

Cosmological apparent and trapping horizons

Valerio Faraoni*

Physics Department and STAR Research Cluster, Bishop's University, 2600 College Street, Sherbrooke, Québec J1M 1Z7, Canada
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The dynamics of particle, event, and apparent horizons in Friedmann-Lemaître-Robertson-Walker space are discussed. The apparent horizon is trapping when the Ricci curvature is positive. This simple criterion coincides with the condition for the Kodama-Hayward apparent horizon temperature to be positive and also discriminates between the timelike and spacelike character of the apparent horizon. We discuss also the entropy of apparent cosmological horizons in extended theories of gravity and we use the generalized 2nd law to discard an exact solution of Brans-Dicke gravity as unphysical.

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I. INTRODUCTION

Black hole thermodynamics [1] links classical gravity and quantum mechanics and constitutes a major advancement of the theoretical physics of the 1970's. The discovery by Bekenstein [2] and Hawking [3,4] that black hole horizons have entropy and temperature associated with them allowed for the formulation of a complete thermodynamics of black holes. It is widely believed that formulating also a statistical mechanics to explain black hole thermodynamics in terms of microscopic degrees of freedom requires a fully developed theory of quantum gravity, which is not yet available.

Soon after the discovery of Hawking radiation [3,4], Gibbons and Hawking discovered that also the de Sitter cosmological event horizon is endowed with a temperature and an entropy, similar to the Schwarzschild horizon [5]. Later, it was realized that the notion of the black hole event horizon, which requires one to know the entire future development and causal structure of spacetime, is essentially useless for practical purposes. The teleological event horizon is not an easy quantity to compute: this feature has been emphasized by the development of numerical relativity. In the sophisticated simulations of black hole collapse available nowadays, outermost marginally trapped surfaces and *apparent* horizons are used as proxies for event horizons [6]. While the early literature on black holes and the development of black hole thermodynamics in the 1970s focused on static and stationary black holes, for which apparent and event horizons coincide, dynamical situations such as the intermediate stages of black hole collapse, black hole evaporation backreacting on its source, and black holes interacting with nontrivial environments (e.g., with another black hole or compact object, or with a cosmological background) require the generalization of the concept of event horizon to situations in which no timelike Killing vector is available. For this purpose, the concepts of apparent, trapping, isolated, dynamical, and slowly evolving horizons were developed (see [7–9] for reviews).

In addition to black hole horizons, also cosmological horizons have been the subject of intense scrutiny. The de Sitter event horizon considered by Gibbons and Hawking as a thermodynamical system [5] is static due to the high symmetry of de Sitter space, which admits a timelike Killing vector, and plays a role analogous to that of the Schwarzschild event horizon among black holes. For more general Friedmann-Lemaître-Robertson-Walker (FLRW) spaces, which do not admit such a Killing vector, the particle and event horizons are familiar from standard cosmology textbooks. However, they do not exist in all FLRW spaces and they do not seem suitable for formulating consistent thermodynamics ([10–15] and references therein). Instead, the FLRW *apparent* horizon, which always exists contrary to the event and particle horizons, seems a better candidate. In this paper, we reconsider our knowledge of this horizon and try to deepen our understanding of it. Specifically, we derive a simple criterion for the apparent horizon to be also a trapping horizon and we show that the Kodama-Hayward temperature, which is based on the Kodama vector playing the role of the timelike Killing vector outside the horizon, is positive if and only if the apparent horizon is trapping. The causal character of this surface is related to this criterion.

The thermodynamics of the FLRW apparent horizon has seen much interest recently, with many authors deriving the temperature of this horizon with the Hamilton-Jacobi variant of the Parikh-Wilczek “tunneling” approach [16]. However, different definitions of surface gravity can be applied to this calculation in order to define the energy of (scalar) particles, corresponding to a background notion of time, and these different prescriptions provide different notions of temperature. The Kodama-Hayward prescription seems to stand out among its competitors because the Kodama vector is associated with a conserved current even in the absence of a timelike Killing vector [17], a fact called the “Kodama miracle” [18] which leads to several interesting results. What is more, the Noether charge associated with the Kodama vector is the Misner-Sharp-Hernandez mass [19,20], which is almost universally adopted as the internal energy U in horizon

*vfaraoni@ubishops.ca

thermodynamics. The Misner-Sharp-Hernandez mass, defined in spherical symmetry, coincides with the Hawking quasilocal energy [21], and, if we insist on using it in thermodynamics, the use of the Kodama-Hayward surface gravity follows naturally (although this point may be considered debatable by some).

In the next section, we review background material while deriving new formulas useful in the study of apparent and trapping horizons. Section III discusses the various notions of horizons in FLRW space and their dynamics and elucidates the causal character of the apparent horizon. The following section raises a question neglected in the literature, namely, the condition under which the apparent horizon is also a trapping horizon. The simple criterion is that the Ricci scalar must be positive. We then show that the Kodama-Hayward temperature is positive when the apparent horizon is trapping. Sections V and VI contain discussions of the thermodynamics of cosmological horizons in general relativity (GR) and in extended theories of gravity and use the generalized 2nd law to reject as unphysical an exact solution of Brans-Dicke theory. Section VII contains the conclusions. We follow the notations of [22]. The speed of light c , reduced Planck constant \hbar , and Boltzmann constant K_B are set equal to unity; however, they are occasionally restored for better clarity.

II. BACKGROUND

The FLRW line element in comoving coordinates (t, r, θ, φ) is

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_{(2)}^2 \right), \quad (1)$$

where k is the curvature index, $a(t)$ is the scale factor, and $d\Omega_{(2)}^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on the unit two-sphere. Sometimes, different coordinates employing the areal radius $R(t, r) \equiv a(t)r$ are useful. Such coordinate systems include the pseudo-Painlevé-Gullstrand and Schwarzschild-like coordinates, which we introduce here for a general FLRW space.

Begin from the metric (1); using the areal radius R , this line element assumes the pseudo-Painlevé-Gullstrand form

$$ds^2 = - \left(1 - \frac{H^2 R^2}{1 - kR^2/a^2} \right) dt^2 - \frac{2HR}{1 - kR^2/a^2} dt dR + \frac{dR^2}{1 - kR^2/a^2} + R^2 d\Omega_{(2)}^2, \quad (2)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, and an overdot denotes differentiation with respect to the comoving time t . We use the word ‘‘pseudo’’ because the coefficient of dR^2 is not unity, as required for Painlevé-Gullstrand coordinates [23], and the spacelike surfaces $t = \text{constant}$ are not flat (unless $k = 0$), which is regarded as the essential property of Painlevé-Gullstrand coordinates [27].

To transform to the Schwarzschild-like form, one first introduces the new time T , defined by

$$dT = \frac{1}{F} (dt + \beta dR), \quad (3)$$

where F is a (generally nonunique) integrating factor satisfying

$$\frac{\partial}{\partial R} \left(\frac{1}{F} \right) = \frac{\partial}{\partial t} \left(\frac{\beta}{F} \right) \quad (4)$$

to guarantee that dT is a locally exact differential, while $\beta(t, R)$ is a function to be determined. Substituting $dt = FdT - \beta dR$ into the line element, one obtains

$$ds^2 = - \left(1 - \frac{H^2 R^2}{1 - kr^2} \right) F^2 dT^2 + \left[- \left(1 - \frac{H^2 R^2}{1 - kr^2} \right) \beta^2 + \frac{2HR\beta + 1}{1 - kr^2} \right] dR^2 + 2 \left(1 - \frac{H^2 R^2}{1 - kr^2} \right) F \beta dT dR - \frac{2HRF}{1 - kr^2} dT dR + R^2 d\Omega_{(2)}^2. \quad (5)$$

By choosing

$$\beta = \frac{HR}{1 - H^2 R^2 - kr^2}, \quad (6)$$

the cross term proportional to $dT dR$ is eliminated, and one obtains the FLRW line element in the Schwarzschild-like form [28]

$$ds^2 = - \left(1 - \frac{H^2 R^2}{1 - kR^2/a^2} \right) F^2 dT^2 + \frac{dR^2}{1 - kR^2/a^2 - H^2 R^2} + R^2 d\Omega_{(2)}^2, \quad (7)$$

where $F = F(T, R)$, a , and H are implicit functions of T .

Horizons in spherical symmetry are discussed in a clear and elegant way by Nielsen and Visser in [29] (see also [15]). These authors consider the most general spherically symmetric metric with a spherically symmetric spacetime slicing, which assumes the form (in Schwarzschild-like coordinates)

$$ds^2 = -e^{-2\phi(t,R)} \left[1 - \frac{2M(t,R)}{R} \right] dt^2 + \frac{dR^2}{1 - \frac{2M(t,R)}{R}} + R^2 d\Omega_{(2)}^2, \quad (8)$$

where $M(t, R)$ *a posteriori* turns out to be the Misner-Sharp-Hernandez mass [19,20]. This form is ultimately inspired by the Morris-Thorne wormhole metric [30]; it compromises between the latter and the widely used gauge $ds^2 = -A(t, R)dt^2 + B(t, R)dR^2 + R^2 d\Omega_{(2)}^2$ and is particularly convenient in the study of both static and time-varying black holes [29,31]. For the metric (7), we have

$$e^{-\phi} = \frac{F(T, R)}{\sqrt{1 - kR^2/a^2}} \quad (9)$$

and

$$1 - \frac{2M}{R} = 1 - \frac{kR^2}{a^2} - H^2R^2 = 1 - \frac{8\pi}{3}\rho R^2, \quad (10)$$

which is consistent with the well-known expression

$$M = \left(H^2 + \frac{k}{a^2}\right) \frac{R^3}{2} = \frac{4\pi}{3}R^3\rho \quad (11)$$

of the Misner-Sharp-Hernandez mass in FLRW space [21].

