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Recursive method for *n*-point tree-level amplitudes in supersymmetric Yang-Mills theories

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We present a recursive method for super Yang-Mills color-ordered n-point tree amplitudes based on the cohomology of pure spinor superspace in ten space-time dimensions. The amplitudes are organized into BRST covariant building blocks with diagrammatic interpretation. Manifestly cyclic expressions (no longer than one line each) are explicitly given up to $n = 10$ and higher leg generalizations are straightforward.

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I. INTRODUCTION

Elementary particle physics relies on the computation of scattering amplitudes in Yang-Mills theory. Parke and Taylor found compact and simple expressions for maximally helicity violating (MHV) amplitudes in four space-time dimensions [1], which provide an important milestone in discovering hidden structures underlying the S-matrix. Many formal, as well as phenomenological advances, followed since then; see [2,3] for some reviews.

Supersymmetric field theories emerge in the low-energy limit of superstring theory, that is why the latter can be used as a powerful tool to gain further insights into field theories; see [4] for a recent example. There are several descriptions for the superstring's world sheet degrees of freedom, and the pure spinor formalism [5] is the only manifestly supersymmetric formulation known so far which can still be quantized covariantly.

In this paper, we use the framework of the pure spinor formalism to reduce the computation of n-point tree amplitudes in ten-dimensional $\mathcal{N} = 1$ super-Yang-Mills theory (SYM) to a recursive cohomology problem in pure spinor superspace. The end result is the compact formula (1) for the supersymmetric color-ordered n-point scattering amplitude at tree level.

Up until now, cohomology arguments have been used to propose SYM amplitudes up to seven-point [6], and they have been identified as the low-energy limit of superstring amplitudes up to six-point [7]. The main idea of [6] and this article is to bypass taking the field theory limit of a superstring computation and to instead fix SYM amplitudes using the BRST cohomology. This is achieved for any number n of external legs in this paper.

Although the pure spinor framework is initially adapted to ten space-time dimensions, one can still dimensionally reduce the results and extract the physics from any lower dimensional point of view. At any rate, the striking simplicity of our results is exhibited without the need of four-dimensional spinor helicity formalism. Moreover, the simplicity is furnished both for MHV and non-MHV helicity configurations in four space-time dimensions.

II. PURE SPINOR COHOMOLOGY FORMULA FOR A_n

The color-ordered tree-level massless SYM amplitudes in ten dimensions will be argued to be determined by the pure spinor superspace cohomology formula,[1]

$$\mathcal{A}_n = \langle E_{i_1 \ldots i_{n-1}} V_n \rangle, \tag{1}$$

where V_n is the vertex operator for the SYM multiplet in the pure spinor approach to superstring theory. The bosonic superfields $E_{i_1 \ldots i_p}$ are closed under the pure spinor BRST charge Q but not BRST exact in the momentum phase space of an n-point massless amplitude where the Mandelstam variables $s_{i_1 \ldots i_p} = \frac{1}{2}(k_{i_1} + \ldots + k_{i_p})^2$ encompassing $n - 1$ momenta vanish, $s_{i_1 \ldots i_{n-1}} = 0$:

$$QE_{i_1 \ldots i_p} = 0, \quad E_{i_1 \ldots i_p} = QM_{i_1 \ldots i_p} \ \text{if}\ s_{i_1 \ldots i_p} \neq 0. \tag{2}$$

The $\langle \ldots \rangle$ bracket denotes a zero mode integration prescription automated in [8], which extracts the superfield components from the enclosed superfields [5]. More precisely, nonvanishing contributions arise from tensor structures of order $\lambda^3 \theta^5$, where λ is the ghost variable of the pure spinor formalism and θ the Grassmann odd superspace variable of ten-dimensional $\mathcal{N} = 1$ SYM.

A. BRST building blocks

The first step in constructing the BRST cohomological objects $E_{i_1 \ldots i_{n-1}}$ in (1) is guided by the world sheet conformal field theory of superstring theory in its pure spinor formulation. Apart from the unintegrated vertex operator

[1]The n-point color-ordered formulas in this paper are all for the ordering $1, 2, \ldots, n$.

