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Non-Abelian current oscillations in harmonic string loops: Existence of throbbing vortons

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It is shown that a string carrying a field of harmonic type can have circular vorton states of a new throbbing kind, for which the worldsheet geometry is stationary but the internal structure undergoes periodic oscillation.

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I. INTRODUCTION

The purpose of the present work is to demonstrate the use of the formalism developed in a preceding article [1] for the treatment of fields in curved target spaces, by applying it to simple but nontrivial examples of the important special case of harmonic and other simply harmonious fields [2,3] on a string worldsheet.

The present investigation will be restricted to geometric configurations of the simplest nontrivial type, namely, circularly symmetric string loops in a flat background spacetime, for which the metric will be conveniently expressible in cylindrical coordinates as

$$ds^2 = -dt^2 + \varrho^2 d\phi^2 + d\varrho^2 + dz^2, \tag{1}$$

so that the string worldsheet will be specifiable by an expression for the radius ϱ as a function of the time t at a fixed value of the longitudinal coordinate z that can be taken without loss of generality to be the origin $z = 0$.

A systematic investigation of the dynamics of such a worldsheet has already been carried out [4] for conducting string models of the simple type for which the current has only a single degree of freedom, in the sense that the target space of the scalar field on the string is just one dimensional—and therefore trivially flat—the outcome being that if its energy is not too high the string will oscillate about a ‘‘vorton’’ type equilibrium state. Such a vorton state will be generically stable with respect to perturbations of the purely axisymmetic kind to which the present analysis will be restricted, but it has been shown [5] that they will be commonly, though not generically, unstable with respect to nonaxisymmetric modes. It is to be expected that qualitatively similar behavior will occur for scalar field models with more degrees of freedom [6,7], so long as all the currents are generated by commuting symmetries of a flat target space.

The novelty in the present work will be to consider a situation of a qualitatively different kind that can arise when the relevant target space is not flat. Of course a curved target space might have no symmetries at all, in which case the currents in question would not even be conserved. The present work will however be concerned with the opposite extreme, in which the target space is highly symmetric, so that there will be many conserved current combinations, but with generators that do not commute. Attention will be focussed here on the simplest nontrivial possibility of this kind, namely, the case in which the target space is just an ordinary 2-sphere, with metric $d\hat{s}^2 = \hat{g}_{AB} dX^A dX^B$ that will be expressible in terms of the usual coordinates $X^1 = \hat{\theta}$ and $X^2 = \hat{\varphi}$ by

$$d\hat{s}^2 = d\hat{\theta}^2 + \sin^2\hat{\theta} d\hat{\varphi}^2. \tag{2}$$

II. EXTRINSIC MOTION OF CIRCULAR STRING WORLDSHEET

It will be convenient to describe the evolution of the worldsheet, within the background characterized by (1), in terms of unit timelike radial and spacelike transverse tangent vectors u^μ and $\tilde{u}^\mu$, and of a unit spacelike radial normal vector λ^μ, that are given in terms of the coordinates $x^0 = t$, $x^1 = \phi$, $x^2 = \varrho$, $x^3 = z$ by

$$u^\mu = \gamma(\delta_0^\mu + \dot{\varrho}\delta_2^\mu), \quad \tilde{u}^\mu = \frac{1}{\varrho}\delta_1^\mu, \quad \lambda^\mu = \gamma(\dot{\varrho}\delta_0^\mu + \delta_2^\mu), \tag{3}$$

where a dot denotes differentiation with respect to the time coordinate t and the Lorentz factor for the radial velocity $\dot{\varrho}$ is defined as usual by $\gamma = 1/\sqrt{1-\dot{\rho}^2}$.

The ensuing derivative formulas

$$u^\nu \nabla_\nu u^\mu = \gamma^3 \ddot{\varrho} \lambda^\mu, \qquad \tilde{u}^\nu \nabla_\nu \tilde{u}^\mu = -\frac{1}{\varrho}\delta_2^\mu,$$

$$\tilde{u}^\nu \nabla_\nu u^\mu = \gamma\frac{\dot{\varrho}}{\varrho}\tilde{u}^\mu, \qquad u^\nu \nabla_\nu \tilde{u}^\mu = 0,$$

can be used to evaluate the second fundamental tensor as given [8] by the prescription

$$K_{\mu\nu}{}^\rho = \eta_\nu{}^\sigma \bar{\nabla}_\mu \eta_\sigma{}^\rho, \qquad \bar{\nabla}_\mu = \eta_\mu{}^\lambda \nabla_\lambda, \tag{4}$$

in which the first fundamental tensor of the worldsheet is specified as

$$\eta_\mu{}^\nu = -u_\mu u^\nu + \tilde{u}_\mu \tilde{u}^\nu. \tag{5}$$

The second fundamental tensor of the time-dependent circular worldsheet is thereby found to be

$$K_{\mu\nu}{}^{\rho} = \gamma\lambda^{\rho}\left(\gamma^2\ddot{\varrho}u_{\mu}u_{\nu} - \frac{1}{\varrho}\tilde{u}_{\mu}\tilde{u}_{\nu}\right). \tag{6}$$

