Amplitudes for the $n = 3$ moment of the Wilson operator at two loops in the RI′/SMOM scheme

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The renormalization of the third moment of the twist-2 flavor nonsinglet Wilson operator is given to two loops in the RI′/SMOM renormalization scheme. This involves renormalizing the operator itself and the two total derivative operators into which it mixes by inserting it into a quark 2-point function and evaluating the amplitudes at the symmetric subtraction point. The corresponding two loop conversion functions are derived from which the three loop Landau gauge anomalous dimension is deduced. The full set of amplitudes for the two loop Green's function for each of the operators are given in both the $\overline{\text{MS}}$ and RI′/SMOM schemes.

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I. INTRODUCTION

Recently a renormalization scheme has been introduced which aids the extraction of accurate measurements of matrix elements or Green's functions from the lattice for strongly interacting fields, [1–3]. The scheme is named RI′/SMOM which stands for the modified regularization invariant (RI′) scheme at the symmetric subtraction point where the infinities are removed in a fashion akin to the momentum subtraction scheme, [4]. The shorthand for the latter is MOM. While this is a précis of the syntax it in effect precisely defines the method for performing the renormalization. The RI′/SMOM scheme is an extension of the original regularization invariant (RI) scheme and its modification (RI′), [5,6]. These were developed earlier to aid the matching of lattice computations with the high energy behavior determined in conventional perturbation theory. In essence those schemes are such that 2-point functions are rendered finite by choosing the renormalization constant so that at the subtraction point there are no $O(a)$ corrections where $a = g^2/(16\pi^2)$ and g is the coupling constant in QCD. By contrast the renormalization of 3-point functions such as vertices or operators inserted into 2-point functions is carried out using the standard $\overline{\text{MS}}$ scheme, [5,6]. The technical difference between RI and RI′ resides in the way the quark 2-point function is treated prior to applying the renormalization condition. More specifically this is how to project out the contribution associated with $\not{p}$ where p is the external momentum of the 2-point function. The scheme which is more widely used of the two is RI′ as it minimizes the number of derivatives required to be taken. This reduces the financial penalty in respect of defining the scheme when lattice regularization is used. By contrast using dimensional regularization in the continuum there is no such cost, only the limitation in the range of validity of the coupling constant, due to its smallness, which is not an issue on the lattice. There the nonperturbative structure can be fully explored. Further, there have been several continuum computations performed in perturbation theory in QCD, [7,8]. For instance, the three and four loop renormalization of QCD in RI′ was carried out in the Landau gauge in [7] and for an arbitrary linear covariant gauge at three loops in [8]. In [8] *all* 2-point functions were made finite in the same way as the quark 2-point function was in the original article, [5,6]. Though it should be noted that in the Landau gauge the coupling constant and gauge parameter variables are the same in $\overline{\text{MS}}$ and RI′.

We have given a resumé of the RI′ scheme partly to contrast with that of RI′/SMOM which is the main focus here, but also since it is retained as part of the latter scheme. This is because RI′/SMOM differs from RI′ in the way 3-point and higher functions are treated. For instance, it was pointed out in [1] that RI′ is sensitive to infrared effects. This stems from the fact that there is an exceptional momentum configuration for 3-point functions and the nullified momentum flowing through an inserted operator leads to unwanted infrared singularities. Examples of where one has to be careful in the treatment are the quark current operators and the Wilson operators used in deep inelastic scattering. Therefore, there is a potential problem with making accurate measurements of matrix elements involving such operator insertions on the lattice which is the reason why RI′/SMOM was developed, [1]. In this scheme there is a momentum flowing through each of the external legs and inserted operators of the Green's function in a way which is consistent with energy-momentum conservation. Therefore in the absence of an exceptional momentum configuration there ought to be no infrared problems. Indeed it was shown in [2,3] that there was an improvement in the convergence of the conversion function associated with the renormalization at two loops when the RI′/SMOM scheme was used instead of RI′ for the case of the scalar and tensor quark currents. More recently the second moment of the twist-2 flavor nonsinglet Wilson operator was studied in [9] at two loops. This built on the initial one loop calculation of [10] and the two loop computations for the quark currents, [2,3,11]. The Wilson

operator was a more involved computation than the quark currents due to the mixing of the operator with a total derivative operator. The latter is completely passive in the RI′ scheme renormalization as the operator insertion is at zero momentum by definition. For RI′/SMOM it cannot be neglected and the mixing, which has been studied at three loops in $\overline{\text{MS}}$ in [12], is crucial. Though comparing similar conversion functions for this operator moment suggested that, if anything, in the RI′ case the perturbative convergence of the series was marginally quicker, [9]. Though it was noted in [9] that as the RI′ scheme does not access the off-diagonal part of the mixing matrix, due to the form of the Green's function momentum configuration, it is in some sense not a full scheme for operators with mixing, in contrast to multiplicatively renormalizable operators such as the quark currents.

While we have concentrated on the renormalization, one ingredient which is crucial for lattice computations is the actual structure of the Green's function with the operator inserted in the quark 2-point function. This is used to measure nonperturbative matrix elements. However, in order to assist matching the lattice results accurately with the continuum, the provision of the same quantity in perturbation theory to as high a loop order as is computationally possible is important. Indeed the RI′/SMOM lends itself naturally to this problem of measuring forward matrix elements and hence generalized parton distribution functions. For instance, there has been significant lattice work in this general area represented by [13–29]. As the second moment of the flavor nonsinglet Wilson operator has been treated in [9], it is the purpose of this article to extend that work to the case of the third moment building on the one loop result of [10]. This is a larger computation as the extra covariant derivative means that there is mixing into *two* total derivative operators, [12]. However, as discussed there these operators are actually related to the total derivative of both second moment operators. In practice this means that the anomalous dimensions should be the same but only some of the amplitudes of the Green's functions will be related. This latter property is due to the Lorentz index imbalance between operators of different moments. In addition to focusing on the RI′/SMOM scheme structure of the Green's function we will also provide the full set of $\overline{\text{MS}}$ results at the symmetric subtraction point. Aside from being the reference renormalization scheme we record these because some lattice groups choose to devise their own RI′/SMOM type scheme and then convert their results to the $\overline{\text{MS}}$ scheme. Therefore, they would perform the high energy matching to perturbation theory in the continuum in $\overline{\text{MS}}$. While all the lattice computations are performed in the Landau gauge, our results will be in an arbitrary linear covariant gauge. This is because the presence of the gauge parameter is partly used as a checking procedure within the symbolic manipulation programs we have used. In addition as was noted in [9,10] when one has two or more free Lorentz indices on the inserted operator, then there is not a unique way of defining RI′/SMOM. This is mainly to do with what combination of basis tensors one uses to write the Green's function itself in. Different basis choices could lead to different RI′/SMOM schemes. Indeed it might be possible to choose one in such a way that the associated conversion functions have a better convergence than the analogous RI′ one. One final point concerning the third moment of the Wilson operator and that is that the vector current is present in the set of operators at this level through one of the total derivative operators, [12]. As this current is physical then its renormalization is determined by general properties of renormalization theory and the Slavnov-Taylor identities. Therefore, we will pay special attention to that to ensure there is consistency.

The article is organized as follows. We devote the next section to the background to the set of operators we will renormalize and the structure of the Green's function we will evaluate to two loops in both schemes as well as a summary of the technical machinery we employed to perform the computation. The details of the two loop renormalization as well as the RI′/SMOM anomalous dimension mixing matrix are given in Sec. III. The explicit forms of all the two loop conversion functions are given in the subsequent section. These are then used in Sec. Vto determine the Landau gauge anomalous dimensions for the diagonal and some off-diagonal entries of the mixing matrix at *three* loops. The expressions for the amplitudes of the Green's function we compute are given in Sec. VI for both the $\overline{\text{MS}}$ and RI′/SMOM schemes with conclusions provided in Sec. VII. An appendix gives the explicit form of the tensor basis used for the Green's function as well as the projection matrix used to extract the amplitudes.

II. BACKGROUND

We begin by recalling the formalism and properties of the operators which we will consider here. The notation we use is the same as in previous articles, [9–12], and we will denote our three operators with the shorthand notation,

$$\begin{aligned} W_3 &\equiv \mathcal{S}\bar{\psi}\gamma^\mu D^\nu D^\sigma \psi, \\ \partial W_3 &\equiv \mathcal{S}\partial^\mu(\bar{\psi}\gamma^\nu D^\sigma \psi), \\ \partial\partial W_3 &\equiv \mathcal{S}\partial^\mu\partial^\nu(\bar{\psi}\gamma^\sigma \psi) \end{aligned} \tag{2.1}$$

where all derivatives act to the right and $\mathcal{S}$ denotes both symmetrization and tracelessness, in d spacetime dimensions, in all Lorentz indices. In particular

$$\begin{aligned} \mathcal{S}\mathcal{O}^i_{\mu\nu\sigma} = \mathcal{O}^i_{S\mu\nu\sigma} - \frac{1}{(d+2)}[&\eta_{\mu\nu}\mathcal{O}^i_{S\sigma}{}^\rho{}_\rho + \eta_{\nu\sigma}\mathcal{O}^i_{S\mu}{}^\rho{}_\rho \\ &+ \eta_{\sigma\mu}\mathcal{O}^i_{S\nu}{}^\rho{}_\rho] \end{aligned} \tag{2.2}$$

with

$$\mathcal{O}^i_{S\,\mu\nu\sigma} = \frac{1}{6}[\mathcal{O}^i_{\mu\nu\sigma} + \mathcal{O}^i_{\nu\sigma\mu} + \mathcal{O}^i_{\sigma\mu\nu} + \mathcal{O}^i_{\mu\sigma\nu} + \mathcal{O}^i_{\sigma\nu\mu} + \mathcal{O}^i_{\nu\mu\sigma}] \tag{2.3}$$

where the basic operators, $\mathcal{O}^i_{\mu\nu\sigma}$, are

$$\begin{aligned} \mathcal{O}^{W_3}_{\mu\nu\sigma} &= \bar{\psi}\gamma_\mu D_\nu D_\sigma \psi, \\ \mathcal{O}^{\partial W_3}_{\mu\nu\sigma} &= \partial_\mu(\bar{\psi}\gamma_\nu D_\sigma \psi), \\ \mathcal{O}^{\partial\partial W_3}_{\mu\nu\sigma} &= \partial_\mu\partial_\nu(\bar{\psi}\gamma_\sigma \psi). \end{aligned} \tag{2.4}$$

As in previous articles the symbol W_3 at times will be referred to as the level and in those instances, which will be clear from the context, indicates the set of the three operators. For example, it appears here in the superscript. The specific choice of level W_3 operators is dictated by the need to have three independent operators. One could have excluded one of the total derivative operators in favor of one with the covariant derivative acting on the antiquark field but we retain consistency with lower moment operators here, [9,10,12]. In (2.1) we have suppressed the flavor indices and emphasize that we are using flavor nonsinglet operators. If the operators were flavor singlet then one would require gluonic and ghost operators to have a closed set under renormalization. In the flavor nonsinglet case there is mixing but only within the set (2.1) which is also closed under renormalization. However, the mixing is only relevant when the parent operator of W_3 is inserted in a Green's function with a momentum flowing through the operator itself, [12]. With the way we have chosen the set of operators then if no momentum flows through the inserted operator there is no contribution to the Green's function from the total derivative operators. For the RI′/SMOM scheme renormalization, [1], where there are two external momentum, p and q, flowing in through the quark legs then there is a net momentum flow of $(p + q)$ out of the operator, as illustrated in Fig. 1 where the form of the Green's function, $\langle\psi(p)\mathcal{O}^i_{\mu\nu\sigma}(-p-q)\bar{\psi}(q)\rangle$, we consider is given. Since we work at the symmetric subtraction point, [1–3], then

$$p^2 = q^2 = (p+q)^2 = -\mu^2 \tag{2.5}$$

which imply

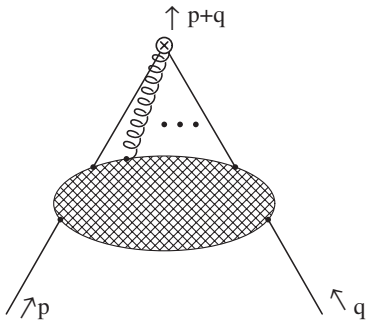

FIG. 1. The Green's function $\langle\psi(p)\mathcal{O}^i_{\mu\nu\sigma}(-p-q)\bar{\psi}(q)\rangle$.

$$pq = \tfrac{1}{2}\mu^2 \tag{2.6}$$

where μ is the common mass scale which in this case is equivalent to the mass scale introduced to ensure that the coupling constant is dimensionless in dimensional regularization which we use throughout.

The renormalization of the operators of (2.1) is accommodated by the mixing matrix of renormalization constants defined by

$$Z^{W_3}_{ij} = \begin{pmatrix} Z^{W_3}_{11} & Z^{W_3}_{12} & Z^{W_3}_{13} \\ 0 & Z^{W_3}_{22} & Z^{W_3}_{23} \\ 0 & 0 & Z^{W_3}_{33} \end{pmatrix} \tag{2.7}$$

where the subscripts label the operators in the order W_3, ∂W_3 and $\partial\partial W_3$. The upper triangularity of the matrix follows from the choice of our operators in (2.1) and simplifies the computation when compared to the extraction of all nine elements for another choice. Associated with the renormalization constants are the anomalous dimensions which are encoded in a parallel matrix with the formal definition

$$\gamma^{W_3}_{ij}(a,\alpha) = -\mu\frac{d}{d\mu}\ln Z^{W_3}_{ij} \tag{2.8}$$

with

$$\mu\frac{d}{d\mu} = \beta(a)\frac{\partial}{\partial a} + \alpha\gamma_\alpha(a,\alpha)\frac{\partial}{\partial\alpha}. \tag{2.9}$$

Here a is the shorthand for the coupling constant via $a = g^2/(16\pi^2)$ and α is the canonical gauge parameter associated with a linear covariant gauge fixing with $\alpha = 0$ corresponding to the Landau gauge. Further, $\beta(a)$ is the β function and $\gamma_\alpha(a,\alpha)$ is the anomalous dimension of the gauge parameter where we retain the conventions of [8]. While the anomalous dimensions for gauge invariant operators are independent of the gauge parameter in massless renormalization schemes, such as $\overline{\text{MS}}$, we have retained it here since the RI′/SMOM scheme is a mass dependent scheme and therefore the anomalous dimensions will depend on the choice of gauge as will be evident later and hence is indicated in (2.8). The explicit values for the $\overline{\text{MS}}$ anomalous dimensions have already been determined in the $\overline{\text{MS}}$ scheme and we record them as they are required for constructing the RI′/SMOM three loop anomalous dimensions later. From [12]

$$\gamma_{11}^{W_3}(a)|_{\overline{\rm MS}} = \frac{25}{6} C_F a + \frac{1}{432}[8560 C_A C_F - 2035 C_F^2 - 3320 C_F T_F N_f] a^2 + \frac{1}{15\,552}[(285\,120\zeta(3) + 1\,778\,866) C_A^2 C_F - (855\,360\zeta(3) + 311213) C_A C_F^2 - (1\,036\,800\zeta(3) + 497\,992) C_A C_F T_F N_f + (570\,240\zeta(3) - 244\,505) C_F^3 + (1\,036\,800\zeta(3) - 814508) C_F^2 T_F N_f - 82\,208 C_F T_F^2 N_f^2] a^3 + O(a^4),$$

$$\gamma_{12}^{W_3}(a)|_{\overline{\rm MS}} = -\frac{3}{2} C_F a + \frac{1}{144}[81 C_F^2 - 848 C_A C_F + 424 C_F T_F N_f] a^2 + O(a^3),$$

$$\gamma_{13}^{W_3}(a)|_{\overline{\rm MS}} = -\frac{1}{2} C_F a + \frac{1}{144}[103 C_F^2 - 388 C_A C_F + 104 C_F T_F N_f] a^2 + O(a^3),$$

$$\gamma_{22}^{W_3}(a)|_{\overline{\rm MS}} = \frac{8}{3} C_F a + \frac{1}{27}[376 C_A C_F - 112 C_F^2 - 128 C_F T_F N_f] a^2 + \frac{1}{243}[(5184\zeta(3) + 20\,920) C_A^2 C_F - (15\,552\zeta(3) + 8528) C_A C_F^2 - (10\,368\zeta(3) + 6256) C_A C_F T_F N_f + (10\,368\zeta(3) - 560) C_F^3 + (10\,368\zeta(3) - 6824) C_F^2 T_F N_f - 896 C_F T_F^2 N_f^2] a^3 + O(a^4),$$

$$\gamma_{23}^{W_3}(a)|_{\overline{\rm MS}} = -\frac{4}{3} C_F a + \frac{1}{27}[56 C_F^2 - 188 C_A C_F + 64 C_F T_F N_f] a^2 + \frac{1}{243}[(7776\zeta(3) + 4264) C_A C_F^2 - (2592\zeta(3) + 10\,460) C_A^2 C_F + (5184\zeta(3) + 3128) C_A C_F T_F N_f - (5184\zeta(3) - 280) C_F^3 - (5184\zeta(3) - 3412) C_F^2 T_F N_f + 448 C_F T_F^2 N_f^2] a^3 + O(a^4),$$

$$\gamma_{33}^{W_3}(a)|_{\overline{\rm MS}} = O(a^4) \tag{2.10}$$

where $\zeta(z)$ is the Riemann zeta function and the color group Casimirs are defined as

$$T^a T^a = C_F, \qquad f^{acd} f^{bcd} = C_A \delta^{ab}, \qquad \mathrm{Tr}(T^a T^b) = \delta^{ab} T_F \tag{2.11}$$

where $1 \le a \le N_A$, N_A is the dimension of the adjoint representation and N_f is the number of massless quarks. However, only the diagonal elements and the 23 elements are known to three loops. The off-diagonal elements for the first row were only determined to two loops. The reason for this is that to extract the mixing matrix renormalization constants, [12], one has to consider various momentum routings of the operator inserted in the Green's function of Fig. 1 as well as exploit the fact that there must be no terms such as $(\ln(p^2/\mu^2))/\epsilon$, where $d = 4 - 2\epsilon$ and p is the sole momentum flowing through external legs. This latter criterion establishes relations between the off-diagonal counterterms. For the $n = 2$ moment operator one had sufficient linear equations in order to determine the three loop mixing matrix, [12]. With the increase in moment one requires a further linear relation in order to resolve the counterterms at three loops. This can only be achieved by a *four* loop renormalization for W_3 which was not considered in [12].