$V^i = \lambda^\alpha A^i_\alpha$, the massless level of the BRST cohomology contains the integral over $U^j = \partial\theta^\alpha A^j_\alpha + \Pi^m A^j_m + d_\alpha W^\alpha_j + \frac{1}{2} N^{mn} \mathcal{F}^j_{mn}$ along the world sheet boundary. The so-called integrated vertex operator U^j is built from $h = 1$ fields $[\partial\theta^\alpha, \Pi^m, d_\alpha, N^{mn}]$ of the pure spinor conformal field theory contracted with SYM superfields $[A^j_\alpha, A^j_m, W^\alpha_j, \mathcal{F}^j_{mn}]$.

Computing scattering amplitudes involves the residues $L_{2131\ldots p1}$ of the operator product expansion (OPE) of $p-1$ integrated vertex operators $U^j(z_j)$ approach their unintegrated counterpart $V^i(z_i)$:

$$\lim_{z_2\to z_1} V^1(z_1)U^2(z_2) \longrightarrow \frac{L_{21}(z_1)}{z_{21}},$$
$$\lim_{z_p\to z_1} L_{2131\ldots(p-1)1}(z_1)U^p(z_p) \longrightarrow \frac{L_{2131\ldots(p-1)1p1(z_1)}}{z_{p1}}. \quad (3)$$

Using the explicit form of V^i, U^j in terms of SYM superfields and their OPEs, we find

$$L_{21} = -A^1_m(\lambda\gamma^m W^2) - V^1(k^1\cdot A^2)$$
$$L_{2131} = -L_{21}((k^1+k^2)\cdot A^3) + (\lambda\gamma^m W^3)[A^1_m(k^1\cdot A^2) + A^{1n}\mathcal{F}^2_{mn} - (W^1\gamma_m W^2)]$$

for two and three legs, respectively.

The p-leg residues $L_{2131\ldots p1}$ by themselves do transform BRST covariantly, e.g.,

$$QL_{ji} = s_{ij}V_iV_j,$$
$$QL_{jiki} = s_{ijk}L_{ji}V_k - s_{ij}[L_{kj}V_i - L_{ki}V_j + L_{ji}V_k],$$

but they do not exhibit any symmetry properties in the labels i, j, k as required for a diagrammatic interpretation. However, many irreducibles of the symmetric group turn out to be BRST exact, e.g., $Q(A_i\cdot A_j) = -2L_{(ij)}$. Only truly BRST cohomological pieces are kept,

$$T_{ij} := L_{[ji]} = L_{ji} - L_{(ji)} = L_{ji} + \tfrac{1}{2}Q(A_i\cdot A_j).$$

Any higher rank residue $L_{21\ldots p1}$ with $p\geq 3$ requires a redefinition in two steps to form the so-called BRST building blocks $T_{12\ldots p}$, which ultimately enter the n-point SYM amplitude (1): $L_{2131\ldots p1} \longrightarrow \tilde{T}_{123\ldots p} \longrightarrow T_{123\ldots p}$. A first step $\tilde{T}_{123\ldots p} = L_{2131\ldots p1} + \ldots$ removes the BRST trivial parts in $Q\tilde{T}_{123\ldots p}$, e.g.,

$$\tilde{T}_{ijk} \equiv L_{jiki} + \frac{s_{ij}}{2}[(A_j\cdot A_k)V_i - (A_i\cdot A_k)V_j + (A_i\cdot A_j)V_k] - \frac{s_{ijk}}{2}(A_i\cdot A_j)V_k$$
$$Q\tilde{T}_{ijk} = s_{ijk}T_{ij}V_k - s_{ij}[T_{jk}V_i - T_{ik}V_j + T_{ij}V_k]$$

such that the BRST variation of $\tilde{T}_{123\ldots p}$ involves $T_{i_1\ldots i_{q<p}}$ rather than $L_{i_2i_1\ldots i_{q<p}i_1}$. But there will be BRST-exact components in $\tilde{T}_{123\ldots p}$, which still have to be subtracted in a second step. For example, there exist superfields $R^{(l)}_{ijk}$ such that [7,9]

$$QR^{(1)}_{ijk} = 2\tilde{T}_{(ij)k}, \qquad QR^{(2)}_{ijk} = 3\tilde{T}_{[ijk]}.$$

The following redefinition yields the hook Young tableau $T_{ijk} = T_{[ij]k}$ with $T_{[ijk]} = 0$

$$T_{ijk} = \tilde{T}_{ijk} - \tfrac{1}{2}QR^{(1)}_{ijk} - \tfrac{1}{3}QR^{(2)}_{ijk}$$

suitable to represent field theory diagrams made of cubic vertices. Similarly, one has to remove $p-1$ BRST trivial irreducibles from $T_{12\ldots p} = \tilde{T}_{12\ldots p} + \ldots$ where the higher order generalizations of $A_i\cdot A_j$, and $R^{(l)}_{ijk}$ superfields are related to z_{ij} double poles in the OPE of $U^i(z_i)U^j(z_j)$.