In the simple case for which the only external force is that of viscous drag by a static external background medium [9], which in this case will give a force density of the form

$$f^{\mu} = f\lambda^{\mu}, \tag{7}$$

with velocity-dependent coefficient f, the corresponding equation of motion of the worldsheet will be given [8] in terms of the second fundamental tensor by an expression of the generic form

$$\bar{T}^{\mu\nu}K_{\mu\nu}{}^{\rho} = f^{\rho}, \tag{8}$$

in which $\bar{T}^{\mu\nu}$ is the relevant surface stress energy tensor, which will of course depend on the internal structure of the string. It can be seen that in this simple circular case, the ensuing differential equation for the radius ϱ will take the form

$$\gamma^3\bar{T}^{\mu\nu}u_{\mu}u_{\nu}\ddot{\varrho} - \frac{\gamma}{\varrho}\bar{T}^{\mu\nu}\tilde{u}_{\mu}\tilde{u}_{\nu} = f. \tag{9}$$

III. ENERGY AND ANGULAR MOMENTUM

The invariance of the background (1) under the action of the time translation Killing vector k^{μ} and the rotation Killing vector ϱ^{μ} defined by

$$k^{\nu} = \delta_0^{\nu} = \gamma(u^{\nu} - \dot{\varrho}\lambda^{\nu}), \qquad \varrho^{\mu} = \delta_1^{\nu} = \varrho\tilde{u}^{\nu}, \tag{10}$$

(so that $k_{\nu}k^{\nu} = 1$ and $\varrho_{\nu}\varrho^{\nu} = \varrho^2$) allows us to construct corresponding energy and angular momentum flux vectors

$$\mathcal{P}^{\mu} = -k^{\nu}\bar{T}_{\nu}{}^{\mu}, \qquad \mathcal{J}^{\mu} = \varrho^{\nu}\bar{T}_{\nu}{}^{\mu}, \tag{11}$$

which, subject to the variational field equations, will, as discussed in the preceding work [1], automatically satisfy the surface divergence conditions of the form

$$\bar{\nabla}_{\nu}\mathcal{P}^{\nu} = -k^{\mu}f_{\mu}, \qquad \bar{\nabla}_{\nu}\mathcal{J}^{\nu} = \varrho^{\mu}f_{\mu}. \tag{12}$$

For an external force density of the postulated form (7), one thus obtains the work rate formula

$$\bar{\nabla}_{\nu}\mathcal{P}^{\nu} = \gamma\dot{\varrho}f, \tag{13}$$

and the angular momentum conservation condition

$$\bar{\nabla}_{\nu}\mathcal{J}^{\nu} = 0. \tag{14}$$

It will be useful for what follows to rewrite these conditions in terms of internal worldsheet coordinates σ^i, with respect to which they will be expressible as

$$\bar{\nabla}_{i}\mathcal{P}^{i} = \gamma\dot{\varrho}f, \qquad \mathcal{P}^{i} = -\gamma u^{j}\bar{T}_{j}{}^{i}, \tag{15}$$

and

$$\bar{\nabla}_{i}\mathcal{J}^{i} = 0, \qquad \mathcal{J}^{i} = \varrho\tilde{u}^{j}\bar{T}_{j}{}^{i}, \tag{16}$$

while the corresponding expression for the extrinsic equation of motion (9) will be

$$\gamma^3\bar{T}^{ij}u_{i}u_{j}\ddot{\varrho} - \frac{\gamma}{\varrho}\bar{T}^{ij}\tilde{u}_{i}\tilde{u}_{j} = f. \tag{17}$$

More particularly, with respect to the internal coordinate system that is induced on the worldsheet by taking $\sigma^0 = t$, $\sigma^1 = \phi$, the corresponding expression for the intrinsic metric of the worldsheet will take the form

$$d\bar{s}^2 = -\frac{1}{\gamma^2}dt^2 + \varrho^2 d\phi^2, \tag{18}$$

and the corresponding expressions for the orthonormal frame vectors will be

$$u^{i} = \gamma\delta_0^{i}, \qquad \tilde{u}^{i} = \frac{1}{\varrho}\delta_1^{i}. \tag{19}$$

It can be seen that, with respect to these particular coordinates, the energy and angular momentum flux vectors will be given by

$$\mathcal{P}^{i} = -\gamma^2\bar{T}_0{}^{i}, \qquad \mathcal{J}^{i} = \bar{T}_1{}^{i}, \tag{20}$$

while the extrinsic equation of motion (17) will be expressible more explicitly as

$$\gamma^5\bar{T}_{00}\ddot{\varrho} - \frac{\gamma}{\varrho^3}\bar{T}_{11} = f. \tag{21}$$

