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Thermal noise and coating optimization in multilayer dielectric mirrors

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Optical multilayer coatings of high-reflective mirrors significantly determine properties of Fabry-Perot resonators. Thermal (Brownian) noise in these coatings produce excess phase noise which can seriously degrade the sensitivity of high-precision measurements using these cavities. In particular, it is one of the main limiting factors at the current stage in laser gravitational-wave detectors (for example, project LIGO). We present a method to calculate this effect accurately and analyze different strategies to diminish it by optimizing the coating. Traditionally, the effect of the Brownian noise is calculated as if the beam is reflected from the very surface of the mirror's coating. However, the beam penetrates the coating, and Brownian expansion of the layers leads to dephasing of interference in the coating and consequently to an additional change in the reflected amplitude and phase. Fluctuations in the thickness of a layer change the strain in the medium and hence, due to a photoelastic effect, change the refractive index of this layer. This additional effect should also be considered. It is possible to reduce the noise by changing the total number and thicknesses of high and low refractive layers preserving the reflectivity. We show how an optimized coating may be constructed analytically rather than numerically as before. We also check the possibility of using internal resonant layers, an optimized cap layer, and double mirrors to decrease the thermal noise.

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I. INTRODUCTION

Any precise measurement faces the challenge of different noises superposing a useful signal. Brownian noise coming from chaotic thermal motion of particles is one of the enemies. A Michelson interferometer is able to detect minor changes in the lengths of its arms: two beams traveling different optical paths interfere with the detector, producing intensity which depends on the difference between the phases of the beams. Thermal (sometimes also called Brownian) noise in coatings and substrates of the interferometer's mirrors results in fluctuations of their surfaces which add a random phase to the waves. This effect is one of the key factors limiting the sensitivity of laser gravitational-wave detectors [1]. Though the thickness of the multilayer coating is just several micrometers, the internal mechanical losses in layers is several orders of magnitude larger than in the substrate. That is why thermal coating noise, in accordance with the fluctuation-dissipation theorem, exceeds other noises produced in the mirrors [2].

In this paper we analyze different effects and strategies aimed at decreasing the thermal coating noise for a generalized multilayer reflective coating. Traditionally, the effect of the Brownian noise is calculated as if the beam is reflected from the surface of the mirror's coating, fluctuating as an incoherent sum of the fluctuations of each layer and of the substrate. However, the beam actually penetrates the coating, and Brownian expansion of the layers leads to dephasing of interference and consequently to an additional change in the reflected phase [3] and amplitude. Fluctuations in the thickness of a layer change the strain in the medium and hence, due to a photoelastic effect, change the refractive index of this layer. This additional effect should also be considered. It was proposed in [4,5] to change the number and thicknesses of high and low refractive layers in order to diminish the noise while preserving the reflectivity. We also check the possibility of using internal resonant layers [6], an optimized cap layer [2], and double mirrors [7] to decrease the thermal noise.

Brownian noise is not the only source of noise produced by the coating. Fluctuations of temperature, which are translated into a displacement of the mirror's surface through thermal expansion (thermoelastic noise) [8,9] and a change of the optical path due to fluctuations of the refraction index (thermorefractive noise) [10], combine to produce generalized thermo-optical noise [2,11]. Brownian fluctuations causing a displacement of the mirror's surface and the previously neglected correlated photoelastic effect produced by these fluctuations form a Brownian branch of noises. The Brownian branch of noises, which is the topic of this paper, and thermo-optical noise are uncorrelated, as they represent uncorrelated fluctuations of volume and temperature.

II. MULTILAYER COATING PHASE NOISE

A. Reflectivity

To calculate the amplitude and phase of a reflected beam, the impedance method [12] will be used below. We found this method more convenient for analytical consideration than the equivalent and more widely used matrix method [13].

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To consider the reflection of light at normal incidence on each boundary, separating the layers, starting from the substrate/coating boundary (see Fig. 1), we introduce an effective impedance $Z(z)$ and an amplitude reflection coefficient $\Gamma(z)$ as follows:

$$Z(z) = \frac{E(z)}{H(z)} = \frac{E_+(z) + E_-(z)}{H_+(z) + H_-(z)} = \eta(z)\frac{1+\Gamma(z)}{1-\Gamma(z)}, \quad (1)$$

$$\Gamma(z) = \frac{E_-(z)}{E_+(z)} = \frac{Z(z) - \eta(z)}{Z(z) + \eta(z)}, \quad (2)$$

$$\eta(z) = \sqrt{\frac{\mu(z)\mu_0}{\epsilon(z)\epsilon_0}} = \frac{\mu(z)}{n(z)} Z_v, \quad (3)$$

where E and H are tangential electric and magnetic fields in the standing wave, while E_+, H_+ and E_-, H_- are forward and backward (reflected) waves, n is the refraction index, μ and ϵ the relative permeability and permittivity, and Z_v is the vacuum impedance ($Z_v = 1$ in the Gaussian cgs system). We assume that in a small neighborhood of the boundary μ and ϵ are piecewise-constant.

