

Dirac equation in Kerr-NUT-(A)dS spacetimes: Intrinsic characterization of separability in all dimensions

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We intrinsically characterize separability of the Dirac equation in Kerr-NUT-(A)dS spacetimes in all dimensions. Namely, we explicitly demonstrate that, in such spacetimes, there exists a complete set of first-order mutually commuting operators, one of which is the Dirac operator, that allows for common eigenfunctions which can be found in a separated form and correspond precisely to the general solution of the Dirac equation found by Oota and Yasui [Phys. Lett. B **659**, 688 (2008)]. Since all the operators in the set can be generated from the principal conformal Killing-Yano tensor, this establishes the (up-to-now) missing link among the existence of hidden symmetry, presence of a complete set of commuting operators, and separability of the Dirac equation in these spacetimes.

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I. INTRODUCTION

The most general known stationary higher-dimensional vacuum (including a cosmological constant) black hole spacetimes with spherical horizon topology [1] possess many remarkable properties, some of which are directly inherited from the four-dimensional Kerr-NUT-(A)dS geometry [2]. The similarity stems from the existence of a hidden symmetry associated with the *principal conformal Killing-Yano (PCKY) tensor* [3]. Such a symmetry generates the whole tower of explicit and hidden symmetries [4], which, in their turn, are responsible for many of the properties, including complete integrability of the geodesic motion [5–7], a special algebraic type of the Weyl tensor [8,9], the existence of a Kerr-Schild form [10], and separability of various field perturbations. For reviews on the subject, we refer to [11,12].

Especially interesting is a relationship between the existence of the PCKY tensor and separability of test field equations in the background of general Kerr-NUT-(A)dS spacetimes [1]. Namely, explicit separation of the Hamilton-Jacobi and Klein-Gordon equations was demonstrated by Frolov *et al.* [13], and the achieved separability was *intrinsically characterized* by Sergyeyev and Krtouš [14]. In their paper, the latter authors demonstrated that, in Kerr-NUT-(A)dS spacetimes, in all dimensions, there exists a complete set of first-order and second-order operators

(one of which is the Klein-Gordon operator) that mutually commute. These operators are constructed from second-rank Killing tensors and Killing vectors that can all be generated from the PCKY tensor. The common eigenfunction of these operators is characterized by operators' eigenvalues and can be found in a separated form—it is precisely the separated solution obtained by Frolov *et al.* In fact, the demonstrated results provide a textbook example of general theory discussed in [15–17].

Higher-spin perturbations of Kerr-NUT-(A)dS spacetimes were also studied. Namely, general separation of the Dirac equation in all dimensions was demonstrated by Oota and Yasui [18], electromagnetic perturbations in $n = 5$ spacetime dimensions were studied in [19], and separability of the linearized gravitational perturbations was with increasing generality studied in [19–24]. (The study of gravitational perturbations is very important, for example, for establishing the (in)stability of higher-dimensional black holes; see, e.g., recent papers [24,25] and references therein; or, for the study of quasinormal modes, e.g., [26] and references therein.) We also mention a recent paper [27] on general perturbation theory in higher-dimensional algebraically special spacetimes which employs the higher-dimensional Geroch-Held-Penrose formalism [28] and attempts to generalize Teukolsky's results [29,30].

The aim of the present paper is to intrinsically characterize the result of Oota and Yasui [18]. Namely, we want to demonstrate that, similar to the Klein-Gordon case [14], separability of the Dirac equation in Kerr-NUT-(A)dS

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spacetimes in all dimensions is underlaid by the existence of a complete set of mutually commuting operators, one of which is the Dirac operator. The corresponding set of operators was already studied and the mutual commutation proved in [31]; the operators are of the first order and correspond to Killing vectors and closed conformal Killing-Yano tensors—all generated from the PCKY tensor. In this paper, we pick up the threads of these results and demonstrate that, in a properly chosen representation, the common eigenfunction of the symmetry operators in the set can be chosen in the tensorial R -separated form and corresponds precisely to the separated solution of the Dirac equation found by Oota and Yasui [18]. Our paper generalizes the $n = 4$ results of Chandrasekhar [32] and Carter and McLenaghan [33] and the $n = 5$ results of Wu [34,35].

A plan of the paper is as follows. In Sec. II, we review the theory of the Dirac equation in curved spacetime while concentrating on the commuting symmetry operators of the Dirac operator. In Sec. III, we introduce the Kerr-NUT-(A) dS spacetimes in all dimensions and summarize their basic properties. Section IV is devoted to the discussion of a complete set of the Dirac symmetry operators; an explicit representation of these operators is found. Section V is the principal section of the paper where the tensorial R separability of the Dirac equation is discussed and the main assertion of the paper is proved. In Sec. VI, we comment on the possibility of introducing a different representation of γ matrices in which the standard tensorial separability occurs. Section VII is devoted to conclusions. In the Appendix, we gather necessary technical results.

II. DIRAC EQUATION IN CURVED SPACE

A. Dirac bundle

In what follows, we write the dimension of spacetime as

$$n = 2N + \varepsilon, \quad (2.1)$$

with $\varepsilon = 0, 1$ parametrizing the even, odd dimension, respectively. The Dirac bundle DM has fiber dimension 2^N . If necessary, we use capital Latin indices for tensors from the Dirac bundle. It is connected with the tangent bundle TM of the spacetime manifold M through the abstract gamma matrices $\gamma^a \in TM \otimes D_1^1 M$, which satisfy

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2g^{ab}. \quad (2.2)$$

They generate an irreducible representation of the abstract Clifford algebra on the Dirac bundle. All linear combinations of products of the abstract gamma matrices (with spacetime indices contracted) form the Clifford bundle CIM , which is thus identified with the space $D_1^1 M$ of all linear operators on the Dirac bundle. The Clifford multiplication (“matrix multiplication”) is denoted by juxtaposition of the Clifford objects. The gamma matrices also provide the Clifford map γ_* and the isomorphism of the exterior algebra ΛM and of the Clifford bundle,

$$\acute{\omega} \equiv \gamma_* \omega \equiv \sum_p \frac{1}{p!} (\omega_p)_{a_1 \dots a_p} \gamma^{a_1 \dots a_p}. \quad (2.3)$$

Here, $\omega = \sum_p \omega_p \in \Lambda M$ is an inhomogeneous form, ω_p its p -form parts, $\omega_p \in \Lambda^p M$, and $\gamma^{a_1 \dots a_p} = \gamma^{[a_1 \dots a_p]}$. For future use, we also define an operator π as $\pi \omega = \sum_p p \omega_p$.

We denote the Dirac operator both in the exterior bundle and Dirac bundle as D :

$$D = e^a \nabla_a, \quad D = \gamma_* e^a \nabla_a = \gamma^a \nabla_a. \quad (2.4)$$

Here, $e^a \in TM \otimes \Lambda M$ is a counterpart of γ_a in the exterior algebra, and ∇ denotes the spinor covariant derivative. We also denote by X_a the object dual to e^a (see, e.g., the Appendix of [31] for details on the notation).

B. First-order symmetry operators

First-order operators commuting with the Dirac operator (2.4) were recently studied in all dimensions [31]. Namely, we have the following result: The most general first-order operator S which commutes with the Dirac operator D , $[D, S] = 0$, splits into the (Clifford) even and odd parts

$$S = S_e + S_o, \quad (2.5)$$

where

$$S_e = K_{f_o} \equiv X^a \lrcorner f_o \nabla_a + \frac{\pi - 1}{2\pi} df_o, \quad (2.6)$$

$$S_o = M_{h_e} \equiv e^a \wedge h_e \nabla_a - \frac{n - \pi - 1}{2(n - \pi)} \delta h_e, \quad (2.7)$$

with f_o being an inhomogeneous odd Killing-Yano form and h_e being an inhomogeneous even closed conformal Killing-Yano form.