The total work rate formula (15) will take the form

$$(\varrho\gamma\bar{T}_0{}^{i})_{,i} = -\varrho\dot{\varrho}f, \tag{22}$$

and the condition of the angular momentum conservation will take the form

$$\left(\frac{\varrho}{\gamma}\bar{T}_1{}^{i}\right)_{,i} = 0. \tag{23}$$

It is to be remarked that (21) can be used to eliminate the force magnitude f from (22) to give an intrinsic energy creation law of the form

$$(\varrho\bar{T}_0{}^{i})_{,i} = \dot{\varrho}\bar{T}_1{}^{1}. \tag{24}$$

IV. GENERIC HARMONIOUS CASE

The formulas of the two preceding sections are applicable to classical string models of any kind. We now restrict attention to the harmonious case, as characterized [1] by a Lagrangian $\bar{L}$ that depends only on the target space metric $\hat{g}_{AB}$ and the symmetric target space tensor defined—in the absence of gauge coupling, as will be assumed here—just by

$$\hat{\mathfrak{w}}^{AB} = \bar{g}^{ij}X^{A}{}_{,i}X^{B}{}_{,j}. \tag{25}$$

This means that it its generic variation will have the form

$$\delta\bar{L} = \frac{\partial\bar{L}}{\partial\hat{\mathfrak{w}}^{AB}}\delta\hat{\mathfrak{w}}^{AB} + \frac{\partial\bar{L}}{\partial\hat{g}_{AB}}\delta\hat{g}_{AB}, \tag{26}$$

in which, as a Noether identity, we must have

$$\frac{\partial \bar{L}}{\partial \hat{\mathfrak{w}}^{BC}}\hat{\mathfrak{w}}^{AC} = \frac{\partial \bar{L}}{\partial \hat{g}_{AC}}\hat{g}_{BC}, \tag{27}$$

so that the coefficients will be specifiable by the expressions

$$\frac{\partial \bar{L}}{\partial \hat{\mathfrak{w}}^{AB}} = -\frac{1}{2}\kappa_{AB}, \qquad \frac{\partial \bar{L}}{\partial \hat{g}_{AB}} = -\frac{1}{2}\kappa_C{}^A\hat{\mathfrak{w}}^{BC} = -\frac{1}{2}\kappa_C{}^B\hat{\mathfrak{w}}^{AC}, \tag{28}$$

in terms of the same symmetric target space tensor κ_{AB}. This tensor can be used to express the generic variation of the Lagrangian in the concise form

$$\delta \bar{L} = -\tfrac{1}{2}\kappa_A{}^B\delta\hat{\mathfrak{w}}_B{}^A, \tag{29}$$

and to express the ensuing surface stress energy tensor as

$$\bar{T}_{ij} = \kappa_{AB}X^A{}_{,i}X^B{}_{,j} + \bar{L}\bar{g}_{ij}. \tag{30}$$

With respect to the coordinates of (11), using a prime for differentiation with respect to ϕ, and a dot (as before) for differentiation with respect to t, we shall obtain

$$\hat{\mathfrak{w}}^{AB} = \frac{1}{\varrho^2}X^{A'}X^{B'} - \gamma^2\dot{X}^A\dot{X}^B, \tag{31}$$

and the stress energy components in (20) will thus be given by

$$\bar{T}_{00} = \kappa_{AB}\dot{X}^A\dot{X}^B - \frac{1}{\gamma^2}\bar{L}, \qquad \bar{T}_{11} = \kappa_{AB}X^{A'}X^{B'} + \varrho^2\bar{L}. \tag{32}$$

Our investigation will be concerned with solutions that are axisymmetric in the strict sense [1], meaning that the gradient fields $\dot{X}^A$ and $X^{A'}$ are independent of ϕ, but not in the strong sense which would require that even the undifferentiated fields X^A should be independent of ϕ. This means that $X^{A'}$ is allowed to be nonzero, but that we require $\dot{X}^{A'} = X^{A''} = 0$. Under these conditions the total work rate formula (22) will take the form

$$(\varrho\gamma^3\kappa_{AB}\dot{X}^A\dot{X}^B - \varrho\gamma L)^{\cdot} = \varrho\dot{\varrho}f, \tag{33}$$

and the angular momentum conservation law (21) will take the form

$$(\varrho\gamma\kappa_{AB}X^{A'}\dot{X}^B)^{\cdot} = 0, \tag{34}$$

while the intrinsic energy creation law (24) will be expressible in the form

$$(\varrho^2\gamma^2\kappa_{AB}\dot{X}^A\dot{X}^B)^{\cdot} + \tfrac{1}{2}\kappa_A{}^B(\varrho^2\hat{\mathfrak{w}}_B{}^A)^{\cdot} = 0. \tag{35}$$

If the target space is only two dimensional, and, in particular, if it is a 2-sphere as in the example dealt with in detail below, the complete system of dynamical evolution equations will be provided just by the pair of internal equations (34) and (35) in conjunction with the extrinsic evolution equation obtained by substitution from (32) in (21). Further input from the set of current pseudoconservation laws constituting the complete system of internal field equations [1,2] will however be needed if the target space dimension is three or more.

