Four-dimensional graphene

L. B. Drissi,[1] E. H Saidi,[1,2,3] and M. Bousmina[1]
[1]*INANOTECH-MAScIR; Institute of Nanomaterials and Nanotechnology, Morocco*
[2]*LPHE-Modélisation et Simulation; Faculty of Science-Rabat, Morocco*
[3]*CPM, Centre of Physics and Mathematics-Rabat, Morocco*
(Received 2 January 2011; published 14 July 2011)

Mimicking pristine 2D graphene, we revisit the BBTW model for 4D lattice QCD given in [P. F. Bedaque *et al.*, Phys. Rev. D **78**, 017502 (2008)] by using the hidden $SU(5)$ symmetry of the 4D hyperdiamond lattice $\mathcal{H}_4$. We first study the link between the $\mathcal{H}_4$ and $SU(5)$; then we refine the BBTW 4D lattice action by using the weight vectors λ_1, λ_2, λ_3, λ_4, and λ_5 of the five-dimensional representation of $SU(5)$ satisfying $\sum_i \lambda_i = 0$. After that, we study explicitly the solutions of the zeros of the Dirac operator $\mathcal{D}$ in terms of the $SU(5)$ simple roots α_1, α_2, α_3, and α_4 generating $\mathcal{H}_4$; and its fundamental weights ω_1, ω_2, ω_3 ω_4 which generate the reciprocal lattice $\mathcal{H}_4^*$. It is shown, among others, that these zeros live at the sites of $\mathcal{H}_4^*$; and the continuous limit $\mathcal{D}$ is given by $\frac{id\sqrt{5}}{2}\gamma^\mu \mathbf{k}_\mu$ with d, γ^μ, and $\mathbf{k}_\mu$ standing, respectively, for the lattice parameter of $\mathcal{H}_4$, the usual 4 Dirac matrices and the 4D wave vector. Other features, such as differences with BBTW model as well as the link between the Dirac operator following from our construction and the one suggested by Creutz using quaternions, are also given.

DOI: 10.1103/PhysRevD.84.014504 PACS numbers: 12.38.Gc

I. INTRODUCTION

In the last few years, there have been attempts to extend results on the relativistic electron system on a 2D honeycomb (graphene) [1–3] to a 4D honeycomb lattice (called 4D hyperdiamond, denoted below as $\mathcal{H}_4$) and apply it to the lattice QCD simulations [4–13]. These attempts try to construct the Dirac fermion on $\mathcal{H}_4$ by keeping all desirable properties; in particular, locality, chiral symmetry, and the minimal number of fermion doublings [4,8]; see also [12] and references therein. In this regard, two remarkable approaches were given, first by Creutz suggesting an extension of graphene dispersion relations by using quaternions [4,8]; and subsequently by Bedaque–Bachoff–Tiburzi–Walker-Loud (BBTW) [5] proposing a 4D hyperdiamond lattice action with enough symmetries to exclude fine tuning. Apparently those two attempts look very similar since both of them extend 2D graphene to 4D; however they have basic differences some of which are discussed in [5]. The Creutz model involves a two-parameter lattice action that lives on a *distorted* 4D lattice, and so loses the high discrete symmetry of the 4D hyperdiamond. The lattice action of the BBTW model extends pristine 2D graphene; it is built on perfect 4D hyperdiamond and has sufficient discrete symmetries for a good continuum limit. Nevertheless, in both Creutz and BBTW constructions, the distorted and perfect 4D hyperdiamonds are thought of as made by the superposition of two sublattices, $\mathcal{A}_4$ and $\mathcal{B}_4$ with massless left- and right-handed fermions as required by the no-go theorems for lattice chiral symmetry [14,15].

Guided by the rich symmetries of the 4D hyperdiamond $\mathcal{H}_4$, we revisit in this paper the BBTW model of Ref. [5] and its higher dimension extensions given in [12] by using the hidden $SU(5)$ [respectively, $SU(d+1)$] symmetry of $\mathcal{H}_4$ [respectively, $\mathcal{H}_{d+1}$] and its reciprocal lattice $\mathcal{H}_4^*$ [respectively, $\mathcal{H}_{d+1}^*$]. Focusing on 4D lattice QCD, we first review the link between BBTW construction and $SU(5)$. Then we refine the hyperdiamond lattice action by using the weight vectors λ_1, λ_2, λ_3, λ_4, and λ_5 of the five-dimensional (fundamental) representation of $SU(5)$ as well as mimicking pristine 2D graphene which, in the language of groups, corresponds precisely to $SU(3)$. After that, we study explicitly the solutions of the zeros of the Dirac operator by using the $SU(5)$ simple roots α_1, α_2, α_3, and α_4 generating $\mathcal{H}_4$, and its fundamental weights ω_1, ω_2, ω_3, and ω_4 generating the reciprocal lattice $\mathcal{H}_4^*$. We also comment on the differences with BBTW construction, and exhibit the link between the Dirac operator, following from our approach and the one suggested by Creutz using quaternions.

The presentation is as follows: In Sec. II, we review briefly the BBTW parametrization of the real 4D hyperdiamond $\mathcal{H}_4$ and comment on some particular discrete symmetries. In Sec. III, we study the link between $\mathcal{H}_4$ and the $SU(5)$ symmetry. It is shown that $\mathcal{H}_4$ is precisely generated by the four simple roots α_1, α_2, α, and α_4 of $SU(5)$, and the reciprocal lattice $\mathcal{H}_4^*$ is generated by its four weight vectors ω_1, ω_2, ω_3, and ω_4. In Sec. IV, we revisit the BBTW model on $\mathcal{H}_4$ given in [5] and propose a refined 4D lattice action mimicking perfectly 2D graphene. In Sec. V, we study explicitly the zeros of the Dirac operator, and in Sec. VI we rederive the Boriçi-Creutz fermions. In the last section, we give a conclusion and make comments regarding other lattice models.

II. 4D HYPERDIAMOND $\mathcal{H}_4$

Having seen that the 4D hyperdiamond $\mathcal{H}_4$ plays a central role in both BBTW and Creutz lattice models [4,5], we start by studying this 4D lattice by exhibiting explicitly its crystallographic structure. In particular, we give the relative positions of the 5 first- and the 20 second-nearest neighbors, and exhibit some particular discrete symmetries of $\mathcal{H}_4$.

This analysis, which is useful in studying the link between the lattices $\mathcal{H}_4$ and the $SU(5)$ simple roots, is important in our construction; it will be used in Sec. III to build the reciprocal lattice $\mathcal{H}_4^*$ and in Sec. V to study the dispersion energy relations as well as the zeros of the Dirac operator.

A. BBTW parametrization of $\mathcal{H}_4$

In order to apply graphene simulation methods to lattice QCD, BBTW generalizes tight binding model of 2D graphene to the 4D diamond $\mathcal{H}_4$ [5,6]; see also [7–10]. Like in the case of 2D honeycomb, this 4D lattice is defined by two superposed sublattices, $\mathcal{A}_4$ and $\mathcal{B}_4$, with the two following basic objects:

First, sites in $\mathcal{A}_4$ and $\mathcal{B}_4$ (L-nodes and R-nodes in the terminology of [5]) are parameterized by the typical 4D-vectors $\mathbf{r_n}$ with $\mathbf{n} = (n_1, n_2, n_3, n_4)$ and n_i's arbitrary integers. These lattice vectors are expanded as follows:

$$\begin{aligned} &\mathcal{A}_4\colon \mathbf{r_n} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3 + n_4\mathbf{a}_4, \\ &\mathcal{B}_4\colon \mathbf{r}'_\mathbf{n} = \mathbf{r_n} + \mathbf{e}, \end{aligned} \tag{2.1}$$

where $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ are primitive vectors generating these sublattices, and $\mathbf{e}$ is a shift vector as described in what follows.

Second, the vector $\mathbf{e}$ is a global vector taking the same value $\forall\ \mathbf{n}$; it is a shift vector giving the relative positions of the $\mathcal{B}_4$ sites with respect to the $\mathcal{A}_4$ ones, i.e $\mathbf{e} = \mathbf{r}'_\mathbf{n} - \mathbf{r_n}$, $\forall\ \mathbf{n}$. In Ref. [5], the $\mathbf{a}_l$'s and the $\mathbf{e}$ have been chosen as given by the following four-component vectors:

$$\begin{aligned} &\mathbf{a}_1 = \mathbf{e}_1 - \mathbf{e}_5, \qquad \mathbf{a}_3 = \mathbf{e}_3 - \mathbf{e}_5, \qquad \mathbf{e} = \mathbf{e}_5, \\ &\mathbf{a}_2 = \mathbf{e}_2 - \mathbf{e}_5, \qquad \mathbf{a}_4 = \mathbf{e}_4 - \mathbf{e}_5 \end{aligned} \tag{2.2}$$

with the representation

$$\begin{aligned} \mathbf{e}_1^\mu &= \frac{1}{4}(+\sqrt{5}, +\sqrt{5}, +\sqrt{5}, +1), \\ \mathbf{e}_2^\mu &= \frac{1}{4}(+\sqrt{5}, -\sqrt{5}, -\sqrt{5}, +1), \\ \mathbf{e}_3^\mu &= \frac{1}{4}(-\sqrt{5}, -\sqrt{5}, +\sqrt{5}, +1), \\ \mathbf{e}_4^\mu &= \frac{1}{4}(-\sqrt{5}, +\sqrt{5}, -\sqrt{5}, +1), \end{aligned} \tag{2.3}$$

and

$$\mathbf{e}_5^\mu = -\mathbf{e}_1^\mu - \mathbf{e}_2^\mu - \mathbf{e}_3^\mu - \mathbf{e}_4^\mu = (\,0, \quad 0, \quad 0, \quad -1\,). \tag{2.4}$$

Notice also that the five vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, and $\mathbf{e}_5$ define the first-nearest neighbors to (0, 0, 0, 0) and satisfy the constraint relations

$$\begin{aligned} &\mathbf{e}_i.\mathbf{e}_i = \sum e_i^\mu.e_i^\mu = \sum e_{i\mu}.e_i^\mu = 1 \\ &\mathbf{e}_i.\mathbf{e}_j = \cos\vartheta_{ij} = -\frac{1}{4}, \qquad i \neq j, \end{aligned} \tag{2.5}$$

showing that the $\mathbf{e}_i$'s are distributed in a symmetric way since all the angles ϑ_{ij} are equal to $\arccos(-\frac{1}{4})$; see also Fig. 1 for illustration.