On the Dirac bundle, these operators read (denoting by $K_{f_o} = \gamma_* K_{f_o}$ and $M_{h_e} = \gamma_* M_{h_e}$)

$$K_{f_o} = \sum_{p \text{ odd}} \frac{1}{(p-1)!} \left[\gamma^{a_1 \dots a_{p-1}} (f_p)_{a_1 \dots a_{p-1}} \nabla_a + \frac{1}{2(p+1)^2} \gamma^{a_1 \dots a_{p+1}} (df_p)_{a_1 \dots a_{p+1}} \right], \quad (2.8)$$

$$M_{h_e} = \sum_{p \text{ even}} \frac{1}{p!} \left[\gamma^{a_1 \dots a_p} (h_p)_{a_1 \dots a_p} \nabla_a - \frac{p(n-p)}{2(n-p+1)} \gamma^{a_1 \dots a_{p-1}} (\delta h_p)_{a_1 \dots a_{p-1}} \right], \quad (2.9)$$

where p -forms f_p and h_p (with $f_o = \sum_{p \text{ odd}} f_p$ and $h_e = \sum_{p \text{ even}} h_p$) are odd Killing-Yano and even closed conformal Killing-Yano tensors, respectively. That is, they satisfy the following equations:

$$\nabla_a (f_p)_{a_1 \dots a_p} = \frac{1}{p+1} (df_p)_{aa_1 \dots a_p}, \quad (2.10)$$

$$\nabla_a (h_p)_{a_1 \dots a_p} = -\frac{p}{n-p+1} g_{a[a_1} (\delta h_p)_{a_2 \dots a_p]}. \quad (2.11)$$

In an odd number of spacetime dimensions, the Hodge duality of Killing-Yano tensors translates into the corresponding relation of symmetry operators K and M . Namely, let z be the Levi-Civita n -form satisfying $z_{a_1 \dots a_n} z^{a_1 \dots a_n} = n!$.¹ Then, the Hodge dual of a p -form ω can be written as

$$* \omega = (-1)^{(n-1)p+[p/2]} z \omega, \quad (2.12)$$

and, for the operators of type K and M , it holds that

$$K_z h = (-1)^{n-1} z M_h, \quad M_z f = (-1)^{n-1} z K_f, \quad (2.13)$$

where f is an odd Killing-Yano form and h an even closed conformal Killing-Yano form.

III. KERR-NUT-(A)DS SPACETIMES

We shall concentrate on the Dirac equation in general rotating Kerr-NUT-(A)dS spacetimes in all dimensions [1]. Slightly more generally, we consider the most general *canonical metric* admitting a PCKY tensor [36,37] and intrinsically characterize separability of the massive Dirac equation in such a background.

The canonical metric is written as²

$$g = \sum_{\mu=1}^N \left[\frac{dx_\mu^2}{Q_\mu} + Q_\mu \left(\sum_{j=0}^{N-1} A_\mu^{(j)} d\psi_j \right)^2 \right] + \varepsilon S \left(\sum_{j=0}^N A^{(j)} d\psi_j \right)^2. \quad (3.1)$$

Here, coordinates x_μ ($\mu = 1, \dots, N$) stand for the (Wick rotated) radial coordinate and longitudinal angles, and Killing coordinates ψ_k ($k=0, \dots, N-1+\varepsilon$) denote time and azimuthal angles associated with Killing vectors $\xi^{(k)}$

$$\xi^{(k)} = \partial_{\psi_k}, \quad \xi^{(k)} \equiv (\partial_{\psi_k})^b. \quad (3.2)$$

We have further defined³ the functions (note that our sign convention for U_μ differs from the one in [18])

¹Note that $\gamma_*(z)$ is the ordered product of all n gamma matrices and, in odd dimensions, it is proportional to a unit matrix. See also Sec. IV A.

²We assume a Euclidean signature of the metric. The physical signature could be obtained by a proper choice of signs of the metric function, a suitable Wick rotation of the coordinates and metric parameters, and a slight modification of various spinor-related conventions.

³In what follows, we assume no implicit summing over μ, ν, \dots and j, k, l, m, \dots indices. The explicit sums have, unless specifically indicated otherwise, ranges $1, \dots, N$ and $0, \dots, N-1+\varepsilon$, respectively.

$$Q_\mu = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\nu \neq \mu} (x_\nu^2 - x_\mu^2), \quad S = \frac{-c}{A^{(N)}}, \quad (3.3)$$

$$A_\mu^{(k)} = \sum_{\substack{\nu_1 \dots \nu_k \\ \nu_1 < \dots < \nu_k, \nu_i \neq \mu}} x_{\nu_1}^2 \dots x_{\nu_k}^2, \quad A^{(j)} = \sum_{\substack{\nu_1 \dots \nu_j \\ \nu_1 < \dots < \nu_j}} x_{\nu_1}^2 \dots x_{\nu_j}^2. \quad (3.4)$$

Functions $A_\mu^{(k)}$ and $A^{(j)}$ can be generated as follows:

$$\prod_{\nu} (t - x_\nu^2) = \sum_{j=0}^N (-1)^j A^{(j)} t^{N-j}, \quad (3.5)$$

$$\prod_{\substack{\nu \\ \nu \neq \mu}} (t - x_\nu^2) = \sum_{j=0}^N (-1)^j A_\mu^{(j)} t^{N-1-j},$$

and satisfy the important relations

$$\sum_{\mu} \frac{A_\mu^{(i)}}{U_\mu} (-x_\mu^2)^{N-1-j} = \delta_j^i, \quad (3.6)$$

$$\sum_j \frac{A_\mu^{(j)}}{U_\nu} (-x_\nu^2)^{N-1-j} = \delta_\mu^\nu.$$

The quantities X_μ are functions of a single variable x_μ , and c is an arbitrary constant. The vacuum (with a cosmological constant) black hole geometry is recovered by setting

$$X_\mu = \sum_{k=\varepsilon}^N c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\varepsilon} + \frac{\varepsilon c}{x_\mu^2}. \quad (3.7)$$

This choice of X_μ describes the most general known Kerr-NUT-(A)dS spacetimes in all dimensions [1]. The constant c_N is proportional to the cosmological constant, and the remaining constants are related to angular momenta, mass, and NUT parameters.

At points with $x_\mu = x_\nu$, with $\mu \neq \nu$, the coordinates are degenerate. We assume a domain where $x_\mu \neq x_\nu$ for $\mu \neq \nu$. In such a domain, we can always order and rescale the coordinates in such a way that

$$x_\mu + x_\nu > 0 \quad \text{and} \quad x_\mu - x_\nu > 0 \quad \text{for} \quad \mu < \nu. \quad (3.8)$$

With this convention and assuming positive signature, we have

$$U_\mu = (-1)^{N-\mu} |U_\mu|, \quad X_\mu = (-1)^{N-\mu} |X_\mu|. \quad (3.9)$$

We introduce the following orthonormal covector frame $E^a = \{E^\mu, E^{\hat{\mu}}, E^0\}$,

$$E^\mu = \frac{dx_\mu}{\sqrt{Q_\mu}}, \quad E^{\hat{\mu}} = \sqrt{Q_\mu} \sum_{j=0}^{N-1} A_\mu^{(j)} d\psi_j, \quad (3.10)$$

$$E^0 = \sqrt{S} \sum_j A^{(j)} d\psi_j,$$

and the dual vector frame $E_a = \{E_\mu, E_{\hat{\mu}}, E_0\}$,

$$\begin{aligned} E_\mu &= \sqrt{Q_\mu} \partial_{x_\mu}, \\ E_{\hat{\mu}} &= \sqrt{Q_\mu} \sum_j \frac{(-x_\mu^2)^{N-1-j}}{X_\mu} \partial_{\psi_j}, \\ E_0 &= \frac{1}{\sqrt{SA^{(N)}}} \partial_{\psi_N}. \end{aligned} \quad (3.11)$$

Note that E^0 and E_0 are defined only in an odd dimension. In this frame, the metric reads

$$g = \sum_\mu (E^\mu \otimes E^\mu + E^{\hat{\mu}} \otimes E^{\hat{\mu}}) + \varepsilon E^0 \otimes E^0, \quad (3.12)$$

and the Ricci tensor is diagonal [9],

$$\text{Ric} = \sum_\mu r_\mu (E^\mu \otimes E^\mu + E^{\hat{\mu}} \otimes E^{\hat{\mu}}) + \varepsilon r_0 E^0 \otimes E^0, \quad (3.13)$$

where

$$\begin{aligned} r_\mu &= -\frac{1}{2x_\mu} \left[\sum_\nu \frac{x_\nu^2 (x_\nu^{-1} \hat{X}_\nu)_{,\nu}}{U_\nu} + \varepsilon \sum_\nu \frac{\hat{X}_\nu}{U_\nu} \right]_{,\mu}, \\ r_0 &= -\sum_\rho \frac{1}{x_\rho} \left(\sum_\sigma \frac{\hat{X}_\sigma}{U_\sigma} \right)_{,\rho}, \end{aligned} \quad (3.14)$$

and $\hat{X}_\mu = X_\mu - \varepsilon c/x_\mu^2$. For the Einstein space, polynomials (3.7) lead to a constant value r_μ .

