitational radius r_g. In this case the frequency spectrum has a cutoff in both low- and high-frequency bands of the spectrum.

Another mechanism of graviton production considered here is the PBHs evaporation. This mechanism is independent on the structure formation during the PBH domination. In Fig. 4 we show the density parameter as a function of frequency for BH masses 1 g and 10^5 g. Having a near blackbody spectrum, the frequency of the emitted gravitons can have any value, but unfortunately the GWs spectrum has a peak in the high-frequency region, which today makes a substantial contribution to the cosmological energy density of the order of $h_0^2\Omega_{\rm GW}(f_{\rm peak}) \sim 10^{-7}$.

The mechanisms considered in this paper could create a rather high cosmological fraction of the energy density of the relic gravitational waves at very high frequencies and gives the opportunity of investigating the high GW spectrum by present and future detectors. Unfortunately, at the lower part of the spectrum $\Omega_{\rm GW}$ significantly drops down. Still the planned interferometers DECIGO/BBO could be sensitive to the predicted GWs. It is noteworthy that the mechanism of GW generation suggested here kills or noticeably diminishes GWs from inflation by the redshift of the earlier generated GWs at the PBH (MD) stage.

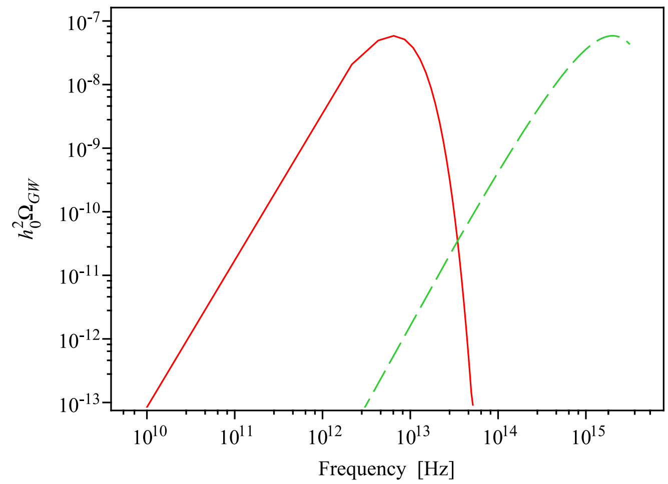

FIG. 4 (color online). Log-log plot of the density parameter per logarithmic frequency, $h_0^2\Omega_{\rm GW}(f; t_0)$, as a function of frequency today, f, for the case $g_S \sim 100$, $N_{\rm eff} \sim 100$, black hole mass $M = 1$ g (solid line), and black hole mass $M = 10^5$ g (dashed line). We can see that the spectrum has a maximum which is sharp and of the order $h_0^2\Omega_{\rm GW}(f_{\rm peak}) \sim 10^{-7}$.

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