In the matrix representation (2.3) and (2.4), the free four vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$ are permuted amongst each other by the typical unimodular matrices $\mathcal{O}_{[ij]}$ acting as

$$\mathbf{e}_i^\mu = \sum_{\nu=1}^{4} (\mathcal{O}_{[ji]})^\mu_\nu \mathbf{e}_j^\nu, \qquad i, j = 1, 2, 3, 4, \tag{2.6}$$

with

$$\begin{aligned} \mathcal{O}_{[21]} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \mathcal{O}_{[32]} &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.7}$$

These transformations leave invariant the vector $\mathbf{e}_5 = -(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)$; they are subsymmetries of the permutation group generated by permutations of the five $\mathbf{e}_i$'s. We also have

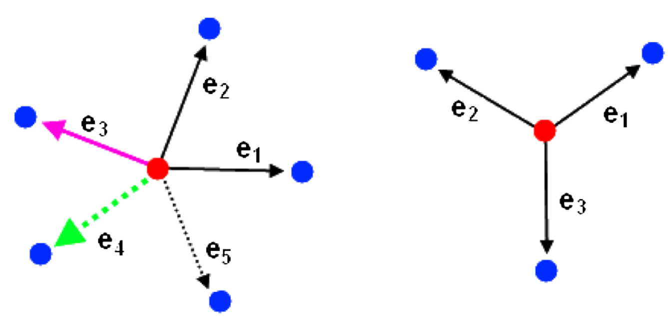

FIG. 1 (color online). On the left, the five first-nearest neighbors in the pristine 4D hyperdiamond with the properties $\|\mathbf{e}_i\| = 1$ and $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 = 0$. On the right, the three first-nearest in pristine 2D graphene with $\|\mathbf{e}_i\| = 1$ and $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = 0$.

$$\mathcal{O}_{[21]} = \mathcal{O}_{[43]}, \qquad \mathcal{O}_{[31]} = \mathcal{O}_{[32]}\mathcal{O}_{[21]}$$
$$\mathcal{O}_{[32]} = \mathcal{O}_{[14]}, \qquad \mathcal{O}_{[41]} = \mathcal{O}_{[43]}\mathcal{O}_{[31]} \tag{2.8}$$

together with other similar relations.

B. Some specific properties

From the Fig. 1 representing the first-nearest neighbors in the 4D hyperdiamond and their analog in 2D graphene, we learn that each $\mathcal{A}_4$-type node at $\mathbf{r_n}$, with some attached wave function $A_{\mathbf{r_n}}$, has the following closed neighbors: five first-nearest neighbors belonging to $\mathcal{B}_4$ with wave functions $B_{\mathbf{r_n}+d\mathbf{e}_i}$; and 20 second-nearest neighbors belonging to the same $\mathcal{A}_4$ with the wave functions $A_{\mathbf{r_n}+d(\mathbf{e}_i-\mathbf{e}_j)}$. The first-nearest neighbors are given by

$$\begin{array}{ccc} \text{lattice position} & & \text{attached wave} \\ \mathbf{r_n}+d\mathbf{e}_1 & \leftrightarrow & B_{\mathbf{r_n}+d\mathbf{e}_1} \\ \mathbf{r_n}+d\mathbf{e}_2 & \leftrightarrow & B_{\mathbf{r_n}+d\mathbf{e}_2} \\ \mathbf{r_n}+d\mathbf{e}_3 & \leftrightarrow & B_{\mathbf{r_n}+d\mathbf{e}_3} \\ \mathbf{r_n}+d\mathbf{e}_4 & \leftrightarrow & B_{\mathbf{r_n}+d\mathbf{e}_4} \\ \mathbf{r_n}+d\mathbf{e}_5 & \leftrightarrow & B_{\mathbf{r_n}+d\mathbf{e}_5} \end{array} \tag{2.9}$$

Using this configuration, the typical tight binding Hamiltonian describing the couplings between the first-nearest neighbors reads as

$$-t\sum_{\mathbf{r_n}}\sum_{i=1}^{5} A_{\mathbf{r_n}}B^{+}_{\mathbf{r_n}+d\mathbf{e}_i} + hc \tag{2.10}$$

where t is the hop energy and where d is the lattice parameter. Notice that in the case where the wave functions at $\mathbf{r_n}$ and $\mathbf{r_n}+d\mathbf{e}_i$ are rather given by two-component Weyl spinors

$$A^{a}_{\mathbf{r_n}} = \begin{pmatrix} A^{1}_{\mathbf{r_n}} \\ A^{2}_{\mathbf{r_n}} \end{pmatrix}, \qquad \bar{B}^{\dot{a}}_{\mathbf{r_n}+d\mathbf{e}_i} = \begin{pmatrix} \bar{B}^{\dot{1}}_{\mathbf{r_n}+d\mathbf{e}_i} \\ \bar{B}^{\dot{2}}_{\mathbf{r_n}+d\mathbf{e}_i} \end{pmatrix} \tag{2.11}$$

together with their adjoints $\bar{A}^{\dot{a}}_{\mathbf{r_n}}$ and $\bar{B}^{a}_{\mathbf{r_n}+d\mathbf{e}_i}$, as in the example of 4D lattice QCD to be described in Sec. IV, the corresponding tight binding model would be

$$-t\sum_{\mathbf{r_n}}\sum_{i=1}^{5}\left[\sum_{\mu=1}^{4}\mathbf{e}_i^{\mu}(A^{a}_{\mathbf{r_n}}\sigma^{\mu}_{a\dot{a}}\bar{B}^{\dot{a}}_{\mathbf{r_n}+d\mathbf{e}_i})\right] + hc \tag{2.12}$$

where the $\mathbf{e}_i^{\mu}$'s are as in (2.3) and where the coefficients $\sigma^{\mu}_{a\dot{a}}$ will be specified later on. Notice moreover that the term $\sum_{i=1}^{5}\mathbf{e}_i^{\mu}(A^{a}_{\mathbf{r_n}}\sigma^{\mu}_{a\dot{a}}\bar{B}^{\dot{a}}_{\mathbf{r_n}})$ vanishes identically due to $\sum_{i=1}^{5}\mathbf{e}_i^{\mu} = 0$. The 20 second nearest neighbors read as

$$\begin{array}{ll} r_{\mathbf{n}} \pm d(\mathbf{e}_1-\mathbf{e}_2), & r_{\mathbf{n}} \pm d(\mathbf{e}_1-\mathbf{e}_3), \\ r_{\mathbf{n}} \pm d(\mathbf{e}_1-\mathbf{e}_4), & r_{\mathbf{n}} \pm d(\mathbf{e}_1-\mathbf{e}_5), \\ r_{\mathbf{n}} \pm d(\mathbf{e}_2-\mathbf{e}_3), & r_{\mathbf{n}} \pm d(\mathbf{e}_2-\mathbf{e}_4), \\ r_{\mathbf{n}} \pm d(\mathbf{e}_2-\mathbf{e}_5), & r_{\mathbf{n}} \pm d(\mathbf{e}_3-\mathbf{e}_4), \\ r_{\mathbf{n}} \pm d(\mathbf{e}_3-\mathbf{e}_5), & r_{\mathbf{n}} \pm d(\mathbf{e}_4-\mathbf{e}_5). \end{array} \tag{2.13}$$

At this order, the standard tight binding Hamiltonian reads as follows:

$$-t'\sum_{\mathbf{r_n}}\sum_{i,j=1}^{5}(A_{\mathbf{r_n}}A^{+}_{\mathbf{r_n}+d(\mathbf{e}_i-\mathbf{e}_j)} + B_{\mathbf{r_n}}B^{+}_{\mathbf{r_n}+d(\mathbf{e}_i-\mathbf{e}_j)}) + hc \tag{2.14}$$

and in the case of Weyl spinors, we have

$$-t'\sum_{\mathbf{r_n}}\sum_{i,j=1}^{5}\left[\sum_{\mu=1}^{4}\mathbf{e}_i^{\mu}(A^{a}_{\mathbf{r_n}}\sigma^{\mu}_{a\dot{a}}\bar{A}^{\dot{a}}_{\mathbf{r_n}+d(\mathbf{e}_i-\mathbf{e}_j)} + B^{a}_{\mathbf{r_n}}\sigma^{\mu}_{a\dot{a}}\bar{B}^{\dot{a}}_{\mathbf{r_n}+d(\mathbf{e}_i-\mathbf{e}_j)})\right] + hc. \tag{2.15}$$

In what follows, we show that the five vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, and $\mathbf{e}_5$ are, up to a normalization factor namely $\frac{\sqrt{5}}{2}$, precisely the weight vectors λ_0, λ_1, λ_2, λ_3, and λ_4 of the five-dimensional representation of $SU(5)$; the 20 vectors $(\mathbf{e}_i-\mathbf{e}_j)$ are, up to a scale factor $\frac{\sqrt{5}}{2}$, their roots $\beta_{ij} = (\lambda_i - \lambda_j)$. We show as well that the particular property $\mathbf{e}_i.\mathbf{e}_j = -\frac{1}{4}$, which is constant $\forall\,\mathbf{e}_i$, $\forall\,\mathbf{e}_j$, has a natural interpretation in terms of the Cartan matrix of $SU(5)$.

III. LINK WITH $SU(5)$ SYMMETRY

For later use, we exhibit here the hidden $SU(5)$ symmetry of the 4D hyperdiamond; we show that $\mathcal{H}_4$ considered above is precisely the lattice $\mathcal{L}_{su(5)}$ studied in [16]. More concretely, we show the three following:

First, the five bond vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, and $\mathbf{e}_5$ (first-nearest neighbors) are given by the five weight vectors λ_1, λ_2, λ_3, λ_4, and λ_5 (below, we set $\lambda_5 \equiv \lambda_0$) of the five-dimensional (fundamental) representation of $SU(5)$ which also satisfy

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0. \tag{3.1}$$

We will show later that $\mathbf{e}_i = \frac{\sqrt{5}}{2}\lambda_i$ with $\lambda_i.\lambda_i = \frac{4}{5}$.

Second, the four primitive ones ($\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$) used in generating $\mathcal{H}_4$ are particular linear combinations of the four simple roots α_1, α_2, α_3, and α_4 of $SU(5)$; see Eq. (3.20) for the explicit relations. Recall that the $SU(5)$ symmetry has 20 roots as given below:

$$\begin{array}{lll} \pm\alpha_1, \pm(\alpha_1+\alpha_2), & \pm(\alpha_1+\alpha_2+\alpha_3), & \\ \pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4) & \pm\alpha_2, \pm(\alpha_2+\alpha_3), & \\ \pm(\alpha_2+\alpha_3+\alpha_4), & \pm\alpha_3, \pm(\alpha_3+\alpha_4), & \pm\alpha_3. \end{array} \tag{3.2}$$

These vectors have all of them the same length $\alpha^2 = 2$, and so they generate the relative lattice positions of the second-nearest neighbors in the 4D hyperdiamond.

Third, the $SU(5)$ has also discrete symmetries given by the so-called Weyl group transformations generated by the σ_α's acting on generic roots β of $SU(5)$ as follows:

$$\sigma_\alpha(\beta) = \beta - 2\frac{\alpha.\beta}{\alpha^2}\beta = \beta - (\alpha.\beta)\beta. \tag{3.3}$$

These discrete transformations permute the roots (3.2) among themselves and are isomorphic to $\mathcal{S}_5$ permutation group transformations. For instance, we have $\sigma_{\alpha_1}(\alpha_1) = -\alpha_1$ and $\sigma_{\alpha_1}(\alpha_2) = \alpha_1 + \alpha_2$.