The canonical metric (3.1) possesses a hidden symmetry of the PCKY tensor [3]. In the basis (3.10), the PCKY two-form reads

$$h = \sum_{\mu=1}^N x^\mu E^\mu \wedge E^{\hat{\mu}}. \quad (3.15)$$

This tensor generates the tower of closed conformal Killing-Yano (2j)-forms (note that this definition differs by the factorial from [31]):

$$h^{(j)} = \frac{1}{j!} h^{\wedge j} = \frac{1}{j!} h \wedge \cdots \wedge h, \quad (3.16)$$

which, in their turn, give rise to Killing-Yano forms

$$f^{(j)} = *h^{(j)} = (-1)^j z h^{(j)}. \quad (3.17)$$

In the second equality, we have used (2.12). Killing-Yano tensors (3.17) “square to” second-rank Killing tensors

$$k^{(j)} = \sum_\mu A_\mu^{(j)} (E^\mu \otimes E^\mu + E^{\hat{\mu}} \otimes E^{\hat{\mu}}) + \varepsilon A^{(j)} E^0 \otimes E^0. \quad (3.18)$$

Obviously, $k^{(0)}$ coincides with the metric, and hence it is a trivial Killing tensor which we include in our tower; so, we take $j = 0, \dots, N-1$.

The PCKY tensor h generates also all the isometries (3.2) of the spacetime. In particular, the *primary Killing vector* $\xi = \xi^{(0)}$ is given by

$$\xi^a = \frac{1}{n-1} \nabla_b h^{ba}, \quad (3.19)$$

which can be written explicitly as

$$\xi = \sum_\mu \sqrt{Q_\mu} E_{\hat{\mu}} + \varepsilon \sqrt{S} E_0 = \partial_{\psi_0}. \quad (3.20)$$

It satisfies the important relation

$$-\frac{1}{n-2j+1} \delta h^{(j)} = \xi^b \wedge h^{(j-1)}. \quad (3.21)$$

In odd dimensions, we also have

$$\xi_{(N)} = (\partial_{\psi_N})^b = \sqrt{-c} * h^{(N)} = \sqrt{-c} f^{(N)}. \quad (3.22)$$

Let us finally mention that the explicit symmetries $\xi^{(k)}$ and hidden symmetries $k^{(j)}$ are responsible for complete integrability of the geodesic motion, as well as for separability of the Hamilton-Jacobi equation in spacetimes (3.1). Moreover, the corresponding operators $\{(\xi^{(k)})^a \nabla_a, \nabla_a (k^{(j)})^{ab} \nabla_b\}$ form a complete set of commuting operators which intrinsically characterize separability of the Klein-Gordon equation in these spacetimes [14].

IV. COMPLETE SET OF DIRAC SYMMETRY OPERATORS

A. Operators of the complete set

The canonical spacetime (3.1) admits a complete set of first-order symmetry operators of the Dirac operator that are mutually commuting [31]. These operators are determined by the tower of symmetries built from the PCKY tensor h . Namely, they are given by $(N + \varepsilon)$ Killing-Yano one-forms $\xi_{(k)}$, (3.2), and N closed conformal Killing-Yano forms $h^{(j)}$, (3.16). In the exterior algebra notation, they read

$$K_k \equiv K_{\xi_{(k)}} = X^a \lrcorner \xi_{(k)} \nabla_a + \frac{1}{4} d\xi_{(k)}, \quad (4.1)$$

for $k = 0, \dots, N-1 + \varepsilon$, and

$$M_j = M_{h^{(j)}} \equiv e^a \wedge h^{(j)} \nabla_a - \frac{n-2j}{2(n-2j+1)} \delta h^{(j)}, \quad (4.2)$$

for $j = 0, \dots, N-1$. Note that the operator M_0 corresponds to the Dirac operator, $M_0 = D$. It is the aim of this section to find an explicit representation of the action of these operators on the Dirac bundle. As usual, we shall denote it by the same letter, i.e., we write $K_j = \gamma_* K_j$ and $M_j = \gamma_* M_j$.

Let us remark here that, in odd dimensions, one has a “different choice” of operators commuting with the Dirac operator—associated with (in this case, odd) Killing-Yano tensors $f^{(j)}$, (3.17). Using first relation (2.13), one finds that

$$K_{f^{(j)}} = (-1)^j z M_j. \quad (4.3)$$

Since, in our representation (introduced below), we shall have $\gamma_*(z) = i^N$, i.e., a trivial matrix, we can, without loss of generality, consider only operators M_j . (Operators $K_{f^{(j)}} = (-1)^j i^N M_j$ have the same eigenvectors.) In particular, due to (3.22), we have the following identification:

$$K_N = (-i)^N \sqrt{-c} M_N, \quad (4.4)$$

which shall be used in Sec. V.

B. Representation of γ matrices and spinors

In the canonical spacetimes (3.1), the geometry determines a special frame E^a , (3.10). This frame can be lifted to the frame ϑ_E in the Dirac bundle by demanding that the abstract gamma matrices γ^a have constant components $(\gamma^a)^A_B$, set to some special values. It was a key observation of [18] that these components can be chosen as a tensor product of N two-dimensional matrices. In other words, the special geometric structure of canonical spacetimes allows us to represent the fiber of the Dirac bundle as a tensor product of N two-dimensional spaces S , $DM = S^N M$, with gamma matrices adjusted to hidden symmetry. We use Greek letters ϵ, s, \dots for tensor indices in these two-dimensional spaces and we use values $\epsilon = \pm 1$ (or just \pm) to distinguish the components.

It means that we choose a frame ϑ_E in the Dirac bundle in a tensor product form:

$$\vartheta_E = \vartheta_{\epsilon_1 \dots \epsilon_N} = \vartheta_{\epsilon_1} \otimes \dots \otimes \vartheta_{\epsilon_N}, \quad (4.5)$$

where ϑ_+ and ϑ_- form a frame in the two-dimensional spinor space S . With such a choice, we have a natural identification of Dirac indices E with the multi-index $\{\epsilon_1, \dots, \epsilon_N\}$.

A generic two-dimensional spinor can thus be written as $\chi = \chi^+ \vartheta_+ + \chi^- \vartheta_- = \chi^\epsilon \vartheta_\epsilon$, with components being two complex numbers

$$\begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix}.$$

Similarly, the Dirac spinors $\psi \in DM$ can be written as $\psi = \psi^{\epsilon_1 \dots \epsilon_N} \vartheta_{\epsilon_1 \dots \epsilon_N}$, with 2^N components $\psi^{\epsilon_1 \dots \epsilon_N}$.

Before we write down the gamma matrices in this frame, let us introduce some useful notation. Let I, ι, σ , and $\hat{\sigma}$ be the unit and, respectively, Pauli operators on S , i.e., their action is given in components by

$$\begin{aligned} (I\chi)^\epsilon &= \chi^\epsilon, & (\iota\chi)^\epsilon &= \epsilon\chi^\epsilon, \\ (\sigma\chi)^\epsilon &= \chi^{-\epsilon}, & (\hat{\sigma}\chi)^\epsilon &= -i\epsilon\chi^{-\epsilon}. \end{aligned} \quad (4.6)$$

In matrix form, they are written as

$$\begin{aligned} I^\epsilon_s &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \iota^\epsilon_s &\equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \sigma^\epsilon_s &\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \hat{\sigma}^\epsilon_s &\equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \end{aligned} \quad (4.7)$$

These operators satisfy the standard relations

$$\begin{aligned} \iota\sigma &= -\sigma\iota = i\hat{\sigma}, \\ \sigma\hat{\sigma} &= -\hat{\sigma}\sigma = i\iota, \\ \hat{\sigma}\iota &= -\iota\hat{\sigma} = i\sigma. \end{aligned} \quad (4.8)$$

Next, for any linear operator $\alpha \in S_1^1 M$, we denote by $\alpha_{\langle\mu\rangle} \in D_1^1 M$ a linear operator on the Dirac bundle

$$\alpha_{\langle\mu\rangle} \equiv I \otimes \dots \otimes I \otimes \alpha \otimes I \otimes \dots \otimes I, \quad (4.9)$$

with α on the μ th place in the tensor product. Similarly, for mutually different indices μ_1, \dots, μ_j , we define

$$\alpha_{\langle\mu_1 \dots \mu_j\rangle} \equiv \alpha_{\langle\mu_1\rangle} \otimes \dots \otimes \alpha_{\langle\mu_j\rangle}. \quad (4.10)$$

Now, we can finally write down the abstract gamma matrices with respect to the frame $E_a = \{E_\mu, E_{\hat{\mu}}, E_0\}$ chosen in tangent space,

$$\begin{aligned} \gamma^\mu &= \iota_{\langle 1 \dots \mu-1 \rangle} \sigma_{\langle \mu \rangle}, \\ \gamma^{\hat{\mu}} &= \iota_{\langle 1 \dots \mu-1 \rangle} \hat{\sigma}_{\langle \mu \rangle}, \\ \gamma^0 &= \iota_{\langle 1 \dots N \rangle}, \end{aligned} \quad (4.11)$$

where γ^0 is defined only in odd dimensions. This definition essentially fixes the relation of the spinor frame ϑ_E to the frame in the tangent space. It is straightforward to check that the matrices (4.11) satisfy the property (2.2).