As tangential fields E and H are continuous in a medium without free currents, the effective impedance is also continuous on all boundaries, while the reflection coefficient experiences jumps. Meanwhile, the reflection coefficient changes continuously between boundaries according to the following expression:

$$\Gamma(z - d_j) = \frac{E_- e^{ik_0 n_j (z-d_j)}}{E_+ e^{-ik_0 n_j (z-d_j)}} = \Gamma(z) e^{-i2k_0 n_j d_j}, \quad (4)$$

where $k_0 = \frac{2\pi}{\lambda}$ is the wave vector of the optical field in vacuum, and λ is the wavelength. This allows us to calculate the reflectivity of any multilayer coating recursively, layer by layer. We start from the substrate, where the impedance is equal to the impedance of the free substrate η_0, and move to the surface, turning from the reflection coefficient $\tilde{\Gamma}_j = \Gamma(-\sum_j d_j + 0)$ to the effective impedance $Z_j = Z(-\sum d_j - 0) = Z(-\sum d_j + 0)$, when facing the boundary and back (to Γ_{j+1}) after crossing it (see Fig. 1). It is possible also to exclude effective impedance from calculations:

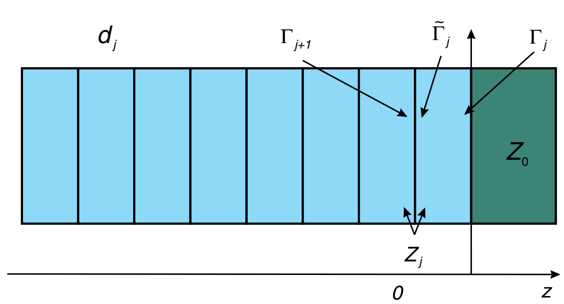

FIG. 1 (color online). A schematic of a multilayer coating.

$$\Gamma_{j+1} = \frac{g_{j+1,j} + \tilde{\Gamma}_j}{1 + g_{j+1,j}\tilde{\Gamma}_j}, \quad (5)$$

where $\tilde{\Gamma}_j = \Gamma_j e^{-i\varphi_j}$ comes from (4), $\varphi_j = 2k_0 n_j d_j$, and $g_{ij} = \frac{n_i - n_j}{n_i + n_j}$. Note that the tilde can be read as "on the left side of the layer" (Fig. 1).

In the case of classical multilayer coating with quarter wavelength layers (QWL) with $\varphi_h = \varphi_l = \pi$, all impedances and reflection coefficients are real.

B. Interference

We now assume that each of the layers experiences a variation of thickness δd_j and a variation of its refraction index δn_j, producing changes in optical thicknesses of layers and in boundary conditions between layers. These variations may be included by changing φ_j for $\varphi_j + 2k_0 \delta n_j d_j + 2k_0 n_j \delta d_j = \varphi_j - \Delta_j$ and assuming

$$\tilde{\Gamma}'_j = \Gamma'_j e^{-i\varphi_j}(1 + i\Delta_j), \quad (6)$$

where the prime means modified reflectivity. We also have to substitute η_i for $\eta_i(1 + \delta\eta_i)$ in (1) and (2), which is a consequence of the refraction index change $\delta\eta_j = -\frac{\delta n_j}{n_j}$. As before, moving layer by layer to the surface, we expand each result into a series to the first order of variations δn_j and δd_j. In this way, we can build a perturbed amplitude reflection coefficient Γ'_m:

$$\Gamma'_m = \Gamma_m(1 + \varepsilon),$$
$$\varepsilon = z_m \frac{\delta n_m}{n_m} + \sum_{j=1}^{m-1} \prod_{k=j+1}^{m} \frac{z_k}{\tilde{z}_{k-1}} \left(i\Delta_j - \zeta_j \frac{\delta n_j}{n_j} \right), \quad (7)$$

$$z_k = \frac{(1 - \Gamma_k^2)}{2\Gamma_k}, \quad \tilde{z}_k = \frac{1 - \Gamma_k^2 e^{-i2\varphi_k}}{2\Gamma_k e^{-i\varphi_k}}, \quad \zeta_k = \tilde{z}_k - z_k. \quad (8)$$

Here m is the index of the layer of interest ($m = N + e$ for the reflectivity of the whole mirror, where N is the total number of layers, and "e" represents the consideration of the top layer—vacuum boundary). Taking into account that $\Delta_j, \frac{\delta n_j}{n_j} \ll 1$, we can find an equivalent phase shift $\delta\varphi$ as well as a variation of reflectivity $\delta\Gamma$ (leading to amplitude noise which cannot be found in a traditional approach), collecting all imaginary and real parts and noting the decomposition $\Gamma e^{\varepsilon} \simeq \Gamma(1 + \varepsilon)$. Total fluctuations may arise both from layer thickness fluctuations δd_j (Brownian and thermoelastic noises) and from deviations of the refraction index δn_j (photoelastic and thermorefractive noises).

In the case of inhomogeneous refraction index deviations, the equations described above can be easily modified. If $\delta n_j(z)$ and its derivative are small enough, Eqs. (5) and (7) will not change their forms, while (8) will require a minor modification, without bringing out any new effects. The only difficulty is then to find the analogue of (4) and $\delta n_j(z)$. However, such an inhomogeneous extension of (7) and (8) is not essential for the Brownian branch, as all spectral density estimations for it are based on the "thin coating approximation" giving a constant strain (and hence δn_j) in a layer.