Throughout we will use the convention that the results will be expressed in the RI′/SMOM scheme, using RI′/SMOM variables, unless otherwise specified as shown in (2.10). As we will be mapping between the $\overline{\rm MS}$ and RI′/SMOM schemes we recall that the variables are not necessarily the same in each scheme. However, the relations are known to three loops, [8], and are

$$a_{\rm RI'} = a_{\overline{\rm MS}} + O(a_{\overline{\rm MS}}^5) \tag{2.12}$$

and

$$\alpha_{\rm RI'} = \Bigg[1 + ((-9\alpha_{\overline{\rm MS}}^2 - 18\alpha_{\overline{\rm MS}} - 97) C_A + 80 T_F N_f) \frac{a_{\overline{\rm MS}}}{36} + ((18\alpha_{\overline{\rm MS}}^4 - 18\alpha_{\overline{\rm MS}}^3 + 190\alpha_{\overline{\rm MS}}^2 - 576\zeta(3)\alpha_{\overline{\rm MS}} + 463\alpha_{\overline{\rm MS}} + 864\zeta(3) - 7143) C_A^2 + (-320\alpha_{\overline{\rm MS}}^2 - 320\alpha_{\overline{\rm MS}} + 2304\zeta(3) + 4248) C_A T_F N_f + (-4608\zeta(3) + 5280) C_F T_F N_f) \frac{a_{\overline{\rm MS}}^2}{288} \Bigg] \alpha_{\overline{\rm MS}} + O(a_{\overline{\rm MS}}^3) \tag{2.13}$$

which implies that the Landau gauge is preserved under mappings between schemes. These relations derive from [8] where the full three loop renormalization of QCD in a linear covariant gauge in the RI′ scheme was recorded. That scheme was defined by ensuring that all the 2-point function renormalizations were performed in such a way that there were no $O(a)$ corrections at the subtraction point. So included within this is the gluon 2-point function which determines the renormalization of α. The coupling constant renormalization was performed using 3-point function renormalization but those functions and higher are renormalized in the $\overline{\rm MS}$ fashion whence the trivial mapping of (2.12). The RI′/SMOM scheme differs from RI′ in that the 3-point and higher functions are rendered finite by ensuring there are no $O(a)$ corrections at the symmetric subtraction point. We should note that in [2,3] the renormalization of α was assumed to have been carried out in the $\overline{\rm MS}$ scheme. While this means that Landau gauge expressions will be similar, if one was comparing the α

dependent parts for the same quantities in the definition of [1–3] and that used here, then they would not be in agreement for nonzero α.

The Green's function we will compute with the level W_3 operators inserted is illustrated in Fig. 1 where there are three Lorentz indices associated with the operator insertion which is denoted by the circle containing the cross. Therefore, at the symmetric subtraction point the Green's function has to be decomposed into a basis of Lorentz tensors which obey the same tracelessness and total symmetry as the original operator. The choice of basis tensors was given in [10] for the one loop computation and we retain the same basis here. More explicitly at the subtraction point we formally write

$$\langle \psi(p) \mathcal{O}^i_{\mu\nu\sigma}(-p-q) \bar{\psi}(q) \rangle|_{p^2=q^2=-\mu^2} = \sum_{k=1}^{14} \mathcal{P}^i_{(k)\mu\nu\sigma}(p,q) \Sigma^{\mathcal{O}^i}_{(k)}(p,q) \tag{2.14}$$

where $\mathcal{P}^i_{(k)\mu\nu\sigma}(p,q)$ are the basis tensors. For W_3 there are 14 of these and as they are cumbersome we have provided the explicit forms in the Appendix. It should be noted that the tensor basis we use is defined at the symmetric subtraction point and is the same basis for each of the operators of level W_3. When the squares of the external momenta of the Green's function are not all equal then the basis of tensors will be much larger. The objects which we will concentrate on computing are the amplitudes associated with each tensor for each W_3 operator insertion. These are $\Sigma^{\mathcal{O}^i}_{(k)}(p,q)$ and defined by the projection method, [10],

$$\Sigma^{\mathcal{O}^i}_{(k)}(p,q) = \mathcal{M}^i_{kl} \mathcal{P}^{i\mu\nu\sigma}_{(l)}(p,q) \langle \psi(p) \times \mathcal{O}^i_{\mu\nu\sigma}(-p-q) \bar{\psi}(q) \rangle|_{p^2=q^2=-\mu^2}. \tag{2.15}$$

Here $\mathcal{M}^i_{kl}$ is a dimension 14 matrix of rational polynomials in d. It is computed by inverting the matrix $\mathcal{N}^i_{kl}$ which is defined by

$$\mathcal{N}^{\,i}_{kl} = \mathcal{P}^i_{(k)\mu\nu\sigma}(p,q) \mathcal{P}^{i\mu\nu\sigma}_{(l)}(p,q)|_{p^2=q^2=-\mu^2}. \tag{2.16}$$

Because of the large number of basis tensors this matrix is equally cumbersome and is also recorded explicitly in the Appendix.

In the choice of basis tensors we used the generalized γ matrices $\Gamma^{\mu_1\dots\mu_m}_{(m)}$, [30–34], which are defined by

$$\Gamma^{\mu_1\dots\mu_n}_{(n)} = \gamma^{[\mu_1} \dots \gamma^{\mu_n]} \tag{2.17}$$

where the notation includes an overall factor $1/n!$. These are totally antisymmetric and form the generalization of the γ matrices in d-dimensional spacetime. Although we are ultimately interested in four dimensions they are a more appropriate object to use to build the tensor basis as the generalized γ space naturally partitions into regions defined by the number of free Lorentz indices. The justification for this lies in the observation, [30–32],

$$\mathrm{tr}\,(\Gamma^{\mu_1\dots\mu_m}_{(m)} \Gamma^{\nu_1\dots\nu_n}_{(n)}) \propto \delta_{mn} I^{\mu_1\dots\mu_m\nu_1\dots\nu_n} \tag{2.18}$$

where $I^{\mu_1\dots\mu_m\nu_1\dots\nu_n}$ is the unit matrix in this Γ space. Indeed this is why (A.2) has a $\Gamma^{\mu}_{(1)}$ and $\Gamma^{\mu\nu\sigma}_{(3)}$ subspace. Though we will always use γ^μ as the object rather than the generalized object. Hence, when constructing the tensor basis one can be confident that a complete set has been used. For instance, the antisymmetry means that at most two external momenta can be contracted with $\Gamma^{\mu_1\dots\mu_n}_{(n)}$ for $n \geq 2$. So the Lorentz aspect of the basis is built from combinations of tensors from the set $\{\eta^{\mu\nu}, p^\mu, q^\mu, \Gamma^{\mu_1\dots\mu_n}_{(n)}\}$. The choice of the 14 given in the Appendix were also derived from the explicit one loop computation, [10]. In other words the one loop Feynman diagrams were expressed in terms of the basis Lorentz tensor Feynman integrals which were evaluated directly. The expressions for these were substituted back into the original computation and the Lorentz indices contracted producing the 14 basis tensors, [10]. Therefore, given that we use the basis here but do not construct all possible two loop basis Lorentz tensor master integrals. Instead we use the projection method.

The machinery used to perform the computations are the symbolic manipulation language FORM, [35]. The 3 one loop and 37 two loop Feynman diagrams are generated in electronic format using the QGRAF package, [36], and then Lorentz and color group indices appended. As the Green's function contains two external momenta then packages such as MINCER, [37], are not applicable. Instead we first rewrite all the diagrams as scalar integrals where the numerator tensor structure is rewritten in terms of the denominator propagators in as far as this is possible. In this form we then used the Laporta algorithm, [38]. This creates a set of linear relations between all integrals with a certain specification for each appropriate momentum topology. The relations are established by integration by parts and Lorentz identities and the integrals of interest are rewritten in terms of a relatively small set of basic scalar master Feynman integrals. For the two loop Green's function these have already been established to the order in ϵ we need in order to find the finite part. This is not an empty statement because in the huge set of linear equations spurious poles in ϵ can emerge in the factor multiplying the master integrals. To handle such an enormous set of equations requires computer technology. Several packages are available but we used the REDUZE program, [39], based on GINAC, [40], which is written in C++ . It turns out that for the 37 two loop diagrams there are only two basic momentum topologies. In other words the momentum routing of all the diagrams could be expressed in terms of one or other of these configurations. Also given that we had constructed the algorithm for the lower moment operators, W_2, it was straightforward to lift the extra integral relations needed for W_3 out of the earlier database created for W_2. Therefore, we are reasonably confident that our procedure is consistent as the same routines were used for the quark

current operator renormalization which agreed with the two loop results of [2,3] for the scalar and tensor operators.

III. RENORMALIZATION

The renormalization of the W_3 operators has been outlined in [12] and we briefly recall the procedure. First, we compute the Green's function with all the parameters being regarded as bare. This is the simplest way to proceed when dealing with automatic symbolic manipulation computer programs. The renormalized variables are then introduced via the associated renormalization constant in the canonical way, [41]. This ensures that the counterterms are correctly embedded within the Feynman diagrams automatically. For the operators themselves the renormalized versions are also introduced by a similar rescaling involving their renormalization constant. However, unlike the fields and parameters this renormalization is not multiplicative due to the mixing, [12]. Therefore, when the renormalized operators are introduced we include the operators into which it mixes. Consequently poles in ϵ will appear in various channels of the tensor basis but not all. They have to be removed by the criterion of the scheme in question. For $\overline{\text{MS}}$ this is achieved in the standard way. However, for the RI$'$/SMOM scheme it is more involved. If, for the moment we concentrate on a generic tensor channel which contains divergences, and denote that channel by 0, then the renormalization condition is, [1],

$$\lim_{\epsilon\to 0}[Z_\psi^{\text{RI}'} Z_{\mathcal{O}}^{\text{RI}'/\text{SMOM}} \Sigma_{(0)}^{\mathcal{O}}(p,q)]|_{p^2=q^2=-\mu^2} = 1. \qquad (3.1)$$

The quark wave function renormalization is performed in the RI$'$ scheme but as we are now dealing with a 3-point function then the operator renormalization constant obeys the same condition. The ethos for the regularization invariant schemes is that after renormalization there are no $O(a)$ corrections at the subtraction point which here is the symmetric point, [1]. So both the quark 2-point function and the Green's function we are considering have no $O(a)$ corrections. For W_3 we need to qualify (3.1) by saying that the generic renormalization constant could represent a combination of counterterms which appear in the mixing matrix. Therefore to determine all the information to compute the anomalous dimension mixing matrix requires the solution of a set of linear equations. This was also a feature of the original $\overline{\text{MS}}$ renormalization of the operators in [12]. While (3.1) is the condition which determines the operator renormalization constant to avoid confusion it is important to note that the *full* Green's function is multiplied by the mixing matrix of renormalization constants. So after the renormalization constants have been fixed the finite parts of all the amplitudes are affected by the renormalization.

Having outlined the general aspects of the renormalization we have to discuss the specific renormalization of $\partial\partial W_3$ which requires special attention. This is because that operator is related to the divergence of the vector current. The vector current is a special operator in that it is physical. Therefore, not only does it not get renormalized to all orders in perturbation theory but the renormalization constant is the *same* in all schemes. See, for example, [42] for a summary of this. This is a consequence of gauge symmetry and effected by the Slavnov-Taylor identities. Therefore, our RI$'$ scheme renormalization of $\partial\partial W_3$ must be consistent with these general principles. So from the definition of $\partial\partial W_3$ we have to project out that piece which corresponds to the total derivative of the divergence of the vector current. Contracting the Green's function containing the $\partial\partial W_3$ operator with $(p+q)_\mu \eta_{\nu\sigma}$ does not produce anything nontrivial due to the traceless condition. Instead we contract with $(p+q)_\mu (p+q)_\nu \times (p+q)_\sigma$ and this combination of vectors is chosen since that is the momentum flow through the operator itself. This produces a nontrivial combination of the amplitudes. Ordinarily this would produce an expression where there are poles in ϵ. Indeed the individual amplitudes have divergences. However, as the underlying operator is finite there are no divergences in ϵ. While this is consistent with earlier remarks one has also to ensure that the Slavnov-Taylor identity is actually satisfied and this relates to the finite part of this projection. Computing in the $\overline{\text{MS}}$ scheme, where the scheme dependence is in effect the quark wave function renormalization from (3.1), we find that the combination of amplitudes of the contraction proportional to $\not{p}$ is, [10],

$$\begin{aligned}
&\frac{3[d-6]}{4[d+2]}\Sigma_{(1)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} + \frac{3}{2}\Sigma_{(2)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} \\
&\quad + \frac{3[d-4]}{4[d+2]}\Sigma_{(3)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} - \frac{[d-10]}{8[d+2]}\Sigma_{(4)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} \\
&\quad - \frac{3}{8}\Sigma_{(5)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} - \frac{3}{8}\Sigma_{(6)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} \\
&\quad - \frac{[d-10]}{8[d+2]}\Sigma_{(7)}^{\partial\partial W_3}(p,q)|_{\overline{\text{MS}}} \\
&= -\frac{1}{2} - \frac{1}{2}\alpha C_F a + \left[\left[\frac{3}{2}\zeta(3) - \frac{41}{8} + \frac{3}{2}\zeta(3)\alpha \right.\right. \\
&\quad \left.\left. - \frac{13}{4}\alpha - \frac{9}{16}\alpha^2\right] C_F C_A + \frac{7}{4} C_F T_F N_f + \frac{5}{16} C_F^2 \right] a^2 + O(a^3).
\end{aligned} \qquad (3.2)$$

A similar expression for the piece involving $\not{q}$ produces the same outcome. Clearly it is proportional to the finite part of the quark 2-point function after renormalization in the $\overline{\text{MS}}$ scheme. Indeed the equality of the $\not{p}$ and $\not{q}$ parts with the unit renormalization constant required for $\partial\partial W_3$, due to the absence of poles in ϵ, as well as the explicit values for the $\overline{\text{MS}}$ amplitudes means that our renormalization is consistent with the Slavnov-Taylor identities. Repeating the process for the RI$'$/SMOM scheme with the RI$'$ scheme quark wave function renormalization constant produces a similar result which is

$$\frac{3[d-6]}{4[d+2]}\Sigma^{\partial\partial W_3}_{(1)}(p,q) + \frac{3}{2}\Sigma^{\partial\partial W_3}_{(2)}(p,q) + \frac{3[d-4]}{4[d+2]}\Sigma^{\partial\partial W_3}_{(3)}(p,q) - \frac{[d-10]}{8[d+2]}\Sigma^{\partial\partial W_3}_{(4)}(p,q) - \frac{3}{8}\Sigma^{\partial\partial W_3}_{(5)}(p,q) - \frac{3}{8}\Sigma^{\partial\partial W_3}_{(6)}(p,q) - \frac{[d-10]}{8[d+2]}\Sigma^{\partial\partial W_3}_{(7)}(p,q) = -\frac{1}{2} + O(a^3). \tag{3.3}$$

Clearly there are no $O(a)$ corrections which is in agreement with the RI′ scheme quark 2-point function after renormalization. The fact that when this combination is computed that there were no poles in ϵ means the renormalization of $\partial\partial W_3$ also has a unit renormalization constant in this scheme consistent with the physicality of the operator itself.