In nonspatially flat FLRW spaces, $k \neq 0$, the quantity $4\pi R^3/3$ is not the proper volume of a sphere of radius R , which is instead

$$V_{\text{proper}} = \int_0^{2\pi} d\varphi \int_0^\pi d\theta \int_0^r dr' \sqrt{g^{(3)}}, \quad (12)$$

where $g^{(3)} = \frac{a^6 r^4 \sin^2 \theta}{1 - kr^2}$ is the determinant of the restriction of the metric g_{ab} to the three-surfaces $r = \text{constant}$. Therefore,

$$V_{\text{proper}} = 4\pi a^3(t) \int_0^r \frac{dr' r'^2}{\sqrt{1 - kr'^2}} = 4\pi a^3(t) \int_0^\chi d\chi' f^2(\chi), \quad (13)$$

where χ is the hyperspherical radius, and

$$f(\chi) = r = \begin{cases} \sinh \chi & \text{if } k < 0, \\ \chi & \text{if } k = 0, \\ \sin \chi & \text{if } k > 0, \end{cases} \quad (14)$$

with $\chi = f^{-1}(r) = \int \frac{dr}{\sqrt{1 - kr^2}}$. Integration gives

$$V_{\text{proper}} = \begin{cases} 2\pi a^3(t)(r\sqrt{1+r^2} - \sinh^{-1} r) & \text{if } k = -1, \\ \frac{4\pi}{3} a^3(t) r^3 & \text{if } k = 0, \\ 2\pi a^3(t)(\sin^{-1} r - r\sqrt{1-r^2}) & \text{if } k = +1. \end{cases} \quad (15)$$

However, it turns out that only the ‘‘areal volume’’

$$V \equiv \frac{4\pi R^3}{3} \quad (16)$$

is used, as a consequence of the use of the Misner-Sharp-Hernandez mass, which is identified as the internal energy U in the thermodynamics of the apparent horizon.

The Misner-Sharp-Hernandez mass (11) of a sphere of radius R does not depend explicitly on the pressure P of the cosmic fluid. Its time derivative, instead, depends explicitly on P . Consider a sphere of proper radius $R = R_s(t)$; then, using $R \equiv ar$ and Eq. (37), one has

$$\dot{M} = 4\pi R_s^3 \left[\frac{\dot{R}_s}{R_s} \rho - H(P + \rho) \right]. \quad (17)$$

If the sphere is comoving, $R_s \propto a(t)$, then $\dot{R}_s/R_s = H$ and

$$\dot{M} = -4\pi H R_s^3 P; \quad (18)$$

in this case, \dot{M} depends explicitly on P but not on ρ . By taking the ratio of Eqs. (18) and (11), one also obtains, in GR,

$$\dot{M} + 3H \frac{P}{\rho} M = 0 \quad (\text{comoving sphere}). \quad (19)$$

However, for the thermodynamics of the apparent horizon (and of the event horizon, as well), the horizon is *not* a comoving surface.

It is now easy to locate the apparent horizon of a general FLRW space. In spherically symmetric spacetimes, the apparent horizon (existence, location, dynamics, surface gravity, etc.) can be studied by using the Misner-Sharp-Hernandez mass M [19,20], which coincides with the Hawking-Hayward quasilocal mass [32,33] for these spacetimes. The Misner-Sharp-Hernandez mass is only defined for spherically symmetric spacetimes. A spherically symmetric line element can always be written as

$$ds^2 = h_{ab} dx^a dx^b + R^2 d\Omega_{(2)}^2, \quad (20)$$

where $a, b = 1, 2$. The Misner-Sharp-Hernandez mass M is defined by [19,20]

$$1 - \frac{2M}{R} \equiv \nabla^c R \nabla_c R \quad (21)$$

or [34]

$$M = \frac{R}{2} (1 - h^{ab} \nabla_a R \nabla_b R), \quad (22)$$

an invariant quantity of the two-space normal to the two-spheres of symmetry. In a FLRW space, setting $g^{RR} = 0$ (equivalent to $h^{ab} \nabla_a R \nabla_b R = 0$ in Schwarzschild-like coordinates or to $R_{\text{AH}} = 2M$) yields the radius of the FLRW apparent horizon

$$R_{\text{AH}} = \frac{1}{\sqrt{H^2 + k/a^2}}. \quad (23)$$

Since the Misner-Sharp-Hernandez mass is defined quasi-locally [21], this derivation illustrates the quasilocal nature of the apparent horizon, as opposed to the global nature of the event and particle horizons.

Equations (21) and (7) yield

$$1 - \frac{2M}{R} = 1 - H^2 R^2 - \frac{kR^2}{a^2}, \quad (24)$$

and the Hamiltonian constraint (35) then implies that

$$M(R) = \frac{4\pi R^3}{3} \rho. \quad (25)$$

The Kodama vector is introduced as follows. Using the metric decomposition (20), let ϵ_{ab} be the volume form associated with the two-metric h_{ab} ; then, the Kodama vector is [17]

$$K^a \equiv \epsilon^{ab} \nabla_b R, \quad (26)$$

with $K^\theta = K^\varphi = 0$. The Kodama vector lies in the two-surface orthogonal to the two-spheres of symmetry and $K^a \nabla_a R = \epsilon^{ab} \nabla_a R \nabla_b R = 0$. In a static spacetime, the

Kodama vector is parallel (in general, not equal) to the timelike Killing vector. In the region in which it is timelike, the Kodama vector defines a class of preferred observers with four-velocity $u^a \equiv K^a/\sqrt{|K^c K_c|}$. It can be proven ([17], see [18] for a simplified proof) that the Kodama vector is divergence-free, $\nabla_a K^a = 0$, which has the consequence that the Kodama energy current $J^a \equiv G^{ab} K_b$ is covariantly conserved, $\nabla^a J_a = 0$, a remarkable property referred to as the ‘‘Kodama miracle’’ [18]. If the spherically symmetric metric is written in the gauge

$$ds^2 = -A(t, R)dt^2 + B(t, R)dR^2 + R^2 d\Omega_{(2)}^2, \quad (27)$$

then the Kodama vector assumes the simple form (e.g., [35])

$$K^a = \frac{1}{\sqrt{AB}} \left(\frac{\partial}{\partial t} \right)^a. \quad (28)$$

It is shown in [21] that the Noether charge associated with the Kodama current is the Misner-Sharp-Hernandez energy [19,20] of spacetime. The Hayward proposal for the horizon surface gravity in spherical symmetry [12] is based on the Kodama vector. This definition is unique because the Kodama vector is unique, and κ_{Kodama} agrees with the surface gravity on the horizon of a Reissner-Nordström black hole but not with other definitions of dynamical surface gravity. The Kodama-Hayward surface gravity can be written as [12]

$$\kappa_{\text{Kodama}} = \frac{1}{2} \square_{(h)} R = \frac{1}{2\sqrt{-h}} \partial_\mu (\sqrt{-h} h^{\mu\nu} \partial_\nu R). \quad (29)$$

The components of the Kodama vector in Schwarzschild-like coordinates are

$$K^\mu = \left(\frac{\sqrt{1 - kR^2/a^2}}{F}, 0, 0, 0 \right), \quad (30)$$

and its norm squared is

$$K_c K^c = - \left(1 - H^2 R^2 - \frac{kR^2}{a^2} \right) = - \left(1 - \frac{R^2}{R_{\text{AH}}^2} \right). \quad (31)$$

The Kodama vector is timelike ($K_c K^c < 0$) if $R < R_{\text{AH}}$, null if $R = R_{\text{AH}}$, and spacelike ($K_c K^c > 0$) outside the apparent horizon $R > R_{\text{AH}}$.

The components of the Kodama vector in pseudo-Painlevé-Gullstrand coordinates are

$$K^\mu = (\sqrt{1 - kR^2/a^2}, 0, 0, 0), \quad (32)$$

while, in comoving coordinates, they are

$$K^\mu = (\sqrt{1 - kr^2}, -Hr\sqrt{1 - kr^2}, 0, 0), \quad (33)$$

with $K^c K_c = -(1 - kr^2 - \dot{a}^2 r^2) = 1 - 2M/R$ (e.g., [36]).