C. Photoelastic effect

A photoelastic effect in layers of the coating may produce additional noise correlated with Brownian noise. Photoelasticity is a phenomenon of the refraction index change under deformation:

$$\Delta B_i = p_{ij} u_j, \tag{9}$$

where B_i is the optical indicatrix, u_j is the strain tensor, p_{ij} is the photoelastic tensor, and indices $i, j \in 1 \ldots 6$ [14]. In the case of cylindrical symmetry we have a longitudinal effect $\Delta B_i = p_{i3} u_3 = p_{i3} \delta d/d$ and a transversal effect $\Delta B_i = p_{i\rho} u_{\rho\rho}$. However, only the longitudinal effect may produce the noise correlated to the Brownian longitudinal surface noise, providing a theoretical possibility of their interference compensation. Variations of refraction indices due to longitudinal photoelasticity are the following:

$$\delta n_x = -\frac{n_0^3}{2} p_{13} \frac{\delta d}{d}, \qquad \delta n_y = -\frac{n_0^3}{2} p_{23} \frac{\delta d}{d}. \tag{10}$$

We neglect a nonzero δn_z component, as we consider normal incidence. It is known that tantalum oxide used in multilayer coatings Ta_2O_5 is a rutile-type crystal with tetragonal symmetry. Rutile (titanium dioxide) has $p_{13} = 0.171$, $p_{23} = 0.16$. From [15] we can also make a rough estimate for tantalum oxide, $p_{Ta_2O_5} < 0.18$. For simplicity, we put $p_{13} = p_{23} = p_{Ta_2O_5} = 0.17$. The other component of the coating—fused silica—has $p_{13} = p_{23} = p_{SiO_2} = 0.27$.

Photoelasticity also produces a transversal effect coupled to $u_{\rho\rho}$, which should be considered separately (as $u_{\rho\rho}$ noise is not correlated with $u_{zz} \propto \delta d$ noise) and added incoherently. This component, producing a small correction, will not be considered here.

D. Brownian branch of noises

The photoelastic effect converts a fluctuation layer thickness into a correlated fluctuation of its refraction index, producing additional phase and boundary variations:

$$\Delta_j = -2k_0 n_j \left(1 - \frac{n_j^2}{2} p_j\right) \delta d_j = -2k_0 n_j \psi_j \delta d_j, \tag{11}$$

$$-\frac{\delta n_j}{n_j} = \frac{n_j^2 p_j}{2} \frac{\delta d_j}{d_j} = -\frac{n_j^2 p_j}{\varphi_j (2 - n_j^2 p_j)} \Delta_j = \gamma_j \Delta_j, \tag{12}$$

where p_j is the effective photoelastic index for a j layer. Thereby, coating induced deviations of the reflected phase and reflection coefficient are easily obtained from (7) and (8):

$$\delta\varphi_c = \sum_{j=1}^{N} \alpha_j \delta d_j, \tag{13}$$

$$\delta\Gamma_c = \sum_{j=1}^{N} \beta_j \delta d_j, \tag{14}$$

where

$$\alpha_j = -2k_0 n_j \psi_j \operatorname{Im}\left[\prod_k \frac{z_k}{\tilde{z}_{k-1}} (i + \zeta_j \gamma_j)\right], \tag{15}$$

$$\beta_j = -2k_0 n_j \psi_j \operatorname{Re}\left[\prod_k \frac{z_k}{\tilde{z}_{k-1}} (i + \zeta_j \gamma_j)\right]. \tag{16}$$

Let us consider one end mirror in an arm of an interferometer. Thermal displacement of the mirror's surface produces phase fluctuations in the interferometer output. It is more intuitive to consider a case of contraction (Fig. 2) of all layers in the mirror. Then the length of the additional gap for the light to travel before entering the mirror is $-\delta d$ (as $\delta d < 0$ for contracting), yielding a phase shift

$$\delta\varphi_g = -2k_0 \sum_{j=1}^{N} (-\delta d_j). \tag{17}$$

The total phase shift produced by the perturbed coating (relative to the unperturbed one) will be

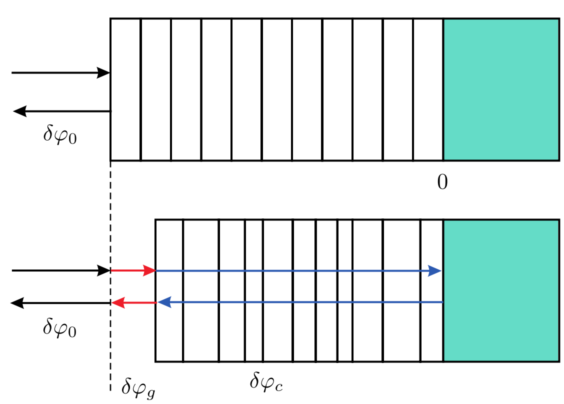

FIG. 2 (color online). Phase shift of the optical wave reflecting from an unperturbed (upper figure) and perturbed mirror. $\delta\varphi_0$, $\delta\varphi_B$, and $\delta\varphi_I$ are the shift in the total phase, the shift due to the surface displacement, and the shift due to interference dephasing in the coating, respectively ($\delta d_j < 0$).

$$\delta\varphi_\Sigma = -2k_0 \sum_{j=1}^{N} [z_{N+e}(-1)^{N-j}\tilde{z}_j^{-1}\psi_j n_j - 1]\delta d_j, \quad (18)$$

where we took into account that inside QWL coating all quantities are real and $\alpha_j = -2k_0 n_j \psi_j z_{N+e}(-1)^{N-j}\tilde{z}_j^{-1}$.

It is also important to admit that in a "good mirror" approximation, when $1 - |\Gamma| \ll 1$ (in this case $Z_N \to 0$ or $Z_N \to \infty$ depending on the topmost layer), the amplitude reflection coefficient correction for QWL coating produced by each layer $\beta_j = (-1)^{N-j} z_{N+e}\gamma_j \tilde{z}_j^{-1}\frac{Z_j}{\eta_j}$ tends to zero.

The term before δd_j can be regarded as a noise coefficient showing a contribution of each layer to the total noise. This coefficient can have any sign, depending on the values of the interferential contribution ("$-$" sign) or surface displacement ("+"), but only its absolute value is significant as noise contributions from different layers are added incoherently.