The procedure to determine the remaining renormalization constants of the mixing matrix is to identify a set of amplitudes or their combinations to which one can apply the condition (3.1). As noted in [9,10] for the RI′/SMOM scheme there is not a unique way of doing this. One could use the same combinations for W_3 and ∂W_3 as $\partial\partial W_3$. However, as there are five renormalization constants to determine one would require four other conditions. Instead we extend to two loops the RI′/SMOM scheme definition given in [10]. Here the first three channels are rendered finite by the generic condition (3.1). In addition to the lack of uniqueness in defining the scheme because of the choice of amplitudes there is also the issue that our basis choice of tensors defining the amplitude channels is not unique. One could have defined a different set of basis tensors, though with the same symmetries as the operators themselves. Then one would have another definition of RI′/SMOM. The choice we make here is in some sense a minimal choice and one could readily make other more exotic ones. However, following this particular path we have determined the renormalization constants and have encoded them in the mixing matrix of anomalous dimensions. We have

$$\begin{aligned}
\gamma^{W_3}_{11}(a,\alpha)|_{\mathrm{RI}'/\mathrm{SMOM}} &= \frac{25}{6}C_F a + \left[\left[(384\alpha^2 + 1152\alpha - 3168)\psi'\left(\frac{1}{3}\right) - (256\alpha^2 + 786\alpha - 2112)\pi^2 - 288\alpha^2 - 864\alpha + 66\,204\right]C_A - 6105C_F + \left[1152\psi'\left(\frac{1}{3}\right) - 768\pi^2 - 24\,696\right]T_F N_f\right]\frac{C_F a^2}{1296} + O(a^3),\\
\gamma^{W_3}_{12}(a,\alpha)|_{\mathrm{RI}'/\mathrm{SMOM}} &= -\frac{3}{2}C_F a + \left[\left[(2112 - 864\alpha - 288\alpha^2)\psi'\left(\frac{1}{3}\right) + (192\alpha^2 + 576\alpha - 1408)\pi^2 + 216\alpha^2 + 648\alpha - 63\,684\right]C_A + 2187C_F + \left[26\,280 + 512\pi^2 - 768\psi'\left(\frac{1}{3}\right)\right]T_F N_f\right]\frac{C_F a^2}{3888} + O(a^3),\\
\gamma^{W_3}_{13}(a,\alpha)|_{\mathrm{RI}'/\mathrm{SMOM}} &= -\frac{1}{2}C_F a + \left[\left[(288\alpha^2 + 864\alpha - 1056)\psi'\left(\frac{1}{3}\right) - (192\alpha^2 + 576\alpha - 704)\pi^2 - 1188\alpha^2 - 3564\alpha - 10\,872\right]C_A + \left[(2592 - 2784\alpha)\psi'\left(\frac{1}{3}\right) + (1856\alpha - 1728)\pi^2 + 8244\alpha - 9207\right]C_F + \left[384\psi'\left(\frac{1}{3}\right) - 256\pi^2 + 2952\right]T_F N_f\right]\frac{C_F a^2}{3888} + O(a^3),\\
\gamma^{W_3}_{22}(a,\alpha)|_{\mathrm{RI}'/\mathrm{SMOM}} &= \frac{8}{3}C_F a + \left[\left[(108\alpha^2 + 324\alpha - 924)\psi'\left(\frac{1}{3}\right) - (72\alpha^2 + 216\alpha - 616)\pi^2 - 81\alpha^2 - 243\alpha + 16\,866\right]C_A - 2016C_F + \left[336\psi'\left(\frac{1}{3}\right) - 224\pi^2 - 5976\right]T_F N_f\right]\frac{C_F a^2}{486} + O(a^3),\\
\gamma^{W_3}_{23}(a,\alpha)|_{\mathrm{RI}'/\mathrm{SMOM}} &= -\frac{4}{3}C_F a + \left[\left[264\psi'\left(\frac{1}{3}\right) - 176\pi^2 - 81\alpha^2 - 243\alpha - 6651\right]C_A + \left[144\psi'\left(\frac{1}{3}\right) - 288\psi'\left(\frac{1}{3}\right)\alpha - 288 + 648\alpha - 96\pi^2 + 192\pi^2\alpha^2\right]C_F + \left[64\pi^2 + 2340 - 96\psi'\left(\frac{1}{3}\right)\right]T_F N_f\right]\frac{C_F a^2}{486} + O(a^3),\\
\gamma^{W_3}_{33}(a,\alpha)|_{\mathrm{RI}'/\mathrm{SMOM}} &= O(a^3),
\end{aligned} \tag{3.4}$$

where $\psi(z)$ is the derivative of the logarithm of the Euler Γ function. As a check on our procedures we first verified that the two loop $\overline{\mathrm{MS}}$ mixing matrix of anomalous dimensions emerged and agreed with [12]. While [12] was a three loop computation it used the MINCER algorithm, [37], designed for massless 2-point function renormalization in dimensional regularization. The one loop terms are clearly gauge parameter independent but the two and higher loop corrections will depend on the gauge parameter α. This is because in mass dependent renormalization schemes, such as RI′/SMOM, the anomalous dimensions of gauge invariant operators are not gauge independent in contrast to mass independent schemes

such as $\overline{\rm MS}$. A final observation on the calculation is that the renormalization of ∂W_3 is the same as that for W_2 which was considered in [9]. This is a nontrivial check since one has a different set of tensors for the basis due to the imbalance of Lorentz indices between the operators. That the same renormalization emerges is partly due to a similar renormalization scheme choice but also because the set of operators W_n are all connected via the tower of operators involving the associated total derivative operators.

IV. CONVERSION FUNCTIONS

Having constructed the RI$'$/SMOM mixing matrix of anomalous dimensions at two loops we can construct the associated conversion functions. These allow one to translate between renormalization schemes. For us the two schemes will be RI$'$/SMOM and $\overline{\rm MS}$ and general background can be found in, for example, [42]. However, as the renormalization of the operators is via a mixing matrix one has to extend the theory to allow for this. So the natural extension is to a matrix of conversion functions which is formally defined by

$$C^{W_3}_{ij}(a,\alpha) = Z^{W_3}_{ik,{\rm RI}'/{\rm SMOM}}[Z^{W_3}_{kj,\overline{\rm MS}}]^{-1}. \tag{4.1}$$

As the parameters a and α are tied to a renormalization scheme we have to make a choice and note that in the argument of the conversion functions the parameters will be in the $\overline{\rm MS}$ scheme. So in this definition those parameters in the RI$'$/SMOM renormalization constants have to be converted to the corresponding $\overline{\rm MS}$ parameters via (2.12) and (2.13). Otherwise aside from being inconsistent one would have a nonfinite expression due to poles in ϵ. For practical purposes, the definition translates into

$$\begin{aligned}
C^{W_3}_{ii}(a,\alpha) &= \frac{Z^{W_3}_{ii,{\rm RI}'/{\rm SMOM}}}{Z^{W_3}_{ii,\overline{\rm MS}}},\\
C^{W_3}_{12}(a,\alpha) &= \frac{Z^{W_3}_{12,{\rm RI}'/{\rm SMOM}}}{Z^{W_3}_{22,\overline{\rm MS}}} - \frac{Z^{W_3}_{11,{\rm RI}'/{\rm SMOM}}Z^{W_3}_{12,\overline{\rm MS}}}{Z^{W_3}_{11,\overline{\rm MS}}Z^{W_3}_{22,\overline{\rm MS}}},\\
C^{W_3}_{13}(a,\alpha) &= \frac{Z^{W_3}_{13,{\rm RI}'/{\rm SMOM}}}{Z^{W_3}_{33,\overline{\rm MS}}} + \frac{Z^{W_3}_{11,{\rm RI}'/{\rm SMOM}}Z^{W_3}_{12,\overline{\rm MS}}Z^{W_3}_{23,\overline{\rm MS}}}{Z^{W_3}_{11,\overline{\rm MS}}Z^{W_3}_{22,\overline{\rm MS}}Z^{W_3}_{33,\overline{\rm MS}}} - \frac{Z^{W_3}_{11,{\rm RI}'/{\rm SMOM}}Z^{W_3}_{13,\overline{\rm MS}}}{Z^{W_3}_{11,\overline{\rm MS}}Z^{W_3}_{33,\overline{\rm MS}}} - \frac{Z^{W_3}_{12,{\rm RI}'/{\rm SMOM}}Z^{W_3}_{23,\overline{\rm MS}}}{Z^{W_3}_{22,\overline{\rm MS}}Z^{W_3}_{33,\overline{\rm MS}}},\\
C^{W_3}_{23}(a,\alpha) &= \frac{Z^{W_3}_{23,{\rm RI}'/{\rm SMOM}}}{Z^{W_3}_{33,\overline{\rm MS}}} - \frac{Z^{W_3}_{22,{\rm RI}'/{\rm SMOM}}Z^{W_3}_{23,\overline{\rm MS}}}{Z^{W_3}_{22,\overline{\rm MS}}Z^{W_3}_{33,\overline{\rm MS}}}
\end{aligned} \tag{4.2}$$

for individual entries of the conversion function matrix where the first term corresponds to the diagonal and there is no sum over i and $i = 1$, 2 or 3. Equipped with these the explicit values of the matrix elements to two loops are

$$\begin{aligned}
C^{W_3}_{11}(a,\alpha) = 1 &+ \left[(192\alpha - 216)\psi'\left(\frac{1}{3}\right) + (144 - 128\alpha)\pi^2 - 144\alpha + 2763\right]\frac{C_F a}{324}\\
&+ \left[\left[(294\,912\alpha^2 - 663\,552\alpha + 248\,832)\left(\psi'\left(\frac{1}{3}\right)\right)^2 + (884\,736\alpha - 393\,216\alpha^2 - 331\,776)\psi'\left(\frac{1}{3}\right)\pi^2\right.\right.\\
&\left.\left. + (13\,360\,896\alpha - 857\,088\alpha^2 - 16\,819\,488)\psi'\left(\frac{1}{3}\right) - (5184 + 82\,944\alpha)\psi'''\left(\frac{1}{3}\right)\right.\right.\\
&\left.\left. + (40\,310\,784\alpha - 94\,058\,496)s_2\left(\frac{\pi}{6}\right) + (188\,116\,992 - 80\,621\,568\alpha)s_2\left(\frac{\pi}{2}\right)\right.\right.\\
&\left.\left. + (156\,764\,160 - 67\,184\,640\alpha)s_3\left(\frac{\pi}{6}\right) + (53\,747\,712\alpha - 125\,411\,328)s_3\left(\frac{\pi}{2}\right)\right.\right.\\
&\left.\left. + (131\,072\alpha^2 - 73\,728\alpha + 124\,416)\pi^4 + (571\,392\alpha^2 - 8907264\alpha + 11\,212\,992)\pi^2 + 616\,896\alpha^2\right.\right.\\
&\left.\left. - 3\,348\,864\alpha + 20\,576\,997 + (1\,492\,992\alpha - 1\,679\,616)\Sigma + (3\,359\,232\alpha + 3\,919\,104)\zeta(3)\right.\right.\\
&\left.\left. + (279\,936\alpha - 653\,184)\frac{\ln^2(3)\pi}{\sqrt{3}}\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+ (7\,838\,208 - 3\,359\,232\alpha)\frac{\ln(3)\pi}{\sqrt{3}} + (701\,568 - 300\,672\alpha)\frac{\pi^3}{\sqrt{3}}\Big]C_F \\
&+ \Big[62\,208\Big(\psi'\Big(\frac{1}{3}\Big)\Big)^2 - 82\,944\psi'\Big(\frac{1}{3}\Big)\pi^2 + (580\,608\alpha^2 - 1\,111\,968\alpha + 974\,592)\psi'\Big(\frac{1}{3}\Big) \\
&+ (145\,152 - 46\,656\alpha)\psi'''\Big(\frac{1}{3}\Big) + (65\,505\,024 - 25\,194\,240\alpha)s_2\Big(\frac{\pi}{6}\Big)(50\,388\,480\alpha - 131\,010\,048)s_2\Big(\frac{\pi}{2}\Big) \\
&+ (41\,990\,400\alpha - 109\,175\,040)s_3\Big(\frac{\pi}{6}\Big) \\
&+ (50\,388\,480\alpha - 131\,010\,048)s_2\Big(\frac{\pi}{2}\Big) + (41\,990\,400\alpha - 109\,175\,040)s_3\Big(\frac{\pi}{6}\Big) \\
&+ (87\,340\,032 - 33\,592\,320\alpha)s_3\Big(\frac{\pi}{2}\Big) + (124\,416\alpha - 359\,424)\pi^4 - (387\,072\alpha^2 - 741\,312\alpha + 649\,728)\pi^2 \\
&- 505\,440\alpha^2 - 1\,714\,608\alpha + 79\,566\,624 + (746\,496\alpha - 1\,213\,056)\Sigma + (1\,679\,616\alpha - 24\,354\,432)\zeta(3) \\
&+ (454\,896 - 174\,960\alpha)\frac{\ln^2(3)\pi}{\sqrt{3}} + (2\,099\,520\alpha - 5\,458\,752)\frac{\ln(3)\pi}{\sqrt{3}} + (187\,920\alpha - 488\,592)\frac{\pi^3}{\sqrt{3}}\Big]C_A \\
&+ \Big[2\,052\,864\psi'\Big(\frac{1}{3}\Big) - 1\,368\,576\pi^2 - 32\,160\,888\Big]T_F N_f\Big]\frac{C_F a^2}{839\,808} + O(a^3),
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
C_{12}^{W_3}(a,\alpha) = &\Big[(48 - 48\alpha)\psi'\Big(\frac{1}{3}\Big) + (32\alpha - 32)\pi^2 + 36\alpha - 927\Big]\frac{C_F a}{324} + \Big[\Big[(276\,480\alpha - 129\,024\alpha^2 - 396\,288) \\
&\times\Big(\psi'\Big(\frac{1}{3}\Big)\Big)^2 + (172\,032\alpha^2 - 368\,640\alpha + 528\,384)\psi'\Big(\frac{1}{3}\Big)\pi^2 + (421\,632\alpha^2 - 2\,847\,744\alpha - 2\,817\,504)\psi'\Big(\frac{1}{3}\Big) \\
&- (155\,520 - 20\,736\alpha)\psi'''\Big(\frac{1}{3}\Big) - 73\,903\,104s_2\Big(\frac{\pi}{6}\Big) + 147\,806\,208s_2\Big(\frac{\pi}{2}\Big) + 123\,171\,840s_3\Big(\frac{\pi}{6}\Big) \\
&- 98\,537\,472s_3\Big(\frac{\pi}{2}\Big) - (57\,344\alpha^2 - 67\,584\alpha - 238\,592)\pi^4 - (281\,088\alpha^2 - 1\,898\,496\alpha - 1\,878\,336)\pi^2 \\
&- 383616\alpha^2 + 440\,640\alpha - 17\,134\,821 - (373\,248\alpha - 373\,248)\Sigma + 13\,996\,800\zeta(3) - 513\,216\frac{\ln^2(3)\pi}{\sqrt{3}} \\
&+ 6\,158\,592\frac{\ln(3)\pi}{\sqrt{3}} + 551\,232\frac{\pi^3}{\sqrt{3}}\Big]C_F + \Big[124\,416\Big(\psi'\Big(\frac{1}{3}\Big)\Big)^2 - 165\,888\psi'\Big(\frac{1}{3}\Big)\pi^2 \\
&- (207\,360\alpha^2 + 147\,744\alpha - 2\,208\,384)\psi'\Big(\frac{1}{3}\Big) + (18\,144 + 15\,552\alpha)\psi'''\Big(\frac{1}{3}\Big) \\
&+ (15\,116\,544 + 5\,038\,848\alpha)s_2\Big(\frac{\pi}{6}\Big) - (10\,077\,696\alpha + 30\,233\,088)s_2\Big(\frac{\pi}{2}\Big) \\
&- (8\,398\,080\alpha + 25\,194\,240)s_3\Big(\frac{\pi}{6}\Big) + (20\,155\,392 + 6\,718\,464\alpha)s_3\Big(\frac{\pi}{2}\Big) + (6912 - 41\,472\alpha)\pi^4 \\
&+ (138\,240\alpha^2 + 98\,496\alpha - 1\,472\,256)\pi^2 + 225\,504\alpha^2 + 594\,864\alpha - 23\,258\,016 + (93\,312 - 186\,624\alpha)\Sigma \\
&+ (839\,808 - 839\,808\alpha)\zeta(3) + (34\,992\alpha + 104\,976)\frac{\ln^2(3)\pi}{\sqrt{3}} - (1\,259\,712 + 419\,904\alpha)\frac{\ln(3)\pi}{\sqrt{3}} \\
&- (112\,752 + 37\,584\alpha)\frac{\pi^3}{\sqrt{3}}\Big]C_A + \Big[373\,248\pi^2 - 559\,872\psi'\Big(\frac{1}{3}\Big) + 11\,611\,512\Big]T_F N_f\Big]\frac{C_F a^2}{839\,808} + O(a^3),
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
C_{13}^{W_3}(a,\alpha) = &\Big[(48\alpha - 24)\psi'\Big(\frac{1}{3}\Big) + (16 - 32\alpha)\pi^2 - 198\alpha - 9\Big]\frac{C_F a}{324} + \Big[\Big[(129\,024\alpha^2 - 193\,536\alpha + 322\,560) \\
&\times\Big(\psi'\Big(\frac{1}{3}\Big)\Big)^2 - (172\,032\alpha^2 - 258\,048\alpha + 430\,080)\psi'\Big(\frac{1}{3}\Big)\pi^2
\end{aligned}
$$