In GR, if the FLRW universe is sourced by a perfect fluid with energy-momentum tensor

$$T_{ab} = (P + \rho)u_a u_b + P g_{ab}, \quad (34)$$

where ρ , P , and u^a are the energy density, pressure, and four-velocity field of the fluid, respectively, one has

$$H^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2}, \quad (35)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3P). \quad (36)$$

The covariant conservation equation $\nabla^b T_{ab} = 0$ yields the energy conservation equation

$$\dot{\rho} + 3H(P + \rho) = 0, \quad (37)$$

which is not independent of Eqs. (35) and (36) and can be derived from them. Another useful relation following from these equations is

$$\dot{H} = -4\pi G(P + \rho) + \frac{k}{a^2}. \quad (38)$$

Let $t = 0$ denote the big bang singularity (in the cases in which it is present). All comoving observers whose world-lines have u^a as tangent are equivalent, and, therefore, the following considerations apply to any of them, although we refer explicitly to a comoving observer located at $r = 0$.

III. FLRW HORIZONS AND THEIR DYNAMICS

Two horizons of FLRW space are familiar from standard cosmology textbooks: the particle and the event horizons [37]. The *particle horizon* [37] at time t is a sphere centered on the comoving observer at $r = 0$ and with radius

$$R_{\text{PH}}(t) = a(t) \int_0^t \frac{dt'}{a(t')}. \quad (39)$$

The particle horizon contains every particle signal that has reached the observer between the time of the big bang $t = 0$ and the time t [38]. For particles travelling radially to the observer at light speed, it is $ds = 0$ and $d\Omega_{(2)} = 0$. The line element can be written using hyperspherical coordinates,

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + f^2(\chi)d\Omega_{(2)}^2]. \quad (40)$$

Along radial null geodesics, $d\chi = -dt/a$, and the infinitesimal proper radius is $a(t)d\chi$. Integrating between the emission of a light signal at χ_e at time t_e and its detection at $\chi = 0$ at time t , one obtains

$$\int_{\chi_e}^0 d\chi = - \int_{t_e}^t \frac{dt'}{a(t')}, \quad (41)$$

and, using $\chi_e = \int_0^{\chi_e} d\chi = - \int_0^{\chi_e} d\chi$, we obtain [40]

$$\chi_e = \int_{t_e}^t \frac{dt'}{a(t')}. \quad (42)$$

The physical (proper) radius R is obtained by multiplication by the scale factor [41],

$$R_e = a(t) \int_{t_e}^t \frac{dt'}{a(t')}. \quad (43)$$

Now, take the limit $t_e \rightarrow 0^+$:

- (i) if the integral $\int_0^t \frac{dt'}{a(t')}$ diverges, it is possible for the observer at $r = 0$ to receive all the light signals emitted at sufficiently early times from any point in the universe. The maximal volume that can be causally connected to the observer at time t is infinite.
- (ii) if the integral $\int_0^t \frac{dt'}{a(t')}$ is finite, the observer at $r = 0$ receives, at time t , only the light signals started within the sphere $r \leq \int_0^t \frac{dt'}{a(t')}$.

The physical (proper) radius of the particle horizon is, therefore, given by Eq. (39). At a given time t , the particle horizon is the boundary between the worldlines that can be seen by the observer and those (“beyond the horizon”) which cannot be seen. This boundary hides events which cannot be known by that observer at time t and it evolves with time. The particle horizon is the horizon commonly studied in inflationary cosmology.

The particle and event horizons depend on the observer: contrary to the event horizon of the Schwarzschild black hole, different comoving observers in FLRW space will see event horizons located at different places. Another difference with respect to a black hole horizon is that the observer is located *inside* the event horizon and cannot be reached by signals sent from the outside.

The cosmological particle horizon is a null surface. This statement is obvious from the fact that the event horizon is a causal boundary and is generated by the null geodesics which barely fail to reach the observer; it can also be checked explicitly. Using hyperspherical coordinates (t, χ) , the equation of the particle horizon is

$$\mathcal{F}(t, \chi) \equiv \chi - \int_0^t \frac{dt'}{a(t')} = 0. \quad (44)$$

The normal to this surface has components

$$N_\mu = \nabla_\mu \mathcal{F}|_{\text{PH}} = \delta_{\mu 1} - \frac{\delta_{\mu 0}}{a}, \quad (45)$$

and it is straightforward to see that $N^a N_a = 0$.

The particle horizon evolves according to the equation (e.g., [42])

$$\dot{R}_{\text{PH}} = HR_{\text{PH}} + 1, \quad (46)$$

which is obtained by differentiating Eq. (39). In an expanding universe with a particle horizon, it is $\dot{R}_{\text{PH}} > 0$, which means that more and more signals emitted between the big bang and time t reach the observer as time progresses. If $R_{\text{PH}}(t)$ does not diverge as $t \rightarrow t_{\text{max}}$, then there will always be a region inaccessible to the comoving observers.

The acceleration of the particle horizon is

$$\ddot{R}_{\text{PH}} = \frac{\ddot{a}}{a} R_{\text{PH}} + H = -\frac{4\pi}{3}(\rho + 3P)R_{\text{PH}} + H. \quad (47)$$

Let us turn now our attention to the *event horizon*. Consider all the events which can be seen by the comoving observer at $r = 0$ between time t and future infinity $t = +\infty$ (in a closed universe which recollapses, or in a big rip universe which ends at a finite time, substitute $+\infty$, with the time t_{max} corresponding to the maximal expansion or the big rip, respectively). The comoving radius of the region which can be seen by this observer is

$$\chi_{\text{EH}} = \int_t^{+\infty} \frac{dt'}{a(t')}; \quad (48)$$

if this integral diverges as the upper limit of integration goes to infinity or to t_{max} , it is said that there is no event horizon in this FLRW space, and events arbitrarily far away can eventually be seen by the observer by waiting a sufficiently long time. If the integral converges, there is an event horizon: events beyond r_{EH} will never be known to the observer [37]. The physical (proper) radius of the event horizon is

$$R_{\text{EH}}(t) = a(t) \int_t^{+\infty} \frac{dt'}{a(t')}. \quad (49)$$

In short, the event horizon can be said to be the “complement” of the particle horizon [39]; it is the (proper) distance to the most distant event that the observer will ever see. Clearly, in order to define the event horizon, one must know the entire future history of the universe from time t to infinity, and the event horizon is defined *globally*, not locally.

The cosmological event horizon is a null surface. Again, the statement follows from the fact that the event horizon is a causal boundary. To check explicitly, use the equation of the event horizon in comoving coordinates

$$\mathcal{F}(t, \chi) \equiv \chi - \int_t^{t_{\text{max}}} \frac{dt'}{a(t')} = 0; \quad (50)$$

the normal to this surface has components

$$N_\mu = \nabla_\mu \mathcal{F}|_{\text{EH}} = \delta_{\mu 1} - \frac{\delta_{\mu 0}}{a}, \quad (51)$$

and it is easy to see that $N^a N_a = 0$.

The event horizon evolves according to the equation [42–44]

$$\dot{R}_{\text{EH}} = HR_{\text{EH}} - 1, \quad (52)$$

which is obtained by differentiating Eq. (49). The acceleration of the event horizon is also straightforward to derive,

$$\ddot{R}_{\text{EH}} = (\dot{H} + H^2)R_{\text{EH}} - H. \quad (53)$$

The event horizon does not exist in every FLRW space. To wit, consider a spatially flat FLRW universe sourced by a perfect fluid with equation of state $P = w\rho$ and $w = \text{const} > -1$; if $w \geq -1/3$ (i.e., in GR, for a decelerating universe), there is no event horizon because

$$a(t) = a_0 t^{[2/3(w+1)]}, \quad (54)$$

and the event horizon has radius

$$R_{\text{EH}} = t^{[2/3(w+1)]} \left[\frac{3(w+1)}{3w+1} t^{[(3w+1)/3(w+1)]} \right]_t^{+\infty}. \quad (55)$$

If $w > -1/3$, the exponent $\frac{3w+1}{3(w+1)}$ is positive, and the integral diverges: there is no event horizon in this case. Indeed, the existence of cosmological event horizons seems to require the violation of the strong energy condition in at least some region of spacetime [45]. We can state that, in GR, with a perfect fluid, the event horizon exists only for accelerated universes with $P < -\rho/3$.

The literature sometimes refers to a ‘‘Hubble horizon’’ of FLRW space with radius

$$R_H \equiv \frac{1}{H}. \quad (56)$$

This quantity only provides the order of magnitude of the radius of curvature of a FLRW space and is used as an estimate of the radius of the event horizon during inflation, when the universe is close to a de Sitter space [46]. The Hubble horizon coincides with the apparent horizon for spatially flat universes [see Eq. (66) below] and with the event horizon of de Sitter space. However, this concept does not add to the discussion of the various types of FLRW horizons and it seems unnecessary.

Let us consider now the *apparent horizon*, which depends on the spacetime slicing (this feature is illustrated by the fact that it is possible to find nonspherical slicings of the Schwarzschild spacetime without any apparent horizon [47,48]). In a FLRW spacetime, it is natural to use a slicing with hypersurfaces of homogeneity and isotropy (surfaces of constant comoving time). FLRW space is spherically symmetric about every point of space, and the outgoing and ingoing radial null geodesics have tangent fields with comoving components

$$\begin{aligned} l^\mu &= \left(1, \frac{\sqrt{1-kr^2}}{a(t)}, 0, 0 \right), \\ n^\mu &= \left(1, -\frac{\sqrt{1-kr^2}}{a(t)}, 0, 0 \right), \end{aligned} \quad (57)$$

respectively, as is immediately obtained by setting $p_c p^c = 0$ for the tangents. There is freedom to rescale a null vector by an arbitrary constant (which must be positive if we want to keep this vector future-oriented). The choice (57) implies that $l^c n_c = -2$. The more common normalization $l^c n_c = -1$ is obtained by dividing both l^a and n^a by $\sqrt{2}$.