Using the acquired formulas we can plot a diagram of the phase shift contribution of each layer and calculate the values of noise spectral densities in the whole. In Fig. 3 such a distribution is plotted, keeping the sign from (18). It can be seen that the interference part of noise plays a role in a few outer layers (order of penetration depth) [3] while Brownian (surface displacement) noise forms the major part. Several layers can even demonstrate nearly complete noise compensation.

The noise contribution of a layer is formally composed of three summands: the main Brownian (surface displacement) contribution, the interferential part, and the photoelastic effect:

$$\delta\varphi_\Sigma = \sum 2k_0\delta d_j + \frac{\partial\varphi}{\partial d_j}\delta d_j + \frac{\partial\varphi}{\partial n_j}\frac{\partial n_j}{\partial d_j}\delta d_j, \quad (19)$$

where φ denotes the phase of total complex reflectivity of the mirror. Formulas (7) and (8) give analytical expressions

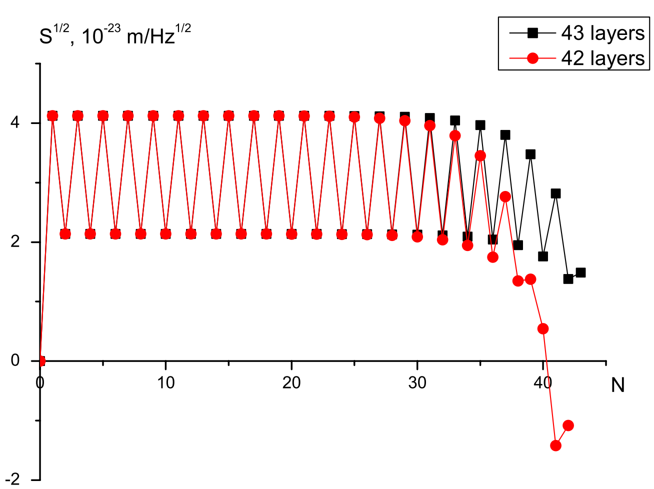

FIG. 3 (color online). Noise coefficient (keeping the sign) from each layer in a coating consisting of 42 (circles) or 43 (squares) layers on silica substrate.

of the derivatives. Their sign distribution may be illustrated as follows. If the coating contracts, then the phase shift produced by each layer is positive due to the change of its thickness, as the Brownian (surface displacement) noise is not really a phase shift acquired by light inside the mirror, but a phase shift acquired outside it (Fig. 2). The contraction of each layer leads, at the same time, to an increase of the refraction index (as in normal materials, it grows with density), providing a positive phase shift. Interference dephasing (phase shift due to reduction of the layer thickness itself), on the other hand, may compensate the phase shift produced by both effects. It looks like the photoelastic effect can play only a negative role; however, it can reduce too high interference dephasing in particular cases.

Equation (19) is quite suitable for numerical calculations, as the partial derivatives in it may be calculated numerically. We used this approach for independent checking of formulas (7) and (8).

E. Noise spectral density

Using (18) one can estimate the total spectral density of the phase noise if the spectral density of each layer thickness noise is known. It is convenient to use the fluctuation–dissipation theorem [16,17] to estimate those noises. In the model of independent thin layers on an infinite half-space substrate, each layer behaves just as if it was the only layer on the substrate. This model was heavily treated and the solution is well known [3,5,18,19]. However, we should split the total surface fluctuations of one layer into two parts for our purpose. The first one represents the fluctuations of the thickness of the coating layer S^c, and the second one represents the fluctuations of the substrate surface induced by losses in the coating S^s. Interference and photoelastic effects influence only the first term. If the losses in the layer responsible for both fluctuations (shear and expansion losses) are equal, then these two spectral densities are uncorrelated. Otherwise, cross correlation terms should be taken into account. We assume the losses to be equal in this paper.

Using the approach presented in [3], this splitting may be easily obtained in the assumption that the noise produced by each layer is independent, $\langle\delta d_j^2\rangle \to S^c(\Omega)_j$, $\langle\delta d_j \delta d_k\rangle = 0$:

$$S(\Omega)_j = S^c(\Omega)_j + S^s(\Omega)_j = (\xi_j^c + \xi_j^s)\phi_j d_j = \xi_j\phi_j d_j, \quad (20)$$

$$\xi_j^c = \frac{4k_BT}{\pi w^2\Omega}\frac{(1+\nu_j)(1-2\nu_j)}{Y_j(1-\nu_j)}, \qquad \xi_j^s = \frac{4k_BT}{\pi w^2\Omega}\frac{Y_j(1+\nu_s)^2(1-2\nu_s)^2}{Y_s^2(1-\nu_j^2)}, \quad (21)$$

where ν_j is the Poisson coefficient of layer j, Y_j—its Young's modulus (Y_s and ν_s are the parameters of the substrate), ϕ_j is the mechanical loss angle, w is the Gaussian beam radius on the mirror, Ω is the frequency of analysis, k_B is Boltzmann's constant, and T is the temperature. Thereby, we obtain spectral densities of phase and amplitude reflection fluctuations,

$$S_\varphi = 4k_0^2 \sum_{j=1}^{N} [(\alpha_j - 1)^2 S^c(\Omega)_j + S^s(\Omega)_j], \tag{22}$$

$$S_\Gamma = 4k_0^2 \sum_{j=1}^{N} \beta_j^2 S^c(\Omega)_j. \tag{23}$$

In assumption of a "good mirror" and QWL layers the second expression is zero and the first one can be simplified:

$$S_\varphi = 4k_0^2 \sum_{m=1}^{2} \bigg[S^c(\Omega)_m \bigg(\frac{a_m^2 \psi_m^2}{|n_1^4 - n_2^4|} - \frac{2a_m \psi_m}{|n_1^2 - n_2^2|} + N \bigg) + S^s(\Omega)_m N \bigg] \tag{24}$$

for $2N$ layers and $S_\varphi + 4k_0^2 S_1$ for $2N + 1$ layers, where $a_m = n_m^2$ for zero outer impedance ($2N$ layers with $n_1 > n_2$) and $a_m = n_2^2 n_1^2$ for infinite outer impedance ($2N + 1$ layers).