In components, the action of these matrices on a spinor $\psi = \psi^{\epsilon_1 \dots \epsilon_N} \vartheta_{\epsilon_1 \dots \epsilon_N}$ is given as

$$\begin{aligned} (\gamma^\mu \psi)^{\epsilon_1 \dots \epsilon_N} &= \left(\prod_{\nu=1}^{\mu-1} \epsilon_\nu \right) \psi^{\epsilon_1 \dots (-\epsilon_\mu) \dots \epsilon_N}, \\ (\gamma^{\hat{\mu}} \psi)^{\epsilon_1 \dots \epsilon_N} &= -i \epsilon_\mu \left(\prod_{\nu=1}^{\mu-1} \epsilon_\nu \right) \psi^{\epsilon_1 \dots (-\epsilon_\mu) \dots \epsilon_N}, \\ (\gamma^0 \psi)^{\epsilon_1 \dots \epsilon_N} &= \left(\prod_{\nu=1}^N \epsilon_\nu \right) \psi^{\epsilon_1 \dots \epsilon_N}. \end{aligned} \quad (4.12)$$

We shall also use the relations

$$\gamma^{\hat{\mu}} = -i \iota_{\langle \mu \rangle} \gamma^\mu, \quad \gamma^{\mu \hat{\mu}} = i \iota_{\langle \mu \rangle} \quad (4.13)$$

and the fact that $\gamma_*(z) = \gamma^{\mu_1 \dots \mu_N \hat{\mu}_1 \dots \hat{\mu}_N 0} = i^N I$.

C. Explicit form of the operators

Symmetry operators determined by Killing vectors are, in general, equivalent to the Lie derivative lifted from the tangent bundle to the Clifford or Dirac bundles. Thanks to (3.2), we can thus write

$$K_k = \mathcal{L}_{\xi^{(k)}} = \frac{\delta}{\partial \psi_k}, \quad (4.14)$$

where $\frac{\delta}{\partial \psi_k}$ is a partial derivative along ψ_k which acts only on the components of the spinor in the frame ∂_E described above. That is, let $\chi = \chi^E \partial_E$ be a spinor; then, $K_k \chi = \frac{\delta \chi}{\partial \psi_k} = \frac{\partial \chi^E}{\partial \psi_k} \partial_E$ (cf. footnote 4).

The operators M_j , (4.2), must be lifted to the Dirac bundle by using (2.3). Let us start with expressing the action of a form given by $\alpha \wedge h^{(j)}$, with α being a one-form:

$$\begin{aligned} \gamma_*(\alpha \wedge h^{(j)}) &= \frac{1}{j!(2j+1)!} (\alpha \wedge h \wedge \cdots \wedge h)_{a_0 a_1 \dots a_{2j}} \gamma^{a_0 a_1 \dots a_{2j}} \\ &= \frac{1}{j! 2^j} \alpha_{a_0} h_{a_1 a_2} \cdots h_{a_{2j-1} a_{2j}} \gamma^{a_0 a_1 \dots a_{2j}} \\ &= \frac{1}{j! 2^j} \sum_{\substack{a_0, a_1, \dots, a_{2j} \\ a_i \text{ all different}}} \alpha_{a_0} h_{a_1 a_2} \cdots h_{a_{2j-1} a_{2j}} \gamma^{a_0} \gamma^{a_1} \cdots \gamma^{a_{2j}}. \end{aligned} \quad (4.15)$$

In the last equality, we assumed that indices a_i correspond to the orthonormal frame $E_\mu, E_{\hat{\mu}}, E_0$, which implies that gamma matrices γ^a with different indices anticommute. In such a frame, however, the PCKY tensor h has only non-zero components $h_{\mu \hat{\mu}} = -h_{\hat{\mu} \mu} = x_\mu$. We can thus write

$$\begin{aligned} \gamma_*(\alpha \wedge h^{(j)}) &= \frac{1}{j!} \sum_{\mu} \sum_{\substack{\mu_1, \dots, \mu_j \\ \mu_i \text{ different} \\ \mu_i \neq \mu}} (\alpha_\mu \gamma^\mu + \alpha_{\hat{\mu}} \gamma^{\hat{\mu}}) x_{\mu_1} \cdots x_{\mu_j} \gamma^{\mu_1 \hat{\mu}_1 \dots \mu_j \hat{\mu}_j} \\ &\quad + \varepsilon \frac{1}{j!} \alpha_0 \gamma^0 \sum_{\substack{\mu_1, \dots, \mu_j \\ \mu_i \text{ different}}} x_{\mu_1} \cdots x_{\mu_j} \gamma^{\mu_1 \hat{\mu}_1 \dots \mu_j \hat{\mu}_j} \\ &= i^j \sum_{\mu} \left(\sum_{\substack{\mu_1, \dots, \mu_j \\ \mu_1 < \dots < \mu_j \\ \mu_i \neq \mu}} x_{\mu_1} \cdots x_{\mu_j} \iota_{(\mu_1 \dots \mu_j)} \right) (\alpha_\mu \gamma^\mu + \alpha_{\hat{\mu}} \gamma^{\hat{\mu}}) \\ &\quad + \varepsilon i^j \left(\sum_{\substack{\mu_1, \dots, \mu_j \\ \mu_1 < \dots < \mu_j}} x_{\mu_1} \cdots x_{\mu_j} \iota_{(\mu_1 \dots \mu_j)} \right) \alpha_0 \gamma^0. \end{aligned} \quad (4.16)$$

In the last equality, we have used (4.13) and the symmetry of the summands with respect to permutation of the μ_i indices. The result can be rewritten as

$$\gamma_*(\alpha \wedge h^{(j)}) = i^j \sum_{\mu} \mathcal{B}_{\mu}^{(j)} (\alpha_{\mu} \gamma^{\mu} + \alpha_{\hat{\mu}} \gamma^{\hat{\mu}}) + \varepsilon i^j \mathcal{B}^{(j)} \alpha_0 \gamma^0, \quad (4.17)$$

where we introduced a spinorial analogue of functions $A_{\mu}^{(j)}$ and $A^{(j)}$, (3.4) and (3.5), given by

$$\mathcal{B}_{\mu}^{(k)} = \sum_{\substack{\nu_1, \dots, \nu_k \\ \nu_1 < \dots < \nu_k, \nu_i \neq \mu}} \iota_{(\nu_1)} x_{\nu_1} \cdots \iota_{(\nu_k)} x_{\nu_k}, \quad (4.18)$$

$$\mathcal{B}^{(k)} = \sum_{\substack{\nu_1, \dots, \nu_k \\ \nu_1 < \dots < \nu_k}} \iota_{(\nu_1)} x_{\nu_1} \cdots \iota_{(\nu_k)} x_{\nu_k}.$$

These functions are elementary symmetric functions of $\{\iota_{(\nu)} x_{\nu}\}$ and $\{\iota_{(\nu)} x_{\nu}\}_{\nu \neq \mu}$, respectively:

$$\prod_{\nu} (t - \iota_{(\nu)} x_{\nu}) = \sum_{j=0}^N (-1)^j \mathcal{B}^{(j)} t^{N-j}, \quad (4.19)$$

$$\prod_{\nu \neq \mu} (t - \iota_{(\nu)} x_{\nu}) = \sum_{j=0}^N (-1)^j \mathcal{B}_{\mu}^{(j)} t^{N-1-j}.$$