The expansions of the null geodesic congruences are computed using the equation

$$\theta_l = \left[g^{ab} + \frac{l^a n^b + n^a l^b}{(-n^c l^d g_{cd})} \right] \nabla_a l_b. \quad (58)$$

Computing first

$$\nabla_c l^c = 3H + \frac{2}{ar} \sqrt{1-kr^2}, \quad (59)$$

$$\nabla_c n^c = 3H - \frac{2}{ar} \sqrt{1-kr^2} \quad (60)$$

and using

$$\begin{aligned} \sqrt{-g} &= \frac{a^3 r^2 \sin^2 \theta}{\sqrt{1-kr^2}}, & g_{cd} l^c n^d &= -2, & \Gamma_{00}^c &= 0, \\ \Gamma_{01}^c &= \Gamma_{10}^c = H \delta^{c1}, & \Gamma_{11}^c &= \frac{kr \delta^{c1} + a \dot{a} \delta^{c0}}{1-kr^2}, \end{aligned}$$

the result is [49]

$$\theta_l = \frac{2(\dot{a}r + \sqrt{1-kr^2})}{ar} = 2 \left(H + \frac{1}{R} \sqrt{1 - \frac{kR^2}{a^2}} \right), \quad (61)$$

$$\theta_n = \frac{2(\dot{a}r - \sqrt{1-kr^2})}{ar} = 2 \left(H - \frac{1}{R} \sqrt{1 - \frac{kR^2}{a^2}} \right). \quad (62)$$

Following [50], the apparent horizon is a surface defined by the conditions on the time slicings

$$\theta_l > 0, \quad (63)$$

$$\theta_n = 0 \quad (64)$$

and is located at

$$r_{\text{AH}} = \frac{1}{\sqrt{\dot{a}^2 + k}} \quad (65)$$

or

$$R_{\text{AH}}(t) = \frac{1}{\sqrt{H^2 + k/a^2}} \quad (66)$$

in terms of the proper radius $R \equiv ar$. The apparent horizon is defined locally using null geodesic congruences and their expansions, and there is no reference to the global causal structure.

Looking at Eqs. (61) and (62), or at their product

$$\theta_l \theta_n = \frac{4}{R^2} \left(\frac{R^2}{R_{\text{AH}}^2} - 1 \right), \quad (67)$$

it is clear that, when $R > R_{\text{AH}}$, it is $\theta_l > 0$ and $\theta_n > 0$, while the region $0 \leq R < R_{\text{AH}}$ has $\theta_l > 0$ and $\theta_n < 0$ (radial null rays coming from the region outside the horizon will not cross it and reach the observer).

For a spatially flat universe, the radius of the apparent horizon R_{AH} coincides with the Hubble radius H^{-1} , while, for a positively curved ($k > 0$) universe, R_{AH} is smaller than the Hubble radius and is larger for an open ($k < 0$) universe. In GR, the Hamiltonian constraint (35) guarantees that the argument of the square root in Eq. (66) is positive for positive densities ρ . The apparent horizon exists in all FLRW spaces.

In general, the apparent horizon is not a null surface, contrary to the event and particle horizons. The equation of the apparent horizon in comoving coordinates is

$$\mathcal{F}(t, r) = a(t)r - \frac{1}{\sqrt{H^2 + k/a^2}} = 0. \quad (68)$$

The normal has components

$$\begin{aligned} N_\mu &= \nabla_\mu \mathcal{F}|_{\text{AH}} = \left\{ \left[\dot{a}r + \frac{H(\dot{H} - k/a^2)}{(H^2 + k/a^2)^{3/2}} \right] \delta_{\mu 0} + a \delta_{\mu 1} \right\}_{\text{AH}} \\ &= HR_{\text{AH}} \left[1 + \left(\dot{H} - \frac{k}{a^2} \right) R_{\text{AH}}^2 \right] \delta_{\mu 0} + a \delta_{\mu 1} \\ &= HR_{\text{AH}}^3 \frac{\ddot{a}}{a} \delta_{\mu 0} + a \delta_{\mu 1}. \end{aligned} \quad (69)$$

In GR, with a perfect fluid, with equation of state $P = w\rho$, Eqs. (35), (36), and (66) yield

$$N_\mu = -\frac{(3w+1)}{2} HR_{\text{AH}} \delta_{\mu 0} + a \delta_{\mu 1}. \quad (70)$$

The norm squared of the normal is

$$\begin{aligned} N^a N_a &= 1 - kr_{\text{AH}}^2 - \frac{H^2(\dot{H} + H^2)^2}{(H^2 + k/a^2)^3} \\ &= H^2 R_{\text{AH}}^2 \left[1 - \left(\frac{\ddot{a}}{a} \right)^2 R_{\text{AH}}^4 \right] \\ &= \frac{3H^2 R_{\text{AH}}^2}{4\rho^2} (\rho + P)(\rho - 3P) \end{aligned} \quad (71)$$

$$= H^2 R_{\text{AH}}^2 (1 - q^2 H^4 R_{\text{AH}}^4), \quad (72)$$

where $q \equiv -\ddot{a}a/\dot{a}^2$ is the deceleration parameter. The horizon is null if and only if $P = -\rho$ or $P = \rho/3$. In GR, with a perfect fluid, the Hamiltonian constraint (35) yields

$$H^2 R_{\text{AH}}^2 = \left(1 + \frac{k}{a^2 H^2} \right)^{-1} = \left(\frac{8\pi G}{3H^2} \rho \right)^{-1} \equiv \frac{\rho_c}{\rho} \equiv \Omega^{-1}, \quad (73)$$

where $\rho_c \equiv \frac{3H^2}{8\pi G}$ is the critical density, and $\Omega \equiv \rho/\rho_c$ is the density parameter, and one obtains

$$N^a N_a = -\frac{3}{4} (w+1)(3w-1) H^2 R_{\text{AH}}^2 = \frac{\Omega^2 - q^2}{\Omega^3}. \quad (74)$$

Equation (74) establishes that:

- (i) if $-1 < w < 1/3$, then $N^c N_c > 0$, and the apparent horizon is *timelike*. For a $k=0$ universe in Einstein's theory, this condition corresponds to $\dot{H} < 0$.
- (ii) If $w = -1$ or $w = 1/3$, then $N^c N_c = 0$, and the apparent horizon is *null* (de Sitter space, which has $\dot{H} = 0$ and $q = -1$, falls into this category but it is not the only space with these properties).
- (iii) If $w < -1$ or $w > 1/3$, then $N^c N_c < 0$, the normal is timelike, and the apparent horizon is *spacelike*. In Einstein's theory, with $k=0$ and a perfect fluid as the source, $w < -1$ corresponds to $\dot{H} > 0$

(“superacceleration”). This is the case of big rip universes and of a phantom fluid which violates the weak energy condition.

The black hole dynamical horizons considered in the literature are usually required to be *spacelike* [9]. However, cosmological horizons can be timelike. In GR, the radius of the apparent horizon can be written as $R_{\text{AH}} = (\sqrt{\Omega}|H|)^{-1}$ in terms of the density parameter Ω by using Eq. (73).

The apparent horizon evolves according to the equation [42,51–54]

$$\dot{R}_{\text{AH}} = HR_{\text{AH}}^3 \left(\frac{k}{a^2} - \dot{H} \right) = 4\pi HR_{\text{AH}}^3 (P + \rho), \quad (75)$$

as is easy to check by differentiating Eq. (66) with respect to t . In GR, with a perfect fluid as a source, the only way to obtain a stationary apparent horizon is when $P = -\rho$. For de Sitter space, Eq. (75) reduces to $\dot{R}_{\text{AH}} = 0$, consistent with $R_H = H^{-1}$ and $H = \text{const}$. For nonspatially flat universes, the equation of state $P = -\rho$ produces other solutions. For example, for $k = -1$ and a cosmological constant $\Lambda > 0$ as the only source of gravity, the scale factor

$$a(t) = \sqrt{\frac{3}{\Lambda}} \sinh \left(\sqrt{\frac{\Lambda}{3}} t \right) \quad (76)$$

is a solution of the Einstein-Friedmann equations. The radius of the event horizon has the time dependence

$$R_{\text{EH}}(t) = \sqrt{\frac{3}{\Lambda}} \sinh \left(\sqrt{\frac{\Lambda}{3}} t \right) \left| \ln \left[\tanh \left(\sqrt{\frac{\Lambda}{3}} \frac{t}{2} \right) \right] \right|. \quad (77)$$

The apparent horizon, instead, has constant radius $R_{\text{AH}} = \sqrt{3/\Lambda}$.