V. QUADRATICALLY AND SIMPLY HARMONIOUS MODELS

Within the extensive category of harmonious models to which the foregoing formulas are applicable, a noteworthy subcategory is that of models that are quadratically harmonious, in the sense of being governed by a Lagrangian whose dependence on $\hat{\mathfrak{w}}_B{}^A$ is just quadratic, so that it will be expressible in terms of fixed parameters m, $\kappa_\star$, $\alpha_\star$, $\beta_\star$ in the form

$$\bar{L} = -m - \tfrac{1}{2}\kappa_\star\hat{\mathfrak{w}} - \tfrac{1}{4}\alpha_\star\hat{\mathfrak{w}}^2 + \tfrac{1}{4}\beta_\star\hat{\mathfrak{w}}_A{}^B\hat{\mathfrak{w}}_B{}^A, \tag{36}$$

with the usual notation $\hat{\mathfrak{w}} = \hat{\mathfrak{w}}_A{}^A$, which gives

$$\kappa_{AB} = (\kappa_\star + \alpha_\star\hat{\mathfrak{w}})g_{AB} - \beta_\star\hat{\mathfrak{w}}_{AB}. \tag{37}$$

An important special case is that for which $\alpha_\star = \beta_\star$, so that the quadratic part is interpretable as a current cross product: this gives what is known as a baby Skyrme model [10,11] when the target space is a 2-sphere, and it gives a fully fledged Skyrme model [12,13] when the target space is a 3-sphere.

The quadratic special case for which $\beta_\star = 0$ belongs to another noteworthy subcategory, namely, that of simply harmonious models [2], which are characterized by a Lagrangian $\bar{L}$ that depends only on the scalar $\hat{\mathfrak{w}}$, as given by the formula

$$\hat{\mathfrak{w}} = \bar{g}^{ij}\hat{g}_{AB}X^A{}_{,i}X^B{}_{,j}, \tag{38}$$

so long as gauge coupling is absent, as is supposed here, so that with respect to the coordinates of (11) it will take the form

$$\hat{\mathfrak{w}} = \frac{1}{\varrho^2}\hat{g}_{AB}X^{A'}X^{B'} - \gamma^2\hat{g}_{AB}\dot{X}^A\dot{X}^B, \tag{39}$$

In this simply harmonious case we shall have

$$\kappa_{AB} = \kappa g_{AB}, \tag{40}$$

with the coefficient κ given by

$$\kappa = -2\frac{d\bar{L}}{d\mathfrak{w}}. \tag{41}$$

In terms of this quantity, the intrinsic energy creation law (35) will be expressible in the form

$$(\varrho^2\gamma^2\kappa^2\hat{g}_{AB}\dot{X}^A\dot{X}^B)^{\cdot} + \kappa^2(\hat{g}_{AB}X^{A'}X^{B'})^{\cdot} = 0. \tag{42}$$

VI. MINIMALLY NON-ABELIAN—SPHERICAL TARGET—CASE

Let us now concentrate on the minimally non-Abelian case, meaning that with the simplest nonflat target space geometry, namely, that of a 2-sphere as given by (2). In such a case there are only two internal degrees of freedom, namely, those of the independent field variables $\hat{\theta}$ and $\hat{\varphi}$; their evolution will be fully determined just by the two preceding conditions (34) and (35) if the loop radius ϱ is given in advance, as, for example, in the artificial case in which f is adjusted to hold the radius at a fixed value with $\dot{\varrho} = 0$. These two intrinsic evolution equations will also be sufficient, in conjunction with the total work rate Eq. (33) if the external force magnitude f is given in advance, and thus, in particular, in the case of most obvious natural interest, namely, that in which it is taken to vanish,

$$f = 0. \tag{43}$$

The possibility of configurations that, with respect to the rotation Killing vector ϱ^{μ}, are symmetric not in the strong sense, which would require $X^{A'} = 0$, but in the less restrictive weak, albeit strict sense [1], as postulated here, depends on the existence of a corresponding symmetry in the target space, with generator V^A such that

$$X^{A'} = V^A. \tag{44}$$

In the spherical case under consideration here, such a vector field could be chosen in many ways as a combination of the set of not just one but three independent target space Killing vector fields, for which the standard basis $a_{\alpha}{}^{A}$ is given [1] for $\alpha = \mathbf{1}, \mathbf{2}, \mathbf{3}$ (corresponding to what are, respectively, interpretable as rotations about the West, East, and North poles) for $X^1 = \hat{\theta}$, $X^2 = \hat{\varphi}$ by

$$\begin{aligned} a_{\mathbf{1}}^{A} &= -\sin\hat{\varphi}\,\delta_1^A - \cot\hat{\theta}\cos\hat{\varphi}\,\delta_2^A, \\ a_{\mathbf{2}}^{A} &= \cos\hat{\varphi}\,\delta_1^A - \cot\hat{\theta}\sin\hat{\varphi}\,\delta_2^A, \\ a_{\mathbf{3}}^{A} &= \delta_2^A. \end{aligned} \tag{45}$$