The explicit construction of BRST building blocks $T_{12\ldots p}$ with higher rank p involves two completely straightforward steps: The residue $L_{2131\ldots p1}$ is determined by the OPEs of the conformal world sheet fields, and the corresponding $\tilde{T}_{12\ldots p}$ follows from replacing the lower rank $L_{2131\ldots q1} \mapsto T_{12\ldots q}$, $q<p$ within $QL_{2131\ldots p1}$. Only the last step of finding "parent superfields" $R^{(i)}_{12\ldots p}$ whose Q variation yields the BRST-exact components of $\tilde{T}_{12\ldots p}$ requires some intuition. We have worked out such higher order generalizations of the $R^{(1)}_{ijk}$ and $R^{(2)}_{ijk}$ above up to $p=5$ (see the Appendix of [9]) on the basis of a "trial and error" analysis.

More generally, each $T_{i_1\ldots i_p}$ inherits all the symmetries of $T_{i_1\ldots i_{p-1}}$ in the first $p-1$ labels, so there is one new identity at each rank p (such as $T_{12[34]} + T_{34[12]} = 0$ at $p=4$) which cannot be inferred from lower order relatives. It can be determined from the symmetries of the diagrams described by $T_{i_1\ldots i_p}$, e.g.,

$$T_{ijklm} - T_{ijkml} + T_{lmijk} - T_{lmjik} - T_{lmkij} + T_{lmkji} = 0 \quad (4)$$

at $p=5$. Higher order generalizations of (4) will be listed in [9].

Just like the OPE residues $L_{2131\ldots p1}$ defined by (3), the BRST building blocks $T_{12\ldots p}$ transform covariantly under the BRST charge,

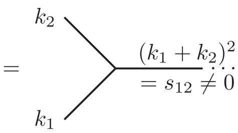
$$QT_{ijk} = s_{ijk}T_{ij}V_k - s_{ij}(T_{ij}V_k + T_{jk}V_i + T_{ki}V_j)$$
$$QT_{ijkl} = s_{ijkl}T_{ijk}V_l + s_{ijk}(T_{ijl}V_k - T_{ijk}V_l + T_{ij}T_{kl}) + s_{ij}(V_iT_{jkl} + T_{ikl}V_j - T_{ijl}V_k + T_{ik}T_{jl} + T_{il}T_{jk} - T_{ij}T_{kl}), \tag{5}$$

once again, we refer the reader to [9] for higher order generalizations.

B. Feynman diagrams and Berends-Giele currents

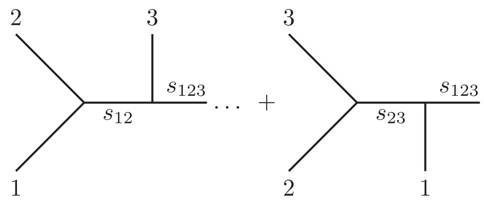
In this subsection, we give a diagrammatic interpretation of the BRST building blocks $T_{12..p}$ and combine them to color-ordered field theory amplitudes with one off-shell leg, so-called Berends-Giele currents [10]. The Mandelstam invariants $s_{ij}, s_{ijk}, s_{ijkl}, \ldots$, which appear in the BRST variation (5), play a crucial role: They must be the propagators associated with the T_{ijkl} to guarantee that each term in $QT_{j_1\ldots j_p}$ cancels one of the poles. This is the only way to combine different terms $T_{j_1\ldots j_p}/(s_{j_1j_2}, s_{j_1j_2j_3}, \ldots, s_{j_1\ldots j_p})$ to an overall BRST closed SYM or superstring amplitude.