$$
\begin{aligned}
&- (743\,040\alpha^2 - 1\,156\,032\alpha + 9\,328\,608)\psi'\left(\frac{1}{3}\right) - (134\,784 + 20\,736\alpha)\psi'''\left(\frac{1}{3}\right) - 80\,621\,568 s_2\left(\frac{\pi}{6}\right) \\
&+ 161\,243\,136 s_2\left(\frac{\pi}{2}\right) + 134\,369\,280 s_3\left(\frac{\pi}{6}\right) - 107\,495\,424 s_3\left(\frac{\pi}{2}\right) + (57\,344\alpha^2 - 30\,720\alpha + 502\,784)\pi^4 \\
&+ (495\,360\alpha^2 - 770\,688\alpha + 6\,219\,072)\pi^2 + 974\,592\alpha^2 + 629\,856\alpha - 2\,984\,931 + (373\,248\alpha - 186\,624)\Sigma \\
&+ 11\,757\,312\zeta(3) - 559\,872\frac{\ln^2(3)\pi}{\sqrt{3}} + 6\,718\,464\frac{\ln(3)\pi}{\sqrt{3}} + 601\,344\frac{\pi^3}{\sqrt{3}}\Bigg]C_F \\
&+ \Bigg[165\,888\psi'\left(\frac{1}{3}\right)\pi^2 - 124\,416\left(\psi'\left(\frac{1}{3}\right)\right)^2 + (129\,600\alpha^2 + 1\,314\,144\alpha + 1\,428\,192)\psi'\left(\frac{1}{3}\right) \\
&+ (20\,736 + 15\,552\alpha)\psi'''\left(\frac{1}{3}\right) + (12\,597\,120 + 10\,077\,696\alpha)s_2\left(\frac{\pi}{6}\right) - (20\,155\,392\alpha + 25\,194\,240)s_2\left(\frac{\pi}{2}\right) \\
&- (16\,796\,160\alpha + 20\,995\,200)s_3\left(\frac{\pi}{6}\right) + (16\,796\,190 + 13\,436\,928\alpha)s_3\left(\frac{\pi}{2}\right) - (110\,592 + 41\,472\alpha)\pi^4 \\
&- (86\,400\alpha^2 + 876\,096\alpha + 952\,128)\pi^2 - 517\,104\alpha^2 - 1\,662\,120\alpha - 1\,840\,968 - (93\,312 - 186\,624\alpha)\Sigma \\
&+ (629\,856 - 1\,679\,616\alpha)\zeta(3) + (69\,984\alpha + 87\,480)\frac{\ln^2(3)\pi}{\sqrt{3}} - (1\,049\,760 + 839\,808\alpha)\frac{\ln(3)\pi}{\sqrt{3}} \\
&- (93\,960 + 75\,168\alpha)\frac{\pi^3}{\sqrt{3}}\Bigg]C_A + \Bigg[311\,040\psi'\left(\frac{1}{3}\right) - 207\,360\pi^2 + 390\,744\Bigg]T_F N_f\Bigg]\frac{C_F a^2}{839\,808} + O(a^3), \qquad (4.5)
\end{aligned}
$$

$$
\begin{aligned}
C_{22}^{W_3}(a,\alpha) = 1 &+ \left[(36\alpha - 42)\psi'\left(\frac{1}{3}\right) + (28 - 24\alpha)\pi^2 - 27\alpha + 459\right]\frac{C_F a}{81} + \Bigg[\Bigg[(5184\alpha^2 - 12\,096\alpha - 4608) \\
&\times \left(\psi'\left(\frac{1}{3}\right)\right)^2 + (16\,128\alpha - 6912\alpha^2 + 6144)\psi'\left(\frac{1}{3}\right)\pi^2 + (328\,536\alpha - 13\,608\alpha^2 - 613\,656)\psi'\left(\frac{1}{3}\right) \\
&- (5022 + 1944\alpha)\psi'''\left(\frac{1}{3}\right) + (1\,259\,712\alpha - 5\,248\,800)s_2\left(\frac{\pi}{6}\right) + (10\,497\,600 - 2\,519\,424\alpha)s_2\left(\frac{\pi}{2}\right) \\
&+ (8\,748\,000 - 2\,099\,520\alpha)s_3\left(\frac{\pi}{6}\right) + (1\,679\,616\alpha - 6\,998\,400)s_3\left(\frac{\pi}{2}\right) + (2304\alpha^2 - 192\alpha + 11\,344)\pi^4 \\
&+ (9072\alpha^2 - 219\,024\alpha + 409\,104)\pi^2 + 7290\alpha^2 - 90\,882\alpha + 107\,568 + (34\,992\alpha - 40\,824)\Sigma \\
&+ (104\,976\alpha + 559\,872)\zeta(3) + (8748\alpha - 36\,450)\frac{\ln^2(3)\pi}{\sqrt{3}} + (437\,400 - 104\,976\alpha)\frac{\ln(3)\pi}{\sqrt{3}} \\
&+ (39150 - 9396\alpha)\frac{\pi^3}{\sqrt{3}}\Bigg]C_F + \Bigg[5832\left(\psi'\left(\frac{1}{3}\right)\right)^2 - 7776\psi'\left(\frac{1}{3}\right)\pi^2 + (11\,664\alpha^2 - 39\,366\alpha + 99\,468)\psi'\left(\frac{1}{3}\right) \\
&+ (5103 - 972\alpha)\psi'''\left(\frac{1}{3}\right) + (2\,519\,424 - 629\,856\alpha)s_2\left(\frac{\pi}{6}\right) + (1\,259\,712\alpha - 5\,038\,848)s_2\left(\frac{\pi}{2}\right) \\
&+ (1\,049\,760\alpha - 41\,990\,40)s_3\left(\frac{\pi}{6}\right) + (3\,359\,232 - 839\,808\alpha)s_3\left(\frac{\pi}{2}\right) + (2592\alpha - 11\,016)\pi^4 \\
&- (7776\alpha^2 - 26\,244\alpha + 66\,312)\pi^2 - 8748\alpha^2 - 34\,992\alpha + 1\,759\,644 + (17\,496\alpha - 34\,992)\Sigma \\
&+ (26\,244\alpha - 734\,832)\zeta(3) + (17\,496 - 4374\alpha)\frac{\ln^2(3)\pi}{\sqrt{3}} + (52\,488\alpha - 209\,952)\frac{\ln(3)\pi}{\sqrt{3}} \\
&+ (4698\alpha - 18\,792)\frac{\pi^3}{\sqrt{3}}\Bigg]C_A + \Bigg[46\,656\psi'\left(\frac{1}{3}\right) - 31\,104\pi^2 - 642\,168\Bigg]T_F N_f\Bigg]\frac{C_F a^2}{26\,244} + O(a^3), \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
C_{23}^{W_3}(a,\alpha) = &\left[24\psi'\left(\frac{1}{3}\right) - 16\pi^2 - 54\alpha - 297\right]\frac{C_F a}{162} + \left[\left[(13824\alpha + 35136)\left(\psi'\left(\frac{1}{3}\right)\right)^2 - (18432\alpha + 46848)\psi'\left(\frac{1}{3}\right)\pi^2\right.\right.\\
&- (342144\alpha + 19440\alpha^2 - 824904)\psi'\left(\frac{1}{3}\right) + 18792\psi'''\left(\frac{1}{3}\right) + (10917504 - 1679616\alpha)s_2\left(\frac{\pi}{6}\right)\\
&+ (3359232 - 21835008\alpha)s_2\left(\frac{\pi}{2}\right) + (2799360 - 18195840\alpha)s_3\left(\frac{\pi}{6}\right) + (14556672 - 2239488\alpha)s_3\left(\frac{\pi}{2}\right)\\
&+ (6144\alpha - 34496)\pi^4 + (12960\alpha^2 + 228096\alpha - 549936)\pi^2 + 37908\alpha^2 + 76788\alpha - 188892\\
&+ 46656\Sigma - (139968\alpha + 1609632)\zeta(3) + (75816 - 11664\alpha)\frac{\ln^2(3)\pi}{\sqrt{3}} + (139968\alpha - 909792)\frac{\ln(3)\pi}{\sqrt{3}}\\
&\left.\left.+ (12528\alpha - 81432)\frac{\pi^3}{\sqrt{3}}\right]C_F + \left[31104\psi'\left(\frac{1}{3}\right)\pi^2 - 23328\left(\psi'\left(\frac{1}{3}\right)\right)^2 + (107892\alpha - 5832\alpha^2 - 122472)\psi'\left(\frac{1}{3}\right)\right.\right.\\
&+ (972\alpha - 9720)\psi'''\left(\frac{1}{3}\right) + (944784\alpha - 4513968)s_2\left(\frac{\pi}{6}\right) + (9027936 - 1889568\alpha)s_2\left(\frac{\pi}{2}\right)\\
&+ (7523280 - 1574640\alpha)s_3\left(\frac{\pi}{6}\right) + (1259712\alpha - 6018624)s_3\left(\frac{\pi}{2}\right) + (15552 - 2592\alpha)\pi^4\\
&+ (3888\alpha^2 - 71928\alpha + 81648)\pi^2 - 21870\alpha^2 - 113724\alpha - 2488482 + 34992\Sigma\\
&+ (1277208 - 52488\alpha)\zeta(3) + (6561\alpha - 31347)\frac{\ln^2(3)\pi}{\sqrt{3}} + (376164 - 78732\alpha)\frac{\ln(3)\pi}{\sqrt{3}}\\
&\left.\left.+ (33669 - 7047\alpha)\frac{\pi^3}{\sqrt{3}}\right]C_A + \left[29376\pi^2 - 44064\psi'\left(\frac{1}{3}\right) + 946080\right]T_F N_f\right]\frac{C_F a^2}{104976} + O(a^3) \qquad (4.7)
\end{aligned}
$$

and

$$C_{33}^{W_3}(a,\alpha) = 1 + O(a^3). \qquad (4.8)$$

The final expression merely reflects the fact that $\partial\partial W_3$ is related to the vector current which is a physical operator. The values for row 2 and 3 correspond to the values of the conversion function matrix for W_2 given in [9].[1] This is consistent with the Wilson operator renormalization being part of a tower of operators. These expressions include the usual transcendental type of numbers such as powers of π and the Riemann zeta function of odd argument as well as rationals. However, given the nature of the subtraction point other classes of basic numbers arise. These include derivatives of the Euler ψ function and the natural logarithm as well as the polylogarithm functions through the specific function

$$s_n(z) = \frac{1}{\sqrt{3}}\Im\left[\mathrm{Li}_n\left(\frac{e^{iz}}{\sqrt{3}}\right)\right]. \qquad (4.9)$$

In addition a combination of various harmonic polylogarithms also appears which is

$$\Sigma = \mathcal{H}_{31}^{(2)} + \mathcal{H}_{43}^{(2)} \qquad (4.10)$$

using the notation of [2]. These particular basic numbers derive from the explicit expressions for the scalar master integrals which the Laporta algorithm produces as a result of the reduction of integrals. They have been evaluated by various authors in [43–46].

For the conversion to all the numerical forms we present here we have used the following numerical values for the various polygamma and polylogarithm functions which are

$$
\begin{aligned}
&\zeta(3) = 1.202\,056\,90, \quad \Sigma = 6.345\,173\,34, \quad \psi'\left(\frac{1}{3}\right) = 10.095\,597\,13, \quad \psi'''\left(\frac{1}{3}\right) = 488.183\,816\,7,\\
&s_2\left(\frac{\pi}{2}\right) = 0.322\,258\,82, \quad s_2\left(\frac{\pi}{6}\right) = 0.224\,596\,02, \quad s_3\left(\frac{\pi}{2}\right) = 0.329\,483\,20, \quad s_3\left(\frac{\pi}{6}\right) = 0.192\,593\,41.
\end{aligned} \qquad (4.11)
$$

Expressing the conversion matrix in numerical form for the color group $SU(3)$ produces

[1]The expression for $C_{23}^{W_3}(a,\alpha)$ corrects a typographical error in the presentation of the two loop term involving C_F^2 of the corresponding equation for $C_{12}^{W_2}(a,\alpha)$ recorded in [9] but which was correct in the attached data file of that article.

$$C_{11}^{W_3}(a,\alpha) = 1 + [2.185\,371\,5\alpha + 8.245\,160\,7]a + [9.959\,421\,6\alpha^2 + 38.424\,142\,4\alpha + 224.426\,596\,3 - 19.800\,798\,8N_f]a^2 + O(a^3)$$
$$C_{12}^{W_3}(a,\alpha) = -[0.546\,342\,9\alpha + 3.120\,323\,8]a - [3.448\,606\,4\alpha^2 + 15.245\,969\,4\alpha + 91.803\,747\,8 - 7.654\,987\,8N_f]a^2 + O(a^3),$$
$$C_{13}^{W_3}(a,\alpha) = -[0.120\,323\,8\alpha + 0.384\,282\,6]a - [0.383\,560\,1\alpha^2 + 1.233\,209\,3\alpha + 9.054\,372\,3 - 1.178\,299\,0N_f]a^2 + O(a^3),$$
$$C_{22}^{W_3}(a,\alpha) = 1 + [1.639\,028\,7\alpha + 5.124\,836\,9]a + [6.510\,815\,1\alpha^2 + 23.178\,173\,0\alpha + 132.622\,848\,6 - 12.145\,811\,0N_f]a^2 + O(a^3),$$
$$C_{23}^{W_3}(a,\alpha) = -[0.444\,444\,4\alpha + 1.749\,953\,4]a - [2.130\,145\,5\alpha^2 + 7.277\,240\,5\alpha + 49.196\,706\,3 - 5.024\,368\,2N_f]a^2 + O(a^3),$$
$$C_{33}^{W_3}(a,\alpha) = 1 + O(a^3). \tag{4.12}$$

Clearly there is a large correction at two loops for W_3 itself compared with ∂W_3 or W_2. As one of the motivations for developing the RI′/SMOM scheme was the hope that the convergence of the conversion function would improve we recall the numerical values for the conversion functions from RI′ to $\overline{\text{MS}}$. From [47] we have the numerical values in the Landau gauge

$$\tilde{C}_{11}^{W_3}(a,0) = 1 + 7.925\,925\,9a + [215.085\,359\,3 - 18.980\,9671N_f]a^2 + O(a^3),$$
$$\tilde{C}_{22}^{W_3}(a,0) = 1 + 4.592\,592\,6a + [119.826\,815\,8 - 10.979\,423\,9N_f]a^2 + O(a^3),$$
$$\tilde{C}_{33}^{W_3}(a,0) = 1 + O(a^3). \tag{4.13}$$

Given the fact the top entry is the key one it appears that the RI′/SMOM scheme has a slightly larger two loop correction compared to the RI′ conversion function. A similar observation was made for W_2, [10]. However, as noted in that article it is not entirely clear whether one can truly compare these conversion functions. This is because the nature of the RI′ scheme is such that it cannot access the off-diagonal elements of the mixing matrix. This stems from the momentum configuration of the underlying Green's function. Indeed in this respect it is not entirely clear whether one can regard RI′ as a full renormalization scheme for the Wilson operators as a result of the mixing with total derivative operators. For operators where there is no mixing such as the quark currents then a comparison of conversion functions would appear more appropriate, [1–3].