(On planet Earth, in the roughly Jerusalem centered system favored by cartographers since the time of Dante, the West pole is in the South Atlantic where the Greenwich meridian intersects the equator in the vicinity of the Gulf of Guinea, and the East pole is in the Indian Ocean, again on the equator but 90° further East in the vicinity of the Bay of Bengal, while the North pole is of course in the middle of the Arctic Ocean. The places referred to in the Old Testament, including Jerusalem and particularly Noah's legendary landing place, Mount Ararat, are near the centroid of these three poles, opposite to what Dante called the antepode, which is about as far as possible from any major land mass, in the middle of the South Pacific.)

There will be no loss of generality in choosing the coordinate system in such a way as to align V^A with the last of these, that is to say with the generator of rotations about the North pole, which means that we shall have

$$V^A = \mathfrak{n}\,\delta_2^A, \tag{46}$$

with a proportionality constant $\mathfrak{n}$ that is evidently interpretable as a winding number, so that it must be an integer (which would have to be zero in the special case of strong symmetry). This simply means that the space gradients involved in the dynamical equations above will be given just by

$$\hat{\theta}' = 0, \qquad \hat{\varphi}' = \mathfrak{n}. \tag{47}$$

As explained in the preceding work [1], the existence of the target space Killing vector fields (45) allows the two independent internal field equations to be expressed as conservation laws for the three currents given, for $\alpha = \mathbf{1}, \mathbf{2}, \mathbf{3}$, by

$$J_{\alpha i} = \kappa a_{\alpha}{}^{A} \hat{g}_{AB} X^{A}{}_{,i}, \tag{48}$$

of which only two are independent. It is to be recalled that κ is specified by the equation of state as a function of the quantity $\mathfrak{w}$, which will be given in this case by

$$\mathfrak{w} = \frac{\mathfrak{n}^2}{\varrho^2}\sin^2\hat{\theta} - \gamma^2(\dot{\hat{\theta}}^2 + \sin^2\hat{\theta}\,\dot{\hat{\varphi}}^2). \tag{49}$$

It can be seen that the equation of conservation for the third of these currents, namely,

$$J_{\mathbf{3}^i} = \kappa\sin^2\hat{\theta}(\dot{\hat{\varphi}}\delta_i^0 + \mathfrak{n}\delta_i^1), \tag{50}$$

will take the form

$$(\varrho\gamma\kappa\sin^2\hat{\theta}\,\dot{\hat{\varphi}})^{\cdot} = 0, \tag{51}$$

which contains just the same information as Eq. (34) for conservation of angular momentum, except in the special case of strong symmetry, $\mathfrak{n} = 0$, for which the angular momentum simply vanishes. It can also be seen that the only independent information obtainable from the conservation of the other two currents $J_{\mathbf{1}^i}$ and $J_{\mathbf{2}^i}$ is that of the internal energy creation Eq. (42), which will take the form

$$(\varrho^2\gamma^2\kappa^2(\dot{\hat{\theta}}^2 + \sin^2\hat{\theta}\,\dot{\hat{\varphi}}^2))^{\cdot} + \kappa^2\mathfrak{n}^2(\sin^2\hat{\theta})^{\cdot} = 0. \tag{52}$$

To obtain the complete system of equations of motion for the three independent variables $\hat{\theta}$, $\hat{\varphi}$, and ϱ, the internal dynamical equations (51) and (52) need to be supplemented by the information about the extrinsic motion that is contained in (33) which, in the force free case characterized by (43), will take the form of the total energy conservation condition

$$(\varrho\gamma^3\kappa(\dot{\hat{\theta}}^2 + \sin^2\hat{\theta}\,\dot{\hat{\varphi}}^2) - \varrho\gamma L)^{\cdot} = 0. \tag{53}$$

VII. HARMONIC SEPARABILITY FOR SPHERICAL TARGET CASE

It is evident that the preceding set of three dynamical equations will immediately provide two constant first integrals of the motion, namely, a current conservation constant

$$\varrho\gamma\kappa\sin^2\hat{\theta}\,\dot{\hat{\varphi}} = \mathfrak{C}, \tag{54}$$

obtained from (51) and a total energy conservation constant

$$\varrho\gamma^3\kappa(\dot{\hat{\theta}}^2 + \sin^2\hat{\theta}\dot{\hat{\varphi}}^2) - \varrho\gamma L = \mathfrak{E}, \tag{55}$$

obtained from (53). However, the third dynamical equation (52) will not be so conveniently integrable in the generic harmonious case, for which the coefficient κ is a variable function of the quantity $\mathfrak{w}$ given by (49).