Relations including $\mathcal{B}_{\mu}^{(k)}$ and $\mathcal{B}^{(k)}$ can be formally obtained from the corresponding expressions valid for $A_{\mu}^{(k)}$ and $A^{(k)}$ by using a simple rule

$$A \leftrightarrow B \Leftrightarrow x_{\mu}^2 \leftrightarrow \iota_{(\mu)} x_{\mu}. \quad (4.20)$$

In particular, we can introduce the analogue of functions U_{μ} , (3.3), by

$$\mathcal{V}_{\mu} = \prod_{\substack{\nu \\ \nu \neq \mu}} (\iota_{(\nu)} x_{\nu} - \iota_{(\mu)} x_{\mu}) \quad (4.21)$$

and derive the relations analogous to Eq. (3.6)

$$\begin{aligned} \sum_{\mu} \frac{\mathcal{B}_{\mu}^{(i)}}{\mathcal{V}_{\mu}} (-\iota_{(\mu)} x_{\mu})^{N-1-j} &= \delta^i_j, \\ \sum_j \frac{\mathcal{B}_{\mu}^{(j)}}{\mathcal{V}_{\mu}} (-\iota_{(\mu)} x_{\mu})^{N-1-j} &= \delta_{\mu}^{\nu}. \end{aligned} \quad (4.22)$$

Additional important relations regarding quantities $\mathcal{B}_{\mu}^{(k)}$ and $\mathcal{B}^{(k)}$ are gathered in Sec. 2 of the Appendix.

After this preliminary work, we are ready to find the action of the operators M_j (4.2). Let us start with the second term in (4.2). Using (3.21), Eq. (4.17), the explicit expression for the primary Killing vector (3.20), and the first relation (4.13), we find

$$\begin{aligned} \gamma_* \left(-\frac{n-2j}{2(n-2j+1)} \delta h^{(j)} \right) &= \frac{1}{2} (n-2j) \gamma_* (\xi^b \wedge h^{(j-1)}) = -i^j (N-j + \varepsilon/2) \\ &\quad \times \left(\sum_{\mu} \sqrt{Q_{\mu}} \mathcal{B}_{\mu}^{(j-1)} \iota_{(\mu)} \gamma^{\mu} + i \varepsilon \sqrt{S} \mathcal{B}^{(j-1)} \gamma^0 \right). \end{aligned} \quad (4.23)$$

Next, we want to find the expression for $\gamma_*(e^a \wedge h^{(j)} \nabla_a)$, where the spin derivative with respect to the chosen frame E^a is

$$\nabla_a = \partial_a + \frac{1}{4}\omega_{abc}\gamma^b\gamma^c. \quad (4.24)$$

Here, ∂_a is the derivative acting only on components of the spinor,⁴ and the connection coefficients ω_{abc} are listed in Sec. 1 of the Appendix. Using (4.17), we have

$$\begin{aligned} \gamma_*(e^\mu \wedge h^{(j)}) &= ijB_\mu^{(j)}\gamma^\mu, \\ \gamma_*(e^{\hat{\mu}} \wedge h^{(j)}) &= ijB_\mu^{(j)}\gamma^{\hat{\mu}}, \\ \gamma_*(e^0 \wedge h^{(j)}) &= ijB^{(j)}\gamma^0. \end{aligned} \quad (4.25)$$

Hence, the derivative term can be expressed, with the help of (3.11), in terms of partial derivatives

$$\begin{aligned} &\gamma_*(e^a \wedge h^{(j)})\partial_a \\ &= ij\sum_\mu \sqrt{Q_\mu} B_\mu^{(j)} \left[\frac{\partial}{\partial x_\mu} - \frac{i\iota_{\langle\mu\rangle}}{X_\mu} \sum_k (-x_\mu^2)^{N-1-k} \frac{\partial}{\partial \psi_k} \right] \gamma^\mu \\ &\quad + \varepsilon ij \frac{B^{(j)}}{\sqrt{SA}^{(N)}} \frac{\partial}{\partial \psi_N} \gamma^0. \end{aligned} \quad (4.26)$$

Moreover, using the explicit form of the connection coefficients, we find

$$\begin{aligned} &\frac{1}{4}\gamma_*(e^a \wedge h^{(j)})\omega_{abc}\gamma^b\gamma^c \\ &= ij\sum_\mu \sqrt{Q_\mu} \left(\frac{X'_\mu}{4X_\mu} B_\mu^{(j)} + \sum_{\nu \neq \mu} \frac{x_\mu + \iota_{\langle\mu\nu\rangle}x_\nu}{x_\mu^2 - x_\nu^2} \right. \\ &\quad \times \left. \left(B_\nu^{(j)} - \frac{1}{2}B_\mu^{(j)} \right) \right) \gamma^\mu + \varepsilon ij \sum_\mu \left(\frac{\sqrt{Q_\mu}}{2x_\mu} B^{(j)} \gamma^\mu \right. \\ &\quad \left. + \frac{i\sqrt{S}}{2x_\mu} (B^{(j)} - 2B_\mu^{(j)})\iota_{\langle\mu\rangle}\gamma^0 \right). \end{aligned} \quad (4.27)$$

Putting all three terms (4.23), (4.26), and (4.27) together and using the identities (A6) and (A7), we derive our final form for the operators M_j :

$$\begin{aligned} M_j &= ij\sum_\mu \sqrt{Q_\mu} B_\mu^{(j)} \left(\frac{\partial}{\partial x_\mu} + \frac{X'_\mu}{4X_\mu} + \frac{1}{2} \sum_{\nu \neq \mu} \frac{1}{x_\mu - \iota_{\langle\mu\nu\rangle}x_\nu} \right. \\ &\quad \left. - \frac{i\iota_{\langle\mu\rangle}}{X_\mu} \sum_k (-x_\mu^2)^{N-1-k} \frac{\partial}{\partial \psi_k} + \frac{\varepsilon}{2x_\mu} \right) \gamma^\mu \\ &\quad + \varepsilon ij^{j+1} \frac{\sqrt{S}}{2} \left[B^{(j-1)} - B^{(j)} \left(\frac{2}{ic} \frac{\partial}{\partial \psi_N} + \sum_\mu \frac{1}{\iota_{\langle\mu\rangle}x_\mu} \right) \right] \gamma^0. \end{aligned} \quad (4.28)$$

⁴The derivative ∂_a annihilates the frames $E^a \equiv \{E^\mu, E^{\hat{\mu}}, E^0\}$ and $\partial_E, \partial_a E^\mu = \partial_a E^{\hat{\mu}} = \partial_a E^0 = 0, \partial_a \partial_E = 0$. It thus acts just on the components, $\partial_a \alpha = (\partial_a \alpha_b)E^b$ and $\partial_a \chi = (\partial_a \chi^E) \partial_E$. The connection coefficients are defined as $\nabla_a E^b = -\omega_a{}^b{}_c E^c$.

V. R SEPARABILITY OF THE DIRAC EQUATION

Now, we can formulate the main result: the commuting symmetry operators K_k and M_j have common spinorial eigenfunctions ψ :

$$K_k \psi = i\Psi_k \psi, \quad (5.1)$$

$$M_j \psi = \mathcal{X}_j \psi, \quad (5.2)$$

which can be found in the tensorial R -separated form

$$\psi = R \exp\left(i \sum_k \Psi_k \psi_k\right) \bigotimes_{\nu} \chi_{\nu}, \quad (5.3)$$

where $\{\chi_{\nu}\}$ is an N -tuple of two-dimensional spinors, and R is the (Clifford-bundle-)valued prefactor⁵

$$R = \prod_{\substack{\kappa, \lambda \\ \kappa < \lambda}} (x_\kappa + \iota_{\langle\kappa\lambda\rangle} x_\lambda)^{-(1/2)}. \quad (5.4)$$

As part of the separation ansatz, we ask that χ_{ν} depends only on the variable x_{ν} , $\chi_{\nu} = \chi_{\nu}(x_{\nu})$. (Hence, we have $\frac{\partial}{\partial \psi_k} \chi_{\nu} = 0$ and $\frac{\partial}{\partial x_\mu} \chi_{\nu} = 0$ for $\nu \neq \mu$.)

In this section, we are going to show that Eqs. (5.1) and (5.2) are satisfied if and only if the spinors χ_{ν} satisfy the ordinary differential equations (5.16) below. These equations are equivalent to the conditions found in [18].