5. THREE LOOP ANOMALOUS DIMENSIONS

One aspect of the conversion functions is that they can be used to determine the anomalous dimensions given knowledge of the anomalous dimensions in one scheme. Although we are working with mixing matrices here it is straightforward to extend the formalism to show that in our case

$$\gamma^{W_3}_{ij,\text{RI}'/\text{SMOM}}(a_{\text{RI}'},\alpha_{\text{RI}'}) = C^{W_3}_{ik}(a_{\overline{\text{MS}}},\alpha_{\overline{\text{MS}}})\gamma^{W_3}_{kl,\overline{\text{MS}}}(a_{\overline{\text{MS}}},\alpha_{\overline{\text{MS}}})(C^{W_3}_{lj}(a_{\overline{\text{MS}}},\alpha_{\overline{\text{MS}}}))^{-1} - \left[\mu\frac{d}{d\mu}C^{W_3}_{ik}(a_{\overline{\text{MS}}},\alpha_{\overline{\text{MS}}})\right](C^{W_3}_{kj}(a_{\overline{\text{MS}}},\alpha_{\overline{\text{MS}}}))^{-1} \tag{5.1}$$

where we now have to explicitly label the scheme the variables correspond to by the subscript. More explicitly for the three diagonal elements of the anomalous dimension mixing matrix we have

$$\gamma^{W_3}_{ii,\text{RI}'/\text{SMOM}}(a_{\text{RI}'},\alpha_{\text{RI}'}) = \gamma^{W_3}_{ii,\overline{\text{MS}}}(a_{\overline{\text{MS}}}) - \beta(a_{\overline{\text{MS}}})\frac{\partial}{\partial a_{\overline{\text{MS}}}}\ln C^{W_3}_{ii}(a_{\overline{\text{MS}}},\alpha_{\overline{\text{MS}}}) - \alpha_{\overline{\text{MS}}}\gamma^{\overline{\text{MS}}}_{\alpha}(a_{\overline{\text{MS}}},\alpha_{\overline{\text{MS}}})\frac{\partial}{\partial \alpha_{\overline{\text{MS}}}}\ln C^{W_3}_{ii}(a_{\overline{\text{MS}}},\alpha_{\overline{\text{MS}}}) \tag{5.2}$$

where again there is no sum over i and $i = 1, 2$ or 3. The off-diagonal elements are more involved since

$$\gamma^{W_3}_{12,\mathrm{RI}'/\mathrm{SMOM}}(a_{\mathrm{RI}'}, \alpha_{\mathrm{RI}'}) = [\gamma^{W_3}_{12,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{11}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) - \beta(a_{\overline{\mathrm{MS}}}) \frac{\partial}{\partial a_{\overline{\mathrm{MS}}}} C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) - \alpha_{\overline{\mathrm{MS}}} \gamma^{\overline{\mathrm{MS}}}_{\alpha}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$
$$\times \frac{\partial}{\partial \alpha_{\overline{\mathrm{MS}}}} C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) - \gamma^{W_3}_{11,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) + \gamma^{W_3}_{22,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$
$$+ C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \beta(a_{\overline{\mathrm{MS}}}) \frac{\partial}{\partial a_{\overline{\mathrm{MS}}}} \ln C^{W_3}_{11}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) + C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \alpha_{\overline{\mathrm{MS}}} \gamma^{\overline{\mathrm{MS}}}_{\alpha}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$
$$\times \frac{\partial}{\partial \alpha_{\overline{\mathrm{MS}}}} \ln C^{W_3}_{11}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})][C^{W_3}_{22}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})]^{-1}, \tag{5.3}$$

$$\gamma^{W_3}_{13,\mathrm{RI}'/\mathrm{SMOM}}(a_{\mathrm{RI}'}, \alpha_{\mathrm{RI}'}) = \Big[\gamma^{W_3}_{13,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{11}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) + \gamma^{W_3}_{23,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) - \gamma^{W_3}_{11,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{13}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$
$$+ \gamma^{W_3}_{33,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{13}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) - \beta(a_{\overline{\mathrm{MS}}}) \frac{\partial}{\partial a_{\overline{\mathrm{MS}}}} C^{W_3}_{13}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$
$$- \alpha_{\overline{\mathrm{MS}}} \gamma^{\overline{\mathrm{MS}}}_{\alpha}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \frac{\partial}{\partial \alpha_{\overline{\mathrm{MS}}}} C^{W_3}_{13}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) + C^{W_3}_{13}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \beta(a_{\overline{\mathrm{MS}}}) \frac{\partial}{\partial a_{\overline{\mathrm{MS}}}} \ln C^{W_3}_{11}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$
$$+ C^{W_3}_{13}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \alpha_{\overline{\mathrm{MS}}} \gamma^{\overline{\mathrm{MS}}}_{\alpha}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \frac{\partial}{\partial \alpha_{\overline{\mathrm{MS}}}} \ln C^{W_3}_{11}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \Big] [C^{W_3}_{33}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})]^{-1}$$
$$+ \Big[\gamma^{W_3}_{11,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) - \gamma^{W_3}_{12,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{11}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) - \gamma^{W_3}_{22,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$
$$+ \beta(a_{\overline{\mathrm{MS}}}) \frac{\partial}{\partial a_{\overline{\mathrm{MS}}}} C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) + \alpha_{\overline{\mathrm{MS}}} \gamma^{\overline{\mathrm{MS}}}_{\alpha}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \frac{\partial}{\partial \alpha_{\overline{\mathrm{MS}}}} C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$
$$- C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \beta(a_{\overline{\mathrm{MS}}}) \frac{\partial}{\partial a_{\overline{\mathrm{MS}}}} \ln C^{W_3}_{11}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) - C^{W_3}_{12}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \alpha_{\overline{\mathrm{MS}}} \gamma^{\overline{\mathrm{MS}}}_{\alpha}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$
$$\times \frac{\partial}{\partial \alpha_{\overline{\mathrm{MS}}}} \ln C^{W_3}_{11}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \Big] C^{W_3}_{23}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) [C^{W_3}_{22}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) C^{W_3}_{33}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})]^{-1}, \tag{5.4}$$

and

$$\gamma^{W_3}_{23,\mathrm{RI}'/\mathrm{SMOM}}(a_{\mathrm{RI}'}, \alpha_{\mathrm{RI}'}) = \Big[\gamma^{W_3}_{23,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{22}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) - \beta(a_{\overline{\mathrm{MS}}}) \frac{\partial}{\partial a_{\overline{\mathrm{MS}}}} C^{W_3}_{23}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) - \alpha_{\overline{\mathrm{MS}}} \gamma^{\overline{\mathrm{MS}}}_{\alpha}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$
$$\times \frac{\partial}{\partial \alpha_{\overline{\mathrm{MS}}}} C^{W_3}_{23}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) - \gamma^{W_3}_{22,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{23}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) + \gamma^{W_3}_{33,\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}}) C^{W_3}_{23}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$
$$+ C^{W_3}_{23}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \beta(a_{\overline{\mathrm{MS}}}) \frac{\partial}{\partial a_{\overline{\mathrm{MS}}}} \ln C^{W_3}_{22}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) + C^{W_3}_{23}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \alpha_{\overline{\mathrm{MS}}} \gamma^{\overline{\mathrm{MS}}}_{\alpha}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})$$
$$\times \frac{\partial}{\partial \alpha_{\overline{\mathrm{MS}}}} \ln C^{W_3}_{22}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}) \Big] [C^{W_3}_{33}(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}})]^{-1}. \tag{5.5}$$

Using these we have first verified that the two loop arbitrary gauge anomalous dimensions, (3.4), are reproduced. However, equipped with the explicit values of the two loop conversion functions as well as various three loop $\overline{\mathrm{MS}}$ anomalous dimensions we have the Landau gauge expressions

$$\gamma_{11}^{W_3}(a,0)|_{\mathrm{RI}'/\mathrm{SMOM}} = \frac{25}{6}C_F a + \left[\left[704\pi^2 - 1056\psi'\left(\frac{1}{3}\right) + 22068\right]C_A - 2035C_F \right.$$
$$\left. + \left[384\psi'\left(\frac{1}{3}\right) - 256\pi^2 - 8232\right]T_F N_f\right]\frac{C_F a^2}{432} + O(a^3)$$
$$+ \left[\left[25344\left(\psi'\left(\frac{1}{3}\right)\right)^2 - 33792\psi'\left(\frac{1}{3}\right)\pi^2 + 44544\psi'\left(\frac{1}{3}\right) + 59136\psi'''\left(\frac{1}{3}\right) + 26687232 s_2\left(\frac{\pi}{6}\right)\right.\right.$$
$$\left.\left. - 53374464 s_2\left(\frac{\pi}{2}\right) - 44478720 s_3\left(\frac{\pi}{6}\right) + 35582976 s_3\left(\frac{\pi}{2}\right) - 146432\pi^4 - 29696\pi^2 - 494208\Sigma\right.\right.$$
$$\left.\left. - 9066816\zeta(3) + 42261846 + 185328\frac{\ln^2(3)\pi}{\sqrt{3}} - 2223936\frac{\ln(3)\pi}{\sqrt{3}} - 199056\frac{\pi^3}{\sqrt{3}}\right]C_A^2\right.$$
$$+ \left[25344\left(\psi'\left(\frac{1}{3}\right)\right)^2 - 33792\psi'\left(\frac{1}{3}\right)\pi^2 - 4907232\psi'\left(\frac{1}{3}\right) - 2112\psi'''\left(\frac{1}{3}\right) - 38320128 s_2\left(\frac{\pi}{6}\right)\right.$$
$$\left. + 76640256 s_2\left(\frac{\pi}{2}\right) + 63866880 s_3\left(\frac{\pi}{6}\right) - 51093504 s_3\left(\frac{\pi}{2}\right) + 16896\pi^4 + 3271488\pi^2\right.$$
$$\left. - 684288\Sigma - 969408\zeta(3) - 4991286 - 266112\frac{\ln^2(3)\pi}{\sqrt{3}} + 3193344\frac{\ln(3)\pi}{\sqrt{3}} + 285824\frac{\pi^3}{\sqrt{3}}\right]C_A C_F$$
$$+ \left[12288\psi'\left(\frac{1}{3}\right)\pi^2 - 9216\left(\psi'\left(\frac{1}{3}\right)\right)^2 + 899328\psi'\left(\frac{1}{3}\right) - 21504\psi'''\left(\frac{1}{3}\right) - 9704448 s_2\left(\frac{\pi}{6}\right)\right.$$
$$\left. + 19408896 s_2\left(\frac{\pi}{2}\right) + 16174080 s_3\left(\frac{\pi}{6}\right) - 12939264 s_3\left(\frac{\pi}{2}\right) + 53248\pi^4 - 599552\pi^2 + 179712\Sigma\right.$$
$$\left. + 497664\zeta(3) - 29036688 - 67392\frac{\ln^2(3)\pi}{\sqrt{3}} + 808704\frac{\ln(3)\pi}{\sqrt{3}} + 72384\frac{\pi^3}{\sqrt{3}}\right]C_A T_F N_f$$
$$+ \left[12288\psi'\left(\frac{1}{3}\right)\pi^2 - 9216\left(\psi'\left(\frac{1}{3}\right)\right)^2 + 1908864\psi'\left(\frac{1}{3}\right) + 768\psi'''\left(\frac{1}{3}\right) + 13934592 s_2\left(\frac{\pi}{6}\right)\right.$$
$$\left. - 27869184 s_2\left(\frac{\pi}{2}\right) - 23224320 s_3\left(\frac{\pi}{6}\right) + 18579456 s_3\left(\frac{\pi}{2}\right) - 6144\pi^4 - 1272576\pi^2 + 248832\Sigma\right.$$
$$\left. + 2529792\zeta(3) - 2559504 + 96768\frac{\ln^2(3)\pi}{\sqrt{3}} - 1161216\frac{\ln(3)\pi}{\sqrt{3}} - 103936\frac{\pi^3}{\sqrt{3}}\right]C_F T_F N_f$$
$$+ \left[202752\pi^2 - 304128\psi'\left(\frac{1}{3}\right) + 4517952\right]T_F^2 N_f^2 + [1710720\zeta(3) - 733515]C_F^2\left]\frac{C_F a^3}{46656} + O(a^4),\right. \tag{5.6}$$

$$\gamma_{22}^{W_3}(a,0)|_{\mathrm{RI}'/\mathrm{SMOM}} = \frac{8}{3}C_F a + \left[\left[308\pi^2 - 462\psi'\left(\frac{1}{3}\right) + 8433\right]C_A - 1008C_F\right.$$
$$\left. + \left[168\psi'\left(\frac{1}{3}\right) - 112\pi^2 - 2988\right]T_F N_f\right]\frac{C_F a^2}{243} + \left[\left[64152\left(\psi'\left(\frac{1}{3}\right)\right)^2 - 85536\psi'\left(\frac{1}{3}\right)\pi^2\right.\right.$$
$$\left.\left. + 862812\psi'\left(\frac{1}{3}\right) + 56133\psi'''\left(\frac{1}{3}\right) + 27713664 s_2\left(\frac{\pi}{6}\right) - 55427328 s_2\left(\frac{\pi}{2}\right) - 46189440 s_3\left(\frac{\pi}{6}\right)\right.\right.$$
$$\left.\left. + 36951552 s_3\left(\frac{\pi}{2}\right) - 121176\pi^4 - 575208\pi^2 - 384912\Sigma - 7243344\zeta(3) + 25273296\right.\right.$$
$$\left.\left. + 192456\frac{\ln^2(3)\pi}{\sqrt{3}}\right.\right.$$