In order to proceed, we now restrict attention to the special case of a model that is not just harmonious but actually harmonic, so that the coefficient κ is just a constant. The harmonic case is characterized in terms of a pair of constants m and $\kappa_\star$ by a Lagrangian of the merely linear form

$$L = -m^2 - \tfrac{1}{2}\kappa_\star\mathfrak{w}, \tag{56}$$

which simply gives

$$\kappa = \kappa_\star. \tag{57}$$

The presence of the (Kibble type) mass term is irrelevant for the purely harmonic equations (51) and (52) that govern the internal fields on the world sheet if the latter is prescribed in advance, but for the actual calculation, via (53), of the evolution of the worldsheet the specification of m is indispensible, as it fixes the value of the string tension in the zero current limit.

In this special harmonic case, the first constant of the motion (54) can be used to eliminate the variable $\dot{\hat{\varphi}}$, which will be given simply by

$$\dot{\hat{\varphi}} = \frac{\mathfrak{c}}{\varrho\gamma\sin^2\hat{\theta}}, \qquad \mathfrak{c} = \frac{\mathfrak{C}}{\kappa_\star}, \tag{58}$$

and it is apparent that the second internal dynamical equation (52) will also provide a constant first integral, which can be specified as the necessarily positive quantity $\mathfrak{a}^2$ given by the formula

$$\varrho^2\mathfrak{w}^\dagger = \mathfrak{a}^2, \tag{59}$$

using the notation $\mathfrak{w}^\dagger$ for the quantity obtained by changing the sign of the second term of the definition (49) of $\mathfrak{w}$, namely,

$$\mathfrak{w}^\dagger = \frac{\mathfrak{n}^2}{\varrho^2}\sin^2\hat{\theta} + \gamma^2(\dot{\hat{\theta}}^2 + \sin^2\hat{\theta}\dot{\hat{\varphi}}^2). \tag{60}$$

In the special harmonic case (56) this notation can be used to rewrite (32) as

$$\bar{T}_{00} = \frac{1}{\gamma^2}\Big(m^2 + \frac{1}{2}\kappa_\star\mathfrak{w}^\dagger\Big),$$
$$\bar{T}_{11} = \varrho^2\Big(-m^2 + \frac{1}{2}\kappa_\star\mathfrak{w}^\dagger\Big), \tag{61}$$

and to rewrite the formula (55) for the energy constant in the form

$$\varrho\gamma(m^2 + \tfrac{1}{2}\kappa_\star\mathfrak{w}^\dagger) = \mathfrak{E}. \tag{62}$$

This leads to the discovery of a remarkably convenient separability property, whereby the external dynamical variable ϱ can be decoupled from the internal field variables $\hat{\theta}$ and $\hat{\varphi}$ by the elimination of $\mathfrak{w}^\dagger$ between (59) and (34). The ensuing separated equation, for the radial variable ϱ by itself, can be seen to take the form

$$\mathfrak{E}\sqrt{1 - \dot{\varrho}^2} = m^2\varrho + \frac{\kappa_\star}{2}\frac{\mathfrak{a}^2}{\varrho}. \tag{63}$$

VIII. THROBBING VORTON STATES

It can be seen that the radial evolution equation (63) will give rise to an evolution that will be qualitatively similar to what has been found [4] for circular strings with just a single independent current variable, which is that the loop will oscillate periodically between finite minimum and maximum values of its radius ϱ.

More particularly, when the energy constant $\mathfrak{E}$ is taken to have the minimum value compatible with a given value of the other constant, $\mathfrak{a}^2$, a vorton type equilibrium state, with fixed radius

$$\varrho = \mathfrak{b}, \qquad \dot{\varrho} = 0, \tag{64}$$

will be obtained. By minimizing the left -hand side of (63), it can be seen that such a vorton will be characterized by

$$\mathfrak{b}^2 = \kappa_\star\mathfrak{a}^2/2m^2, \qquad \mathfrak{E} = \sqrt{2\kappa_\star}m\mathfrak{a}. \tag{65}$$

Unlike the stationary, meaning strictly time-independent (though not static, meaning strongly time-independent) vorton states of the kind that are familiar in cases when there is only a single current—or even when there are several currents if their generators commute—a vorton state of the non-Abelian kind considered here has the remarkable feature describable (using a term borrowed from the medical context of blood circulation) as *throbbing*. What this means is that, although the stress tensor and worldsheet geometry of the string loop are time independent, its internal fields undergo nonstationary oscillations. By substituting from (58) and (59) in (61), the field $\hat{\theta}$ can be seen to have a nontrivial time evolution given by

$$\mathfrak{b}^2\dot{\hat{\theta}}^2 = \mathfrak{a}^2 - \mathfrak{c}^2/\sin^2\hat{\theta} - \mathfrak{n}^2\sin^2\hat{\theta}. \tag{66}$$