As another example, consider, for $k = +1$ and cosmological constant $\Lambda > 0$, the scale factor

$$a(t) = \sqrt{\frac{3}{\Lambda}} \cosh \left(\sqrt{\frac{\Lambda}{3}} t \right); \quad (78)$$

the event horizon has radius

$$R_{\text{EH}}(t) = \sqrt{\frac{3}{\Lambda}} \cosh \left(\sqrt{\frac{\Lambda}{3}} t \right) \cdot \left[\frac{\pi}{2} + n\pi - \tan^{-1} \left(\sinh \left(\sqrt{\frac{\Lambda}{3}} t \right) \right) \right], \quad (79)$$

where $n = 0, \pm 1, \pm 2, \dots$. The multiple possible values of n correspond to the infinite possible branches which one can consider when inverting the tangent function and to the fact that, in a closed universe, light rays can travel multiple times around the universe. In this situation, it is problematic to regard the event horizon as a true horizon [55]. The apparent horizon has constant radius $R_{\text{AH}} = \sqrt{3/\Lambda}$, according to the fact that $\rho_\Lambda + P_\Lambda = 0$ in Eq. (75) [56].

In a $k = 0$ FLRW universe, with a perfect fluid and constant equation of state $P = w\rho$ and $-1 < w < -1/3$ (accelerating but not superaccelerating universe), the event horizon is always outside the apparent horizon and is, therefore, unobservable [14,57].

Let us summarize the dynamical evolution of the FLRW horizons and compare their evolutionary laws. The first question to ask is whether these horizons are comoving: they almost never are. The difference between the expansion rate of a horizon \dot{R}/R and that of the expanding matter H is, for the particle, event, and apparent horizons,

$$\frac{\dot{R}_{\text{PH}}}{R_{\text{PH}}} - H = \frac{1}{R_{\text{PH}}}, \quad (80)$$

$$\frac{\dot{R}_{\text{EH}}}{R_{\text{EH}}} - H = -\frac{1}{R_{\text{EH}}}, \quad (81)$$

$$\frac{\dot{R}_{\text{AH}}}{R_{\text{AH}}} - H = H \left[\frac{(\frac{k}{a^2} - \dot{H})}{H^2 + \frac{k}{a^2}} - 1 \right] = -\left(\frac{\ddot{a}}{a}H\right)R_{\text{AH}}^2 = \frac{(3w+1)H}{2}, \quad (82)$$

respectively. Taking into consideration only expanding FLRW universes ($H > 0$), when it exists, the particle horizon always expands faster than comoving. The event horizon (which only exists for accelerated universes) always expands slower than comoving. The apparent horizon expands faster than comoving for decelerated universes ($\ddot{a} < 0$); slower than comoving for accelerated universes ($\ddot{a} > 0$); and comoving for coasting universes [$a(t) \propto t$].

An even simpler way of looking at the evolution is by using the comoving radius of the horizon: if this radius is constant, then the horizon is comoving. We have

TABLE I. Causal character of the FLRW cosmological horizons.

Horizon	Causal character
Event horizon	Null
Particle horizon	Null
Apparent horizon	Timelike if $-\rho < P < \rho/3$, Null if $P = -\rho$ or $\rho/3$, Spacelike if $P < -\rho$ or $P > \rho/3$
de Sitter horizon	Null

TABLE II. FLRW cosmological horizons and their dynamical behavior.

Horizon	Location	Velocity	Acceleration
Event horizon	$R_{\text{EH}} = a(t) \int_t^{+\infty} \frac{dt'}{a(t')}$	$\dot{R}_{\text{EH}} = HR_{\text{EH}} - 1$	$\ddot{R}_{\text{EH}} = \frac{\ddot{a}}{a}R_{\text{EH}} - H$
Particle horizon	$R_{\text{PH}} = a(t) \int_0^t \frac{dt'}{a(t')}$	$\dot{R}_{\text{PH}} = HR_{\text{PH}} + 1$	$\ddot{R}_{\text{PH}} = \frac{\ddot{a}}{a}R_{\text{PH}} + H$
Apparent horizon	$R_{\text{AH}} = \frac{1}{\sqrt{H^2 + k/a^2}}$	$\dot{R}_{\text{AH}} = HR_{\text{AH}}^3 \left(\frac{k}{a^2} - 1\right)$ $= 4\pi HR_{\text{AH}}^2 (P + \rho)$	$\ddot{R}_{\text{AH}} = R_{\text{AH}}^3 \cdot \left[\frac{k}{a^2}(\dot{H} - H^2) - \dot{H}\right]$ $+ 3H^2 R_{\text{AH}}^2 \cdot \left(\frac{k}{a^2} - 1\right)$
de Sitter horizon	$R_{\text{dS}} = H^{-1}$	$\dot{R}_{\text{dS}} = 0$	$\ddot{R}_{\text{dS}} = 0$

$$\dot{r}_{\text{PH}} = \frac{1}{a} > 0, \quad (83)$$

$$\dot{r}_{\text{EH}} = -\frac{1}{a} < 0, \quad (84)$$

$$\dot{r}_{\text{AH}} = -\frac{\dot{a}\ddot{a}}{(\dot{a}^2 + k)^{3/2}}, \quad (85)$$

respectively. The causal character and the dynamics of the various FLRW horizons are summarized in Tables I and II.

IV. TRAPPING HORIZON OF FLRW SPACE

Let us now ask the question: When is the FLRW apparent horizon also a trapping horizon? According to Hayward's definition, when $\mathcal{L}_l\theta_n > 0$, which gives the coordinate- (but not slicing) invariant criterion,

$$\mathcal{L}_l\theta_n = \frac{R^a_a}{3} > 0, \quad (86)$$

where R^a_a is the Ricci scalar of FLRW space, and \mathcal{L}_l is the Lie derivative along l^a . In fact, using Eqs. (57) and (62), we have

$$\begin{aligned} \mathcal{L}_l\theta_n &= l^a\nabla_a\theta_n = l^a\partial_a\theta_n \\ &= 2\left(\partial_t + \frac{\sqrt{1-kr^2}}{a}\partial_r\right)\left(H - \frac{\sqrt{1-kr^2}}{ar}\right) \\ &= \frac{2}{R^2}\left(\dot{H}R^2 + HR\sqrt{1-\frac{kR^2}{a^2}} + 1\right). \end{aligned}$$

At the apparent horizon $R = R_{\text{AH}}$, it is

$$\begin{aligned} \mathcal{L}_l\theta_n|_{\text{AH}} &= 2\left(H^2 + \frac{k}{a^2}\right)\left[\frac{\dot{H}}{H^2 + \frac{k}{a^2}} + \frac{H\sqrt{1-\frac{k}{a^2(H^2+k/a^2)}}}{\sqrt{H^2+k/a^2}} + 1\right] \\ &= 2\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right) = \frac{R^a_a}{3}. \end{aligned}$$

This result is independent of the field equations. If we assume Einstein's theory and a perfect fluid as the sole source of gravity, we obtain

$$\mathcal{L}_l\theta_n|_{\text{AH}} = \frac{8\pi G}{3}(\rho - 3P), \quad (87)$$

and, therefore, *the apparent horizon is also a trapping horizon if $R^a_a > 0$ (equivalent to $P < \rho/3$ in GR with a perfect fluid).*

Note that, in a radiation-dominated universe, which is decelerated, the event horizon does not exist.

V. THERMODYNAMICS OF COSMOLOGICAL HORIZONS IN GR

Originally developed for static or stationary *event* horizons, black hole thermodynamics has now been extended to apparent, trapping, isolated, dynamical, and slowly evolving horizons [7–9]. Similarly, the cosmological thermodynamics associated with cosmological horizons has been extended from the static de Sitter event horizon [5] to FLRW dynamical *apparent* horizons.

The thermodynamic formulas valid for the de Sitter event (and apparent) horizon are generalized to the nonstatic apparent horizon of FLRW space. The apparent horizon is argued to be a causal horizon associated with gravitational temperature, entropy, and surface gravity in dynamical spacetimes ([10–15] and references therein), and these arguments apply also to cosmological horizons. That thermodynamics are ill-defined for the event horizon of FLRW space was argued in [14,55,57,58]. The Hawking radiation of the FLRW apparent horizon was computed in [59,60]. The authors of [24,26] rederived it using the Hamilton-Jacobi method [29,61,62] in the Parikh-Wilczek approach originally developed for black hole horizons [16]. In this context, the particle emission rate in the WKB approximation is the tunneling probability for the classically forbidden trajectories from inside to outside the horizon,

$$\Gamma \sim \exp\left(-\frac{2\text{Im}(I)}{\hbar}\right) \simeq \exp\left(-\frac{\hbar\omega}{K_B T}\right), \quad (88)$$

where I is the Euclideanized action with imaginary part $\text{Im}(I)$, ω is the angular frequency of the radiated quanta (taken, for simplicity, to be those of a massless scalar field, which is the simplest field to perform Hawking effect calculations), and the Hawking temperature is read off the expression of the Boltzmann factor, $K_B T = \frac{\hbar\omega}{2\text{Im}(I)}$. The particle energy $\hbar\omega$ is defined in an invariant way as $\omega = -K^a \nabla_a I$, where K^a is the Kodama vector, and the action I satisfies the Hamilton-Jacobi equation

$$h^{ab} \nabla_a I \nabla_b I = 0. \quad (89)$$

Although the definition of energy is coordinate-invariant, it depends on the choice of time, here defined as the Kodama time.