Let us first note that rewriting the tensorial R -separability ansatz (5.3) in terms of components, we recover the separated solution of the massive Dirac equation [equivalent to (5.2) with $j = 0$] which was found in [18]:

$$\psi^{\varepsilon_1 \dots \varepsilon_N} = \phi_{\varepsilon_1 \dots \varepsilon_N} \exp\left(i \sum_k \Psi_k \psi_k\right) \prod_{\nu} \chi_{\nu}^{\varepsilon_{\nu}}. \quad (5.5)$$

Here, $\phi_{\varepsilon_1 \dots \varepsilon_N}$ is a diagonal element of the prefactor R ,

$$\phi_{\varepsilon_1 \dots \varepsilon_N} = \prod_{\substack{\kappa, \lambda \\ \kappa < \lambda}} (x_\kappa + \varepsilon_\kappa \varepsilon_\lambda x_\lambda)^{-(1/2)}. \quad (5.6)$$

To derive the announced results, we shall work directly with the tensorial multiplicative ansatz (5.3) and, only in the end, we shall make contact with the work of [18] by finding an equation for the components of each χ_{ν} .

The spinor ψ given by (5.3) satisfies (5.1). To show (5.2), we need to calculate $M_j \psi$, with M_j given by (4.28). We have

⁵Note that, thanks to our convention (3.8), the operators under the square root in R are positively definite.

$M_j \psi$

$$\begin{aligned}
 &= i^j \exp\left(i \sum_k \Psi_k \psi_k\right) \left[\sum_\mu \sqrt{Q_\mu} \mathcal{B}^{(j)} \left(\frac{\partial}{\partial x_\mu} + \frac{X'_\mu}{4X_\mu} \right. \right. \\
 &\quad \left. \left. + \frac{1}{2} \sum_{\nu \neq \mu} \frac{1}{x_\mu - \iota_{(\mu\nu)} x_\nu} + \frac{\tilde{\Psi}_\mu}{X_\mu} \iota_{(\mu)} + \frac{\varepsilon}{2x_\mu} \right) \gamma^\mu \right. \\
 &\quad \left. + \varepsilon \frac{i\sqrt{S}}{2} \left(\mathcal{B}^{(j-1)} - \mathcal{B}^{(j)} \left(\frac{2\tilde{\Psi}_N}{c} + \sum_\mu \frac{\iota_{(\mu)}}{x_\mu} \right) \right) \gamma^0 \right] R \otimes_\nu \chi_\nu,
 \end{aligned} \tag{5.7}$$

where we have performed the derivative with respect to angles ψ_k and introduced the functions of one variable $\tilde{\Psi}_\mu$, given by

$$\tilde{\Psi}_\mu = \sum_k \Psi_k (-x_\mu^2)^{N-1-k}. \tag{5.8}$$

Let us concentrate now on the derivatives of the prefactor R . Using Eq. (A14) and relation (A13), we can bring the operator R to the front to get

$$\begin{aligned}
 M_j \psi &= i^j \exp\left(i \sum_k \Psi_k \psi_k\right) R \left[\sum_\mu \frac{\sqrt{|X_\mu|}}{\sqrt{V_\mu}} (-\iota_{(\mu)})^{N-\mu} \mathcal{B}_\mu^{(j)} \right. \\
 &\quad \times \left(\frac{\partial}{\partial x_\mu} + \frac{X'_\mu}{4X_\mu} + \frac{\tilde{\Psi}_\mu}{X_\mu} \iota_{(\mu)} + \frac{\varepsilon}{2x_\mu} \right) \sigma_{(\mu)} + \varepsilon \frac{i\sqrt{S}}{2} \\
 &\quad \left. \times \left(\mathcal{B}^{(j-1)} - \mathcal{B}^{(j)} \left(\frac{2\tilde{\Psi}_N}{c} + \sum_\mu \frac{\iota_{(\mu)}}{x_\mu} \right) \right) \gamma^0 \right] \otimes_\nu \chi_\nu.
 \end{aligned} \tag{5.9}$$

This expression is to be compared with

$$\mathcal{X}_j \psi = \mathcal{X}_j \exp\left(i \sum_k \Psi_k \psi_k\right) R \otimes_\nu \chi_\nu. \tag{5.10}$$

To simplify the following expressions, we introduce the functions $\tilde{\mathcal{X}}_\nu$ of a single variable x_ν :

$$\tilde{\mathcal{X}}_\nu = \sum_j (-i)^j \mathcal{X}_j (-\iota_{(\nu)} x_\nu)^{N-1-j}. \tag{5.11}$$

In odd dimensions, the constant \mathcal{X}_N , defined by $M_N \psi = \mathcal{X}_N \psi$, is not independent. In fact, using Eqs. (4.4), (5.1), and (5.2), we have

$$\mathcal{X}_N = \frac{i^{N+1}}{\sqrt{-c}} \Psi_N. \tag{5.12}$$

We are now ready to derive the differential equations for χ_ν , so that (5.2) is satisfied. We can cancel the common $\exp(i \sum_k \Psi_k \psi_k) R$ prefactor in (5.9) and (5.10) (in the coordinate domain which we consider, the operator R is

never zero when it is acting on any spinor), multiply both equations by $(-i)^j (-\iota_{(\nu)} x_\nu)^{N-1-j}$, and sum over j to obtain

$$\begin{aligned}
 \tilde{\mathcal{X}}_\nu \otimes_\kappa \chi_\kappa &= \left[\sqrt{|X_\nu|} (-\iota_{(\nu)})^{N-\nu} \left(\frac{\partial}{\partial x_\nu} + \frac{X'_\nu}{4X_\nu} + \frac{\tilde{\Psi}_\nu}{X_\nu} \iota_{(\nu)} \right. \right. \\
 &\quad \left. \left. + \frac{\varepsilon}{2x_\nu} \right) \sigma_{(\nu)} - \varepsilon \frac{i\sqrt{S}}{2x_\nu^2} \mathcal{B}^{(N)} \gamma^0 \right] \otimes_\kappa \chi_\kappa,
 \end{aligned} \tag{5.13}$$

where we have used the latter Eq. (4.22) and identities (A8) and (A9). Using further the formula

$$\gamma^0 \sqrt{S} = \frac{\sqrt{-c}}{\mathcal{B}^{(N)}}, \tag{5.14}$$

we can rewrite Eq. (5.13) as

$$\begin{aligned}
 &\left[\sqrt{|X_\nu|} (-\iota_{(\nu)})^{N-\nu} \left(\frac{\partial}{\partial x_\nu} + \frac{X'_\nu}{4X_\nu} + \frac{\tilde{\Psi}_\nu}{X_\nu} \iota_{(\nu)} + \frac{\varepsilon}{2x_\nu} \right) \sigma_{(\nu)} \right. \\
 &\quad \left. - \varepsilon \frac{i\sqrt{-c}}{2x_\nu^2} - \tilde{\mathcal{X}}_\nu \right] \otimes_\kappa \chi_\kappa = 0.
 \end{aligned} \tag{5.15}$$

We finally note that the operators act only on the χ_ν spinor, leaving invariant all the other spinors in the tensor product. So, we are left with the following ordinary differential equation for each spinor χ_ν :

$$\begin{aligned}
 &\left[\left(\frac{d}{dx_\nu} + \frac{X'_\nu}{4X_\nu} + \frac{\tilde{\Psi}_\nu}{X_\nu} \iota_{(\nu)} + \frac{\varepsilon}{2x_\nu} \right) \sigma_{(\nu)} \right. \\
 &\quad \left. - \frac{(-\iota_{(\nu)})^{N-\nu}}{\sqrt{|X_\nu|}} \left(\varepsilon \frac{i\sqrt{-c}}{2x_\nu^2} + \tilde{\mathcal{X}}_\nu \right) \right] \chi_\nu = 0.
 \end{aligned} \tag{5.16}$$

To make contact with the formalism of [18], we redefine χ_ν in an odd dimension by a suitable rescaling,

$$\tilde{\chi}_\nu = (x_\nu)^{\varepsilon/2} \chi_\nu. \tag{5.17}$$

Taking the \mathfrak{s} component of the spinorial equation (5.16), we then get

$$\begin{aligned}
 &\left(\frac{d}{dx_\nu} + \frac{X'_\nu}{4X_\nu} - \mathfrak{s} \frac{\tilde{\Psi}_\nu}{X_\nu} \right) \tilde{\chi}_\nu^{-\mathfrak{s}} - \frac{(-\mathfrak{s})^{N-\nu}}{\sqrt{|X_\nu|}} \\
 &\quad \times \left(\varepsilon \frac{i\sqrt{-c}}{2x_\nu^2} + \tilde{\mathcal{X}}_\nu \right) \tilde{\chi}_\nu^{\mathfrak{s}} = 0.
 \end{aligned} \tag{5.18}$$

For each ν , these are two coupled ordinary differential equations for components $\tilde{\chi}_\nu^+$ and $\tilde{\chi}_\nu^-$, which can be easily decoupled by substituting one into another.