A. Exhibiting the link $\mathcal{H}_4/SU(5)$

To exhibit explicitly the link between pristine lattice $\mathcal{H}_4$ and the simple roots and the basic weight vectors of $SU(5)$, we start by recalling some of its features; in particular, the following useful ingredients: $SU(5)$ is a 24-dimensional symmetry group; it has rank 4, that is four simple roots α_1, α_2, α_3, and α_4; it has 20 roots $\pm\beta_{ij}$ given by Eq. (3.2). The simple roots α_1, α_2, α_3, and α_4 capture most of the algebraic properties of $SU(5)$, and as a consequence, those of the 4D hyperdiamond crystal. In particular, they generate the 20 roots $\pm\beta_{ij}$ as shown on (3.2) and they have a symmetric intersection matrix $\mathbf{K}_{ij} = \alpha_i.\alpha_j$ with inverse $\mathbf{K}_{ij}^{-1}$ given by

$$\mathbf{K}_{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \qquad \mathbf{K}_{ij}^{-1} = \frac{1}{5}\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \tag{3.4}$$

that encode the algebraic data of the underlying Lie algebra of the $SU(5)$ symmetry. These simple roots define as well the four fundamental weights ω_1, ω_2, ω_3, and ω_4 through the following duality relation:

$$\omega_i.\alpha_j = \delta_{ij}, \qquad i, j = 1, \ldots, 4. \tag{3.5}$$

These fundamental weights are important for us; first because they allow us to build the reciprocal 4D hyperdiamond $\mathcal{H}_4^*$, and second they can be used to expand any wave vector in $\mathcal{H}_4^*$ as follows:

$$\mathbf{k} = k_1\omega_1 + k_2\omega_2 + k_3\omega_3 + k_4\omega_4. \tag{3.6}$$

From this expansion, we read the relations $k_i = \mathbf{k}.\alpha_i$ showing that the k_i's are precisely the wave vector components propagating along the α_i-directions; thanks to Eqs. (3.5).

B. Other useful relations

Using the matrices $\mathbf{K}_{ij}$ and $\mathbf{K}_{ij}^{-1}$, one can express the simple roots α_i in terms of the fundamental weight vectors ω_i, and inversely the ω_i's as linear combinations of the simple roots as given below:

$$\begin{aligned} \omega_1 &= \frac{4}{5}\alpha_1 + \frac{3}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{1}{5}\alpha_4 \\ \omega_2 &= \frac{3}{5}\alpha_1 + \frac{6}{5}\alpha_2 + \frac{4}{5}\alpha_3 + \frac{2}{5}\alpha_4 \\ \omega_3 &= \frac{2}{5}\alpha_1 + \frac{4}{5}\alpha_2 + \frac{6}{5}\alpha_3 + \frac{3}{5}\alpha_4 \\ \omega_4 &= \frac{1}{5}\alpha_1 + \frac{2}{5}\alpha_2 + \frac{3}{5}\alpha_3 + \frac{4}{5}\alpha_4. \end{aligned} \tag{3.7}$$

Using these relations, it is not difficult to check that they satisfy (3.5). For instance, we have $\omega_1.\alpha_1 = \frac{8}{5} - \frac{3}{5} = 1$ and $\omega_1.\alpha_2 = -\frac{4}{5} + \frac{6}{5} - \frac{2}{5} = 0$, and similarly for the others ω_2, ω_3, ω_4, and the intersections $\omega_i.\alpha_j$. Notice, moreover, that the fundamental weight vector ω_1 defines a highest weight representation of $SU(5)$ of dimension 5 with weight vectors λ_0, λ_1, λ_2, λ_3, and λ_4 related to ω_1 as follows:

$$\begin{gathered} \lambda_0 = \omega_1, \quad \lambda_1 = \omega_1 - \alpha_1, \quad \lambda_2 = \omega_1 - \alpha_1 - \alpha_2, \\ \lambda_3 = \omega_1 - \alpha_1 - \alpha_2 - \alpha_3, \quad \lambda_4 = \omega_1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4. \end{gathered} \tag{3.8}$$

By using (3.7), one may also express these vectors weights in terms of the ω_i's as follows:

$$\begin{gathered} \lambda_0 = \omega_1, \quad \lambda_1 = \omega_2 - \omega_1, \quad \lambda_2 = \omega_3 - \omega_2, \\ \lambda_3 = \omega_4 - \omega_3, \quad \lambda_4 = -\omega_4. \end{gathered} \tag{3.9}$$

Furthermore, substituting ω_1 by its expression (3.7), we get the following values of the λ_i's in terms of the simple roots:

$$\begin{aligned} \lambda_0 &= +\frac{4}{5}\alpha_1 + \frac{3}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{1}{5}\alpha_4, \\ \lambda_1 &= -\frac{1}{5}\alpha_1 + \frac{3}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{1}{5}\alpha_4, \\ \lambda_2 &= -\frac{1}{5}\alpha_1 - \frac{2}{5}\alpha_2 + \frac{2}{5}\alpha_3 + \frac{1}{5}\alpha_4, \\ \lambda_3 &= -\frac{1}{5}\alpha_1 - \frac{2}{5}\alpha_2 - \frac{3}{5}\alpha_3 + \frac{1}{5}\alpha_4, \\ \lambda_4 &= -\frac{1}{5}\alpha_1 - \frac{2}{5}\alpha_2 - \frac{3}{5}\alpha_3 - \frac{4}{5}\alpha_4. \end{aligned} \tag{3.10}$$

These weight vectors satisfy remarkable properties that will be used later on, in particular, the three following: First, these λ_i's obey the constraint relation $\sum_{i=0}^4 \lambda_i = 0$ which agrees with (3.1) and which should be compared with the identity $\mathbf{e}_1^\mu + \mathbf{e}_2^\mu + \mathbf{e}_3^\mu + \mathbf{e}_4^\mu + \mathbf{e}_4^\mu = 0$. Second, they have the intersection matrix

$$\begin{gathered} \lambda_i.\lambda_i = \frac{4}{5}, \qquad \lambda_i.\lambda_j = -\frac{1}{5}, \\ \cos\vartheta_{ij} = \frac{\lambda_i.\lambda_j}{|\lambda_i||\lambda_j|} = -\frac{1}{4}, \end{gathered} \tag{3.11}$$

leading to Eq. (2.5). The third point concerns the zeros of the Dirac operator; see Eq. (4.11) to fix the ideas. They are given by solving the following constraint relations:

$$e^{i(d\sqrt{5}/2)p_0} = e^{i(d\sqrt{5}/2)p_1} = e^{i(d\sqrt{5}/2)p_2} = e^{i(d\sqrt{5}/2)p_3} = e^{i(d\sqrt{5}/2)p_4} = e^{i\varphi}, \tag{3.12}$$

where we have set

$$p_0 = \mathbf{k}.\lambda_0, \qquad p_1 = \mathbf{k}.\lambda_1, \qquad p_2 = \mathbf{k}.\lambda_2 \qquad p_3 = \mathbf{k}.\lambda_3, \qquad p_4 = \mathbf{k}.\lambda_4 \tag{3.13}$$

and where the phase $\varphi = \frac{2\pi N}{5}$, with N an integer. The values of this phase are due to equiprobability in hops from a generic site at $\mathbf{r}$ to the five first-nearest neighbors at $\mathbf{r} + \frac{d\sqrt{5}}{2}\lambda_i$. This equiprobability requires

$$\prod_{l=0}^{5} e^{i(d\sqrt{5}/2)p_l} = 1 = e^{5i\varphi}. \tag{3.14}$$

Solutions of the constraint Eqs. (3.12) are then given by

$$p_i = \frac{4\pi N}{5d\sqrt{5}}, \qquad i = 0, 1, 2, 3, 4. \tag{3.15}$$

Notice, moreover, the two useful features: First, Eqs. (3.13) imply in turn that the wave vector $\mathbf{k}$ may be also written as

$$\mathbf{k} = p_0\lambda_0 + p_1\lambda_1 + p_2\lambda_2 + p_3\lambda_3 + p_4\lambda_4. \tag{3.16}$$

Multiplying both sides of this relation by λ_i and using (3.11), we find $\mathbf{k}.\lambda_i = p_i - \frac{1}{5}(p_0 + p_1 + p_2 + p_3 + p_4) = p_i$, thanks to the identity $\sum_i p_i = 0$ following from $\sum_i \lambda_i = 0$. Second, expressing this vector $\mathbf{k}$ in terms of the basis ω_1, ω_2, ω_3, and ω_4 of the reciprocal lattice, then using Eqs. (3.9) giving the λ_l's in terms of the ω_l's, we get

$$\mathbf{k} = (p_0 - p_1)\omega_1 + (p_1 - p_2)\omega_2 + (p_2 - p_3)\omega_3 + (p_3 - p_4)\omega_4. \tag{3.17}$$

Putting back (3.15), we find that the zeros of the Dirac operator are precisely located at the sites of the reciprocal lattice $\mathcal{H}_4^*$.

C. Link with BBTW parametrization of $\mathcal{H}_4$

From Eq. (3.8), we can also determine the expression of the simple roots α_i's in terms of the weight vectors λ_i's. We have

$$\alpha_1 = \lambda_0 - \lambda_1, \qquad \alpha_3 = \lambda_2 - \lambda_3, \qquad \alpha_2 = \lambda_1 - \lambda_2, \qquad \alpha_4 = \lambda_3 - \lambda_4. \tag{3.18}$$

By comparing these equations with Eqs. (2.3) and (2.5), we obtain the relation between the $\mathbf{e}_i$'s used in [5] and the weight vectors of the fundamental representation of $SU(5)$:

$$\mathbf{e}_i = \frac{\sqrt{5}}{2}\lambda_i, \qquad \lambda_i = \frac{2\sqrt{5}}{5}\mathbf{e}_i. \tag{3.19}$$

Putting Eqs. (3.18) and (3.19) back into (2.2), we find that the four primitive vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ generating the sublattice $\mathcal{A}_4$ (respectively, $\mathcal{B}_4$) are nothing but linear combinations of the four simple roots of $SU(5)$,

$$\begin{aligned} \mathbf{a}_1 &= -\frac{\sqrt{5}}{2}\alpha_1 \\ \mathbf{a}_2 &= -\frac{\sqrt{5}}{2}(\alpha_1 + \alpha_2) \\ \mathbf{a}_3 &= -\frac{\sqrt{5}}{2}(\alpha_1 + \alpha_2 + \alpha_3) \\ \mathbf{a}_4 &= -\frac{\sqrt{5}}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4). \end{aligned} \tag{3.20}$$

From these relations, we read the identities of [5]

$$\mathbf{a}_i.\mathbf{a}_i = \frac{10}{4}, \qquad \mathbf{a}_i.\mathbf{a}_j = \frac{5}{4}, \qquad i \neq j. \tag{3.21}$$

These relations are just a property of the Cartan matrix of $SU(5)$.