A review of the thermodynamical properties of the FLRW apparent horizon, as well as the computation of the Kodama vector, Kodama-Hayward surface gravity, and Hawking temperature in various coordinate systems, are given in Ref. [36]. The Kodama-Hayward temperature of the FLRW apparent horizon is given by

$$\begin{aligned} K_B T &= \left(\frac{\hbar}{c}\right) \frac{R_{\text{AH}}(H^2 + \frac{\dot{H}}{2} + \frac{k}{2a^2})}{2\pi} = \left(\frac{\hbar}{24\pi c}\right) R_{\text{AH}} R^a{}_a \\ &= \left(\frac{\hbar G}{c}\right) \frac{R_{\text{AH}}}{3} (\rho - 3P). \end{aligned} \quad (90)$$

The expression of the temperature depends on the choice of surface gravity κ , since $T = |\kappa|/2\pi$ in geometrized units, and there are several inequivalent prescriptions for this quantity (see [63,64] for reviews). The choice of κ giving the temperature reported here is the Kodama-Hayward prescription (29) [36]. In fact, this equation yields

$$\kappa_{\text{Kodama}} = -\frac{R_{\text{AH}}}{2} \left(2H^2 + \dot{H} + \frac{k}{a^2}\right) = -\frac{R_{\text{AH}}}{2} R^a{}_a, \quad (91)$$

as can be quickly assessed by using comoving coordinates and the decomposition (20) of the metric, where $h_{ab} = \text{diag}(-1, \frac{a^2}{1-kr^2})$. The entropy of the FLRW apparent horizon is

$$S_{\text{AH}} = \left(\frac{K_B c^3}{\hbar G}\right) \frac{A_{\text{AH}}}{4} = \left(\frac{K_B c^3}{\hbar G}\right) \frac{\pi}{H^2 + k/a^2}, \quad (92)$$

where

$$A_{\text{AH}} = 4\pi R_{\text{AH}}^2 = \frac{4\pi}{H^2 + k/a^2} \quad (93)$$

is the area of the event horizon. The Hamiltonian constraint (35) gives

$$S_{\text{AH}} = \frac{3}{8\rho}, \quad \dot{S}_{\text{AH}} = \frac{9H}{8\rho^2} (P + \rho). \quad (94)$$

In an expanding universe, the apparent horizon entropy increases if $P + \rho > 0$, stays constant if $P = -\rho$, and decreases if the weak energy condition is violated, $P < -\rho$.

It seems to have gone unnoticed in the literature that the horizon temperature is positive if and only if the Ricci scalar is, which is equivalent to equations of state satisfying $P < \rho/3$ for a perfect fluid in Einstein's theory. This is the condition for the apparent horizon to be also a trapping horizon. A ‘‘cold horizon’’ with $T = 0$ is obtained for the vanishing Ricci scalar, but the entropy is positive for such an horizon, a situation analogous to that of extremal black hole horizons in GR. Note also that, if the weak energy condition (which implies $P + \rho \geq 0$) is assumed, the boundary $P = \rho/3$ between positive and negative Kodama-Hayward temperatures corresponds to the boundary between the timelike and spacelike character of the apparent horizon. It is not obvious *a priori* that a null apparent horizon, obtained for $R^a{}_a = 0$ ($P = \rho/3$ for a perfect fluid in GR), should occur when the universe is filled with conformal matter.

The natural choice of surface gravity seems to be that of Kodama-Hayward, which produces the apparent horizon temperature (90). The apparent horizon entropy is $S_{\text{AH}} = \frac{A_{\text{AH}}}{4} = \pi R_{\text{AH}}^2$ (this can be obtained using Wald's Noether charge method—see the discussion of the next section), and the internal energy U should be identified with the Misner-Sharp-Hernandez mass $M_{\text{AH}} = \frac{4\pi R_{\text{AH}}^3}{3} \rho$ contained inside the apparent horizon. The factor $4\pi R_{\text{AH}}^3/3$ is not the proper volume of a sphere of proper (areal) radius R_{AH} , unless the universe has flat spatial sections. It is the use of

the Misner-Sharp-Hernandez mass which points us to use the areal volume $V_{\text{AH}} \equiv \frac{4\pi R_{\text{AH}}^3}{3}$ instead of the proper volume when discussing thermodynamics (failing to do so would jeopardize the possibility of writing the 1st law consistently). However, even with this *caveat*, the 1st law does not assume the form

$$T_{\text{AH}}\dot{S}_{\text{AH}} = \dot{M}_{\text{AH}} + P\dot{V}_{\text{AH}} \quad (95)$$

that one might expect. Let us review now the laws of thermodynamics for cosmological horizons.

0th law. The temperature (or, equivalently, the surface gravity) is constant on the horizon. This law ensures that all points of the horizon are at the same temperature or that there is no temperature gradient on it. The 0th law is a rather trivial consequence of spherical symmetry.

1st law. The 1st law of thermodynamics for apparent horizons is more complicated than (95) and was given in Refs. [11,12] under the name of “unified 1st law.” While using the Misner-Sharp-Hernandez mass M_{AH} as internal energy, the Kodama-Hayward horizon temperature (90), and the areal volume, one introduces further quantities as follows [11,12]. Decompose the metric as in Eq. (20); then, the *work density* is

$$w \equiv -\frac{1}{2}T_{ab}h^{ab}; \quad (96)$$

$$\psi_a \equiv T_a{}^b\nabla_b R + w\nabla_a R \quad (97)$$

is the energy flux across the apparent horizon, when computed on this hypersurface. The quantity $A_{\text{AH}}\psi_a$ is called the *energy supply vector*. The quantity

$$j_a \equiv \psi_a + wK_a \quad (98)$$

is a divergence-free energy-momentum vector which can be used in lieu of ψ_a . The Einstein equations then give [11,12]

$$M = \kappa R^2 + 4\pi R^3 w, \quad (99)$$

$$\nabla_a M = A j_a. \quad (100)$$

The last equation is rewritten as [11,12]

$$A\psi_a = \nabla_a M - w\nabla_a V_{\text{AH}} \quad (101)$$

(“unified 1st law”). The energy supply vector is then written as

$$A\psi_a = \frac{\kappa}{2\pi}\nabla_a\left(\frac{A}{4}\right) + R\nabla_a\left(\frac{M}{R}\right). \quad (102)$$

Along the apparent horizon, it is $M_{\text{AH}} = R_{\text{AH}}/2$ and

$$A_{\text{AH}}\psi_a = \frac{\kappa}{2\pi}\nabla_a S_{\text{AH}} = T_{\text{AH}}\nabla_a S_{\text{AH}}. \quad (103)$$

This equation is interpreted by saying that the energy supply across the apparent horizon $A_{\text{AH}}\psi_a$ is the “heat” $T_{\text{AH}}\nabla_a S_{\text{AH}}$ gained. Writing the energy supply explicitly gives

$$T_{\text{AH}}\nabla_a S_{\text{AH}} = \nabla_a M_{\text{AH}} - w\nabla_a V_{\text{AH}}, \quad (104)$$

and $-w\nabla_a V_{\text{AH}}$ is a work term. The heat entering the apparent horizon goes into changing the internal energy M_{AH} and performing work due to the change in size of this horizon.

Let us compute now the time component of Eq. (104) in comoving coordinates for a FLRW space sourced by a perfect fluid in GR. We have

$$w \equiv -\frac{1}{2}[(P + \rho)u_a u_b + P g_{ab}]h^{ab} = \frac{\rho - P}{2}, \quad (105)$$

$$\dot{V}_{\text{AH}} = 3HV_{\text{AH}}\left(1 - \frac{\ddot{a}}{a}R_{\text{AH}}^2\right) = \frac{9HV_{\text{AH}}}{2\rho}(P + \rho), \quad (106)$$

$$\dot{M}_{\text{AH}} = \frac{d}{dt}(V_{\text{AH}}\rho) = \frac{3HV_{\text{AH}}}{2}(P + \rho), \quad (107)$$

and

$$\dot{S}_{\text{AH}} = 2\pi R_{\text{AH}}\dot{R}_{\text{AH}} = \frac{3\pi R_{\text{AH}}^2}{\rho}H(P + \rho), \quad (108)$$

so that

$$\begin{aligned} T_{\text{AH}}\dot{S}_{\text{AH}} &= \frac{HV_{\text{AH}}}{2}\left(1 - \frac{\ddot{a}}{a}R_{\text{AH}}^2\right)(\rho - 3P) \\ &= \frac{3HV_{\text{AH}}}{4\rho}(P + \rho)(\rho - 3P). \end{aligned} \quad (109)$$

Therefore, it is [11,12]

$$T_{\text{AH}}\dot{S}_{\text{AH}} = \dot{M}_{\text{AH}} + \frac{(P - \rho)}{2}\dot{V}_{\text{AH}}. \quad (110)$$