From now on, we convert the phase noise into the noise of effective reflecting surface displacement $S_x = \frac{S_\varphi}{4k_0^2}$, in units of m^2/Hz, at 100 Hz frequency to simplify the comparison of this type of noise with other types of noises and Fabry-Perot coordinate sensitivity.

Calculations were made for a silica-tantala mirror of 42-43 layers (21 pairs of $SiO_2 - Ta_2O_5 \lambda/4$ layers on a fused silica substrate with or without an additional silica $\lambda/4$ layer) with the following parameters:

$$\nu_l = 0.17, \quad n_l = 1.45, \quad \nu_h = 0.23, \quad n_h = 2.06,$$
$$Y_l = 7.2 \times 10^{10}\,\text{Pa}, \quad \phi_l = 0.4 \times 10^{-4},$$
$$Y_h = 14 \times 10^{10}\,\text{Pa}, \quad \phi_h = 2.3 \times 10^{-4},$$
$$\lambda = 1.064 \times 10^{-6}\,\text{m}, \quad w = 0.06\,\text{m}, \quad T = 290\,\text{K}.$$

The results are shown in Table I as a correction to Brownian (displacement) noise $\chi = \frac{\sqrt{S_{\rm Br}} - \sqrt{S}}{\sqrt{S_{\rm Br}}} \times 100\%$. Numerical estimates for the relative power transmittance noise $\delta\tau/\tau = 2\Gamma\sqrt{S_\Gamma}/(1 - |\Gamma|^2)$ is less than $10^{-12}\ \text{Hz}^{-1/2}$.

The interference correction to thermal coating thickness noise is about 6%, or 7.5% when taking photoelasticity into account. The thickness fluctuations of the tantala layer are much smaller than its bending ($\xi_h^c = 0.36\xi_h^s$). That is why the interference correction to the total coating Brownian (displacement) noise is only about 2.0%, or 2.3% when taking photoelasticity into account.

TABLE I. Silica-tantala mirror efficiencies relative to the Brownian noise. The standard LIGO coating consists of 41 layers + $\lambda/2$ cap mirror. The modified cap has an optical width $\lambda/4$ (42 layers case).

Type	$42 \times \lambda/4$	$41 \times \lambda/4 + \lambda/2$	$43 \times \lambda/4$
Transmittance τ, ppm	2.28	1.08	0.54
Brownian 10^{-20} m/$\sqrt{\text{Hz}}$	0.632	0.635	0.645
χ, with interference	1.96%	2.34%	1.75%
χ, with photoelasticity	2.33%	1.85%	1.31%
χ, modified cap	2.33%	2.76%	0.81%

III. OPTIMIZATION STRATEGIES

A. Additional top layer corrector

One may alter the thickness of the topmost "correcting" layer in an attempt to minimize the noises using interference effects. This method proved to be useful for thermoelastic and thermorefractive noises [2]. Using formulas (7) and (8) we can obtain

$$S_\varphi = \sum_{m=1}^{2} \bigg[S^c(\Omega)_m \bigg(\frac{a_m^2 \psi_m^2}{|n_1^4 - n_2^4|} - \frac{2a_m \psi_m}{|n_1^2 - n_2^2|} + N \bigg) + S^s(\Omega)_m N \bigg] + S_c',$$
$$S_c' = \bigg[\text{Re}\bigg(\frac{z_{2N+c+e}}{\tilde{z}_{2N+c}} \bigg) (1 \pm \gamma_c \sin(\phi_c)) n_c \psi_c - n_e \bigg]^2 S^c(\Omega)_c + S^s(\Omega)_c \tag{25}$$

for $2N$ layers and $S_\varphi + 4k_0^2 S_1$ for $2N + 1$ layers, where $a_m = n_m^2 n_c \,\text{Re}(\frac{z_{2N+c+e}}{\tilde{z}_{2N+c}})$ and "+" are used for zero impedance of the layer under the cap ($2N + c$ layers with $n_2 < n_1$), and $a_m = \frac{n_2^2 n_1^2}{n_c} \text{Re}(\frac{z_{2N+1+c+e}}{\tilde{z}_{2N+1+c}})$ and "−" are used for infinite impedance of the layer under the cap ($2N + 1 + c$ layers). The index "c" represents one cap layer corrector.

Results are quite unfavorable: for an even number of layers + cap, the minimum of noise is at $n_c < 1$, while its suppression $\chi = \frac{\delta\sqrt{S}}{\sqrt{S_{\rm unmod}}} \times 100\%$ is only 0.04%. For odd layers + cap, the absolute value of noise does not become lower than 6.198×10^{-20} m/$\sqrt{\text{Hz}}$, which means that the suppression is less than 0.69% (for $n_c = 3.6$; $d_c = 0.42\lambda/4$). Even after removing a pair of layers, the noise is about 6.04×10^{-20} m/$\sqrt{\text{Hz}}$, which is more than in the case of an even number of layers.