The λ ghost number one of the $T_{j_1\ldots j_p}$ implies that it just represents a subdiagram with p on-shell legs and one off-shell leg. Adding all the color-ordered diagrams contributing to a $p+1$ point amplitude gives rise to a Berends-Giele current $M_{j_1\ldots j_p}$, these objects were first considered in the context of gluon scattering [10]

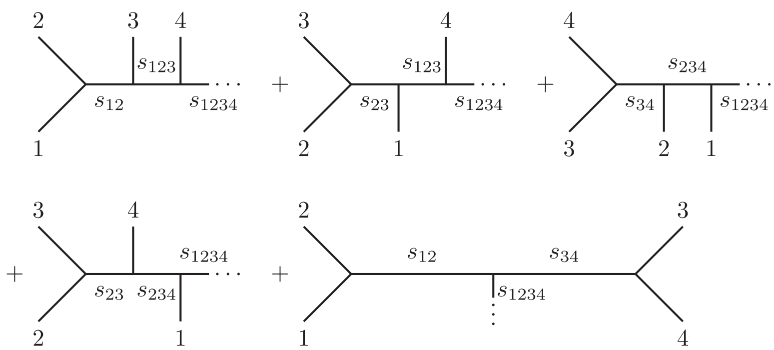
$M_{12} =$ 

$= \frac{T_{12}}{s_{12}}, \qquad k_1^2 = k_2^2 = 0.$

Let us give explicit lower order examples of $M_{j_1\ldots j_p}$ at $p = 2, 3, 4, 5$: The $p = 2$ case $M_{i_1i_2} := T_{i_1i_2}/s_{i_1i_2}$ just represents the cubic vertex of an off-shell three-point amplitude. The next examples $p \geq 3$ involve $P_{p+1} = 2, 5, 14, \ldots$ terms according to the color-ordered $(p+1)$ point amplitudes[2]:

$M_{123} =$ 

$= \frac{1}{s_{123}}\left(\frac{T_{123}}{s_{12}} + \frac{T_{321}}{s_{23}}\right)$

$M_{1234} =$ 

$$= \frac{1}{s_{1234}}\left(\frac{T_{1234}}{s_{12}s_{123}} + \frac{T_{3214}}{s_{23}s_{123}} + \frac{T_{3421}}{s_{34}s_{234}} + \frac{T_{3241}}{s_{23}s_{234}} + \frac{2T_{12[34]}}{s_{12}s_{34}}\right).$$

According to $P_5 = 5$, there are five diagrams collected in M_{1234} and the last one makes use of the fact that $QT_{12[34]}$ cancels poles in s_{12}, s_{34}, and s_{1234}. As we have mentioned before, the diagrammatic interpretation of the BRST building blocks rests on their symmetry properties such as $T_{(ij)} = T_{(ij)k} = T_{[ijk]} = 0$ at $p = 2, 3$. In the $p = 4$ case at hand, $T_{12[34]} + T_{34[12]} = 0$ is crucial to preserve the reflection symmetry $(1, 2, 3, 4) \leftrightarrow (4, 3, 2, 1)$ of the last diagram in the figure above.

[2]The number P_n of pole channels in an n point amplitude will be recursively and explicitly given in Eq. (9) and the line after.

As a last explicit example, we shall display M_{12345} here:

$$\begin{aligned} M_{12345} \equiv \frac{1}{s_{12345}}\Big[& \frac{T_{12345}}{s_{12}s_{123}s_{1234}} - \frac{T_{23145}}{s_{23}s_{123}s_{1234}} - \frac{T_{23415}}{s_{23}s_{234}s_{1234}} \\ & + \frac{T_{34215}}{s_{34}s_{234}s_{1234}} - \frac{T_{23451}}{s_{23}s_{234}s_{2345}} + \frac{T_{34251}}{s_{34}s_{234}s_{2345}} \\ & + \frac{T_{34521}}{s_{34}s_{345}s_{2345}} - \frac{T_{45321}}{s_{45}s_{345}s_{2345}} + \frac{(T_{34215} - T_{34125})}{s_{12}s_{34}s_{1234}} \\ & + \frac{(T_{45231} - T_{45321})}{s_{23}s_{45}s_{2345}} + \frac{(T_{12345} + T_{21354})}{s_{12}s_{45}s_{123}} \\ & - \frac{(T_{23145} + T_{32154})}{s_{23}s_{45}s_{123}} - \frac{(T_{34512} + T_{43521})}{s_{12}s_{34}s_{345}} \\ & + \frac{(T_{45312} + T_{54321})}{s_{12}s_{45}s_{345}}\Big]. \end{aligned} \tag{6}$$

The 14 cubic graphs encompassed by M_{12345}, as well as higher rank currents, can be found in the Appendix of [9]. Apart from this diagrammatic method to construct $M_{i_1 \ldots i_p}$, we will give a string-inspired formula in Sec. IV.