It can be seen that, whenever $\mathfrak{a}^2 > \mathfrak{c}^2 + \mathfrak{n}^2$, the colatitudinal field $\hat{\theta}$ will oscillate symmetrically between a minimum, where $2\mathfrak{n}^2\sin^2\hat{\theta} = \mathfrak{a}^2 - \sqrt{\mathfrak{a}^4 - 4\mathfrak{n}^2\mathfrak{c}^2}$, in the northern hemisphere, $0 \le \theta < \pi/2$, and a maximum with the same value of $\sin^2\hat{\theta}$ in the southern hemisphere, $\pi/2 \le \theta < \pi$. If $\mathfrak{c}^2 > \mathfrak{n}^2$, such oscillating non-Abelian configurations can be viewed as perturbations of a strictly stationary single current configuration with longitude variable $\hat{\varphi} = \mathfrak{n}\phi \pm \mathfrak{c}t/\mathfrak{b}$, for fixed equatorial colatitude, $\cos\hat{\theta} = 0$. For small amplitudes, the perturbations will have $\cos\hat{\theta} \propto \cos\{\omega t\}$ with $\omega^2 = (\mathfrak{c}^2 - \mathfrak{n}^2)/\mathfrak{b}^2$.

There can be no solution at all with $\mathfrak{a}^2 < 2|\mathfrak{n}\mathfrak{c}|$, but solutions of a rather weird kind will be possible for the intermediate range $2|\mathfrak{n}\mathfrak{c}| < \mathfrak{a}^2 < \mathfrak{c}^2 + \mathfrak{n}^2$, provided the supplementary condition $\mathfrak{c}^2 < \mathfrak{n}^2$ is also satisfied. For this parameter range, the field $\hat{\theta}$ will be asymmetrically confined to a single one of the target space hemispheres, oscillating between values where $2\mathfrak{n}^2\sin^2\hat{\theta} = \mathfrak{a}^2 \pm \sqrt{\mathfrak{a}^4 - 4\mathfrak{n}^2\mathfrak{c}^2}$ without ever crossing the equator where $\hat{\theta} = \pi/2$. Such oscillating non-Abelian configurations can be considered as perturbations of a strictly stationary single current configuration having longitude variable of the chiral form $\hat{\varphi} = \mathfrak{n}(\phi \pm t/\mathfrak{b})$, with fixed colatitude $\hat{\theta} = \arcsin\sqrt{|\mathfrak{c}/\mathfrak{n}|}$. In the small amplitude limit, the perturbations will have $\omega^2 = 4(\mathfrak{n}^2 - |\mathfrak{n}\mathfrak{c}|)/\mathfrak{b}^2$.

IX. HARMONIC SEPARABILITY FOR AXISYMMETRIC TARGET

The main motive for the preceding work was to exhibit the behavior of currents generated by target space symmetries that do not commute—and so cannot be made simultaneously manifest—by considering the simplest case for which noncommuting symmetries are present, namely, that for which the target space is spherical. However it has turned out that whereas the one-parameter Abelian subgroup corresponding to axisymmetry is essential, the existence of the other noncommuting symmetries has played no qualitatively important role in the foregoing results, for which the indispensible postulate was that the field equations in question should be not just harmonious but of the strictly harmonic form (56), which is all that is needed to obtain the constants $\mathfrak{a}^2$ and $\mathfrak{C}$ as given by (59) and (62) in terms of the quantity

$$\hat{\mathfrak{w}}^{\dagger} = \frac{1}{\varrho^2}\hat{g}_{AB}X^{A'}X^{B'} + \gamma^2\hat{g}_{AB}\dot{X}^A\dot{X}^B. \tag{67}$$

Elimination of this quantity then gives a separated radial evolution equation for ϱ of exactly the same form (63) as for the special case of a target space that is spherical.

For complete separability of the system when the target space is two dimensional, it is sufficient that its metric should have the general axisymmetric form

$$d\hat{s}^2 = \hat{q}^2 d\hat{\varpi}^2 + \hat{\varpi}^2 d\hat{\varphi}^2, \tag{68}$$

with $\hat{q}$ given as an arbitrary function of $\hat{\varpi}$. By setting $\hat{\varpi} = \sin\hat{\theta}$ it can be seen that this metric will take the spherical form (2) in the special case for which $\hat{q} = 1/\sqrt{1-\hat{\varpi}^2}$, and it will simply be flat if $\hat{q} = 1$.