This means that standard coating with a top silica $\lambda/2$ layer is reflectivity optimized, and "all $\lambda/4$" coating (cap = $\lambda/4$) is noise optimized (see Table I).

B. Layer corrector inside the mirror

The idea of inserting a resonant layer into the mirror is proposed in [6]. This case was studied numerically (Fig. 4). The maximum suppression of 4.4% was shown by a layer

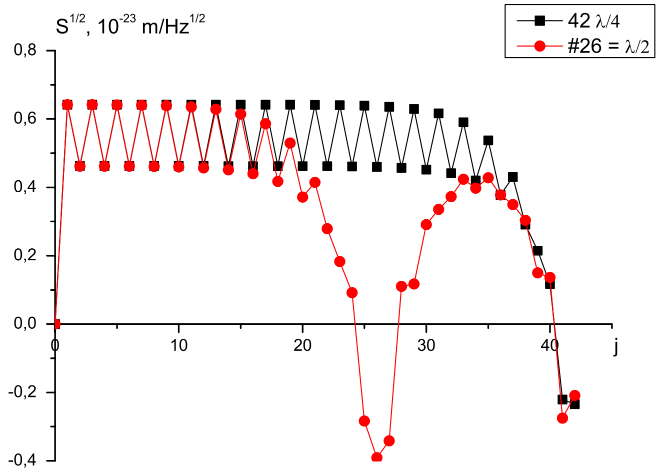

FIG. 4 (color online). Noise coefficient distribution in a coating with 42 layers, keeping the sign. Silica substrate, vacuum medium [squares—simple mirror; circles—a mirror with modified layer #26 (16 from top) $d_m = 0.98\lambda/2$].

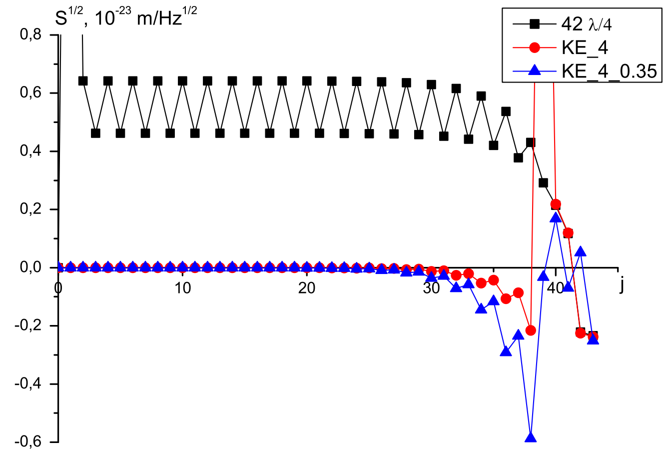

FIG. 5 (color online). Noise distribution in a two-sided mirror, keeping the sign. Silica substrate, vacuum medium (squares—$\lambda/4$ general coating; circles—corresponding etalon with $\lambda/4$ substrate; triangles—etalon, optimized for interference).

corrector close to $d = \lambda/2$, which is the resonant cavity. But such a modification increases power transmittance by more than two orders. If we add eight bilayers to restore transmittance, suppression will be more than eliminated (-14%).

C. Two-sided and double mirror

A novel combined structure was proposed in [7]. This composite mirror has just a few layers on the front side of a big silica substrate, and other layers at the bottom (two-sided mirror or the Khalili etalon). The idea is that only top layers can imprint Brownian (displacement) noise on the phase of reflected light, while bottom layers do not contribute as they do not directly reflect the incoming beam (just some residual power). In this case we should pay attention to interference effects, because the first layers and the substrate are well penetrated by light. This also means that coating noise and substrate noise in combined structures should be treated simultaneously, as there is a possibility of interferential compensation (Figs. 5 and 6).

The main difficulty with the Khalili etalon is its high sensitivity to the manufacturing precision and fluctuations of its optical thickness produced by other sources. Namely, the imprecision of the substrate optical length by $0.07\lambda/4$, corresponding to the mirror's temperature variation of 6 mK, increases noise by 5%.

The same idea may be realized in another geometry (double mirror or the Khalili cavity) with a combined end mirror consisting of two individually suspended mirrors separated by a controlled gap. The first mirror has a small number of layers and hence low noise level, while the layers of the second one provide the required reflectivity. The sensitivity to the gap length is 2 times higher, though it may be controlled with actuators in real time yielding the desired conditions. Our calculations show encouraging suppression of noise in both schemes. The deficiency of both schemes, however, is high power circulating in the mirror's substrate, which leads to various thermal lensing and detuning effects.

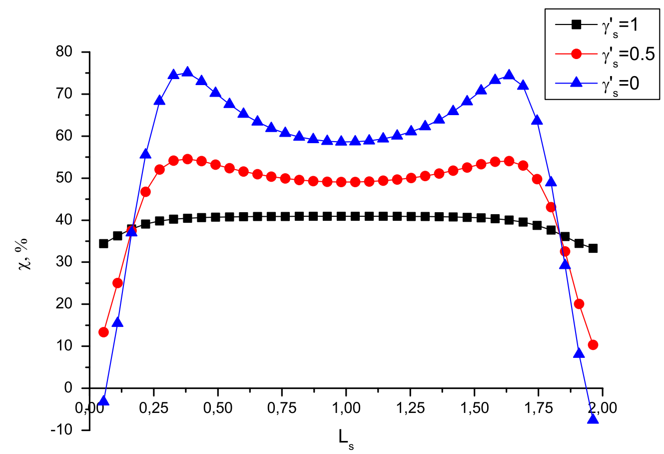

FIG. 6 (color online). Suppression as a function of excess substrate optical thickness (in units of $\lambda/4$) in addition to the integer number of half wavelengths for different noise ratios $\gamma'_s = \frac{\xi_s^c}{\xi_s}$.