It can be checked that these are equivalent to the differential equations given in [18] with a proper identification of coefficients. In particular, for the eigenvalue of the Dirac equation, the identification is $q_{N-1} = \mathcal{X}_0$, as expected.

VI. STANDARD SEPARABILITY

In this section, we shall comment on how to achieve the standard tensorial separability, without the prefactor R . For this purpose, it is first instructive to prove directly the commutativity of operators M_j . This will give us a hint how to “upgrade” our representation to achieve the standard tensorial separability.

A. Direct proof of commutativity

Let us start from the expression for M_j (4.28) and apply the identity (5.14) and (A12), to obtain

$$\begin{aligned} M_j &= i^j \sum_{\mu} \sqrt{Q_{\mu}} \mathcal{B}_{\mu}^{(j)} \left(\frac{\partial}{\partial x_{\mu}} + \frac{X'_{\mu}}{4X_{\mu}} + \frac{1}{2} \sum_{\nu \neq \mu} \frac{1}{x_{\mu} - \iota_{\langle \mu \nu \rangle} x_{\nu}} \right. \\ &\quad \left. - \frac{i \iota_{\langle \mu \rangle}}{X_{\mu}} \sum_k (-x_{\mu}^2)^{N-1-k} \frac{\partial}{\partial \psi_k} + \frac{\varepsilon}{2x_{\mu}} \right) \gamma^{\mu} \\ &\quad - \frac{1}{2} \varepsilon^{ij+1} \sqrt{-c} \sum_{\mu} \frac{\mathcal{B}_{\mu}^{(j)}}{V_{\mu}} \left(\frac{1}{x_{\mu}^2} + \frac{2}{ic} \frac{1}{\iota_{\langle \mu \rangle} x_{\mu}} \frac{\partial}{\partial \psi_N} \right). \end{aligned} \quad (6.1)$$

In order to prove commutativity of these operators, we introduce new “auxiliary” operators

$$\tilde{M}_j \equiv R^{-1} M_j R, \quad (6.2)$$

with R given by (5.4). Then, obviously, if

$$[\tilde{M}_j, \tilde{M}_k] = R^{-1} [M_j, M_k] R = 0, \quad (6.3)$$

the same is true for operators without tilde.

We calculate

$$\begin{aligned} M_j R &= i^j \sum_{\mu} \sqrt{Q_{\mu}} \mathcal{B}_{\mu}^{(j)} R R^{-1} \gamma^{\mu} R \left(\frac{\partial}{\partial x_{\mu}} + \frac{X'_{\mu}}{4X_{\mu}} + \frac{\varepsilon}{2x_{\mu}} \right. \\ &\quad \left. + \frac{i \iota_{\langle \mu \rangle}}{X_{\mu}} \sum_k (-x_{\mu}^2)^{N-1-k} \frac{\partial}{\partial \psi_k} \right) - \frac{1}{2} \varepsilon^{ij+1} \sqrt{-c} \\ &\quad \times \sum_{\mu} \frac{\mathcal{B}_{\mu}^{(j)}}{V_{\mu}} \left(\frac{1}{x_{\mu}^2} + \frac{2}{ic} \frac{1}{\iota_{\langle \mu \rangle} x_{\mu}} \frac{\partial}{\partial \psi_N} \right) R, \end{aligned} \quad (6.4)$$

where we have used (A14). The first R on the right-hand side can be brought to the front, whereas, for the product $R^{-1} \gamma^{\mu} R$, we use (A13). So, we get

$$\tilde{M}_j = i^j \sum_{\mu} \frac{\mathcal{B}_{\mu}^{(j)}}{V_{\mu}} \tilde{M}_{\mu}, \quad (6.5)$$

where the operators

$$\begin{aligned} \tilde{M}_{\mu} &= \sqrt{|X_{\mu}|} \left(\frac{\partial}{\partial x_{\mu}} + \frac{1}{4} \frac{X'_{\mu}}{X_{\mu}} + \frac{\varepsilon}{2x_{\mu}} \right. \\ &\quad \left. - \frac{i \iota_{\langle \mu \rangle}}{X_{\mu}} \sum_k (-x_{\mu}^2)^{N-1-k} \frac{\partial}{\partial \psi_k} \right) (-\iota_{\langle \mu \rangle})^{N-\mu} \sigma_{\langle \mu \rangle} \\ &\quad - \frac{i}{2} \varepsilon \sqrt{-c} \left(\frac{1}{x_{\mu}^2} + \frac{2}{ic} \frac{1}{\iota_{\langle \mu \rangle} x_{\mu}} \frac{\partial}{\partial \psi_N} \right) \end{aligned} \quad (6.6)$$

act only on spinor χ_{μ} , and hence $[\tilde{M}_{\mu}, \tilde{M}_{\nu}] = 0$. Using (4.22), we can invert the relation (6.5),

$$\tilde{M}_{\mu} = \sum_{j=0}^{N-1} (-i)^j (-\iota_{\langle \mu \rangle} x_{\mu})^{N-1-j} \tilde{M}_j. \quad (6.7)$$

Following now the procedure in [14], and using the trivial fact that $[\tilde{M}_{\mu}, (-\iota_{\langle \nu \rangle} x_{\nu})^{N-1-j}] = 0$, we establish that

$$\sum_{j,k=0}^{N-1} (-i)^{j+k} (-\iota_{\langle \mu \rangle} x_{\mu})^{N-1-j} (-\iota_{\langle \nu \rangle} x_{\nu})^{N-1-k} [\tilde{M}_j, \tilde{M}_k] = 0, \quad (6.8)$$

from which Eq. (6.3) follows.

B. R representation and standard separability

We have seen that the new operators \tilde{M}_j (6.2) possess a remarkable property—they can be expressed in the form (6.5), where the operators \tilde{M}_{μ} act only on the spinor χ_{μ} . Hence, such operators are directly related to standard tensorial separability. Indeed, a solution of

$$K_k \psi = i \Psi_k \psi, \quad \tilde{M}_j \psi = \mathcal{X}_j \psi \quad (6.9)$$

can be found in the standard tensorial separated form

$$\psi = \exp\left(i \sum_k \Psi_k \psi_k\right) \bigotimes_{\nu} \chi_{\nu}, \quad (6.10)$$

where χ_{ν} satisfy Eq. (5.16). This can be easily seen by calculating $\tilde{M}_{\mu} \psi$ while using Eq. (6.7). Note also that the R separability discussed in Sec. V is recovered by applying R on the left-hand side of (6.9).

Moreover, operators \tilde{M}_j are nothing else but operators (4.2) in the “ R representation” in which we take

$$\tilde{\gamma}^a = R^{-1} \gamma^a R, \quad (6.11)$$

with γ^a defined earlier.

VII. CONCLUSIONS

The Dirac equation in Kerr-NUT-(A)dS spacetimes in all dimensions possesses a truly remarkable property. Namely, its solution can be found by separating variables, and the resulting ordinary differential equations can be completely decoupled.

We have demonstrated that behind the separability stands a complete set of first-order mutually commuting operators that can be generated from the PCKY tensor, present in the spacetime geometry. These results directly generalize the corresponding results on separability of the Hamilton-Jacobi and Klein-Gordon equations and further establish the unique role which the PCKY tensor plays in determining the remarkable properties of the Kerr-NUT-(A)dS geometry in all dimensions.

A very important open question left for the future is whether the PCKY tensor is also intrinsically linked to other higher-spin perturbations. In particular, can the electromagnetic and gravitational perturbations in general rotating higher-dimensional Kerr-NUT-(A)dS spacetimes be decoupled and separated?