We end this section by giving the following summary:

The 4D hyperdiamond $\mathcal{H}_4$ is made of two superposed sublattices, $\mathcal{A}_4$ and $\mathcal{B}_4$. These sublattices are generated by the simple roots α_1, α_2, α_3, and α_4 of $SU(5)$. The relative shift vector between $\mathcal{A}_4$ and $\mathcal{B}_4$ is a weight vector of the five-dimensional representation of $SU(5)$. Each site in $\mathcal{H}_4$ has five first-nearest neighbors forming a dimension 5 representation of $SU(5)$, and 20 second-nearest ones, forming together with the "four zero roots" the adjoint representation of $SU(5)$. The reciprocal space of the 4D hyperdiamond is generated by the fundamental weight vectors ω_1, ω_2, ω_3, and ω_4 of $SU(5)$. Generic wave vectors $\mathbf{k}$ in this lattice read as

$$\mathbf{k} = k_1\omega_1 + k_2\omega_2 + k_3\omega_3 + k_4\omega_4 \tag{3.22}$$

where $k_i = (p_{i-1} - p_i)$, where p_i is the momentum along the λ_i-direction and $(p_{i-1} - p_i)$ the momentum along the α_i-direction in the real 4D hyperdiamond lattice $\mathcal{H}_4$. In the particular case where all the momenta $p_i = \frac{4\pi N}{5d\sqrt{5}}$, we have

$$\sum_{l=0}^{4} \lambda_l^\mu e^{\pm id(\sqrt{5}/2)p_l} = e^{\pm i(2\pi N/5)}\left(\sum_{l=0}^{4} \lambda_l^\mu\right) = 0. \tag{3.23}$$

This property will be used later on.

IV. BBTW LATTICE ACTION REVISITED

A. Correspondence 2D/4D

We begin to notice that a generic bond vector $\mathbf{e}_i$ in $\mathcal{H}_4$ links two sites in the same unit cell of the hyperdiamond as shown on the typical coupling term $A_{\mathbf{r_n}} B^+_{\mathbf{r_n}+d\mathbf{e}_i}$. This property is quite similar to the action of the usual γ^μ matrices on 4D (Euclidean) space time spinors. Mimicking the tight binding model of 2D graphene, BBTW proposed in [5] an analogous model for 4D lattice QCD. Their construction relies on the use of the following: First, the naive correspondence between the bond vectors $\mathbf{e}_i$ and the γ^i matrices

$$\mathbf{e}_i \leftrightarrow \gamma_i, \qquad i = 1, \dots, 5, \tag{4.1}$$

with

$$\begin{aligned} -\mathbf{e}_5 &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4, \\ -\Gamma_5 &= \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4. \end{aligned} \tag{4.2}$$

Recall that the four γ^μ matrices satisfy the Clifford algebra $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}$, $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$ gives the $\pm$ chiralities of the two possible Weyl spinors in 4D, and Γ_5 is precisely the matrix Γ used in the Boriçi-Creutz fermions [17,18]; see also Eq. (2.5) of Ref. [19] for a rigorous derivation using $SU(5)$ symmetry. Second, as in the case of 2D graphene, $\mathcal{A}_4$-type sites are occupied by left $\phi_L = (\phi^a_{\mathbf{r}})$ and right $\phi_R = (\bar{\phi}^{\dot{a}}_{\mathbf{r}})$ two-component Weyl spinors. $\mathcal{B}_4$-type sites are occupied by right $\chi_R = (\bar{\chi}^{\dot{a}}_{\mathbf{r}+d\mathbf{e}_i})$ and left $\chi_L = (\chi^a_{\mathbf{r}+d\mathbf{e}_i})$ Weyl spinors.

	2D graphene	4D hyperdiamond
$\mathcal{A}_4$ − sites at $\mathbf{r}_n$	$A_{\mathbf{r}}$	$\phi^a_{\mathbf{r}}, \bar{\phi}^{\dot{a}}_{\mathbf{r}}$
$\mathcal{B}_4$ − sites at $\mathbf{r}_n + d\mathbf{e}_i$	$B^+_{\mathbf{r}+de_i}$	$\bar{\chi}^{\dot{a}}_{\mathbf{r}+d\mathbf{e}_i}, \chi^a_{\mathbf{r}+d\mathbf{e}_i}$
couplings [21]	$A_{\mathbf{r}}B^+_{\mathbf{r}+de_i}$	$\sum_{\mu=1}^4 \mathbf{e}^\mu_i(\phi^a_{\mathbf{r}}\sigma^\mu_{a\dot{a}}\bar{\chi}^{\dot{a}}_{\mathbf{r}+d\mathbf{e}_i})$
	$B_{\mathbf{r}+de_i}A^+_{\mathbf{r}}$	$\sum_{\mu=1}^4 \mathbf{e}^\mu_i(\chi^a_{\mathbf{r}+d\mathbf{e}_i}\bar{\sigma}^\mu_{a\dot{a}}\bar{\phi}^{\dot{a}}_{\mathbf{r}})$

where the indices $a = 1, 2$ and $\dot{a} = \dot{1}, \dot{2}$, and where summation over μ is in the Euclidean sense. For later use, it is interesting to notice the two following: In 2D graphene, the wave functions $A_{\mathbf{r}}$ and $B_{\mathbf{r}+de_i}$ describe polarized electrons in the first-nearest sites of the 2D honeycomb. As the spin up and spin down components of the electrons contribute equally, the effect of spin couplings in 2D graphene is ignored. In the 4D hyperdiamond, we have $4 + 4$ wave functions at each $\mathcal{A}_4$-type site or $\mathcal{B}_4$-type one. These wave functions are given by the doublets $\phi^a = (\phi^1_{\mathbf{r}_n}, \phi^2_{\mathbf{r}_n})$ and $\bar{\phi}^{\dot{a}}_{\mathbf{r}} = (\bar{\phi}^{\dot{1}}_{\mathbf{r}_n}, \bar{\phi}^{\dot{2}}_{\mathbf{r}_n})$ having, respectively, positive and negative γ^5 chirality; these are the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the $SO(4) \simeq SU(2) \times SU(2)$ (the doublets $\bar{\chi}^{\dot{a}}_{\mathbf{r}+d\mathbf{e}_i} = (\bar{\chi}^{\dot{1}}_{\mathbf{r}+d\mathbf{e}_i}, \bar{\chi}^{\dot{2}}_{\mathbf{r}+d\mathbf{e}_i})$ and $\chi^a_{\mathbf{r}+d\mathbf{e}_i} = (\chi^1_{\mathbf{r}+d\mathbf{e}_i}, \chi^2_{\mathbf{r}+d\mathbf{e}_i})$ having, respectively, negative and positive γ^5 chirality).

By mimicking the 2D graphene study, we expect therefore to have four kinds of polarized particles together with the four corresponding "holes" as shown on the typical tight binding couplings

$$\begin{aligned} &\mathbf{e}^\mu_i\sigma^\mu_{1\dot{1}}(\phi^1_{\mathbf{r}}\bar{\chi}^{\dot{1}}_{\mathbf{r}+d\mathbf{e}_i}), \qquad &&\mathbf{e}^\mu_i\sigma^\mu_{2\dot{2}}(\phi^2_{\mathbf{r}}\bar{\chi}^{\dot{2}}_{\mathbf{r}+d\mathbf{e}_i}) \\ &\mathbf{e}^\mu_i\bar{\sigma}^\mu_{1\dot{1}}(\chi^1_{\mathbf{r}+d\mathbf{e}_i}\bar{\phi}^{\dot{1}}_{\mathbf{r}}), \qquad &&\mathbf{e}^\mu_i\bar{\sigma}^\mu_{2\dot{2}}(\chi^2_{\mathbf{r}+d\mathbf{e}_i}\bar{\phi}^{\dot{2}}_{\mathbf{r}}). \end{aligned} \tag{4.3}$$

B. Building the action

Following [5], the BBTW action is a naive lattice QCD action preserving the symmetries of $\mathcal{H}_4$. To describe the spinor structures of the lattice fermions, one considers 4D space time Dirac spinors together with the following γ^μ matrices realizations:

$$\begin{aligned} &\gamma^1 = \tau^1 \otimes \sigma^1, \qquad \gamma^2 = \tau^1 \otimes \sigma^2, \qquad \gamma^3 = \tau^1 \otimes \sigma^3, \\ &\qquad \gamma^4 = \tau^2 \otimes I_2, \qquad \gamma^5 = \tau^3 \otimes I_2, \end{aligned} \tag{4.4}$$

where the τ^i's are the Pauli matrices acting on the sublattice structure of the hyperdiamond lattice $\mathcal{H}_4$,

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.5}$$

The 2×2 matrices σ^i satisfy as well the Clifford algebra $\sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij}I_2$ and act through the coupling of left ϕ_L (respectively, ϕ_R) and right χ_R (respectively, left χ_L) two-component Weyl spinors at neighboring $\mathcal{A}_4$- and $\mathcal{B}_4$-sites

$$\begin{aligned} &\phi^a_{\mathbf{r}}\sigma^\mu_{a\dot{a}}\bar{\chi}^{\dot{a}}_{\mathbf{r}+d(\sqrt{5}/2)\lambda_i} - \chi^a_{\mathbf{r}}\bar{\sigma}^\mu_{a\dot{a}}\bar{\phi}^{\dot{a}}_{\mathbf{r}-d(\sqrt{5}/2)\lambda_i} \\ &\quad = (\phi_{\mathbf{r}}\sigma^\mu\bar{\chi}_{\mathbf{r}+d(\sqrt{5}/2)\lambda_i} - \chi_{\mathbf{r}}\bar{\sigma}^\mu\bar{\phi}_{\mathbf{r}-d(\sqrt{5}/2)\lambda_i}) \end{aligned} \tag{4.6}$$

where $\sigma^\mu = (\sigma^1, \sigma^2, \sigma^3, +iI_2)$ and $\bar{\sigma}^\mu = (\sigma^1, \sigma^2, \sigma^3, -iI_2)$. For later use, it is interesting to set

$$\begin{aligned} \sigma^\mu.\mathbf{e}^\mu_1 &= \frac{\sqrt{5}}{4}\sigma^1 + \frac{\sqrt{5}}{4}\sigma^2 + \frac{\sqrt{5}}{4}\sigma^3 + \frac{i}{4}I_2, \\ \bar{\sigma}^\mu.\mathbf{e}^\mu_1 &= \frac{\sqrt{5}}{4}\sigma^1 + \frac{\sqrt{5}}{4}\sigma^2 + \frac{\sqrt{5}}{4}\sigma^3 - \frac{i}{4}I_2, \end{aligned} \tag{4.7}$$

and similar relations for the other $\sigma.\mathbf{e}_i$ and $\bar{\sigma}.\mathbf{e}_i$.