$$
\begin{aligned}
&- 2309472\frac{\ln(3)\pi}{\sqrt{3}} - 206712\frac{\pi^3}{\sqrt{3}}\Big]C_A^2 + \Big[119328\psi'\Big(\frac{1}{3}\Big)\pi^2 - 89496\Big(\psi'\Big(\frac{1}{3}\Big)\Big)^2 - 5901984\psi'\Big(\frac{1}{3}\Big) \\
&- 55242\psi'''\Big(\frac{1}{3}\Big) - 57736800s_2\Big(\frac{\pi}{6}\Big) + 115473600s_2\Big(\frac{\pi}{2}\Big) + 96228000s_3\Big(\frac{\pi}{6}\Big) - 76982400s_3\Big(\frac{\pi}{2}\Big) \\
&+ 107536\pi^4 + 3934656\pi^2 - 449064\Sigma + 3639168\zeta(3) - 4833270 - 400950\frac{\ln^2(3)\pi}{\sqrt{3}} \\
&+ 4811400\frac{\ln(3)\pi}{\sqrt{3}} + 430650\frac{\pi^3}{\sqrt{3}}\Big]C_A C_F + \Big[31104\Big(\psi'\Big(\frac{1}{3}\Big)\Big)^2 - 23328\psi'\Big(\frac{1}{3}\Big)\pi^2 + 251424\psi'\Big(\frac{1}{3}\Big) \\
&- 20412\psi'''\Big(\frac{1}{3}\Big) - 10077696s_2\Big(\frac{\pi}{6}\Big) + 20155392s_2\Big(\frac{\pi}{2}\Big) + 16796160s_3\Big(\frac{\pi}{6}\Big) - 13436928s_3\Big(\frac{\pi}{2}\Big) \\
&+ 44064\pi^4 - 167616\pi^2 + 139968\Sigma + 1259712\zeta(3) - 16603056 - 69984\frac{\ln^2(3)\pi}{\sqrt{3}} \\
&+ 839808\frac{\ln(3)\pi}{\sqrt{3}} + 75168\frac{\pi^3}{\sqrt{3}}\Big]C_A T_F N_f + \Big[32544\Big(\psi'\Big(\frac{1}{3}\Big)\Big)^2 - 43392\psi'\Big(\frac{1}{3}\Big)\pi^2 \\
&+ 2227824\psi'\Big(\frac{1}{3}\Big) + 20088\psi'''\Big(\frac{1}{3}\Big) + 20995200s_2\Big(\frac{\pi}{6}\Big) - 41990400s_2\Big(\frac{\pi}{2}\Big) - 34992000s_3\Big(\frac{\pi}{6}\Big) \\
&+ 27993600s_3\Big(\frac{\pi}{2}\Big) - 39104\pi^4 - 1485216\pi^2 + 163296\Sigma - 559872\zeta(3) - 742608 \\
&+ 145800\frac{\ln^2(3)\pi}{\sqrt{3}} - 1749600\frac{\ln(3)\pi}{\sqrt{3}} - 156600\frac{\pi^3}{\sqrt{3}}\Big]C_F T_F N_f + \Big[124416\pi^2 - 186624\psi'\Big(\frac{1}{3}\Big) \\
&+ 2423520\Big]T_F^2 N_f^2 + [1679616\zeta(3) - 90720]C_F^2\Big]\frac{C_F a^3}{39366} + O(a^4),
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
\gamma_{23}^{W_3}(a,0)|_{\mathrm{RI}'/\mathrm{SMOM}} = &-\frac{4}{3}C_F a + \Big[\Big[264\psi'\Big(\frac{1}{3}\Big) - 176\pi^2 - 6651\Big]C_A + \Big[144\psi'\Big(\frac{1}{3}\Big) - 96\pi^2 - 288\Big]C_F \\
&+ \Big[64\pi^2 - 96\psi'\Big(\frac{1}{3}\Big) + 2340\Big]T_F N_f\Big]\frac{C_F a^2}{486} + \Big[\Big[342144\psi'\Big(\frac{1}{3}\Big)\pi^2 \\
&- 256608\Big(\psi'\Big(\frac{1}{3}\Big)\Big)^2 - 1082808\psi'\Big(\frac{1}{3}\Big) - 106920\psi'''\Big(\frac{1}{3}\Big) - 49653648s_2\Big(\frac{\pi}{6}\Big) + 99307296s_2\Big(\frac{\pi}{2}\Big) \\
&+ 82756080s_3\Big(\frac{\pi}{6}\Big) - 66204864s_3\Big(\frac{\pi}{2}\Big) + 171072\pi^4 + 721872\pi^2 + 384912\Sigma + 12369672\zeta(3) \\
&- 37423134 - 344817\frac{\ln^2(3)\pi}{\sqrt{3}} + 4137804\frac{\ln(3)\pi}{\sqrt{3}} + 370359\frac{\pi^3}{\sqrt{3}}\Big]C_A^2 \\
&+ \Big[477504\Big(\psi'\Big(\frac{1}{3}\Big)\Big)^2 - 636672\psi'\Big(\frac{1}{3}\Big)\pi^2 + 7978176\psi'\Big(\frac{1}{3}\Big) + 204768\psi'''\Big(\frac{1}{3}\Big) + 117993024s_2\Big(\frac{\pi}{6}\Big) \\
&- 235986048s_2\Big(\frac{\pi}{2}\Big) - 196655040s_3\Big(\frac{\pi}{6}\Big) + 157324032s_3\Big(\frac{\pi}{2}\Big) - 333824\pi^4 - 5318784\pi^2 \\
&+ 653184\Sigma - 11897280\zeta(3) + 367416 + 819396\frac{\ln^2(3)\pi}{\sqrt{3}} - 9832752\frac{\ln(3)\pi}{\sqrt{3}} - 880092\frac{\pi^3}{\sqrt{3}}\Big]C_A C_F \\
&+ \Big[93312\Big(\psi'\Big(\frac{1}{3}\Big)\Big)^2 - 124416\psi'\Big(\frac{1}{3}\Big)\pi^2 - 150336\psi'\Big(\frac{1}{3}\Big) + 38880\psi'''\Big(\frac{1}{3}\Big) + 18055872s_2\Big(\frac{\pi}{6}\Big) \\
&- 36111744s_2\Big(\frac{\pi}{2}\Big) - 30093120s_3\Big(\frac{\pi}{6}\Big) + 24074496s_3\Big(\frac{\pi}{2}\Big) - 62208\pi^4 + 100224\pi^2 - 139968\Sigma \\
&- 1749600\zeta(3) + 24312312
\end{aligned}
$$

$$+ 125\,388\frac{\ln^2(3)\pi}{\sqrt{3}} - 1\,504\,656\frac{\ln(3)\pi}{\sqrt{3}} - 134\,676\frac{\pi^3}{\sqrt{3}}\Big]C_A T_F N_f + \Big[208\,896\psi'\Big(\frac{1}{3}\Big)\pi^2$$
$$- 156\,672\Big(\psi'\Big(\frac{1}{3}\Big)\Big)^2 - 3\,297\,024\psi'\Big(\frac{1}{3}\Big) - 75\,168\psi'''\Big(\frac{1}{3}\Big) - 43\,670\,016 s_2\Big(\frac{\pi}{6}\Big) + 87\,340\,032 s_2\Big(\frac{\pi}{2}\Big)$$
$$+ 72\,783\,360 s_3\Big(\frac{\pi}{6}\Big) - 58\,226\,688 s_3\Big(\frac{\pi}{2}\Big) + 130\,816\pi^4 + 2\,198\,016\pi^2 - 186\,624\Sigma + 3\,079\,296\zeta(3)$$
$$+ 4\,039\,632 - 303\,264\frac{\ln^2(3)\pi}{\sqrt{3}} + 3\,639\,168\frac{\ln(3)\pi}{\sqrt{3}} + 325\,728\frac{\pi^3}{\sqrt{3}}\Big]C_F T_F N_f$$
$$+ \Big[176\,256\psi'\Big(\frac{1}{3}\Big) - 117\,504\pi^2 - 3\,494\,016\Big]T_F^2 N_f^2 + \Big[138\,240\psi'\Big(\frac{1}{3}\Big)\pi^2 - 103\,680\Big(\psi'\Big(\frac{1}{3}\Big)\Big)^2$$
$$+ 1\,537\,056\psi'\Big(\frac{1}{3}\Big) - 34\,992\psi'''\Big(\frac{1}{3}\Big) - 1\,679\,616 s_2\Big(\frac{\pi}{6}\Big) + 3\,359\,232 s_2\Big(\frac{\pi}{2}\Big) + 2\,799\,360 s_3\Big(\frac{\pi}{6}\Big)$$
$$- 2\,239\,488 s_3\Big(\frac{\pi}{2}\Big) + 47\,232\pi^4 - 1\,024\,704\pi^2 + 139\,968\Sigma - 1\,399\,680\zeta(3) + 729\,648$$
$$- 11\,664\frac{\ln^2(3)\pi}{\sqrt{3}} + 139\,968\frac{\ln(3)\pi}{\sqrt{3}} + 12\,528\frac{\pi^3}{\sqrt{3}}\Big]C_F^2\Big]\frac{C_F a^3}{157\,464} + O(a^4) \tag{5.8}$$

and

$$\gamma_{33}^{W_3}(a,0)|_{\mathrm{RI}'/\mathrm{SMOM}} = O(a^4). \tag{5.9}$$

Again the final expression is the same as that for the vector operator. We note that we have not given the complete matrix at three loops. This is because the complete three loop $\overline{\mathrm{MS}}$ matrix is not known. The 12 and 13 elements were not determined in [12] as there was not sufficient information to disentangle the relation between the two counterterms. As was noted in [12] only a four loop computation could resolve this. However, the diagonal elements as well as the 23 element are now available in the RI′/SMOM scheme for comparison with the $\overline{\mathrm{MS}}$ expressions. In numerical form these anomalous dimensions are

$$\begin{aligned}
\gamma_{11}^{W_3}(a,0)|_{\mathrm{RI}'/\mathrm{SMOM}} &= 5.555\,555\,6a + [161.581\,541\,5 - 10.620\,230\,6N_f]a^2 + [6275.495\,687\,5 - 993.607\,231\,7N_f \\
&\quad + 24.639\,062\,3N_f^2]a^3 + O(a^4), \\
\gamma_{22}^{W_3}(a,0)|_{\mathrm{RI}'/\mathrm{SMOM}} &= 3.555\,555\,6a + [104.702\,424\,4 - 6.577\,051\,8N_f]a^2 + [4010.980\,382\,9 - 624.881\,767\,1N_f \\
&\quad + 14.965\,333\,7N_f^2]a^3 + O(a^4), \\
\gamma_{23}^{W_3}(a,0)|_{\mathrm{RI}'/\mathrm{SMOM}} &= -1.777\,777\,8a - [46.302\,861\,2 - 2.746\,882\,5N_f]a^2 - [1692.014\,351\,3 - 265.333\,971\,5N_f \\
&\quad + 6.084\,617\,1N_f^2]a^3 + O(a^4), \\
\gamma_{33}^{W_3}(a,0)|_{\mathrm{RI}'/\mathrm{SMOM}} &= O(a^4)
\end{aligned} \tag{5.10}$$

for $SU(3)$. For comparison the corresponding $\overline{\mathrm{MS}}$ values are

$$\begin{aligned}
\gamma_{11}^{W_3}(a,0)|_{\overline{\mathrm{MS}}} &= 5.555\,555\,6a + [70.884\,773\,7 - 5.123\,456\,8N_f]a^2 + [1244.913\,602\,4 - 199.637\,388\,3N_f \\
&\quad - 1.762\,002\,7N_f^2]a^3 + O(a^4), \\
\gamma_{22}^{W_3}(a,0)|_{\overline{\mathrm{MS}}} &= 3.555\,555\,6a + [48.329\,218\,1 - 3.160\,493\,8N_f]a^2 + [859.447\,837\,2 - 133.438\,161\,7N_f \\
&\quad - 1.229\,080\,9N_f^2]a^3 + O(a^4), \\
\gamma_{23}^{W_3}(a,0)|_{\overline{\mathrm{MS}}} &= -1.777\,777\,8a - [24.164\,609\,1 - 1.580\,246\,9N_f]a^2 - [429.723\,918\,6 - 66.719\,080\,9N_f \\
&\quad - 0.614\,540\,5N_f^2]a^3 + O(a^4), \\
\gamma_{33}^{W_3}(a,0)|_{\overline{\mathrm{MS}}} &= O(a^4)
\end{aligned} \tag{5.11}$$

which illustrates that the higher loop corrections are numerically smaller in the $\overline{\mathrm{MS}}$ scheme.

VI. AMPLITUDES

We devote this section to the main results which are the explicit forms of the two loop amplitudes in both the $\overline{\text{MS}}$ and RI′/SMOM schemes. Given the large number of amplitudes which have to be recorded we have chosen to do this in numerical form for an arbitrary color group. In [9,11] we also recorded the exact two loop expressions in the form of tables using the notation

$$\Sigma^{\mathcal{O}^i}_{(j)}(p,q) = \left(\sum_n c^{\mathcal{O}^i,(1)}_{(j)n} a^{(1)}_n\right) C_F a + \left(\sum_n c^{\mathcal{O}^i,(21)}_{(j)n} a^{(21)}_n\right) C_F T_F N_f a^2 + \left(\sum_n c^{\mathcal{O}^i,(22)}_{(j)n} a^{(22)}_n\right) C_F C_A a^2 + \left(\sum_n c^{\mathcal{O}^i,(23)}_{(j)n} a^{(23)}_n\right) C_F^2 a^2 + O(a^3). \tag{6.1}$$

Here $a^{(l)}_n$ correspond to a basis set of numbers which naturally arise in the computation. These include the transcendental type of numbers which have appeared in the conversion functions and anomalous dimensions earlier. In addition the set $a^{(l)}_n$ also included the gauge parameter dependence. The labelling is chosen in such a way that l indicates the loop order and in addition at two loops a second digit is appended to reference the color group Casimir associated with that set. The other quantities, $c^{\mathcal{O}^i,(1)}_{(j)n}$, are the actual rational coefficients of interest. The main reason for recalling this notation here is that attached to this article is an electronic file containing all the coefficients for all the amplitudes in both schemes in their exact forms in terms of the basis of numbers $a^{(l)}_n$. Therefore this is intended to allow for comparison with other results such as the relation of ∂W_3 and $\partial\partial W_3$ with W_2 and ∂W_2 respectively.

Therefore we now carry out the mundane task of recording the results. First, in the $\overline{\text{MS}}$ scheme we have [48]

$$\Sigma^{W_3}_{(1)}(p,q)|_{\overline{\text{MS}}} = -[0.128\,094\,2 + 0.040\,107\,9\alpha]a - [3.573\,963\,3 + 0.590\,932\,9\alpha + 0.144\,533\,0\alpha^2 - 0.392\,766\,3N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(2)}(p,q)|_{\overline{\text{MS}}} = -[0.648\,148\,1 + 0.131\,165\,1\alpha]a - [11.921\,467\,6 + 0.619\,004\,8\alpha + 0.492\,471\,3\alpha^2 - 1.668\,597\,6N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(3)}(p,q)|_{\overline{\text{MS}}} = -0.333\,333\,3 + [1.580\,184\,8 + 0.061\,790\,5\alpha]a + [23.390\,933\,0 - 0.0454\,454\alpha + 0.358\,724\,2\alpha^2 - 2.878\,059\,6N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(4)}(p,q)|_{\overline{\text{MS}}} = [0.342\,546\,0 + 0.048\,421\,2\alpha]a + [4.534\,238\,8 + 1.129\,214\,7\alpha + 0.170\,327\,7\alpha^2 - 0.236\,191\,1N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(5)}(p,q)|_{\overline{\text{MS}}} = [0.423\,847\,6 + 0.075\,159\,8\alpha]a + [4.593\,542\,3 + 1.275\,975\,1\alpha + 0.261\,965\,5\alpha^2 - 0.366\,053\,5N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(6)}(p,q)|_{\overline{\text{MS}}} = [0.584\,279\,3 + 0.150\,319\,6\alpha]a + [4.857\,474\,4 + 1.708\,1275\alpha + 0.523\,720\,4\alpha^2 - 0.684\,187\,2N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(7)}(p,q)|_{\overline{\rm MS}} = [1.067\,745\,9 + 0.664\,867\,9\alpha]a + [23.377\,608\,2 + 5.551\,735\,0\alpha + 2.410\,8591\alpha^2 - 2.126\,160\,4N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(8)}(p,q)|_{\overline{\rm MS}} = [0.222\,222\,2 + 0.155\,375\,7\alpha]a + [7.102\,707\,6 + 2.480\,654\,5\alpha + 0.542\,721\,0\alpha^2 - 0.163\,383\,8N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(9)}(p,q)|_{\overline{\rm MS}} = [0.282\,927\,0 + 0.208\,852\,9\alpha]a + [9.254\,603\,7 + 3.030\,048\,3\alpha + 0.738\,342\,4\alpha^2 - 0.264\,558\,4N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(10)}(p,q)|_{\overline{\rm MS}} = [0.477\,324\,8 + 0.337\,490\,0\alpha]a + [15.440\,212\,3 + 3.837\,351\,9\alpha + 1.223\,842\,7\alpha^2 - 0.724\,016\,1N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(11)}(p,q)|_{\overline{\rm MS}} = [0.561\,511\,0 + 0.150\,319\,6\alpha]a + [7.607\,960\,0 + 1.776\,502\,3\alpha + 0.543\,955\,3\alpha^2 - 0.606\,188\,8N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(12)}(p,q)|_{\overline{\rm MS}} = -0.060\,704\,8a - [1.275\,919\,6 + 0.244\,317\,9\alpha + 0.005\,058\,7\alpha^2 - 0.049\,904\,9N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(13)}(p,q)|_{\overline{\rm MS}} = -0.121\,409\,5a - [2.553\,703\,9 + 0.318\,815\,0\alpha + 0.010\,117\,5\alpha^2 - 0.127\,070\,6N_f]a^2 + O(a^3),$$

$$\Sigma^{W_3}_{(14)}(p,q)|_{\overline{\rm MS}} = -0.458\,899\,5a - [8.646\,026\,2 - 0.523\,998\,7\alpha + 0.038\,241\,6\alpha^2 - 0.757\,806\,6N_f]a^2 + O(a^3), \tag{6.2}$$