In the infinitesimal interval of comoving time dt , the changes in the thermodynamical quantities are related by

$$T_{\text{AH}}dS_{\text{AH}} = dM_{\text{AH}} + dW_{\text{AH}}, \quad dW_{\text{AH}} = \frac{(P - \rho)}{2}dV_{\text{AH}}. \quad (111)$$

The coefficient of dV_{AH} , i.e., $-w = (P - \rho)/2$ equals the pressure P (the naively expected coefficient) only if $P = -\rho$ (which includes de Sitter space in which dM_{AH} , dV_{AH} , and dS_{AH} all vanish). The fact that the coefficient appearing in the work term is not simply P can be understood as a consequence of the fact that the apparent horizon is not comoving. For a comoving sphere of radius R_s , it is $\dot{R}_s/R_s = H$ and $\dot{V}_s = 3HV_s$, while

$$\dot{M}_s = \dot{V}_s\rho + V_s\dot{\rho} = 3HV_s\rho - 3HV_s(P + \rho) = -3HV_sP; \quad (112)$$

hence, $\dot{M}_s + P\dot{V}_s = 0$. Indeed, the covariant conservation Eq. (37) is often presented as the 1st law of thermodynamics for a comoving volume V . Because of spatial homogeneity and isotropy, there can be no preferred directions and physical spatial vectors in FLRW space; therefore, the heat

flux through a comoving volume must be zero. In fact, consider a comoving volume V_c (which, by definition, is constant in time) and the corresponding proper volume at time t , $V = a^3(t)V_c$. Multiplying Eq. (37) by V , one obtains

$$V\dot{\rho} + \dot{V}(P + \rho) = \frac{d}{dt}(\rho V) + P\dot{V} = 0. \quad (113)$$

By interpreting $U \equiv \rho V$ as the total internal energy of matter in V , one obtains the relation between variations in the time dt ,

$$dU + PdV = 0, \quad (114)$$

and the 1st law (with work term coefficient P) then gives $TdS = 0$, which is consistent with the abovementioned absence of entropy flux vectors and with the well-known fact that, in curved space, there is no entropy generation in a perfect fluid (the entropy along fluid lines remains constant, and there is no exchange of entropy between neighboring fluid lines [65]). Indeed, Eq. (19) for the evolution of the Misner-Sharp-Hernandez mass contained in a *comoving* sphere reduces to $\dot{M} + P\dot{V} = 0$ or $\dot{\rho} + 3H(P + \rho) = 0$. However, for a noncomoving volume, the work term is more complicated than PdV .

Attempts to write the 1st law for the event, instead of the apparent, horizon lead to inconsistencies [14,55,57,58]. This fact supports the belief that it is the apparent horizon which is the relevant quantity in the thermodynamics of cosmological horizons.

(Generalized) 2nd law. A second law of thermodynamics for the event horizon of de Sitter space was given already in the original Gibbons-Hawking paper [5] and reproposed in [66]. Davies [55] has considered the event horizon of FLRW space and, for GR, with a perfect fluid as the source, has proved the following theorem: if the cosmological fluid satisfies $P + \rho \geq 0$ and $a(t) \rightarrow +\infty$ as $t \rightarrow +\infty$, then the area of the event horizon is nondecreasing. The entropy of the event horizon is taken to be $S_{\text{EH}} = \frac{(K_B c^3)}{4\hbar G} A_{\text{EH}}$, where A_{EH} is its area. The validity of the generalized 2nd law for certain radiation-filled universes was established in [67,68].

Because of the difficulties with the event horizon, one is led to consider the apparent horizon instead. Then, Eq. (108) tells us that, in an expanding universe, in Einstein's theory, with perfect fluid, the apparent horizon area increases, except for the quantum vacuum equation of state $P = -\rho$ (for which S_{AH} stays constant) and for phantom fluids with $P < -\rho$, in which case S_{AH} *decreases*, adding another element of weirdness to the behavior of phantom matter (e.g., [69] and references therein).

The generalized 2nd law states that the total entropy of matter and of the horizon $S_{\text{total}} = S_{\text{matter}} + S_{\text{AH}}$ cannot decrease in any physical process,

$$\delta S = \delta S_{\text{matter}} + \delta S_{\text{AH}} \geq 0. \quad (115)$$

(We refer here to the apparent horizon, but several authors refer instead to the event or particle horizons. The apparent horizon is more appropriate, since it is a quasilocally defined quantity.)

VI. ENTROPY OF THE APPARENT HORIZON AND SCALAR-TENSOR GRAVITY

Black hole thermodynamics has been studied in scalar-tensor and other theories of gravity (see [70] for a summary and a list of references). Numerical studies show that, in the intermediate stages of collapse of dust to a black hole in Brans-Dicke gravity, the horizon area decreases, and the apparent horizon is located *outside* the event horizon [71]. Horizon entropy in Brans-Dicke gravity was analyzed by Kang [72], who pointed out that black hole entropy in this theory is not simply one quarter of the horizon area, but rather

$$S_{\text{BH}} = \frac{1}{4} \int_{\Sigma} d^2x \sqrt{g^{(2)}} \phi = \frac{\phi A}{4}, \quad (116)$$

where ϕ is the Brans-Dicke scalar field (assumed to be constant on the horizon), and $g^{(2)}$ is the determinant of the restriction $g_{\mu\nu}^{(2)} \equiv g_{\mu\nu}|_{\Sigma}$ of the metric $g_{\mu\nu}$ to the horizon Σ . Naively, this expression can be understood by replacing the Newton constant G with the effective gravitational coupling

$$G_{\text{eff}} = \phi^{-1} \quad (117)$$

of Brans-Dicke theory; then, the quantity S_{BH} is nondecreasing during black hole collapse [72]. Equation (116) has now been derived using various procedures [73–75].

As done in [72], consider the Einstein frame representation of Brans-Dicke theory given by the conformal rescaling of the metric

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} \equiv \Omega^2 g_{\mu\nu}, \quad \Omega = \sqrt{G\phi}, \quad (118)$$

accompanied by the scalar field redefinition $\phi \rightarrow \tilde{\phi}$, with $\tilde{\phi}$ given by

$$d\tilde{\phi} = \sqrt{\frac{2\omega + 3}{16\pi G}} \frac{d\phi}{\phi}. \quad (119)$$

The Brans-Dicke action [76]

$$I_{\text{BD}} = \int d^4x \frac{\sqrt{-g}}{16\pi} \left[\phi R - \frac{\omega}{2} g^{\mu\nu} \nabla_{\mu} \phi \nabla_{\nu} \phi - V(\phi) + \mathcal{L}^{(m)} \right] \quad (120)$$

(where $\mathcal{L}^{(m)}$ is the matter Lagrangian density) is mapped to its Einstein frame form

$$I_{\text{BD}} = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_{\mu} \tilde{\phi} \tilde{\nabla}_{\nu} \tilde{\phi} - U(\tilde{\phi}) + \frac{\mathcal{L}^{(m)}}{(G\phi)^2} \right], \quad (121)$$

where a tilde denotes Einstein frame quantities, and

$$U(\tilde{\phi}) = \frac{V(\phi(\tilde{\phi}))}{(G\phi(\tilde{\phi}))^2}, \quad (122)$$

where $\phi = \phi(\tilde{\phi})$. In the Einstein frame, the gravitational coupling is constant, but matter couples explicitly to the scalar field, and massive test particles following geodesics of the Jordan frame $g_{\mu\nu}$ no longer follow geodesics of $\tilde{g}_{\mu\nu}$ in the Einstein frame. Null geodesics are not changed by the conformal transformation. A cosmological or black hole event horizon, being a null surface, is also unchanged. The area of an event horizon is not, and the change in the entropy formula $S_{\text{BH}} = \frac{A}{4G} \rightarrow \frac{A}{4G_{\text{eff}}} = \frac{\phi A}{4}$ is merely the change of the horizon area due to the conformal rescaling of $g_{\mu\nu}$. In fact, $\tilde{g}_{\mu\nu}^{(2)} = \Omega^2 g_{\mu\nu}^{(2)}$, and, since the event horizon is not changed, the Einstein frame area is

$$\tilde{A} = \int_{\Sigma} d^2x \sqrt{\tilde{g}^{(2)}} = \int_{\Sigma} d^2x \Omega^2 \sqrt{g^{(2)}} = G\phi A, \quad (123)$$

assuming that the scalar field is constant on the horizon (if this is not true, the zeroth law of black hole thermodynamics will not be satisfied). Therefore, the entropy-area relation for the event horizon $\tilde{S}_{\text{BH}} = \tilde{A}/4G$ still holds in the Einstein frame. This is expected on dimensional grounds, since, *in vacuo*, the theory reduces to GR with varying units of length $\tilde{l}_u \sim \Omega l_u$, time $\tilde{t}_u \sim \Omega t_u$, and mass $\tilde{m}_u = \Omega^{-1} m_u$ (where t_u , l_u , and m_u are the constant units of time, length, and mass in the Jordan frame, respectively). Derived units vary accordingly [77]. An area must scale as $A \sim \Omega^2 = G\phi$, and, in units in which $c = \hbar = 1$, the entropy is dimensionless and, therefore, is not rescaled. As a result, the Jordan frame and Einstein frame entropies coincide [72].