Now extending the tight binding model of 2D graphene to the 4D hyperdiamond;, and using the weight vectors λ_i instead of $\mathbf{e}_i$, we can build a free fermion action on the lattice $\mathcal{H}_4$ by attaching a two-component left-handed spinor $\phi^a(\mathbf{r})$ and right-handed spinor $\bar{\phi}^{\dot{a}}_{\mathbf{r}}$ to each $\mathcal{A}_4$-node $\mathbf{r}$, and a right-handed spinor $\bar{\chi}^{\dot{a}}_{\mathbf{r}+d(\sqrt{5}/2)\lambda_i}$ and left-handed spinor $\chi^a_{\mathbf{r}+d(\sqrt{5}/2)\lambda_i}$ to every $\mathcal{B}_4$-node at $\mathbf{r} + d\frac{\sqrt{5}}{2}\lambda_i$. The action, describing hopping to first-nearest-neighbor sites with equal probabilities in all five directions λ_i, reads as follows:

$$\begin{aligned} \mathcal{S}_{\text{BBTW}} = \sum_{\mathbf{r}}\sum_{i=0}^{4}(&\phi_{\mathbf{r}}\sigma^\mu\bar{\chi}_{\mathbf{r}+d(\sqrt{5}/2)\lambda_i} \\ &- \chi_{\mathbf{r}}\bar{\sigma}^\mu\bar{\phi}_{\mathbf{r}-d(\sqrt{5}/2)\lambda_i})\lambda^\mu_i. \end{aligned} \tag{4.8}$$

Clearly, this action is invariant under the following discrete transformations:

$$\begin{aligned} &\sigma^\mu\bar{\xi}_{\mathbf{r}\pm d(\sqrt{5}/2)\lambda_i} \to \sigma^\nu\bar{\xi}_{\mathbf{r}\pm d(\sqrt{5}/2)\lambda_j}(\mathcal{O}^T_{ji})^\mu_\nu, \\ &\qquad \lambda^\mu_i \to (\mathcal{O}_{ji})^\mu_\rho\lambda^\rho_j. \end{aligned} \tag{4.9}$$

Expanding the various spinorial fields $\xi_{\mathbf{r}\pm\mathbf{v}}$ in Fourier sums as $\int \frac{d^4k}{(2\pi)^4} e^{-i\mathbf{k}.\mathbf{r}}(e^{\mp i\mathbf{k}.\mathbf{v}}\xi_{\mathbf{k}})$ with $\mathbf{k}$ standing for a generic wave vector in $\mathcal{H}_4^*$, we can put the field action $\mathcal{S}_{\mathrm{BBTW}}$ into the form

$$\mathcal{S}_{\mathrm{BBTW}} = i\sum_{\mathbf{k}} (\bar{\phi}_{\mathbf{k}}, \bar{\chi}_{\mathbf{k}}) \begin{pmatrix} 0 & -iD \\ i\bar{D} & 0 \end{pmatrix} \begin{pmatrix} \phi_{\mathbf{k}} \\ \chi_{\mathbf{k}} \end{pmatrix} \tag{4.10}$$

where we have set

$$D = \sum_{l=0}^{4} D_l e^{id(\sqrt{5}/2)\mathbf{k}.\lambda_l} = \sum_{\mu=1}^{4} \sigma^\mu \Big(\sum_{l=0}^{4} \lambda_l^\mu e^{id(\sqrt{5}/2)\mathbf{k}.\lambda_l} \Big), \tag{4.11}$$

with

$$D_l = \sum_{\mu=1}^{4} \sigma^\mu \lambda_l^\mu = \begin{pmatrix} \lambda_l^3 + i\lambda_l^4 & \lambda_l^1 - i\lambda_l^2 \\ \lambda_l^1 + i\lambda_l^2 & \lambda_l^3 - i\lambda_l^4 \end{pmatrix}, \tag{4.12}$$

and $p_l = \mathbf{k}.\lambda_l = \sum_\mu \mathbf{k}_\mu \lambda_l^\mu$. Similarly, we have

$$\bar{D} = \sum_{l=0}^{4} \bar{D}_l e^{-id(\sqrt{5}/2)\mathbf{k}.\lambda_l} = \sum_{\mu=1}^{4} \bar{\sigma}^\mu \Big(\sum_{l=0}^{4} \lambda_l^\mu e^{-id(\sqrt{5}/2)\mathbf{k}.\lambda_l} \Big). \tag{4.13}$$

We end this subsection by making three remarks: the first one deals with the continuous limit, the second one regards the zeros of the Dirac operator, and the third concerns the link with the Creutz fermions. In the continuous limit where the lattice parameter $d \to 0$, we have

$$\sum_{l=0}^{4} \lambda_l^\mu e^{\pm id(\sqrt{5}/2)\mathbf{k}.\lambda_l} \to \Big(\sum_{l=0}^{4} \lambda_l^\mu \Big) \pm i\frac{d\sqrt{5}}{2} \times \Big[\sum_{l=0}^{4} \lambda_l^\mu (\mathbf{k}.\lambda_l) \Big] + \ldots. \tag{4.14}$$

Moreover, since $\sum_{l=0}^{4} \lambda_l^\mu = 0$ and because of the identity $\sum_{l=0}^{4} \lambda_l^\mu (\mathbf{k}.\lambda_l) = \mathbf{k}^\mu$ following from Eqs. (3.16) and (3.17), this limit reduces to

$$\sum_{l=0}^{4} \lambda_l^\mu e^{\pm id(\sqrt{5}/2)\mathbf{k}.\lambda_l} \to \pm i\frac{d\sqrt{5}}{2}\mathbf{k}^\mu + \ldots. \tag{4.15}$$

So, we have

$$D \to i\frac{d\sqrt{5}}{2} \sum_{\mu=1}^{4} \sigma^\mu \mathbf{k}_\mu, \qquad \bar{D} \to -i\frac{d\sqrt{5}}{2} \sum_{\mu=1}^{4} \bar{\sigma}^\mu \mathbf{k}_\mu. \tag{4.16}$$

The operators D and $\bar{D}$ have zeros for wave vectors $\mathbf{k}$ satisfying the following constraint relation:

$$\mathbf{k}.\lambda_l = \frac{4\pi N}{5d\sqrt{5}}, \tag{4.17}$$

with N an arbitrary integer. The point is that for these values, the phases $e^{id(\sqrt{5}/2)\mathbf{k}.\lambda_l} = e^{i\varphi}$ and the operators D and $\bar{D}$ get reduced to

$$D = e^{i\varphi} \sum_{\mu=1}^{4} \sigma^\mu \Big(\sum_{l=0}^{4} \lambda_l^\mu \Big), \qquad \bar{D} = e^{-i\varphi} \sum_{\mu=1}^{4} \bar{\sigma}^\mu \Big(\sum_{l=0}^{4} \lambda_l^\mu \Big) \tag{4.18}$$

which vanish identically due to the property $\sum_{l=0}^{4} \lambda_l^\mu = 0$. Following [8,11], the Dirac operator (4.10) in the Creutz lattice model reads as follows:

$$\begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} \tag{4.19}$$

where $z = \theta_0 I + i\theta_1\sigma^1 + i\theta_2\sigma^2 + i\theta_3\sigma^3$ with

$$\begin{aligned} \theta_1 &= \sin p_1 + \sin p_2 - \sin p_3 - \sin p_4 \\ \theta_2 &= \sin p_1 - \sin p_2 - \sin p_3 + \sin p_4 \\ \theta_3 &= \sin p_1 - \sin p_2 + \sin p_3 - \sin p_4 \\ \theta_0 &= B(4C - \cos p_1 - \cos p_2 - \cos p_3 - \cos p_4) \end{aligned} \tag{4.20}$$

and B and C two real parameters. In the Creutz lattice model, the zero energy states correspond to $z = 0$; this leads to the constraints $\theta_i = 0$ which are solved by taking one of the momenta as $p_1 = p$ and the others as $p_i = p$ or $\pi - p$. To make contact with our construction, the analogous of Eqs. (4.20) are given by

$$\begin{aligned} \theta_1 &= \lambda_0^1 e^{-id(\sqrt{5}/2)p_0} + \lambda_1^1 e^{-id(\sqrt{5}/2)p_1} + \lambda_2^1 e^{-id(\sqrt{5}/2)p_2} \\ &\quad + \lambda_3^1 e^{-id(\sqrt{5}/2)p_3} + \lambda_4^1 e^{-id(\sqrt{5}/2)p_4} \\ \theta_2 &= \lambda_0^2 e^{-id(\sqrt{5}/2)p_0} + \lambda_1^2 e^{-id(\sqrt{5}/2)p_1} + \lambda_2^2 e^{-id(\sqrt{5}/2)p_2} \\ &\quad + \lambda_3^2 e^{-id(\sqrt{5}/2)p_3} + \lambda_4^2 e^{-id(\sqrt{5}/2)p_4} \\ \theta_3 &= \lambda_0^3 e^{-id(\sqrt{5}/2)p_0} + \lambda_1^3 e^{-id(\sqrt{5}/2)p_1} + \lambda_2^3 e^{-id(\sqrt{5}/2)p_2} \\ &\quad + \lambda_3^3 e^{-id(\sqrt{5}/2)p_3} + \lambda_4^3 e^{-id(\sqrt{5}/2)p_4} \\ \theta_0 &= \lambda_0^4 e^{-id(\sqrt{5}/2)p_0} + \lambda_1^4 e^{-id(\sqrt{5}/2)p_1} + \lambda_2^4 e^{-id(\sqrt{5}/2)p_2} \\ &\quad + \lambda_3^4 e^{-id(\sqrt{5}/2)p_3} + \lambda_4^4 e^{-id(\sqrt{5}/2)p_4} \end{aligned} \tag{4.21}$$

where $p_l = \mathbf{k}.\lambda_l$. These relations are complex and are, in some sense, more general than the Creutz ones (4.20). The zeros of these solutions requires $e^{id(\sqrt{5}/2)p_i} = e^{i\varphi}\ \forall\ l = 0, 1, 2, 3, 4$ as anticipated in (3.12).

V. ENERGY DISPERSION AND ZERO MODES

To get the dispersion energy relations of the four wave components $\phi_{\mathbf{k}}^1$, $\phi_{\mathbf{k}}^2$, $\chi_{\mathbf{k}}^1$, and $\chi_{\mathbf{k}}^2$ and their corresponding four holes, one has to solve the eigenvalues of the Dirac operator (4.10). To that purpose, we first write the four-dimensional wave equation as follows:

$$\begin{pmatrix} 0 & -iD \\ i\bar{D} & 0 \end{pmatrix} \begin{pmatrix} \phi_{\mathbf{k}} \\ \chi_{\mathbf{k}} \end{pmatrix} = E \begin{pmatrix} \phi_{\mathbf{k}} \\ \chi_{\mathbf{k}} \end{pmatrix}, \tag{5.1}$$

where $\phi_{\mathbf{k}} = (\phi_{\mathbf{k}}^1, \phi_{\mathbf{k}}^2)$, $\chi_{\mathbf{k}} = (\chi_{\mathbf{k}}^1, \chi_{\mathbf{k}}^2)$ are Weyl spinors, and where the 2×2 matrices D, $\bar{D}$ are as in Eqs. (4.11) and (4.13). Then determine the eigenstates and eigenvalues of the 2×2 Dirac operator matrix by solving the following characteristic equation:

$$\det\begin{pmatrix} -E & 0 & D_{11} & D_{12} \\ 0 & -E & D_{21} & D_{22} \\ \bar{D}_{11} & \bar{D}_{21} & -E & 0 \\ \bar{D}_{12} & \bar{D}_{22} & 0 & -E \end{pmatrix} = 0 \tag{5.2}$$

from which one can learn the four dispersion energy eigenvalues $E_1(\mathbf{k})$, $E_2(\mathbf{k})$, $E_3(\mathbf{k})$, and $E_4(\mathbf{k})$, and therefore their zeros.