The choice of the function $\hat{q}$ has no effect on the condition (58), which will simply go over to the form

$$\dot{\hat{\varphi}} = \frac{\mathfrak{c}}{\varrho\gamma\hat{\varpi}^2}, \tag{69}$$

and implementation, as before, of the strict but weak axisymmetry postulate, to the effect that $\hat{\varpi}' = 0$ but $\hat{\varphi}' = \mathfrak{n}$, for some nonvanishing integral value of $\mathfrak{n}$, will reduce (67) to the form

$$\mathfrak{w}^{\dagger} = \frac{\mathfrak{n}^2}{\varrho^2}\hat{\varpi}^2 + \gamma^2\hat{q}^2\dot{\hat{\varpi}}^2 + \frac{\mathfrak{c}^2}{\varrho^2\hat{\varpi}^2}. \tag{70}$$

It follows that for all such cases there will be geometrically stationary throbbing vorton states characterized via (59) by the same equations, (64) and (65), as before, and thus with internal structure governed by a dynamical equation of the form

$$\mathfrak{b}^2\hat{q}^2\dot{\hat{\varpi}}^2 = \mathfrak{a}^2 - \frac{\mathfrak{c}^2}{\hat{\varpi}^2} - \mathfrak{n}^2\hat{\varpi}^2, \tag{71}$$

which is soluble by quadrature to give

$$t = \int \frac{\mathfrak{b}\hat{q}\,\hat{\varpi}\,d\hat{\varpi}}{\sqrt{\mathfrak{a}^2\hat{\varpi}^2 - \mathfrak{c}^2 - \mathfrak{n}^2\hat{\varpi}^4}}. \tag{72}$$

The simplest example is of course the one provided by the case $\hat{q} = 1$, namely, the model having just a single complex scalar field, with amplitude $\hat{\varpi}$ and phase $\hat{\varphi}$, for which the target space is flat,

$$d\hat{s}^2 = d\hat{\chi}^{1\,2} + d\hat{\chi}^{2\,2}, \qquad \hat{\chi}^1 = \hat{\varpi}\cos\hat{\varphi}, \qquad \hat{\chi}^2 = \hat{\varpi}\sin\hat{\varphi}. \tag{73}$$

This is the case for which the internal field model is purely linear, so that it will admit multiple conducting vorton states of the ordinary strictly stationary kind, in which the conserved currents are generated by the Abelian algebra of the target space translation group. This Abelian algebra is, however, just a subalgebra of the complete symmetry group: although the target space is flat, its symmetry group is non-Abelian because it also includes rotations, which of course do not commute with translations. The presence of the conserved currents generated by such noncommuting rotations is what allows this familiar simple model to provide vortons not just of the usual strictly stationary kind, but also of the throbbing kind considered here. For the linear field model characterized by (73) the corresponding quadrature (72) with $\hat{q} = 1$ can be evaluated explicitly: the internal field amplitude $\hat{\varpi}$ will throb in a manner given by the formula

$$\hat{\varpi}^2 = \frac{\mathfrak{a}^2 + \varepsilon^2\cos\{\omega t\}}{2\mathfrak{n}^2}, \qquad \varepsilon^2 = \sqrt{\mathfrak{a}^4 - 4\mathfrak{c}^2\mathfrak{n}^2}, \qquad \omega = \frac{2\mathfrak{n}}{\mathfrak{b}}. \tag{74}$$

The concomitant formula for the throbbing of the internal phase variable $\hat{\varphi}$ can be seen from (69) to take the form

$$\hat{\varphi} = \mathfrak{n}\phi + \arctan\left\{\frac{2\mathfrak{n}\mathfrak{c}}{\mathfrak{a}^2 + \varepsilon^2}\tan\left\{\frac{\omega t}{2}\right\}\right\}. \tag{75}$$

It follows that the Cartesian field components (73) will be given by the complex combination

$$2\mathfrak{n}(\hat{\chi}^1 + i\hat{\chi}^2)\exp\{-i\mathfrak{n}\phi\} = \sqrt{\mathfrak{a}^2 + 2\mathfrak{n}\mathfrak{c}}\exp\{i\mathfrak{n}\mathfrak{b}t\} + \sqrt{\mathfrak{a}^2 - 2\mathfrak{n}\mathfrak{c}}\exp\{-i\mathfrak{n}\mathfrak{b}t\}. \tag{76}$$

When $\varepsilon^2 \ll \mathfrak{a}^2$ (near the limits $2\mathfrak{n}\mathfrak{c} \to \pm\mathfrak{a}^2$), such a throbbing solution can be regarded as a perturbation of an ordinary stationary vorton configuration of the special chiral type, as given by $\hat{\varphi} = \mathfrak{n}(\phi \pm \mathfrak{b}t)$, with $\hat{\varpi} = \sqrt{|\mathfrak{c}/\mathfrak{n}|}$.

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