$$\Sigma^{\partial W_3}_{(1)}(p,q)|_{\overline{\rm MS}} = -[0.583\,317\,8 + 0.148\,141\,5\alpha]a - [12.461\,642\,8 + 0.809\,893\,9\alpha + 0.553\,628\,5\alpha^2 - 1.674\,789\,4N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(2)}(p,q)|_{\overline{\rm MS}} = -0.166\,666\,7 + [0.270\,821\,7 - 0.097\,198\,9\alpha]a + [1.020\,805\,3 - 0.537\,150\,2\alpha - 0.301\,999\,9\alpha^2 + 0.0393\,764N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(3)}(p,q)|_{\overline{\rm MS}} = -0.333\,333\,3 + [1.124\,961\,7 - 0.046\,249\,7\alpha + [14.503\,253\,5 - 0.264\,406\,4\alpha^2 - 1.596\,036\,5N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(4)}(p,q)|_{\overline{\rm MS}} = [0.626\,558\,7 + 0.101\,898\,4\alpha + [7.861\,884\,8 + 1.974\,225\,71\alpha + 0.360\,890\,3\alpha^2 - 0.479\,977\,1N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(5)}(p,q)|_{\overline{\rm MS}} = [0.760\,251\,8 + 1\,698\,307\alpha]a + [8.239\,259\,5 + 2.269\,413\,9\alpha + 0.597\,629\,7\alpha^2 - 0.741\,702\,3N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(6)}(p,q)|_{\overline{\rm MS}} = [0.988\,615\,8 + 0.419\,877\,3\alpha]a + [15.727\,878\,9 + 4.141\,353\,4\alpha + 1.502\,121\,5\alpha^2 - 1.386\,205\,6N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(7)}(p,q)|_{\overline{\rm MS}} = [1.311\,650\,7 + 0.852\,038\,2\alpha]a + [30.327\,743\,0 + 7.590\,065\,5\alpha + 3.074\,365\,8\alpha^2 - 2.413\,487\,1N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(8)}(p,q)|_{\overline{\rm MS}} = [0.466\,127\,0 + 0.342\,546\,0\alpha]a + [14.052\,842\,5 + 4.518\,985\,0\alpha + 1.206\,227\,7\alpha^2 - 0.450\,710\,5N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(9)}(p,q)|_{\overline{\rm MS}} = [0.687\,263\,5 + 0.478\,410\,6\alpha]a + [20.125\,008\,3 + 5.463\,274\,2\alpha + 1.716\,743\,5\alpha^2 - 0.966\,576\,7N_f]a^2 + O(a^3),$$
$$\Sigma^{\partial W_3}_{(10)}(p,q)|_{\overline{\rm MS}} = [0.813\,729\,1 + 0.432\,160\,9\alpha]a + [19.085\,929\,5 + 4.830\,790\,7\alpha + 1.559\,506\,9\alpha^2 - 1.099\,664\,8N_f]a^2 + O(a^3),$$
$$\Sigma^{\partial W_3}_{(11)}(p,q)|_{\overline{\rm MS}} = [0.845\,523\,7 + 0.203\,796\,9\alpha]a + [10.935\,606\,0 + 2.621\,534\,7\alpha + 0.734\,517\,9\alpha^2 - 0.849\,974\,7N_f]a^2 + O(a^3),$$
$$\Sigma^{\partial W_3}_{(12)}(p,q)|_{\overline{\rm MS}} = -0.148\,148\,1a - [2.946\,175\,0 + 0.442\,236\,0\alpha + 0.012\,345\,7\alpha^2 - 0.1476260N_f]a^2 + O(a^3),$$
$$\Sigma^{\partial W_3}_{(13)}(p,q)|_{\overline{\rm MS}} = -0.347\,245\,5a - [6.631\,228\,3 + 0.058\,077\,7\alpha + 0.0289\,371\alpha^2 - 0.501\,576\,9N_f]a^2 + O(a^3),$$
$$\Sigma^{\partial W_3}_{(14)}(p,q)|_{\overline{\rm MS}} = -0.546\,342\,9a - [10.316\,281\,5 - 0.326\,080\,6\alpha + 0.045\,528\,6\alpha^2 - 0.855\,527\,7N_f]a^2 + O(a^3) \quad (6.3)$$

and

$$\Sigma^{\partial\partial W_3}_{(1)}(p,q)|_{\overline{\rm MS}} = \Sigma^{\partial\partial W_3}_{(2)}(p,q)|_{\overline{\rm MS}} = \Sigma^{\partial\partial W_3}_{(3)}(p,q)|_{\overline{\rm MS}} = -0.333\,333\,3 + [0.541\,643\,4 - 0.194\,397\,9\alpha]a + [2.041\,610\,7 - 1.074\,300\,3\alpha - 0.603\,999\,7\alpha^2 + 0.0787\,529N_f]a^2 + O(a^3),$$
$$\Sigma^{\partial\partial W_3}_{(4)}(p,q)|_{\overline{\rm MS}} = \Sigma^{\partial\partial W_3}_{(11)}(p,q)|_{\overline{\rm MS}} = [1.472\,082\,5 + 0.305\,695\,3\alpha]a + [18.797\,490\,8 + 4.595\,781\,8\alpha + 1.095\,408\,3\alpha^2 - 1.329\,951\,8N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial\partial W_3}_{(5)}(p,q)|_{\overline{\rm MS}} = \Sigma^{\partial\partial W_3}_{(10)}(p,q)|_{\overline{\rm MS}} = [1.573\,980\,9 + 0.601\,991\,6\alpha]a + [27.325\,189\,0 + 7.100\,204\,7\alpha + 2.157\,136\,6\alpha^2 - 1.841\,367\,1N_f]a^2 + O(a^3),$$
$$\Sigma^{\partial\partial W_3}_{(6)}(p,q)|_{\overline{\rm MS}} = \Sigma^{\partial\partial W_3}_{(9)}(p,q)|_{\overline{\rm MS}} = [1.675\,879\,3 + 0.898\,287\,9\alpha]a + [35.852\,887\,2 + 9.604\,627\,6\alpha + 3.218\,865\,0\alpha^2 - 2.352\,782\,3N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial\partial W_3}_{(7)}(p,q)|_{\overline{\rm MS}} = \Sigma^{\partial\partial W_3}_{(8)}(p,q)|_{\overline{\rm MS}} = [1.777\,777\,8 + 1.194\,584\,2\alpha]a + [44.380\,585\,5 + 12.109\,050\,4\alpha + 4.280\,593\,4\alpha^2 - 2.864\,197\,5N_f]a^2 + O(a^3),$$
$$\Sigma^{\partial\partial W_3}_{(12)}(p,q)|_{\overline{\rm MS}} = \Sigma^{\partial\partial W_3}_{(13)}(p,q)|_{\overline{\rm MS}} = \Sigma^{\partial\partial W_3}_{(14)}(p,q)|_{\overline{\rm MS}} = -0.694\,491\,0a - [13.262\,456\,5 + 0.116\,155\,4\alpha + 0.057\,874\,3\alpha^2 - 1.003\,153\,7N_f]a^2 + O(a^3). \quad (6.4)$$

With the two loop corrections computed we note that the results for some of the ∂W_3 and $\partial\partial W_3$ amplitudes are proportional to those of lower moment operators. These relations were noted at one loop in [10] and extend to two loops now. For instance, channels 4 and 7 of $\partial\partial W_3$ are related to channels 3 and 5, respectively, of ∂W_2, [9,10], or equally channels 2 and 3 of the vector current, [10,11]. Also channels 4 and 7 of ∂W_3 are proportional to channels 5 and 7, respectively, of W_2. That not all channels have a similar relation is due to the imbalance of indices of the operator inserted in the Green's function. Also the absence of direct equality stems from the difference in the two bases used for each operator moment. Though this underlying agreement is a partial check on our computation. Equally the equality of several of the $\partial\partial W_3$ channels with themselves reflects the symmetric nature of this particular operator and is another minor calculational check.

The expressions for the RI′/SMOM scheme amplitudes have a similar structure to the $\overline{\rm MS}$ ones. However, given the nature of the scheme definition which we have used then there are no corrections to channels 1 and 2 of W_3 as well as 1 of ∂W_3. For channels 3 of W_3 and 2 and 3 of ∂W_3 only the tree term is present which we record with the rest of the amplitudes. We have

$$
\begin{aligned}
\Sigma^{W_3}_{(3)}(p,q) &= -0.333\,333\,3 + O(a^3),\\
\Sigma^{W_3}_{(4)}(p,q) &= [0.342\,546\,0 + 0.048\,421\,2\alpha]a + [4.837\,823\,9 + 1.256\,850\,4\alpha + 0.191\,762\,3\alpha^2 + 0.036\,315\,9\alpha^3\\
&\quad - (0.236\,191\,1 + 0.053\,801\,3\alpha)N_f]a^2 + O(a^3),\\
\Sigma^{W_3}_{(5)}(p,q) &= [0.423\,847\,6 + 0.075\,159\,8\alpha]a + [5.111\,148\,7 + 1.498\,349\,0\alpha + 0.273\,524\,6\alpha^2 + 0.056\,369\,7\alpha^3\\
&\quad - (0.366\,053\,5 + 0.083\,510\,9\alpha)N_f]a^2 + O(a^3),\\
\Sigma^{W_3}_{(6)}(p,q) &= [0.584\,279\,3 + 0.150\,319\,6\alpha]a + [5.946\,138\,5 + 2.263\,328\,3\alpha + 0.539\,795\,6\alpha^2 + 0.112\,739\,7\alpha^3\\
&\quad - (0.684\,187\,2 + 0.167\,021\,8\alpha)N_f]a^2 + O(a^3),\\
\Sigma^{W_3}_{(7)}(p,q) &= [1.067\,745\,9 + 0.664\,867\,9\alpha]a + [27.405\,401\,0 + 13.296\,573\,7\alpha + 3.365\,411\,9\alpha^2 + 0.498\,650\,9\alpha^3\\
&\quad - (2.126\,160\,4 + 0.738\,742\,1\alpha)N_f]a^2 + O(a^3),\\
\Sigma^{W_3}_{(8)}(p,q) &= [0.222\,222\,2 + 0.155\,375\,7\alpha]a + [6.797\,329\,3 + 3.210\,560\,7\alpha + 0.577\,286\,1\alpha^2 + 0.116\,531\,8\alpha^3\\
&\quad - (0.163\,383\,8 + 0.172\,639\,6\alpha)N_f]a^2 + O(a^3),\\
\Sigma^{W_3}_{(9)}(p,q) &= [0.282\,927\,0 + 0.208\,852\,9\alpha]a + [8.798\,886\,4 + 4.266\,244\,4\alpha + 0.860\,110\,8\alpha^2 + 0.156\,639\,7\alpha^3\\
&\quad - (0.264\,558\,4 + 0.232\,058\,8\alpha)N_f]a^2 + O(a^3),\\
\Sigma^{W_3}_{(10)}(p,q) &= [0.477\,324\,8 + 0.337\,490\,0\alpha]a + [16.231\,880\,8 + 7.540\,974\,8\alpha + 1.709\,090\,1\alpha^2 + 0.253\,117\,5\alpha^3\\
&\quad - (0.724\,016\,1 + 0.3749886\alpha)N_f]a^2 + O(a^3),\\
\Sigma^{W_3}_{(11)}(p,q) &= [0.561\,511\,0 + 0.150\,319\,6\alpha]a + [9.033\,704\,9 + 3.316\,966\,5\alpha + 0.749\,387\,5\alpha^2 + 0.112\,739\,7\alpha^3\\
&\quad - (0.606\,188\,8 + 0.167\,021\,8\alpha)N_f]a^2 + O(a^3),\\
\Sigma^{W_3}_{(12)}(p,q) &= -0.060\,704\,8a - [1.047\,289\,2 + 0.131\,5372\alpha + 0.005\,058\,7\alpha^2 - 0.049\,904\,9N_f]a^2 + O(a^3),\\
\Sigma^{W_3}_{(13)}(p,q) &= -0.121\,409\,5a - [2.204\,345\,8 + 0.148\,981\,6\alpha + 0.010\,117\,5\alpha^2 - 0.127\,070\,6N_f]a^2 + O(a^3),\\
\Sigma^{W_3}_{(14)}(p,q) &= -0.458\,899\,5a - [10.458\,078\,8 - 0.515\,053\,1\alpha + 0.038\,241\,6\alpha^2 - 0.757\,806\,6N_f]a^2 + O(a^3),
\end{aligned}
\tag{6.5}
$$

$$
\begin{aligned}
\Sigma^{\partial W_3}_{(2)}(p,q) &= -0.166\,666\,7 + O(a^3),\\
\Sigma^{\partial W_3}_{(3)}(p,q) &= -0.333\,333\,3 + O(a^3),\\
\Sigma^{\partial W_3}_{(4)}(p,q) &= [0.626\,5587 + 0.101\,898\,4\alpha]a + [8.496\,820\,5 + 2.322\,463\,7\alpha + 0.409\,023\,3\alpha^2 + 0.076\,423\,8\alpha^3\\
&\quad - (0.479\,977\,1 + 0.113\,220\,5\alpha)N_f]a^2 + O(a^3),\\
\Sigma^{\partial W_3}_{(5)}(p,q) &= [0.760\,251\,8 + 0.169\,830\,7\alpha]a + [9.381\,033\,0 + 2.991\,968\,3\alpha + 0.636\,740\,5\alpha^2 + 0.127\,373\,1\alpha^3\\
&\quad - (0.741\,702\,3 + 0.188\,700\,8\alpha)N_f]a^2 + O(a^3),\\
\Sigma^{\partial W_3}_{(6)}(p,q) &= [0.988\,615\,8 + 0.419\,877\,3\alpha]a + [17.861\,663\,1 + 7.672\,582\,6\alpha + 1.861\,053\,0\alpha^2 + 0.314\,908\,0\alpha^3\\
&\quad - (1.386\,205\,6 + 0.466\,530\,4\alpha)N_f]a^2 + O(a^3),
\end{aligned}
$$

$$\Sigma^{\partial W_3}_{(7)}(p,q) = [1.311\,650\,7 + 0.852\,038\,2\alpha]a + [33.938\,710\,9 + 16.364\,306\,6\alpha + 4.081\,960\,9\alpha^2 + 0.639\,028\,7\alpha^3 - (2.413\,487\,1 + 0.946\,709\,1\alpha)N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(8)}(p,q) = [0.466\,127\,0 + 0.342\,546\,0\alpha]a + [13.330\,639\,2 + 6.305\,293\,6\alpha + 1.293\,835\,1\alpha^2 + 0.256\,909\,5\alpha^3 - (0.450\,710\,5 + 0.380\,606\,7\alpha)N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(9)}(p,q) = [0.687\,263\,5 + 0.478\,410\,6\alpha]a + [20.714\,410\,9 + 9.675\,498\,7\alpha + 2.181\,368\,2\alpha^2 + 0.358\,807\,9\alpha^3 - (0.966\,576\,7 + 0.531\,567\,3\alpha)N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(10)}(p,q) = [0.813\,729\,1 + 0.432\,160\,9\alpha]a + [20.501\,765\,0 + 9.034\,594\,1\alpha + 2.072\,306\,0\alpha^2 + 0.324\,120\,7\alpha^3 - (1.099\,664\,8 + 0.480\,178\,8\alpha)N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(11)}(p,q) = [0.845\,523\,7 + 0.203\,796\,9\alpha]a + [12.692\,701\,5 + 4.382\,579\,8\alpha + 0.966\,648\,4\alpha^2 + 0.152\,847\,7\alpha^3 - (0.849\,974\,7 + 0.226\,441\,0\alpha)N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(12)}(p,q) = -0.148\,148\,1a - [2.490\,083\,1 + 0.178\,861\,5\alpha + 0.012\,345\,7\alpha^2 - 0.147\,626\,0N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(13)}(p,q) = -0.347\,245\,5a - [7.195\,478\,0 - 0.144\,433\,6\alpha + 0.028\,937\,1\alpha^2 - 0.501\,576\,9N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial W_3}_{(14)}(p,q) = -0.546\,342\,9a - [11.900\,872\,8 - 0.467\,728\,8\alpha + 0.045\,528\,6\alpha^2 - 0.855\,527\,7N_f]a^2 + O(a^3) \tag{6.6}$$

and

$$\Sigma^{\partial\partial W_3}_{(1)}(p,q) = \Sigma^{\partial\partial W_3}_{(2)}(p,q) = \Sigma^{\partial\partial W_3}_{(3)}(p,q) = -0.333\,333\,3 + [0.541\,643\,3 - 0.250\,046\,6\alpha]a + [10.529\,679\,4 + 4.083\,157\,5\alpha + 0.937\,674\,7\alpha^2 + 0.187\,534\,9\alpha^3 - (0.699\,024\,9 + 0.277\,829\,5\alpha)N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial\partial W_3}_{(4)}(p,q) = \Sigma^{\partial\partial W_3}_{(11)}(p,q) = [1.472\,082\,5 + 0.305\,695\,3\alpha]a + [18.797\,490\,8 + 5.104\,042\,4\alpha + 1.146\,357\,5\alpha^2 + 0.229\,271\,5\alpha^3 - (1.329\,951\,8 + 0.339\,661\,5\alpha)N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial\partial W_3}_{(5)}(p,q) = \Sigma^{\partial\partial W_3}_{(10)}(p,q) = [1.573\,980\,9 + 0.601\,991\,6\alpha]a + [27.325\,189\,0 + 9.867\,662\,4\alpha + 2.257\,468\,6\alpha^2 + 0.451\,493\,7\alpha^3 - (1.841\,367\,1 + 0.668\,879\,6\alpha)N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial\partial W_3}_{(6)}(p,q) = \Sigma^{\partial\partial W_3}_{(9)}(p,q) = [1.675\,879\,3 + 0.898\,287\,9\alpha]a + [35.852\,887\,3 + 14.631\,282\,4\alpha + 3.368\,579\,7\alpha^2 + 0.673\,715\,9\alpha^3 - (2.352\,782\,3 + 0.998\,097\,7\alpha)N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial\partial W_3}_{(7)}(p,q) = \Sigma^{\partial\partial W_3}_{(8)}(p,q) = [1.777\,777\,8 + 1.194\,584\,2\alpha]a + [44.380\,585\,5 + 19.394\,902\,5\alpha + 4.479\,690\,8\alpha^2 + 0.895\,938\,2\alpha^3 - (2.864\,197\,5 + 1.327\,231\,8\alpha)N_f]a^2 + O(a^3),$$

$$\Sigma^{\partial\partial W_3}_{(12)}(p,q) = \Sigma^{\partial\partial W_3}_{(13)}(p,q) = \Sigma^{\partial\partial W_3}_{(14)}(p,q) = -0.694\,491\,0a - [13.262\,456\,5 - 0.809\,832\,6\alpha + 0.057\,874\,3\alpha^2 - 1.003\,153\,7N_f]a^2 + O(a^3). \tag{6.7}$$

The same relations between the various channels noted earlier for the $\overline{\text{MS}}$ scheme apply to the corresponding amplitudes in the RI′/SMOM scheme.