A. Computing the energy dispersion

An interesting way to do these calculations is to act on (5.1) once more by the Dirac operator to bring it to the following diagonal form:

$$\begin{pmatrix} D\bar{D} & 0 \\ 0 & D\bar{D} \end{pmatrix}\begin{pmatrix} \phi_{\mathbf{k}} \\ \chi_{\mathbf{k}} \end{pmatrix} = E^2\begin{pmatrix} \phi_{\mathbf{k}} \\ \chi_{\mathbf{k}} \end{pmatrix}. \tag{5.3}$$

Then solve separately the eigenvalues problem of the two-dimensional equations $D\bar{D}\phi_{\mathbf{k}} = E^2\phi_{\mathbf{k}}$ and $\bar{D}D\chi_{\mathbf{k}} = E^2\chi_{\mathbf{k}}$. To do so, it is useful to set

$$u(\mathbf{k}) = \vartheta^1 + i\vartheta^2, \qquad v(\mathbf{k}) = \vartheta^3 + i\vartheta^4 \tag{5.4}$$

with

$$\vartheta^\mu = \sum_{l=0}^{4} \lambda_l^\mu e^{id(\sqrt{5}/2)\mathbf{k}.\lambda_l}, \qquad \mu = 1,2,3,4. \tag{5.5}$$

Notice that in the continuous limit, we have

$$\begin{aligned} \vartheta^\mu &\to id\frac{\sqrt{5}}{2}\mathbf{k}^\mu, \\ u(\mathbf{k}) &\to id\frac{\sqrt{5}}{2}(\mathbf{k}^1 + i\mathbf{k}^2), \\ v(\mathbf{k}) &\to id\frac{\sqrt{5}}{2}(\mathbf{k}^3 + i\mathbf{k}^4). \end{aligned} \tag{5.6}$$

Substituting (5.4) back into (4.11) and (4.13), we obtain the following expressions:

$$D\bar{D} = \begin{pmatrix} |u|^2 + |v|^2 & 2\bar{u}v \\ 2u\bar{v} & |u|^2 + |v|^2 \end{pmatrix}, \tag{5.7}$$

and

$$\bar{D}D = \begin{pmatrix} |u|^2 + |v|^2 & 2\bar{u}\,\bar{v} \\ 2uv & |u|^2 + |v|^2 \end{pmatrix}. \tag{5.8}$$

By solving the characteristic equations of these 2×2 matrix operators, we get the following eigenstates $\phi_{\mathbf{k}}^{a\prime}$, $\chi_{\mathbf{k}}^{a\prime}$ with their corresponding eigenvalues $E_\pm^2$:

$$\begin{array}{ll} \text{eigenstates} & \text{eigenvalues} \\ \phi_{\mathbf{k}}^{1\prime} = \sqrt{\frac{v\bar{u}}{2|u||v|}}\phi_{\mathbf{k}}^1 + \sqrt{\frac{u\bar{v}}{2|u||v|}}\phi_{\mathbf{k}}^2 & E_+^2 = |u|^2 + |v|^2 + 2|u||v| \\ \phi_{\mathbf{k}}^{2\prime} = -\sqrt{\frac{v\bar{u}}{2|u||v|}}\phi_{\mathbf{k}}^1 + \sqrt{\frac{u\bar{v}}{2|u||v|}}\phi_{\mathbf{k}}^2 & E_-^2 = |u|^2 + |v|^2 - 2|u||v| \end{array} \tag{5.9}$$

and

$$\begin{array}{ll} \text{eigenstates} & \text{eigenvalues} \\ \chi_{\mathbf{k}}^{1\prime} = \sqrt{\frac{\bar{u}\,\bar{v}}{2|u||v|}}\chi_{\mathbf{k}}^1 + \sqrt{\frac{uv}{2|u||v|}}\chi_{\mathbf{k}}^2 & E_+^2 = |u|^2 + |v|^2 + 2|u||v| \\ \chi_{\mathbf{k}}^{2\prime} = -\sqrt{\frac{\bar{u}\,\bar{v}}{2|u||v|}}\chi_{\mathbf{k}}^1 + \sqrt{\frac{uv}{2|u||v|}}\chi_{\mathbf{k}}^2 & E_-^2 = |u|^2 + |v|^2 - 2|u||v|. \end{array} \tag{5.10}$$

By taking square roots of $E_\pm^2$, we obtain two positive and two negative dispersion energies; these are

$$E_\pm = +\sqrt{(|u| \pm |v|)^2} \tag{5.11}$$

which correspond to particles, and

$$E_\pm^* = -\sqrt{(|u| \pm |v|)^2} \tag{5.12}$$

corresponding to the associated holes.

B. Determining the zeros of $E_\pm$ and $E_\pm^*$

From the above energy dispersion relations, one sees that the zero modes are of two kinds as listed here:

$$\text{zeros of both } E_+^2 = 0, \qquad E_-^2 = 0.$$

They are given by those wave vectors $\mathbf{K}_F$ solving the constraint relations $u(\mathbf{K}_F) = v(\mathbf{K}_F) = 0$, which can be also put in the form

$$\begin{aligned} &\lambda_0^\mu e^{id(\sqrt{5}/2)\mathbf{K}_F.\lambda_0} + \lambda_1^\mu e^{id(\sqrt{5}/2)\mathbf{K}_F.\lambda_1} + \lambda_2^\mu e^{id(\sqrt{5}/2)\mathbf{K}_F.\lambda_2} \\ &\quad + \lambda_3^\mu e^{id(\sqrt{5}/2)\mathbf{K}_F.\lambda_3} + \lambda_4^\mu e^{id(\sqrt{5}/2)\mathbf{K}_F.\lambda_4} = 0 \end{aligned} \tag{5.13}$$

for all values of $\mu = 1, 2, 3, 4$; or equivalently like

$$d\frac{\sqrt{5}}{2}\mathbf{K}_F.\lambda_l = \frac{2\pi}{5}N + 2\pi N_l. \tag{5.14}$$

The solutions of these constraint equations have been studied in Sec. III; they are precisely given by Eqs. (3.15) and (3.17). Now, setting $\mathbf{k} = \mathbf{K}_F + \mathbf{q}$ with small $q = \|\mathbf{q}\|$ and expanding D and $\bar{D}$, Eq. (5.1) gets reduced to

$$\frac{d\sqrt{5}}{2}\sum_{\mu=1}^{4}\mathbf{q}_\mu\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} \phi_{\mathbf{k}} \\ \chi_{\mathbf{k}} \end{pmatrix} = E\begin{pmatrix} \phi_{\mathbf{k}} \\ \chi_{\mathbf{k}} \end{pmatrix}. \tag{5.15}$$

case $E_-^2 = 0$, but $E_+^2 = E_{+\,\min}^2 \neq 0$

These minima are given by those wave vectors $\mathbf{K} = \mathbf{k}_{\min}$ solving the following constraint relation $|u(\mathbf{K})| = |v(\mathbf{K})|$, or equivalently

$$\begin{aligned} &\sum_{m,n=0}^{4}(\lambda_m^1 + i\lambda_m^2)(\lambda_n^1 - i\lambda_n^2)e^{id(\sqrt{5}/2)\mathbf{K}.\beta_{mn}} \\ &\quad = \sum_{m,n=0}^{4}(\lambda_m^3 + i\lambda_m^4)(\lambda_n^3 - i\lambda_n^4)e^{id(\sqrt{5}/2)\mathbf{K}.\beta_{mn}}. \end{aligned} \tag{5.16}$$

Expanding this equality, we get the following condition on the wave vector:

$$\sum_{m,n=0}^{4} \mathcal{A}_{nm} \cos\left(d\frac{\sqrt{5}}{2}\mathbf{K}.\beta_{mn}\right) \times \left[\tan\left(d\frac{\sqrt{5}}{2}\mathbf{K}.\beta_{mn}\right) - \frac{\mathcal{B}_{nm}}{\mathcal{A}_{nm}}\right] = 0, \quad (5.17)$$

with

$$\begin{aligned} \mathcal{A}_{nm} &= (\lambda_n^1\lambda_m^2 - \lambda_m^1\lambda_n^2) - (\lambda_n^3\lambda_m^4 - \lambda_m^3\lambda_n^4) \\ \mathcal{B}_{nm} &= (\lambda_m^1\lambda_n^1 + \lambda_m^2\lambda_n^2) - (\lambda_m^3\lambda_n^3 + \lambda_m^4\lambda_n^4). \end{aligned} \quad (5.18)$$

A possible solution is given by those wave vectors $\mathbf{K}$ obeying the relation $\mathbf{K}.\beta_{mn} = \frac{2}{d\sqrt{5}} \arctan(\mathcal{B}_{nm}/\mathcal{A}_{nm})$.

VI. REDERIVING BC FERMIONS

In this section, we give the link between the above study based on $SU(5)$ symmetry and the so-called Boriçi-Creutz (BC) model having two zero modes associated with the light quarks up and down of QCD. Recall that one of the important things in lattice QCD is the need to have a fermion action with a Dirac operator $\mathcal{D}$ having two zero modes at points K and K' of the reciprocal space, so that they could be interpreted as the two light quarks. From this view, one may ask [20] whether there exists a link between the present analysis and the BC fermions [17,18]. In answering this question, we have found that the BC model can be indeed recovered from the analysis developed in this paper. In what follows, we give the main lines of the derivation.

A. More on lattice action (4.8)

One of the interesting lessons we have learned from the analysis developed in the previous sections is that the lattice action for 4D hyperdiamond fermions may generally be written like

$$\mathcal{S} \sim \frac{i}{4a}\sum_{\mathbf{r}}\left(\sum_{l=0}^{4} \bar{\Psi}_{\mathbf{r}}\Gamma^l\Psi_{\mathbf{r}+a\lambda_l} + \sum_{l=0}^{4} \bar{\Psi}_{\mathbf{r}}\bar{\Gamma}^l\Psi_{\mathbf{r}-a\lambda_l}\right), \quad (6.1)$$

where $a = d\frac{\sqrt{5}}{2}$, the weight vectors λ_l as in Eqs. (2.3) and (3.19) and where Γ^l and their complex adjoints $\bar{\Gamma}^l$ are 4×4 complex matrices given by linear combinations of the Dirac matrices γ^μ as follows:

$$\Gamma^l = \left(\sum_{\mu=1}^{4} \gamma^\mu \Omega_\mu^l\right), \qquad \bar{\Gamma}^l = \left(\sum_{\mu=1}^{4} \gamma^\mu \bar{\Omega}_\mu^l\right) \quad (6.2)$$

with Ω_μ^l linking the lattice Euclidean space time index μ and the index l of the five-dimensional representation of the $SU(5)$ symmetry of the hyperdiamond. As such, the lattice action (6.1) depends on the coefficients Ω_μ^l capturing 20 complex numbers that form a 5×4 matrix representing the bi-fundamental of $SO(4) \times SU(5)$

$$\Omega_\mu^l = \begin{pmatrix} \Omega_1^0 & \Omega_1^1 & \Omega_1^2 & \Omega_1^3 & \Omega_1^4 \\ \Omega_2^0 & \Omega_2^1 & \Omega_2^2 & \Omega_2^3 & \Omega_2^4 \\ \Omega_3^0 & \Omega_3^1 & \Omega_3^2 & \Omega_3^3 & \Omega_3^4 \\ \Omega_4^0 & \Omega_4^1 & \Omega_4^2 & \Omega_4^3 & \Omega_4^4 \end{pmatrix}. \quad (6.3)$$

This rank two tensor, which we decompose as $(\omega_\mu, \Omega_\mu^\nu)$ with $\omega_\mu = \Omega_\mu^0$ a complex 4 component vector and Ω_μ^ν a complex 4×4 matrix, gives enough freedom to engineer Dirac operators with a definite number of zero modes. Below, we derive the constraint equations for the zero modes of the Dirac operator; and in next subsection we apply the analysis to the BC model.