C. Berends-Giele recursions for SYM amplitudes

Remarkably, the BRST variation of Berends-Giele currents $M_{12\ldots p}$ introduces bilinears of lower rank $M_{12\ldots j<p}$. Up to $p = 4$, these are

$$\begin{aligned} QM_{ij} &= V_i V_j =: E_{ij}, \\ QM_{ijk} &= V_i M_{jk} + M_{ij} V_k =: E_{ijk} \\ QM_{ijkl} &= V_i M_{jkl} + M_{ij} M_{kl} + M_{ijk} V_l =: E_{ijkl}. \end{aligned} \tag{7}$$

More generally, the BRST charge cuts $M_{12\ldots p}$ into all color-ordered partitions of its p on-shell legs among two lower rank Berends-Giele currents

$$QM_{12\ldots p} = \sum_{j=1}^{p-1} M_{12\ldots j} M_{j+1\ldots p} =: E_{12\ldots p}, \tag{8}$$

where the one-index version is defined to be the unintegrated SYM vertex operator $M_i = V_i$. We have explicitly obtained solutions to (8) up to $M_{12\ldots 7}$ [9].

Let us denote the number of kinematic pole configurations in $M_{i_1 \ldots i_p}$ or $E_{i_1 \ldots i_p}$ by P_{p+1}, then (8) implies the recursion relation

$$P_n = \sum_{i=2}^{n-1} P_i P_{n-i+1}, \quad P_2 = P_3 \equiv 1, \quad n \geq 4. \tag{9}$$

Its explicit solution $P_n = 2^{n-2} \frac{(2n-5)!!}{(n-1)!}$ agrees with the formula for the number of cubic diagrams in the color-ordered n-point SYM amplitude; see, e.g., [11]. Hence, our expression $\mathcal{A}_n = \langle E_{i_1 \ldots i_{n-1}} V_n \rangle$ passes the consistency check to encompass the right number of diagrams.

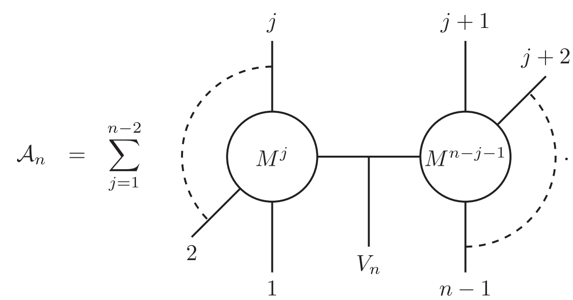
We have defined the rank p Berends-Giele currents $M_{i_1 \ldots i_p}$ to contain $p - 1$ inverse powers of Mandelstam invariants $s_{i_1 \ldots i_q} = \frac{1}{2}(k_{i_1} + \ldots + k_{i_q})^2$ and, in particular, an overall propagator $M_{i_1 \ldots i_p} \sim (s_{i_1 \ldots i_p})^{-1}$. The latter cancels under action (8) of the BRST charge such that the resulting λ ghost number two superfield $QM_{i_1 \ldots i_p} = E_{i_1 \ldots i_p}$ is well defined even if $s_{i_1 \ldots i_p} = 0$.

Actually, this is the crucial reason why $\mathcal{A}_n = \langle E_{i_1 \ldots i_{n-1}} V_n \rangle$ lies in the BRST cohomology: Massless n-particle kinematics imply that $s_{i_1 \ldots i_{n-1}} = 0$. The resulting rank $n - 1$ Berends-Giele current $M_{i_1 \ldots i_{n-1}}$ diverges due to the overall propagator and we cannot write $E_{i_1 \ldots i_{n-1}}$ as a BRST variation. The $s_{i_1 \ldots i_{n-1}} = 0$ constraint saves $\mathcal{A}_n$ from being BRST exact. Expressing the n-point amplitude in terms of $E_{i_1 \ldots i_{n-1}}$ amounts to removing the overall pole before putting the rank $n - 1$ Berends-Giele current on shell.