The equality between black hole entropies in the Jordan and Einstein frames is indeed extended to all theories with action $\int d^4x \sqrt{-g} f(g_{\mu\nu}, R_{\mu\nu}, \phi, \nabla_\alpha \phi)$ which admit an Einstein frame representation [78].

As a by-product of this observation, the Jordan and the Einstein frames are physically equivalent with respect to the entropy of the event horizon. A debate on whether these two frames are physically equivalent seems to flare up now and again; it is pretty well-established that, at the classical level, the two frames are simply different representations of the same physics [77,79,80]. Potential problems arise from the representation dependence of fundamental properties of theories of gravity (including the equivalence principle), but this is not an argument against the equivalence of the two conformal frames: rather, it means that such fundamental properties should, ideally, be reformulated in a representation-independent way ([81] and references therein). We will not address this problem here.

The classical equivalence is expected to break down at the quantum level; in fact, already in the absence of gravity, the quantization of canonically related Hamiltonians

produces inequivalent energy spectra and eigenfunctions [82]. However, it is not clear that this happens at the semiclassical level for conformally related frames [79]. Black hole thermodynamics is not purely classical (the Planck constant \hbar appears in the expressions of the entropy and temperature of black hole and cosmological horizons). It is not insignificant that the physical equivalence between conformal frames holds for the (semiclassical) entropy of event horizons.

Contrary to event horizons, the location of *apparent* horizons (which, in general, are not null surfaces) is changed by conformal transformations. The problem of relating Einstein frame apparent horizons to their Jordan frame counterparts, raised in [83], has been solved in [84].

At this point, one may object that there was a logical gap in our previous discussion: following common practice, we took the formula $S = A/4G$ for the (black hole or cosmological) *event* horizon and we used it for the *apparent* horizon. Let us consider stationary black hole or cosmological event horizons first. The area formula can be derived (in GR or in other theories of gravity) by using Wald's Noether charge method [73–75,85,86] or other methods [5,73]. As a result, the usual entropy-area relation remains valid, provided that the gravitational coupling G is replaced by the corresponding effective gravitational coupling G_{eff} of the theory, the identification of which follows from the inspection of the action or of the field equations rewritten in the form of effective Einstein equations. More rigorously, in [87], G_{eff} is identified by using the matrix of coefficients of the kinetic terms for metric perturbations [87]. The metric perturbations contributing to the Noether charge in Wald's formula are identified with specific metric perturbation polarizations associated with fluctuations of the area density on the bifurcation surface Σ of the horizon (in D spacetime dimensions, this is the $(D-2)$ -dimensional spacelike cross section of a Killing horizon, on which the Killing field vanishes and coincides with the intersection of the two null hypersurfaces comprising this horizon). The horizon entropy is

$$S_{\text{BH}} = \frac{A}{4G_{\text{eff}}}, \quad (124)$$

for a theory described by the action

$$I = \int d^4x \sqrt{-g} \mathcal{L}(g_{\mu\nu}, R_{\alpha\beta\gamma\delta}, \nabla_\sigma R_{\alpha\beta\gamma\delta}, \phi, \nabla_\alpha \phi, \dots), \quad (125)$$

where ϕ is a gravitational scalar field. The Noether charge is

$$S = -2\pi \int_{\Sigma} d^2x \sqrt{g^{(2)}} \left(\frac{\delta \mathcal{L}}{\delta R_{\mu\nu ab}} \right)_{(0)} \hat{\epsilon}_{\mu\nu} \hat{\epsilon}_{\mu\nu}, \quad (126)$$

where $\hat{\epsilon}_{\rho\sigma}$ is the (antisymmetric) binormal vector to the bifurcation surface Σ (which satisfies $\nabla_\mu \chi_\nu = \hat{\epsilon}_{\mu\nu}$ on the bifurcation surface Σ , where χ^μ is the Killing field

vanishing on the horizon) and is normalized to $\hat{\epsilon}^{ab}\hat{\epsilon}_{ab} = -2$. The subscript (0) denotes the fact that the quantity in brackets is evaluated on solutions of the equations of motion. The effective gravitational coupling is then calculated to be [87]

$$G_{\text{eff}}^{-1} = -2\pi \left(\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \right)_{(0)} \hat{\epsilon}_{\mu\nu} \hat{\epsilon}_{\rho\sigma}. \quad (127)$$

For dynamical black holes, there is no timelike Killing vector to provide a bifurcate Killing horizon. However, it was shown in [11] that, for apparent horizons, the Kodama vector can replace the Killing vector in Wald's entropy formula, and the result is one quarter of the area in GR [or the corresponding generalization in theories of the form (125)]. We do not repeat the calculation here, but we simply note that the same calculation applies to cosmological apparent horizons, as well, a point that seems to not have been noted in the literature on cosmological horizons.

Finally, let us see how thermodynamics can restrict the range of physical solutions of a theory of gravity. In Brans-Dicke cosmology, consider the exact solution representing a spatially flat universe, with parameter $\omega = -4/3$ and no matter [88]

$$a(t) = a_0 \exp(Ht), \quad (128)$$

$$\phi(t) = \phi_0 \exp(-3Ht), \quad (129)$$

where a_0 , ϕ_0 , and H are positive constants. In GR, de Sitter spaces are obtained with constant scalar fields, but this is not always the case in scalar-tensor gravity. Since the entropy of the apparent/event horizon in this case is $S = \phi A_H/4 = \phi_0 H^{-2} \exp(-3Ht)$, it is always decreasing. The scalar field plays the role of an effective fluid, with density and pressure given by the equations of Brans-Dicke cosmology, which, in the spatially flat case and for vacuum and a free Brans-Dicke scalar, reduce to [76,89,90]

$$H^2 = \frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi} \right)^2 - \frac{H\dot{\phi}}{\phi}, \quad (130)$$

$$\dot{H} = -\frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi} \right)^2 + 2H \frac{\dot{\phi}}{\phi}, \quad (131)$$

$$\ddot{\phi} + 3H\dot{\phi} = 0. \quad (132)$$

In our case, the energy density and pressure of the effective fluid are $\rho_{(\phi)} = -P_{(\phi)} \simeq H^2$. The entropy density of this effective fluid is

$$s_{(\phi)} = \frac{P_{(\phi)} + \rho_{(\phi)}}{T} = 0; \quad (133)$$

therefore, the horizon associated with this solution violates the generalized 2nd law and should be regarded as unphysical. More generally, the use of entropic considerations to select extended theories of gravity was suggested in [91].

VII. CONCLUSIONS

The apparent horizon suffers from the dependence on the spacetime slicing. In FLRW space, it would be unnatural to choose a slicing unrelated to the hypersurfaces of spatial homogeneity and isotropy and constant comoving time, because the latter identify physical comoving observers who see the cosmic microwave background homogeneous and isotropic around them (apart from small temperature anisotropies of the order 5×10^{-5}). The problem of the slicing dependence, therefore, does not seem so pressing in FLRW spaces; however, it is not completely eliminated. Nevertheless, apparent horizons seem better candidates for thermodynamical considerations than event or particle horizons.

Similar to dynamical black hole horizons, the thermodynamics of FLRW cosmological horizons is not completely free of problems: the choice of surface gravity determines the horizon temperature, and there are many inequivalent proposals for surface gravity. A natural choice of internal energy contained within the apparent (or even event) horizon is given by the Misner-Sharp-Hernandez mass, and then it is natural to choose the Kodama time and Kodama-Hayward surface gravity because the Misner-Sharp-Hernandez mass is intimately associated with the Kodama vector as a Noether charge. However, doing so produces a Kodama-Hayward temperature, which is negative for GR universes with stiff equations of state $P > \rho/3$, and it is not obvious that one should give up these universes as unphysical. Different choices of temperature would produce different forms of the 1st law, corresponding to different coefficients for the work term dW appearing there. What is certain, though, is that the apparent (and also the event and particle) horizons are not comoving, and one should not necessarily expect a simple PdV term to appear. Perhaps other choices of quasilocal energy can produce consistent forms of the 1st law and be applicable to a larger variety of universes; one could turn the argument around and use cosmology to help selecting the ‘‘correct’’ surface gravity also for dynamical black hole horizons (this possibility will be the subject of a separate publication).

We have elucidated the causal character of the apparent horizon and given a simple criterion for the apparent FLRW horizon to be trapping. This criterion coincides with the one for the Kodama-Hayward temperature to be positive-definite, and the threshold between trapping and untrapping horizons is a FLRW universe filled with conformal matter which, if the weak energy condition is assumed, also marks the transition between the timelike and spacelike nature of the apparent horizon. At the moment, we are unable to offer a simple and consistent physical interpretation of this fact and we will refrain from doing so. We have also considered the extension of the thermodynamics of cosmological horizons to alternative theories of gravity and, within Brans-Dicke theory, we

have seen how thermodynamical considerations can help in judging how physical a certain solution can be.

Overall, it appears that the thermodynamics of cosmological apparent horizons exhibits features which are not yet fully understood. Because of the extremely simplified nature of FLRW spacetime, understanding these aspects for cosmological apparent horizons should be more fruitful

and rapid than understanding the corresponding aspects of black hole dynamical horizons.

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