1. Dirac operator

In the reciprocal space, the lattice action (6.1) reads as

$$\mathcal{S} \sim \sum_{\mathbf{k}}\left(\sum_{\mu=1}^{4} \bar{\Psi}_{\mathbf{k}}\mathcal{D}\Psi_{\mathbf{k}}\right) \quad (6.4)$$

with the Dirac operator reading as follows:

$$\mathcal{D} = \frac{i}{4a}\sum_{\mu=1}^{4} \gamma^\mu(D_\mu + \bar{D}_\mu), \quad (6.5)$$

and where D_μ and its complex adjoint $\bar{D}_\mu$ are given by

$$D_\mu = \sum_{l=0}^{4} \Omega_\mu^l e^{ia\mathbf{k}.\lambda_l}, \qquad \bar{D}_\mu = \sum_{l=0}^{4} \bar{\Omega}_\mu^l e^{-ia\mathbf{k}.\lambda_l}. \quad (6.6)$$

These operators depend on $40 = 2(4 + 16)$ real numbers

$$\omega_\mu = \frac{1}{2}(u_\mu + iv_\mu), \qquad \Omega_\mu^\nu = \frac{1}{2}R_\mu^\nu + \frac{i}{2}J_\mu^\nu \quad (6.7)$$

and also on the five momenta $p_l = \hbar k_l$ along the λ_l-directions. Since $k_l = \mathbf{k}.\lambda_l$ and because of $SU(5)$ symmetry, we have moreover the constraint relation

$$k_0 + k_1 + k_2 + k_3 + k_4 = 0, \qquad \mathrm{mod}\frac{2\pi}{a}, \quad (6.8)$$

allowing to express one of the five k_l's in terms of the four others. For instance, we can express k_0 as follows:

$$k_0 = -(k_1 + k_2 + k_3 + k_4), \qquad \mathrm{mod}\frac{2\pi}{a}. \quad (6.9)$$

The next step is to find the set of wave vectors $k_\mu = (k_1, k_2, k_3, k_4)$ that give the zeros of the Dirac operator. These zeros depend on the numbers u_μ, v_μ, R_μ^ν, and J_μ^ν, which can be tuned in order to get the desired number of zeros.

2. Zero modes

The zero modes of the Dirac operator $\mathcal{D}$ given by Eqs. (6.5) and (6.6) are obtained by solving the following constraint equations:

$$\sum_{\mu=1}^{4}\sum_{l=0}^{4}\gamma^{\mu}(\Omega_{\mu}^{l}+\bar{\Omega}_{\mu}^{l})\cos ak_{l} + i\sum_{\mu=1}^{4}\sum_{l=0}^{4}\gamma^{\mu}(\Omega_{\mu}^{l}-\bar{\Omega}_{\mu}^{l})\sin ak_{l}=0, \qquad (6.10)$$

together with the constraint Eq. (6.8). Using the decomposition $\Omega_{\mu}^{l}=(\omega_{\mu},\Omega_{\mu}^{\nu})$, we can decompose these constraints as follows:

$$\Lambda+\sum_{\mu=1}^{4}\left(\sum_{\nu=1}^{4}\gamma^{\mu}(\Omega_{\mu}^{\nu}+\bar{\Omega}_{\mu}^{\nu})\cos ak_{\nu} + i\sum_{\nu=1}^{4}\gamma^{\mu}(\Omega_{\mu}^{\nu}-\bar{\Omega}_{\mu}^{\nu})\sin ak_{\nu}\right)=0, \qquad (6.11)$$

where we have set

$$\Lambda=\cos ak_{0}\sum_{\mu=1}^{4}\gamma^{\mu}(\omega_{\mu}+\bar{\omega}_{\mu}) + i\sin ak_{0}\sum_{\mu=1}^{4}\gamma^{\mu}(\omega_{\mu}-\bar{\omega}_{\mu}). \qquad (6.12)$$

Moreover, using (6.7) we can put the above constraint relations into the following equivalent form:

$$\Lambda+\sum_{\mu=1}^{4}\left(\sum_{\nu=1}^{4}\gamma^{\mu}R_{\mu}^{\nu}\cos ak_{\nu}-\sum_{\nu=1}^{4}\gamma^{\mu}J_{\mu}^{\nu}\sin ak_{\nu}\right)=0, \qquad (6.13)$$

and

$$\Lambda=\cos ak_{0}\left(\sum_{\mu=1}^{4}\gamma^{\mu}u_{\mu}\right)-\sin ak_{0}\left(\sum_{\mu=1}^{4}\gamma^{\mu}v_{\mu}\right)=0, \qquad (6.14)$$

with k_0 given by Eq. (6.8). Equations (6.13) and (6.14) define a highly nonlinear system of coupled equations in the four k_ν's, and are difficult to solve in the generic case. To overcome this difficulty, one may deal with these equations by focusing on adequate solutions for the k_ν's, and engineer the corresponding Ω_{μ}^{l} tensor. Below, we apply this idea to the BC model.

B. BC fermions

1. Deriving the model

The Boriçi-Creutz model [17] is a simple lattice QCD fermions for modeling and simulating the interacting dynamics of the two light quarks up and down. The Dirac operator of this model reads in the reciprocal space as follows:

$$\mathcal{D}_{BC}\sim\frac{i}{a}\sum_{\mu=1}^{4}\gamma^{\mu}\sin ak_{\mu}-\frac{i}{a}\sum_{\mu=1}^{4}\gamma^{\mu}\cos ak_{\mu} +\frac{i}{a}\sum_{\mu=1}^{4}\Gamma\cos ak_{\mu}-\frac{2i}{a}\Gamma, \qquad (6.15)$$

with $\Gamma=\frac{1}{2}(\gamma^{1}+\gamma^{2}+\gamma^{3}+\gamma^{4})$. From this expression, one can check that this operator has two zero modes given by the two following wave vectors:

$$\begin{aligned}&(1)\colon (k_1,k_2,k_3,k_4)=(0,0,0,0),\\ &(2)\colon (k_1,k_2,k_3,k_4)=\left(\frac{\pi}{2a},\frac{\pi}{2a},\frac{\pi}{2a},\frac{\pi}{2a}\right),\end{aligned} \qquad (6.16)$$

satisfying the remarkable property

$$k_1+k_2+k_3+k_4=\frac{2\pi}{a}, \qquad \mathrm{mod}\frac{2\pi}{a}. \qquad (6.17)$$

Clearly, the operator $\mathcal{D}_{\mathrm{BC}}$ corresponds to a particular configuration of the complex tensor Ω_{μ}^{ν} and the vector ω_{μ}. To see that is indeed the case, notice first that the matrix Γ can be conveniently rewritten as $\Gamma=\frac{1}{2}\vartheta_{\mu}\gamma^{\mu}$, with

$$\vartheta_{\mu}=(1,1,1,1). \qquad (6.18)$$

The same feature is valid for the sum $\sum_{\nu=1}^{4}\cos ak_{\nu}$ which can be also put in the form $\sum_{\nu=1}^{4}\vartheta^{\nu}\cos ak_{\nu}$. Putting these expressions back into the above $\mathcal{D}_{\mathrm{BC}}$ relation, we get

$$\mathcal{D}_{BC}\sim\frac{i}{a}\sum_{\mu,\nu=1}^{4}\gamma^{\mu}\delta_{\mu}^{\nu}\sin ak_{\nu} -\frac{i}{a}\sum_{\mu,\nu=1}^{4}\gamma^{\mu}M_{\mu}^{\nu}\cos ak_{\nu}-\frac{2i}{a}\Gamma, \qquad (6.19)$$

with

$$M_{\mu}^{\nu}=\delta_{\mu}^{\nu}-\frac{1}{2}\vartheta_{\mu}\vartheta^{\nu} \qquad (6.20)$$

or more explicitly,

$$M_{\mu}^{\nu}=\begin{pmatrix}+\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}\\ -\frac{1}{2}&+\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}\\ -\frac{1}{2}&-\frac{1}{2}&+\frac{1}{2}&-\frac{1}{2}\\ -\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&+\frac{1}{2}\end{pmatrix}. \qquad (6.21)$$

Now, comparing Eq. (6.19) with the general form of the Dirac operator of Eq. (6.5) and (6.6), which also reads like

$$\Lambda+\sum_{\mu=1}^{4}\left(\sum_{\nu=1}^{4}\gamma^{\mu}R_{\mu}^{\nu}\cos ak_{\nu}-\sum_{\nu=1}^{4}\gamma^{\mu}J_{\mu}^{\nu}\sin ak_{\nu}\right)=0, \qquad (6.22)$$

we see that $\mathcal{D}_{\mathrm{BC}}$ can be recovered by taking

$$R_{\mu}^{\nu}=-M_{\mu}^{\nu}, \qquad J_{\mu}^{\nu}=-\delta_{\mu}^{\nu}, \qquad (6.23)$$

and

$$\Lambda = -2\Gamma = -(\gamma^1 + \gamma^2 + \gamma^3 + \gamma^4). \tag{6.24}$$

Equation (6.23) leads to $\Omega_\mu^\nu = -\frac{1}{2}(M_\mu^\nu + i\delta_\mu^\nu)$; by substituting M_μ^ν by its expression given above, the tensor Ω_μ^ν reads more explicitly like

$$\Omega_\mu^\nu = -\frac{(1+i)}{2}\delta_\mu^\nu + \frac{1}{4}\vartheta_\mu\vartheta^\nu. \tag{6.25}$$

The second constraint relation (6.24) requires

$$(u_\mu \cos ak_0 - v_\mu \sin ak_0) = -\vartheta_\mu, \tag{6.26}$$

with ϑ_μ as in (6.18). Moreover, using Eqs. (6.17) and (6.9), we end with

$$u_\mu = -\vartheta_\mu, \tag{6.27}$$

and v_μ a free vector which, for simplicity, we set to zero. Thus, the tensor $\Omega_\mu^l = (\omega_\mu, \Omega_\mu^\nu)$ describing the BC fermions is given by

$$\Omega_\mu^l = \begin{pmatrix} -\frac{1}{2} & -\frac{1+2i}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1+2i}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & -\frac{1+2i}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1+2i}{4} \end{pmatrix}, \tag{6.28}$$

with the trace property $\sum_l \Omega_\mu^l = -\frac{i}{2}\vartheta_\mu$.