The representation of the SYM n-point amplitude as a bilinear in Berends-Giele currents

$$\mathcal{A}_n = \sum_{j=1}^{n-2} \langle M_{12\ldots j} M_{j+1\ldots n-1} V_n \rangle \tag{10}$$

makes its factorization into $(j + 1)$-point and $(n - j)$-point subamplitudes manifest; see the following figure



Equations (10) and (8) can be viewed as a supersymmetric generalization of Berends-Giele recursion relations for gluon amplitudes [10]. As an additional bonus, our $M_{12\ldots j}$ do not receive contributions from quartic vertices.

D. BRST equivalent expressions for $\mathcal{A}_n$ and cyclic invariance

It follows from (10) that $p = n - 2$ is the maximum rank of $M_{i_1 \ldots i_p}$ appearing in the n-point amplitude cohomology formula (1). However, these terms are of the form $\langle M_{i_1 \ldots i_{n-2}} V_{i_{n-1}} V_{i_n} \rangle$ and can be rewritten as $\langle E_{i_1 \ldots i_{n-2}} M_{i_{n-1} i_n} \rangle$ due to $V_i V_j = E_{ij} = QM_{ij}$ and BRST integration by parts

$$\langle M_{i_1 \ldots i_p} E_{i_1 \ldots i_q} \rangle = \langle E_{i_1 \ldots i_p} M_{i_1 \ldots i_q} \rangle. \tag{11}$$

The decomposition of $E_{i_1 \ldots i_{n-2}}$ involves at most $M_{i_1 \ldots i_{n-3}}$, so BRST integration by parts reduces the maximum rank p of $M_{i_1 \ldots i_p}$ by one. It turns out that the n-point cohomology formula (1) allows enough BRST integrations by parts as to reduce the maximum rank to $p = [n/2]$. The $[\cdot]$ bracket denotes the Gauss bracket $[x] = \max_{n\in\mathbb{Z}} n \leq x$, which picks out the nearest integer smaller than or equal to its argument. This yields a more economic expression for $\mathcal{A}_n$.

Another benefit of the BRST equivalent $\mathcal{A}_n$ representation in terms of $M_{i_1\ldots i_p}$ with $p \leq [n/2]$ lies in the manifest cyclic symmetry. The last leg V_n being singled out in (1) obscures the amplitudes' cyclicity. Performing k integrations by parts includes V_n into bigger blocks $M_{i_1\ldots i_{k+1}}$ such that the nth leg appears on the same footing as any other one in the end. We will give examples in Sec. III.

III. THE n-POINT AMPLITUDES UP TO $n = 10$

The three-point amplitude [5] is trivially reproduced by (1) and (8),

$$A_3 = \langle E_{12} V_3\rangle = \langle V_1 V_2 V_3\rangle. \tag{12}$$

Similarly, (1) and (8) reproduce the results of [6,12,13] for the four-point amplitude:

$$\begin{aligned}\mathcal{A}_4 &= \langle E_{123} V_4\rangle = \langle V_1 M_{23} V_4\rangle + \langle M_{12} V_3 V_4\rangle \\ &= \frac{1}{s_{23}}\langle V_1 T_{23} V_4\rangle + \frac{1}{s_{12}}\langle T_{12} V_3 V_4\rangle. \end{aligned}\tag{13}$$

For $n = 5$, the formulas (1) and (8) lead to

$$\begin{aligned}\mathcal{A}_5 &= \langle E_{1234} V_5\rangle \\ &= \langle V_1 M_{234} V_5\rangle + \langle M_{12} M_{34} V_5\rangle + \langle M_{123} V_4 V_5\rangle \\ &= \frac{\langle T_{123} V_4 V_5\rangle}{s_{12} s_{45}} - \frac{\langle T_{234} V_1 V_5\rangle}{s_{23} s_{51}} + \frac{\langle T_{12} T_{34} V_5\rangle}{s_{12} s_{34}} \\ &\quad - \frac{\langle T_{231} V_4 V_5\rangle}{s_{23} s_{45}} + \frac{\langle T_{342} V_1 V_5\rangle}{s_{34} s_{51}}. \end{aligned}\tag{14}$$