The absolute value of the maximum effect is highly dependent on the ratio of thickness and bend noise spectral densities, which is yet unknown. So far we can only say that noise suppression and amplification effects decrease practically linearly with $\gamma'_s = \frac{\xi_s^c}{\xi_s}$ (Fig. 6).

D. Modifying silica-tantala ratio

A promising way to reduce the thermal noise in the coating was proposed in [4,20], which suggests decreasing the thickness of lossy high-index (tantalum-oxide) layers, presumably preserving the total bilayer optical thickness to be $\lambda/2$ ($n_l d_l + n_h d_h = \lambda/2$). To keep the required reflectivity, more bilayers should be used. It was found numerically that there is an optimal combination of the ratio of the layers' thickness and number of layers providing minimal noise at the given reflectivity.

It appears that noise suppression χ is highly dependent on the noise ratio in layers

$$\chi \propto \frac{S_h/d_h}{S_l/d_l} = \frac{\xi_h \phi_h n_l}{\xi_l \phi_l n_h} = \gamma. \tag{26}$$

For the LIGO parameters [21] $\gamma = 4.56$. In [4] coating was optimized for the chosen parameter $\gamma = 7$. A model silica-tantala mirror of $27 \times \lambda/4$ layers + $\lambda/2$ cap was numerically optimized. The resulting coating had 16 silica-tantala bilayers with $n_l d_l = 1.383\lambda/4$, $n_h d_h = 0.617\lambda/4$, a thin cap $n_l d_l = 0.162\lambda/4$, and the first layer $n_h d_h = 0.556\lambda/4$ on the substrate (34 layers total). In the experiment with this mirror design, noise suppression of $\chi_{\rm exp} = (8.8 \pm 2.0)\%$ was observed. Our calculations with all material parameters taken from [4] yield $\chi_{{\rm th}_7} = 8.2\%$, and if $\gamma = 9.23$ estimated from the same experiment is used [22], one gets $\chi_{\rm th} = 9.1\%$.

IV. OPTIMAL COATING

It is well known that for a fixed number of bilayers, a multilayer coating with QWL with $\varphi_h = \varphi_l = \pi$ provides the largest reflectivity [13]. The LIGO interferometers, however, require not only a large reflectivity but also a small noise added by the coating. The coating usually consists of two different materials having distinctly different mechanical losses. This fact stimulated mostly numerical attempts to construct more optimal coating which could have smaller noise with the increased number of layers but decreased total thickness of the "bad" component, while preserving the desired reflectivity [4,20]. Such coatings were found numerically, and it is a common assumption that $\varphi_h + \varphi_l$ should be equal to 2π. It is possible, however, to construct a nearly perfectly optimized coating analytically, and we will show that the "common knowledge" is incorrect. In fact, previous numerical simulations clearly demonstrate that a small correction is required (see Fig. 6 in [4]).

We would like to find optimal thicknesses of the components of a bilayer for a given thickness φ_h. Suppose we have a bilayer inside a coating and the amplitude reflectivity on the boundary to this bilayer from the side of the substrate is $\Gamma_{\rm in} = \Gamma_0 e^{i\varphi_0}$, where Γ_0 is the real amplitude and φ_0 is some initial phase. Let us introduce the following notations: $\Gamma_{\rm in} = \tilde{\Gamma}_0$—initial reflectivity, Γ_1—intermediate reflectivity, $\Gamma_{\rm out} = \Gamma_2$—output reflectivity, and $\Gamma_{\rm in+1} = \tilde{\Gamma}_2$—reflectivity that will be initial for the next pair (see Fig. 7). Using formulas (4) and (5) twice we can find $\Gamma_{\rm in+1}$ as a function of (Γ_0, φ_0, φ_h, φ_l).

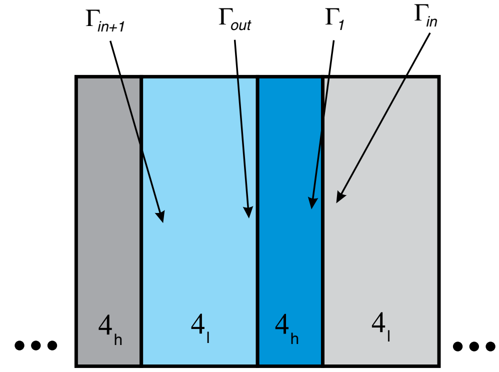

FIG. 7 (color online). A bilayer inside a multilayer coating.