2. Symmetries

Here, we want to make a comment on particular symmetries of 4D lattice QCD fermions by following the analysis of Ref. [6], where the study of the renormalization of this class of models has been explicitly done. There, it has been found that the breaking of discrete symmetries, such as parity $\mathcal{P}$: $\Psi(\vec{k}, k_4) \to \gamma_4\Psi(-\vec{k}, k_4)$ and time-reversal $\mathcal{T}$: $(\vec{k}, k_4) \to \gamma_5\gamma_4\Psi(\vec{k}, -k_4)$, is behind the appearance of relevant dimension 3 operators $\mathcal{O}_3^{(i)}$ and marginal dimension 4 ones $\mathcal{O}_4^{(j)}$ in the analysis of the Symanzik effective theory with Lagrangian $\mathcal{L}_{\text{eff}} = \frac{1}{a^4}\sum_n a^n \sum_j c_n^{(j)}\mathcal{O}_n^{(j)}$. Following the above-mentioned work, one starts from the 4D lattice action,

$$\mathcal{S} \sim \frac{1}{2}\sum_{\mathbf{x}}\sum_{\mu=1}^{4}[\Psi_{\mathbf{x}}^{+} A^\mu \Psi_{\mathbf{x}+av_\mu} - \Psi_{\mathbf{x}}^{+}\bar{A}^\mu\Psi_{\mathbf{x}-av_\mu} - 2iBC\Psi_{\mathbf{x}}^{+}\gamma^4\Psi_{\mathbf{x}}], \tag{6.29}$$

which depends on two real parameters, B and C, that are fixed by physical requirements and symmetries. This typical action depends also on particular combinations of gamma matrices $A^\mu = \sum_{\nu=1}^{4}\gamma^\nu A_\nu^\mu$ where the coefficients A_ν^μ, given in [6], form an invertible 4×4 matrix with $\det(A_\nu^\mu) = -16iB$. Notice that setting $B = 1$, $C = \frac{\sqrt{2}}{2}$, one recovers the Boriçi action. By performing transformations of (6.29) using Fourier integrals to move to the reciprocal space, similarity operations to exhibit particular symmetries, expansion in powers of the lattice spacing parameter a to use the Symanzik effective theory, and switching on the usual gauge interactions $\partial_\mu \to \mathfrak{D}_\mu = \partial_\mu - ig\mathcal{A}_\mu$ with field strength $\mathcal{F}_{\mu\nu} = \frac{i}{g}\times[\mathfrak{D}_\mu, \mathfrak{D}_\nu]$, we get up to the first order in the parameter a the following effective field action:

$$\mathcal{L}_{\text{eff}} = \sum_{\mathbf{x}}\Big[\bar{Q}(\gamma^\mu \otimes I)\mathfrak{D}_\mu Q - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + a\mathcal{O}_5 + \text{ord}(a^2)\Big],$$

with Q_α standing for the quark isodoublet (u_α, d_α) and $\mathcal{O}_5$ some dimension 5 operator that can be found in [6]. This effective theory has several symmetries in particular: (1) the manifest gauge invariance, (2) $U_B(1)$ baryon number, (3) $U_L(1) \times U_R(1)$ chiral symmetry, (4) CPT invariance, and (5) symmetry under S_4 permutation of the four hyperplane axis corresponding to $C = BS$ with $e^{iK} = C + iS$. The authors of [6] concluded their work by two remarkable results: (i) the engineering of a chirally symmetric action with minimal fermion doubling which does not generate dimension 3 operators $\mathcal{O}_3$ is possible as far as $\mathcal{PT}$ symmetry is preserved. This invariance is sufficient to forbid the relevant dimension 3 operators $\mathcal{O}_3^{(i)}$ whose typical forms are listed below:

$$\begin{aligned} \text{broken } \mathcal{P}\text{: } \mathcal{O}_3^{(1)} &= i\bar{\Psi}_{\vec{k},k_4}\gamma_j\Psi_{\vec{k},k_4}, \\ \mathcal{O}_3^{(2)} &= i\bar{\Psi}_{\vec{k},k_4}\gamma_4\gamma_5\Psi_{\vec{k},k_4} \\ \text{broken } \mathcal{T}\text{: } \mathcal{O}_3^{(3)} &= i\bar{\Psi}_{\vec{k},k_4}\gamma_j\gamma_5\Psi_{\vec{k},k_4}, \\ \mathcal{O}_3^{(4)} &= i\bar{\Psi}_{\vec{k},k_4}\gamma_j\gamma_5\Psi_{\vec{k},k_4} \end{aligned}$$

with γ_j standing for γ_1, γ_2, and γ_3. (ii) For particular values of parameters of the theory, there may emerge some additional nonstandard symmetries which could be used to eliminate the relevant operators. These results are important and may serve as guidelines in dealing with this problem by using the hyperdiamond symmetries based on roots and weights of $SU(5)$. Below, we give a comment on this matter; an exact answer, however, needs a deeper analysis. In the $SU(5)$ framework, the previous action (6.29) gets extended as follows:

$$\mathcal{S}_{su_5} \sim \frac{i}{a}\sum_{\mathbf{x}}\sum_\mu\Big(\sum_{l=0}^{4}[\bar{\Psi}_{\mathbf{x}}\gamma^\mu\Omega_\mu^l\Psi_{\mathbf{x}+a\lambda_l} + \bar{\Psi}_{\mathbf{x}}\gamma^\mu\bar{\Omega}_\mu^l\Psi_{\mathbf{x}-a\lambda_l}]\Big), \tag{6.30}$$

where Ω_μ^l as before and the λ_l's are the weight vectors of the five-dimensional representation of $SU(5)$. Clearly, this lattice action is more general than Eq. (6.29), and has two interesting features that are useful in dealing with the study of underlying symmetries and renormalization of $\mathcal{S}_{su_5}$. First, the $SU(5)$ property (3.1) on the weight vectors, namely $\sum_l \lambda_l^\mu = 0$, induces in turns the following constraint relation on the wave vectors k_μ:

$$\sum_{l=0}^{5} k_l = 0, \quad \text{with } k_l = \sum_{\mu=1}^{5} k_\mu . \lambda_l^\mu .$$

This constraint is invariant under $\mathcal{PT}$ symmetry acting on wave vectors as $k_\mu \to -k_\mu$, but not preserved under parity $\mathcal{P}$ nor time-reversal $\mathcal{T}$ separately. Second, the generalized action $\mathcal{S}_{su_5}$ depends on 20 complex (40 real) moduli carried by the tensor Ω_μ^l. This number gives quite enough freedom to engineer QCD-like models with two zeros for the Dirac operator as we have done in case of the BC model. It may lead to desired symmetries of the Symanzik effective theory that follow from the expansion of the action $\mathcal{S}_{su_5}$ in powers of the lattice parameter, and may allow to make appropriate choices to eliminate relevant operators. Progress in this matter will be reported in a future occasion.

VII. CONCLUSION

In this paper, we have studied the lattice fermion action for pristine 4D hyperdiamond $\mathcal{H}_4$ with desired properties for 4D lattice QCD simulations. Using the $SU(5)$ hidden symmetry of $\mathcal{H}_4$, we have constructed a BBTW-like lattice model by mimicking 2D graphene model. To that purpose, we first studied the link between the construction of [5] and $SU(5)$, then we refined the BBTW lattice action by using the weight vectors λ_0, λ_1, λ_2, λ_3, and λ_4 of the five-dimensional representation of $SU(5)$. After that, we studied explicitly the solutions of the zeros of the Dirac operator in terms of the $SU(5)$ simple roots α_1, α_2, α_3, and α_4, and its fundamental weights ω_1, ω_2, ω_3, and ω_4. We have found that the zeros of the Dirac operator live at the sites $\mathbf{k} = \frac{4\pi}{d\sqrt{5}}(N_1\omega_1 + N_2\omega_2 + N_3\omega_3 + N_4\omega_4)$ of the reciprocal lattice $\mathcal{H}_4^*$, with N_i integers. In addition to their quite similar continuum limit, we have also studied the link between the Dirac operator following from our construction and the one suggested by Creutz using quaternions; the Dirac operator in our approach may be viewed as a *complexification* of the Creutz one where the role played by the $\sin(p_i)$'s and the $\cos(p_i)$'s is now played by e^{ip_i} as shown in Eqs. (4.20) and (4.21). The exact link between our approach and the Boriçi-Creutz fermions has been worked out with details in Sec. VI, where it is shown that the BC action follows exactly from (6.1) with Eqs. (6.2) and (6.28), giving the linear combinations of the Dirac matrices of the model.

It is also interesting to notice that our approach is general, and applies straightforwardly to lattice systems in diverse dimensions. The fact that the 4D hyperdiamond is related to $SU(5)$ fundamental weights ω_1, ω_2, ω_3, and ω_4, and its simple roots α_1, α_2, α_3, and α_4 is not specific for four dimensions; it can be extended to generic dimensions D where the underlying D-dimensional hyperdiamond lattice has a hidden $SU(D+1)$ symmetry with simple roots $\alpha_1, \ldots, \alpha_D$ and fundamental weights $\omega_1, \ldots, \omega_D$. From this view, the 2D graphene has therefore a hidden $SU(3)$ symmetry as reported in detail in [3]. Our construction applies as well to the fermion actions given in [12].

ACKNOWLEDGMENTS

L. B. Drissi would like to thank ICTP and E. H. Saidi thanks URAC-09, CNRST.

[1] P. R. Wallace, Phys. Rev. **71**, 622 (1947).
[2] A. H. Castro-Neto *et al.*, Rev. Mod. Phys. **81**, 109 (2009).
[3] L. B. Drissi, E. H. Saidi, and M. Bousmina, **B829**, 523 (2010).
[4] Michael Creutz, J. High Energy Phys. 04 (2008) 017.
[5] P. F. Bedaque, M. I. Buchoff, B. C. Tiburzi, and A. Walker-Loud, Phys. Rev. D **78**, 017502 (2008).
[6] P. F. Bedaque, M. I. Buchoff, B. C. Tiburzi, and A. Walker-Loud, Phys. Lett. B **662**, 449 (2008).
[7] T. Kimura and T. Misumi, Prog. Theor. Phys. **123**, 63 (2010).
[8] M. Creutz and T. Misumi, Phys. Rev. D **82**, 074502 (2010).
[9] S. Catterall, D. B. Kaplan, and M. Unsal, Phys. Rep. **484**, 71 (2009).
[10] S. Catterall, E. Dzienkowski, J. Giedt, A. Joseph, and R. Wells, J. High Energy Phys. 04 (2011) 074.
[11] A. Borici, Phys. Rev. D **78**, 074504 (2008).
[12] T. Kimura and T. Misumi, Prog. Theor. Phys. **123**, 63 (2010).
[13] T. Kimura and T. Misumi, Prog. Theor. Phys. **124**, 415 (2010).
[14] J. B. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975).
[15] H. B. Nielson and M. Ninomiya, Nucl. Phys. **B185**, 20 (1981); **195**, 541 (1982).
[16] L. B. Drissi, E. H. Saidi, and M. Bousmina, J. Math. Phys. (N.Y.) **52**, 022306 (2011).
[17] S. Capitani, J. Weber, and H. Wittig, Phys. Lett. B **681**, 105 (2009).
[18] S. Capitani, M. Creutz, J. Weber, and H. Wittig, J. High Energy Phys. 09 (2010) 027.
[19] L. B. Drissi and E. H. Saidi, arXiv:1103.1316 [Phys. Rev. D (to be published)].
[20] We thank the referee for pointing out this question, which allowed us to exhibit the relationship between our approach and BC fermions; see [19] for explicit details.
[21] This correspondence differs from the one given by BBTW in [5].