As discussed in the previous section, identifying E_{ij} in (14) and using (11) leads to a manifestly cyclic-invariant form proved in [6]

$$\begin{aligned}\mathcal{A}_5 &= \langle M_{12} V_3 M_{45}\rangle + \text{cyclic}\,(12345) \\ &= \frac{\langle T_{12} V_3 T_{45}\rangle}{s_{12} s_{45}} + \text{cyclic}\,(12345). \end{aligned}\tag{15}$$

For $n = 6$, the formula (1) reads

$$\begin{aligned}\mathcal{A}_6 &= \langle E_{12345} V_6\rangle \\ &= \langle V_1 M_{2345} V_6\rangle + \langle M_{12} M_{345} V_6\rangle \\ &\quad + \langle M_{123} M_{45} V_6\rangle + \langle M_{1234} V_5 V_6\rangle. \end{aligned}\tag{16}$$

Integrating the BRST charge by parts in the first and last terms using (11) leads to

$$\begin{aligned}\mathcal{A}_6 &= \langle M_{12} M_{34} M_{56}\rangle + \langle M_{23} M_{45} M_{61}\rangle \\ &\quad + \langle M_{123}(M_{45} V_6 + V_4 M_{56})\rangle \\ &\quad + \langle M_{234}(V_5 M_{61} + M_{56} V_1)\rangle \\ &\quad + \langle M_{345}(V_6 M_{12} + M_{61} V_2)\rangle \\ &= \frac{\langle T_{12} T_{34} T_{56}\rangle}{3 s_{12} s_{34} s_{56}} + \frac{1}{2}\left\langle\left(\frac{T_{123}}{s_{12} s_{123}} - \frac{T_{231}}{s_{23} s_{123}}\right) \right. \\ &\quad \left. \times \left(\frac{T_{45} V_6}{s_{45}} + \frac{V_4 T_{56}}{s_{56}}\right)\right\rangle + \text{cyclic}\,(1\ldots 6). \end{aligned}\tag{17}$$

The amplitude (17) was first proposed in [6] by using BRST cohomology arguments and proved by the field theory limit of the six-point superstring amplitude in [7]. For $n = 7$,

$$\begin{aligned}\mathcal{A}_7 &= \langle V_1 M_{23456} V_7\rangle + \langle M_{12} M_{3456} V_7\rangle + \langle M_{123} M_{456} V_7\rangle \\ &\quad + \langle M_{1234} M_{56} V_7\rangle + \langle M_{12345} V_6 V_7\rangle. \end{aligned}$$

Identifying $V_i V_j = E_{ij} = Q M_{ij}$ and using (11) leads to

$$\begin{aligned}\mathcal{A}_7 &= \langle M_{123} M_{45} M_{67}\rangle + \langle M_{123} M_{456} V_7\rangle + \langle M_{234} M_{56} M_{71}\rangle \\ &\quad + \langle M_{345} M_{67} M_{12}\rangle + \langle M_{456} M_{71} M_{23}\rangle \\ &\quad + \langle M_{1234}(V_5 M_{67} + M_{56} V_7)\rangle + \langle M_{2345}(V_6 M_{71} \\ &\quad + M_{67} V_1)\rangle + \langle M_{3456}(V_7 M_{12} + M_{71} V_2)\rangle, \end{aligned}$$

where the generated factors of E_{12345} and E_{23456} have been replaced by M's using the definition (8). The maximum rank $M_{i_1\ldots i_4}$ only appears in combination with the BRST-exact superfield $E_{ijk} = V_i M_{jk} + M_{ij} V_k = Q M_{ijk}$. Using (11) once again leads to a more compact expression with manifest cyclic symmetry,

$$\mathcal{A}_7 = \langle M_{123} M_{45} M_{67}\rangle + \langle V_1 M_{234} M_{567}\rangle + \text{cyclic}\,(1\ldots 7). \tag{18}$$

Plugging the solutions (6) in (18) leads to the Ansatz of [6],

$$\begin{aligned}\mathcal{A}_7 &= \left\langle V_1\left(\frac{T_{234}}{s_{23} s_{234}} - \frac{T_{342}}{s_{34} s_{234}}\right)\left(\frac{T_{567}}{s_{56} s_{567}} - \frac{T_{675}}{s_{67} s_{567}}\right)\right\rangle \\ &\quad + \left\langle\left(\frac{T_{123}}{s_{12} s_{123}} - \frac{T_{231}}{s_{23} s_{123}}\right)\frac{T_{45} T_{67}}{s_{45} s_{67}}\right\rangle + \text{cyclic}\,(1\ldots 7). \end{aligned}\tag{19}$$

It is easy to check that (19) is expanded in terms of 42 kinematic poles.