VII. DISCUSSION

We conclude our discussions with brief remarks. Clearly we have provided the full two loop structure of the Green's function with level W_3 operators inserted in a quark 2-point function in two renormalization schemes. The underlying

renormalization allowed for the computation of the conversion functions from the RI′/SMOM scheme to $\overline{\text{MS}}$ and we have come to the same conclusion as [9,10] that the series convergence is marginally worse than the conversion functions of the RI′ scheme which has potential infrared issues. Though as we have argued that in some sense RI′ is not as full a scheme as RI′/SMOM as there is no access to off-diagonal elements of the mixing matrix. Although one possibility of improving the convergence rests in the redefinition of the RI′/SMOM scheme to take account of the structure of more amplitudes. This could be achieved by a different choice of tensor basis for the Green's function. However, in providing the full structure of the Green's function at the symmetric point in $\overline{\text{MS}}$ one is free to perform the renormalization in any scheme of their choosing before converting to $\overline{\text{MS}}$ as the reference scheme. Therefore, our results are reasonably comprehensive so as to allow others to be flexible in how they choose to define their own version of RI′/SMOM.

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APPENDIX: BASIS TENSORS AND PROJECTION MATRIX

The explicit forms of the 14 basis tensors in d dimensions are

$$\begin{aligned}
\mathcal{P}^{W_3}_{(1)\mu\nu\sigma}(p,q) &= \frac{1}{\mu^2}[\gamma_\mu p_\nu p_\sigma + \gamma_\nu p_\sigma p_\mu + \gamma_\sigma p_\mu p_\nu] + \frac{1}{[d+2]}[\gamma_\mu \eta_{\nu\sigma} + \gamma_\nu \eta_{\sigma\mu} + \gamma_\sigma \eta_{\mu\nu}] \\
&\quad - \frac{2\not{p}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma + \eta_{\nu\sigma} p_\mu + \eta_{\sigma\mu} p_\nu], \\
\mathcal{P}^{W_3}_{(2)\mu\nu\sigma}(p,q) &= \frac{1}{\mu^2}[\gamma_\mu p_\nu q_\sigma + \gamma_\nu p_\sigma q_\mu + \gamma_\sigma p_\mu q_\nu + \gamma_\mu q_\nu p_\sigma + \gamma_\nu q_\sigma p_\mu + \gamma_\sigma q_\mu p_\nu] \\
&\quad - \frac{1}{[d+2]}[\gamma_\mu \eta_{\nu\sigma} + \gamma_\nu \eta_{\sigma\mu} + \gamma_\sigma \eta_{\mu\nu}] - \frac{2\not{p}}{[d+2]\mu^2}[\eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} q_\mu + \eta_{\sigma\mu} q_\nu] \\
&\quad - \frac{2\not{q}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma + \eta_{\nu\sigma} p_\mu + \eta_{\sigma\mu} p_\nu], \\
\mathcal{P}^{W_3}_{(3)\mu\nu\sigma}(p,q) &= \frac{1}{\mu^2}[\gamma_\mu q_\nu q_\sigma + \gamma_\nu q_\sigma q_\mu + \gamma_\sigma q_\mu q_\nu] + \frac{1}{[d+2]}[\gamma_\mu \eta_{\nu\sigma} + \gamma_\nu \eta_{\sigma\mu} + \gamma_\sigma \eta_{\mu\nu}] \\
&\quad - \frac{2\not{q}}{[d+2]\mu^2}[\eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} q_\mu + \eta_{\sigma\mu} q_\nu], \\
\mathcal{P}^{W_3}_{(4)\mu\nu\sigma}(p,q) &= \frac{\not{p}}{\mu^4} p_\mu p_\nu p_\sigma + \frac{\not{p}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma + \eta_{\nu\sigma} p_\mu + \eta_{\sigma\mu} p_\nu], \\
\mathcal{P}^{W_3}_{(5)\mu\nu\sigma}(p,q) &= \frac{\not{p}}{\mu^4}[p_\mu p_\nu q_\sigma + p_\mu q_\nu p_\sigma + q_\mu p_\nu p_\sigma] - \frac{\not{p}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma - \eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} p_\mu \\
&\quad - \eta_{\nu\sigma} q_\mu + \eta_{\sigma\mu} p_\nu - \eta_{\sigma\mu} q_\nu], \\
\mathcal{P}^{W_3}_{(6)\mu\nu\sigma}(p,q) &= \frac{\not{p}}{\mu^4}[p_\mu q_\nu q_\sigma + q_\mu p_\nu q_\sigma + q_\mu q_\nu p_\sigma] + \frac{\not{p}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma - \eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} p_\mu - \eta_{\nu\sigma} q_\mu \\
&\quad + \eta_{\sigma\mu} p_\nu - \eta_{\sigma\mu} q_\nu], \\
\mathcal{P}^{W_3}_{(7)\mu\nu\sigma}(p,q) &= \frac{\not{p}}{\mu^4} q_\mu q_\nu q_\sigma + \frac{\not{p}}{[d+2]\mu^2}[\eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} q_\mu + \eta_{\sigma\mu} q_\nu], \\
\mathcal{P}^{W_3}_{(8)\mu\nu\sigma}(p,q) &= \frac{\not{q}}{\mu^4} p_\mu p_\nu p_\sigma + \frac{\not{q}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma + \eta_{\nu\sigma} p_\mu + \eta_{\sigma\mu} p_\nu], \\
\mathcal{P}^{W_3}_{(9)\mu\nu\sigma}(p,q) &= \frac{\not{q}}{\mu^4}[p_\mu p_\nu q_\sigma + p_\mu q_\nu p_\sigma + q_\mu p_\nu p_\sigma] - \frac{\not{q}}{[d+2]\mu^2}[\eta_{\mu\nu} p_\sigma - \eta_{\mu\nu} q_\sigma + \eta_{\nu\sigma} p_\mu - \eta_{\nu\sigma} q_\mu \\
&\quad + \eta_{\sigma\mu} p_\nu - \eta_{\sigma\mu} q_\nu],
\end{aligned}$$

$$
\begin{aligned}
\mathcal{P}^{W_3}_{(10)\mu\nu\sigma}(p,q) &= \frac{q}{\mu^4}[p_\mu q_\nu q_\sigma + q_\mu p_\nu q_\sigma + q_\mu q_\nu p_\sigma] + \frac{q}{[d+2]\mu^2}[\eta_{\mu\nu}p_\sigma - \eta_{\mu\nu}q_\sigma + \eta_{\nu\sigma}p_\mu - \eta_{\nu\sigma}q_\mu \\
&\quad + \eta_{\sigma\mu}p_\nu - \eta_{\sigma\mu}q_\nu], \\
\mathcal{P}^{W_3}_{(11)\mu\nu\sigma}(p,q) &= \frac{q}{\mu^4}q_\mu q_\nu q_\sigma + \frac{q}{[d+2]\mu^2}[\eta_{\mu\nu}q_\sigma + \eta_{\nu\sigma}q_\mu + \eta_{\sigma\mu}q_\nu], \\
\mathcal{P}^{W_3}_{(12)\mu\nu\sigma}(p,q) &= \frac{1}{\mu^4}[\Gamma_{(3)\mu pq}p_\nu p_\sigma + \Gamma_{(3)\nu pq}p_\mu p_\sigma + \Gamma_{(3)\sigma pq}p_\mu p_\nu] \\
&\quad + \frac{1}{[d+2]\mu^2}[\Gamma_{(3)\mu pq}\eta_{\nu\sigma} + \Gamma_{(3)\nu pq}\eta_{\mu\sigma} + \Gamma_{(3)\sigma pq}\eta_{\mu\nu}], \\
\mathcal{P}^{W_3}_{(13)\mu\nu\sigma}(p,q) &= \frac{1}{\mu^4}[\Gamma_{(3)\mu pq}p_\nu q_\sigma + \Gamma_{(3)\nu pq}p_\mu q_\sigma + \Gamma_{(3)\sigma pq}p_\mu q_\nu + \Gamma_{(3)\mu pq}q_\nu p_\sigma + \Gamma_{(3)\nu pq}q_\mu p_\sigma + \Gamma_{(3)\sigma pq}q_\mu p_\nu] \\
&\quad - \frac{1}{[d+2]\mu^2}[\Gamma_{(3)\mu pq}\eta_{\nu\sigma} + \Gamma_{(3)\nu pq}\eta_{\mu\sigma} + \Gamma_{(3)\sigma pq}\eta_{\mu\nu}], \\
\mathcal{P}^{W_3}_{(14)\mu\nu\sigma}(p,q) &= \frac{1}{\mu^4}[\Gamma_{(3)\mu pq}q_\nu q_\sigma + \Gamma_{(3)\nu pq}q_\mu q_\sigma + \Gamma_{(3)\sigma pq}q_\mu q_\nu] + \frac{1}{[d+2]\mu^2}[\Gamma_{(3)\mu pq}\eta_{\nu\sigma} + \Gamma_{(3)\nu pq}\eta_{\mu\sigma} + \Gamma_{(3)\sigma pq}\eta_{\mu\nu}].
\end{aligned}
\tag{A.1}
$$

We use the convention that when a Lorentz index of $\Gamma^{\mu_1\cdots\mu_n}_{(n)}$ is contracted with an external momentum then that momentum appears in place of the associated Lorentz index. The designation of W_3 in the superscript here indicates level. For ∂W_3 and $\partial\partial W_3$ the same basis is used. Given this choice we explicitly record the associated 14×14 matrix $\mathcal{M}^{W_3}$, [10]. To save space we set

$$
\mathcal{M}^{W_3} = \frac{1}{2106d(d-2)}\begin{pmatrix} \mathcal{M}^{W_3}_{11} & \mathcal{M}^{W_3}_{12} & \mathcal{M}^{W_3}_{13} & 0 \\ \mathcal{M}^{W_3}_{21} & \mathcal{M}^{W_3}_{22} & \mathcal{M}^{W_3}_{23} & 0 \\ \mathcal{M}^{W_3}_{31} & \mathcal{M}^{W_3}_{32} & \mathcal{M}^{W_3}_{33} & 0 \\ 0 & 0 & 0 & \mathcal{M}^{W_3}_{44} \end{pmatrix},
$$

and emphasize that the $\Gamma_{(3)}$ sector corresponds to the lower outer corner submatrix or subspace. The clear partition is a reflection of the trace, (2.18). The explicit elements of each submatrix are

$$
\mathcal{M}^{W_3}_{11} = \begin{pmatrix} 312(d+1) & 156(d+1) & 78(d+4) & 1248(d+1) \\ 156(d+1) & 39(5d+2) & 156(d+1) & 624(d+1) \\ 78(d+4) & 156(d+1) & 312(d+1) & 312(d+4) \\ 1248(d+1) & 624(d+1) & 312(d+4) & 1664(d+3)(d+1) \end{pmatrix},
$$

$$
\mathcal{M}^{W_3}_{12} = \begin{pmatrix} 624(d+1) & 312(d+2) & 156(d+4) & 624(d+1) \\ 312(2d+1) & 156(3d+2) & 312(d+1) & 312(d+1) \\ 156(3d+4) & 624(d+1) & 624(d+1) & 156(d+4) \\ 832(d+3)(d+1) & 416(d+6)(d+1) & 208(d+12)(d+1) & 832(d+3)(d+1) \end{pmatrix},
$$

$$
\mathcal{M}^{W_3}_{13} = \begin{pmatrix} 624(d+1) & 156(3d+4) & 312(d+4) \\ 156(3d+2) & 312(2d+1) & 624(d+1) \\ 312(d+2) & 624(d+1) & 1248(d+1) \\ 416(d+6)(d+1) & 208(d+12)(d+1) & 104(d^2+22d+48) \end{pmatrix},
$$

$$
\mathcal{M}^{W_3}_{21} = \begin{pmatrix} 624(d+1) & 312(2d+1) & 156(3d+4) & 832(d+3)(d+1) \\ 312(d+2) & 156(3d+2) & 624(d+1) & 416(d+6)(d+1) \\ 156(d+4) & 312(d+1) & 624(d+1) & 208(d+12)(d+1) \\ 624(d+1) & 312(d+1) & 156(d+4) & 832(d+3)(d+1) \end{pmatrix},
$$

$$\mathcal{M}^{W_3}_{22} = (d+1)\begin{pmatrix} 416(2d+3) & 624(d+2) & 104\frac{(4d^2+25d+12)}{(d+1)} & 416(d+3) \\ 624(d+2) & 208\frac{(4d^2+7d+6)}{(d+1)} & 416(2d+3) & 208(d+6) \\ 104\frac{(4d^2+25d+12)}{(d+1)} & 416(2d+3) & 416(4d+3) & 104(d+12) \\ 416(d+3) & 208(d+6) & 104(d+12) & 416(4d+3) \end{pmatrix},$$

$$\mathcal{M}^{W_3}_{23} = \begin{pmatrix} 416(d+3)(d+1) & 312(d^2+7d+4) & 208(d+12)(d+1) \\ 312(d+2)^2 & 416(d+3)(d+1) & 416(d+6)(d+1) \\ 208(d+6)(d+1) & 416(d+3)(d+1) & 832(d+3)(d+1) \\ 416(2d+3)(d+1) & 104(4d^2+25d+12) & 208(d+12)(d+1) \end{pmatrix},$$

$$\mathcal{M}^{W_3}_{31} = \begin{pmatrix} 624(d+1) & 156(3d+2) & 312(d+2) & 416(d+6)(d+1) \\ 156(3d+4) & 312(2d+1) & 624(d+1) & 208(d+12)(d+1) \\ 312(d+4) & 624(d+1) & 1248(d+1) & 104(d^2+22d+48) \end{pmatrix},$$

$$\mathcal{M}^{W_3}_{32} = \begin{pmatrix} 416(d+3)(d+1) & 312(d+2)^2 & 208(d+6)(d+1) & 416(2d+3)(d+1) \\ 312(d^2+7d+4) & 416(d+3)(d+1) & 416(d+3)(d+1) & 104(4d^2+25d+12) \\ 208(d+12)(d+1) & 416(d+6)(d+1) & 832(d+3)(d+1) & 208(d+12)(d+1) \end{pmatrix},$$

$$\mathcal{M}^{W_3}_{33} = \begin{pmatrix} 208(4d^2+7d+6) & 624(d+2)(d+1) & 416(d+6)(d+1) \\ 624(d+2)(d+1) & 416(2d+3)(d+1) & 832(d+3)(d+1) \\ 416(d+6)(d+1) & 832(d+3)(d+1) & 1664(d+3)(d+1) \end{pmatrix},$$

$$\mathcal{M}^{W_3}_{44} = \begin{pmatrix} -416(d+1) & -208(d+1) & -104(d+4) \\ -208(d+1) & -52(5d+2) & -208(d+1) \\ -104(d+4) & -208(d+1) & -416(d+1) \end{pmatrix}. \tag{A.2}$$

These have been given in d dimensions because we used dimensional regularization in our two loop computation.

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[48] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.84.016002 containing the explicit analytic forms for all the amplitudes presented in Sec. VI.