We can now find the optimal phase φ_0 maximizing $|\Gamma_{\rm in+1}|$. Note that $|\Gamma_{\rm in+1}| = |\Gamma_{\rm out}|$ and does not depend on φ_l. After some math we find

$$\tan\varphi_0 = \frac{1 - g_{hl}^2}{1 + g_{hl}^2} \cot\frac{\varphi_h}{2}, \tag{27}$$

$$\varphi_0 \approx \frac{\pi - \varphi_h}{2} - g^2 \sin(\varphi_h). \tag{28}$$

In the last approximation we used the fact that $g_{hl} \simeq 0.17$ is small. It is also important that the reflectivity increases with a new pair of layers only if $\varphi_0 \in [-\frac{\pi}{2}; \frac{\pi}{2}]$ (this will be explained later). To optimize the next layer, we should provide the same input phase $\varphi_{\rm in}$ for it, which can be provided by φ_l, as $|\Gamma_{\rm in+1}|$ does not depend on it. Finally, we obtain

$$\varphi_{l_j} = \varphi_{0_{j+1}} - \varphi_{0_j} - \varphi_{h_j} - 2\sin(\varphi_{h_j}) g_{hl_j}^2 + (\pi m), \tag{29}$$

where j stands for the bilayer number and m is the integer number. For a series of identical bilayers that means

$$\varphi_l + \varphi_h = \pi m - 2\sin(\varphi_h) g^2. \tag{30}$$

It can be shown that in our case ($\varphi_0 \in [-\frac{\pi}{2}; \frac{\pi}{2}]$, $\varphi_h \in [0; 2\pi]$) $m > 0$ and even. As we need to shorten the coating, we should take $m = 2$. From the last equation it is clear that, only in the case of QWL coating, $\varphi_H + \varphi_L = 2\pi$. In other cases, however, there should be a small correction to maximize the reflection. Note that for the first layer $\Gamma_{\rm in} = 0$ with an undefined phase, thus satisfying the requirement on φ_0.

To get an analytical approach consider a bilayer somewhere in the middle of the coating. Suppose that the incoming reflectivity is close to 1:

$$|\Gamma_{\rm in}| = |\tilde{\Gamma}_{2k}| = 1 - \varepsilon. \tag{31}$$

Then, expanding formulas for $|\Gamma_{\rm out}|$ into series to the second order of ϵ and using (28), we obtain

$$|\Gamma_{\rm out}|^2 = 1 - 2\alpha\varepsilon - \alpha(1-2\alpha)\varepsilon^2, \qquad (32)$$

where

$$\alpha = \frac{(1-g^2)^2}{(g\sqrt{2(1-\cos(\varphi_1))} \pm \sqrt{1-2g^2\cos(\varphi_1)+g^4})^2}. \qquad (33)$$

Here we have a "+" sign, when $\varphi_0 \in [-\pi/2; \pi/2]$. It can be shown that, only in this case, $\alpha \le 1$, which means an increasing reflectivity.

Assuming $|2\alpha\varepsilon| \ll 1$, $|(1-2\alpha)\varepsilon| \ll 2$, we can rewrite the local reflectivity as $|\Gamma_{\rm out}| = 1 - \alpha\varepsilon$ and get the total power transmittance in the following form:

$$\tau = \beta(\varphi_h, \varphi_c)\alpha^N(\varphi_h)\varepsilon(\varphi_e), \qquad (34)$$

where

$$\beta = 2\frac{1-g_e^2}{1+2g_e\cos(\varphi_c + \frac{\pi+\varphi_h}{2} + g^2\sin(\varphi_h)) + g_e^2} \qquad (35)$$

describes the coating-air boundary ($g_e = \frac{n_e - n_l}{n_e + n_l}$). For this formula to work we need to satisfy the assumptions we made. Calculations for $g = 0.17$ give $\alpha \in [0.55; 1]$, ($\alpha(\pi) = 0.55$), and $\varepsilon \ll 0.5$. This also requires $\varphi_h \in [\pi/4; 7\pi/4]$. It can be shown numerically that all those requirements can be satisfied with just three initial layers on the substrate.

Now we can eliminate the total number of layers from the equations to design an optimal mirror for a given power transmittance τ_0. As calculations by (7) and (8) for the total noise are rather complicated and provide a small correction only, we consider the simplified formula (17). For the normalized spectral density we obtain

$$\frac{S_{\rm Br}}{A} = E\gamma\varphi_{\varepsilon h} + E\varphi_{\varepsilon l} - \varphi_{0\varepsilon} + \varphi_0 + \frac{\ln\tau_0 - \ln\beta - \ln\varepsilon}{\ln\alpha} \times (\gamma\varphi_h + \varphi_l) - \varphi_l + \varphi_c, \qquad (36)$$

where $A = \frac{\xi_2\varphi_l}{2k_0 n_l}$ m^2/Hz is the dimensional constant. Here $\varphi_{\varepsilon h}$, $\varphi_{\varepsilon l}$ denote the phase thicknesses of the initial (first E from substrate) bilayers; $\varphi_{0\varepsilon}$, φ_0 are the initial phases for initial and regular bilayers; φ_h, φ_l are regular layer thicknesses; and φ_c is the cap layer thickness.

The obtained results are very close to the numerical optimization in [4] (see Table II).

TABLE II. Optimized coating results: $n_h d_{\varepsilon h} = 0.609\lambda/4$, $n_l d_{\varepsilon l} = 1.375\lambda/4$, $n_h d_h = 0.611\lambda/4$, $n_l d_l = 1.373\lambda/4$, $n_c d_c = 0.118\lambda/4$ ($\gamma = 7$).

Type	$25 + \lambda/2$	[4]	Our Method
Power transmittance τ_0, ppm	277.5	277.7	277.7
χ, Brownian (displacement)	0	8.16%	8.4%
χ, with interference	3.37%	11.03%	11.27%
χ, with photoelasticity	2.63%	10.29%	10.5%
Real suppression	0	7.87%	8.08%

We found explicit formulas for the spectral density of phase noise produced by Brownian fluctuations in arbitrary multilayer coatings, taking into account interference effects and photoelasticity. These effects play a role only in a few top layers and give out corrections of the order of 2%. Some optimization methods taking into account interference were considered. The modifying silica-tantala ratio method was found to be the most efficient so far. Another promising approach is compound mirrors.

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