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APPENDIX

1. A spin connection

In even dimensions, the only nonzero connection (Ricci) coefficients with respect to the frame $E^\mu, E^{\hat{\mu}}$ are

$$\begin{aligned}
 \omega_{\mu\mu\nu} &= -\omega_{\nu\nu\mu} = \sqrt{\frac{X_\nu}{U_\nu}} \frac{x_\nu}{x_\nu^2 - x_\mu^2}, \\
 \omega_{\mu\hat{\mu}\hat{\nu}} &= -\omega_{\hat{\nu}\hat{\mu}\hat{\mu}} = \sqrt{\frac{X_\nu}{U_\nu}} \frac{x_\mu}{x_\nu^2 - x_\mu^2}, \\
 \omega_{\hat{\mu}\hat{\mu}\nu} &= -\omega_{\hat{\nu}\nu\hat{\mu}} = \sqrt{\frac{X_\nu}{U_\nu}} \frac{x_\nu}{x_\nu^2 - x_\mu^2}, \\
 \omega_{\hat{\mu}\hat{\nu}\hat{\mu}} &= -\omega_{\hat{\mu}\hat{\mu}\hat{\nu}} = \sqrt{\frac{X_\nu}{U_\nu}} \frac{x_\mu}{x_\nu^2 - x_\mu^2}, \\
 \omega_{\hat{\mu}\nu\hat{\nu}} &= -\omega_{\hat{\nu}\hat{\nu}\mu} = \sqrt{\frac{X_\mu}{U_\mu}} \frac{x_\nu}{x_\nu^2 - x_\mu^2}, \\
 \omega_{\hat{\mu}\hat{\mu}\mu} &= -\omega_{\hat{\mu}\mu\hat{\mu}} = \frac{1}{2} \sqrt{\frac{X_\mu}{U_\mu}} \frac{X'_\mu}{X_\mu} + \sqrt{\frac{X_\mu}{U_\mu}} \sum_{\nu \neq \mu} \frac{x_\mu}{x_\nu^2 - x_\mu^2}.
 \end{aligned} \tag{A1}$$

Here, indices μ and ν are different. In odd dimensions, the same Ricci coefficients apply, plus the following extra terms:

$$\begin{aligned}
 \omega_{\mu\hat{\mu}0} &= -\omega_{\mu0\hat{\mu}} = -\frac{\sqrt{S}}{x_\mu}, \\
 \omega_{\hat{\mu}\mu0} &= -\omega_{\hat{\mu}0\mu} = \frac{\sqrt{S}}{x_\mu}, \\
 \omega_{0\mu0} &= -\omega_{00\mu} = -\sqrt{\frac{X_\mu}{U_\mu}} \frac{1}{x_\mu}, \\
 \omega_{0\hat{\mu}\mu} &= -\omega_{0\mu\hat{\mu}} = -\frac{\sqrt{S}}{x_\mu}.
 \end{aligned} \tag{A2}$$

2. Useful identities

We generalize the definition of functions $\mathcal{B}_\mu^{(k)}$ and $\mathcal{B}^{(k)}$, (4.18), as follows:

$$\mathcal{B}_{\mu_1 \dots \mu_j}^{(k)} = \sum_{\substack{\nu_1 \dots \nu_k \\ \nu_i \leq \nu_k \\ \nu_i \neq \mu_1 \dots \mu_j}} \iota_{\langle \nu_1 \rangle} x_{\nu_1} \cdots \iota_{\langle \nu_k \rangle} x_{\nu_k}. \tag{A3}$$

Such functions obey

$$\mathcal{B}_{\mu_1 \dots \mu_j}^{(k)} = \mathcal{B}_{\mu_1 \dots \mu_j \nu}^{(k)} + \iota_{\langle \nu \rangle} x_\nu \mathcal{B}_{\mu_1 \dots \mu_j \nu}^{(k-1)}. \tag{A4}$$

Therefore, we can write

$$\begin{aligned}
 \sum_{\nu \neq \mu_1 \dots \mu_j} \mathcal{B}_{\mu_1 \dots \mu_j \nu}^{(k)} &= \sum_{\nu \neq \mu_1 \dots \mu_j} (\mathcal{B}_{\mu_1 \dots \mu_j}^{(k)} - \iota_{\langle \nu \rangle} x_\nu \mathcal{B}_{\mu_1 \dots \mu_j \nu}^{(k-1)}) \\
 &= (N - j - k) \mathcal{B}_{\mu_1 \dots \mu_j}^{(k)}.
 \end{aligned} \tag{A5}$$

As a direct consequence of Eqs. (A4) and (A5), we can derive the following important relations used in the main text:

$$\sum_{\mu} \mathcal{B}_\mu^{(j-1)} = (N - j + 1) \mathcal{B}^{(j-1)}, \tag{A6}$$

$$\sum_{\nu \neq \mu} \frac{x_\mu + \iota_{\langle \mu \nu \rangle} x_\nu}{x_\mu^2 - x_\nu^2} (\mathcal{B}_\nu^{(j)} - \mathcal{B}_\mu^{(j)}) = (N - j) \iota_{\langle \mu \rangle} \mathcal{B}_\mu^{(j-1)}, \tag{A7}$$

$$\sum_{j=0}^N \mathcal{B}^{(j)} (-\iota_{\langle \mu \rangle} x_\mu)^{N-1-j} = 0, \tag{A8}$$

$$\sum_{j=0}^N \mathcal{B}^{(j-1)} (-\iota_{\langle \mu \rangle} x_\mu)^{N-1-j} = -\frac{\mathcal{B}^{(N)}}{x_\mu^2}. \tag{A9}$$

Another important relation is

$$\sum_{\mu} \frac{\mathcal{B}_\mu^{(j)}}{\iota_{\langle \mu \rangle} x_\mu \mathcal{V}_\mu} = \frac{\mathcal{B}^{(j)}}{\mathcal{B}^{(N)}}, \tag{A10}$$

which, together with (A4) and the equality

$$\sum_{\mu} \frac{1}{x_\mu^2 \mathcal{V}_\mu} = \sum_{\mu} \frac{1}{(\iota_{\langle \mu \rangle} x_\mu)^2 \mathcal{V}_\mu} = \frac{1}{\mathcal{B}^{(N)}} \sum_{\mu} \frac{1}{\iota_{\langle \mu \rangle} x_\mu} \tag{A11}$$

mentioned in [18], can be used to prove that $\mathcal{B}^{(j-1)}$ can be expressed as

$$\mathcal{B}^{(j-1)} = \mathcal{B}^{(j)} \sum_{\mu} \frac{1}{\iota_{\langle\mu\rangle} x_{\mu}} - \mathcal{B}^{(N)} \sum_{\mu} \frac{\mathcal{B}_{\mu}^{(j)}}{\sqrt{V_{\mu} x_{\mu}^2}}. \quad (\text{A12})$$

Let us finally state two important relations including the R factor. Using the fact that $U_{\mu} = (\sqrt{V_{\mu}} \sigma_{\langle\mu\rangle})^2$, we can derive that

$$R^{-1} \gamma^{\mu} R = \frac{\sqrt{|U_{\mu}|}}{\sqrt{V_{\mu}}} (-\iota_{\langle\mu\rangle})^{N-\mu} \sigma_{\langle\mu\rangle}, \quad (\text{A13})$$

which is an operator analogue of Eq. (20) in [18]. For the derivative of the factor R , one gets

$$\begin{aligned} \left(\frac{\partial}{\partial x_{\mu}} \gamma^{\mu} R \right) &= \gamma^{\mu} \frac{\partial}{\partial x_{\mu}} R \\ &= \gamma^{\mu} \frac{\partial}{\partial x_{\mu}} \left(\prod_{\nu < \mu} (x_{\nu} + \iota_{\langle\mu\nu\rangle} x_{\nu})^{-1/2} \right) \\ &\quad \times \prod_{\nu < \nu} (x_{\mu} + \iota_{\langle\mu\nu\rangle} x_{\nu})^{-1/2} \\ &\quad \times \prod_{\substack{\kappa, \lambda \\ \kappa < \lambda; \kappa, \lambda \neq \mu}} (x_{\kappa} + \iota_{\langle\kappa\lambda\rangle} x_{\lambda})^{-1/2} \\ &= \left(-\frac{1}{2} \sum_{\substack{\nu \\ \nu \neq \mu}} \frac{1}{x_{\mu} - \iota_{\langle\mu\nu\rangle} x_{\nu}} \right) \gamma^{\mu} R. \end{aligned} \quad (\text{A14})$$

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