The procedure to obtain manifestly cyclic symmetric higher-point amplitudes using (1) and (8) is straightforward and follows the same steps as above. Increasing the number of legs allows further BRST integrations by parts to be performed by identifying and integrating $E_{ij}, E_{ijk}, \ldots$ successively at each step, leading to

$$\begin{aligned}\mathcal{A}_8 &= \langle M_{123} M_{456} M_{78}\rangle + \tfrac{1}{2}\langle M_{1234} E_{5678}\rangle \\ &\quad + \text{cyclic}\,(1\ldots 8), \end{aligned}\tag{20}$$

$$\mathcal{A}_9 = \frac{1}{3}\langle M_{123}M_{456}M_{789}\rangle + \langle M_{1234}(M_{567}M_{89} + M_{56}M_{789} + M_{5678}V_9)\rangle + \text{cyclic}\,(1\ldots 9), \tag{21}$$

$$\mathcal{A}_{10} = \langle M_{1234}(M_{567}M_{89;10} + M_{5678}M_{9;10})\rangle + \frac{1}{2}\langle M_{12345}E_{6789;10}\rangle + \text{cyclic}\,(1\ldots 10). \tag{22}$$

IV. RELATION TO SUPERSTRING THEORY

Supersymmetric field theory tree amplitudes can also be obtained from the low-energy limit of superstring theory where the dimensionless combinations $\alpha' s_{i_1\ldots i_p}$ of Regge slope α' and Mandelstam bilinears are formally sent to zero. Using the pure spinor formalism [5], we will argue in [9,14] that the full superstring n-point amplitude at tree level is given by

$$\mathcal{A}_n^{\text{string}}(\alpha') = (2\alpha')^{n-3}\prod_{i=2}^{n-2}\int_{z_{i-1}}^{1} dz_i \prod_{j<k}|z_{jk}|^{-2\alpha' s_{jk}} \sum_{p=1}^{n-2}\frac{\langle T_{12\ldots p}T_{n-1,p+1,\ldots,n-2}V_n\rangle}{(z_{12}z_{23}\ldots z_{p-1,p})(z_{n-1,p+1}z_{p+1,p+2}\ldots z_{n-3,n-2})} + \mathcal{P}(2,3,\ldots,n-2), \tag{23}$$

where $SL(2,\mathbb{R})$ invariance of the tree-level world sheet admits to fix $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$ and $\mathcal{P}(2, 3, \ldots, n-2)$ denotes a sum over all permutations of $(2, 3, \ldots, n-2)$. The full superstring amplitude is determined by BRST building blocks $T_{12\ldots p}$ and $n-3$ world sheet integrals over $z_{jk} = z_j - z_k$. The $\alpha' \to 0$ limit of (23) reproduces $\mathcal{A}_n = \sum_{p=1}^{n-2}\langle M_{i_1\ldots i_p}M_{i_{p+1}\ldots i_{n-1}}V_n\rangle$ term by term in the individual p sums. Therefore considering $p = n-2 \equiv q$ yields an explicit formula for $M_{i_1\ldots i_p}$

$$M_{12\ldots q} = \lim_{\alpha'\to 0}(2\alpha')^{q-1}\prod_{i=2}^{q}\int_{z_{i-1}}^{1} dz_i \prod_{j<k}^{q+1}|z_{jk}|^{-2\alpha' s_{jk}} \times\left(\frac{T_{12\ldots q}}{z_{12}z_{23}\ldots z_{q-1,q}} + \mathcal{P}(2,3,\ldots,q)\right) \tag{24}$$

in the fixing $z_1 = 0$ and $z_{q+1} = 1$. It has been checked up to $q = 7$ that the string-inspired computation (24) of $M_{12\ldots q}$ agrees with its construction from the color-ordered diagrams in $\mathcal{A}_{q+1}$.

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