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Effective approach to the problem of time: General features and examples

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The effective approach to quantum dynamics allows a reformulation of the Dirac quantization procedure for constrained systems in terms of an infinite-dimensional constrained system of classical type. For semiclassical approximations, the quantum constrained system can be truncated to finite size and solved by the reduced phase space or gauge-fixing methods. In particular, the classical feasibility of local internal times is directly generalized to quantum systems, overcoming the main difficulties associated with the general problem of time in the semiclassical realm. The key features of local internal times and the procedure of patching global solutions using overlapping intervals of local internal times are described and illustrated by two quantum mechanical examples. Relational evolution in a given choice of internal time is most conveniently described and interpreted in a corresponding choice of gauge at the effective level and changing the internal clock is, therefore, essentially achieved by a gauge transformation. This article complements the conceptual discussion in [M. Bojowald, P. A. Höhn, and A. Tsobanjan, Classical Quantum Gravity **28**, 035006 (2011).].

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I. INTRODUCTION

One of the most pressing issues in the development of a consistent theory of quantum gravity is the problem of time [1–4]. As a generally covariant theory, its dynamics is fully constrained, without a true Hamiltonian generating evolution with respect to a distinguished or absolute time. Within the classical treatment, using the conventional spacetime (manifold) picture, this does not immediately pose a serious problem since there are different notions of time available in general relativity. The physical notion of time as experienced by a specific observer is supplied in an invariant and unambiguous manner by the proper time along that observer's worldline. The second notion appears in the context of the canonical initial-value formulation, often constructed by introducing a foliation of spacetime by spatial hypersurfaces. However, the time coordinate that labels these hypersurfaces, in contrast to proper time, has no invariant physical meaning. It is simply the gauge parameter for orbits of the Hamiltonian constraint and, classically, these orbits lie entirely within the constraint surface. Evolution along the orbits may be interpreted with respect to this time coordinate which provides an ordering to physical relations. When quantizing the theory via the Dirac procedure, however, physical states are to be annihilated by the quantum constraints and are, therefore, gauge invariant by construction. The gauge flow, along with the gauge parameters of the constraints, is absent in the physical Hilbert space. In the presence of a Hamiltonian constraint this means that physical states are timeless. Furthermore, physical observables should be gauge invariant and must thus be constant along classical dynamical trajectories and commute with the constraints in the quantum theory.[1] It appears as if ''nothing moves,'' or, as if ''dynamics is frozen.''

Change and dynamics, however, can be untangled from this static world by taking the underlying principles of general relativity seriously, according to which physics is purely relational. Evolution is not measured with respect to an absolute external parameter but time can be chosen among the internal degrees of freedom. Evolution is then interpreted relative to such an internal clock, where internal time is more general than and not necessarily directly linked to the proper time of any observer. While proper time is practical for describing dynamics *in* a gravitational field since it depends on the worldlines of observers and has meaning only after solving the Einstein equations, in

[1] The viewpoint that physically observable quantities in parametrized systems should commute with all constraints, including the Hamiltonian constraint, has been challenged by Kuchař (and, more recently, by Barbour and Foster [5]). For instance, in [6] he argues for a difference between conventional gauge systems and parametrized systems, leading to the proposal that states along the orbit of the Hamiltonian constraint should not be identified since this would stand in contradiction to our everyday experience of the flow of time. He advocates that, instead, in general relativity physically observable quantities should only commute with the diffeomorphism constraints, but not necessarily with the Hamiltonian constraint. Nevertheless, in this article we take the conventional standpoint of requiring that physically observable quantities should commute with all constraints and, consequently, that in this sense no distinction ought to be made between the Hamiltonian and the other constraints.

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quantum gravity one is rather interested in the dynamics *of* the gravitational field, for which internal time is useful. This concept has led to the so-called *evolving constants of motion* [4,7], which are relational Dirac observables measuring physical correlations between the internal clock and other degrees of freedom. Significant progress in this direction and generalizations of such relational observables have been undertaken in [8–10], and some criticism concerning their capability of solving the problem of time has been raised in [1,2,6,11]. In the sequel, we will adopt the relational viewpoint and employ internal clocks as measures of a relational time. (Some interesting real-world aspects also relevant to internal clocks have been discussed, for instance, in [12].) As regards evolution, the choice and corresponding notion of time are inherently connected to the choice of the internal clock variable.

Apart from this conceptual issue, the problem of time usually comes with a whole plethora of technical problems [1–3], of which the ones touched upon in this article may be summarized as follows:

(i) *The multiple-choice problem*. Which internal time should one choose as a clock? There is no natural choice of an internal clock variable and different internal times may provide different quantum theories [1,2,13]. Furthermore, one must impose restrictions on the choice of internal time functions, since some choices lead to inconsistent probabilistic predictions in the quantum theory and time orderings which are not well defined [11].

(ii) *The Hilbert-space problem*. Which Hilbert-space representation is one to choose and how is one to construct a positive-definite physical inner product on the space of solutions to the quantum constraints?

(iii) *The operator-ordering problem*. The usual ordering problems arise upon promoting classical constraints to operator equivalents. The choice of a time variable also plays a role in the ordering problem [1].

(iv) *The global time problem*. Similarly to the Gribov problem in non-Abelian gauge theories, there may exist global obstructions to singling out good internal clock variables which provide good parametrizations of the gauge orbits in the sense that each classical trajectory intersects every hypersurface of constant clock time once and only once [1,2,7,10,14].

(v) *The problem of observables*. It is very difficult to construct a sufficient set of explicit observables for gravitational and parametrized theories and even the existence of a sufficient set has been questioned [3,6,10]. In fact, no general Dirac observables are known for general relativity. While classically significant progress has been made in this area [8–10], the problem worsens in the quantum theory due to the previous technical issues since no general scheme exists for converting such observables—if found at all—into suitable operators.

The relational interpretation of evolution is complicated by the fact that internal clock functions are neither universal nor perfect. A globally valid choice of internal time is difficult to find and, due to the *global time problem*, may not exist. For specific matter systems, such as a free massless scalar field or pressureless dust, deparametrizations with a matter clock can be performed, but these models seem rather special. In order to evaluate the dynamics of quantum gravity and derive potentially observable information from first principles, the various problems of time must be overcome without requiring specific adaptations.

The imperfect nature of internal clocks does not constitute a problem at the classical level, however, since, in principle, we can always make use of the gauge parameter along the flow of the Hamiltonian constraint and evolve in this coordinate time with respect to which the internal clock, say $T(x)$, and the other variables of interest, say $Q_i(x)$, have a given evolution. Comparing the values of the internal clock and the $Q_i(x)$ along the coordinate time then gives a relational evolution. If $T(x)$ fails to be a good global clock, the system will eventually go backwards in it, the observable correlations $Q_i(T(x))$ will, in general, be multivalued and, consequently, the evolution of the correlations $Q_i(T)$ will be "patched up," where on each patch T will be a good clock. Thus, classically, in principle, we do not even need to switch clocks if one takes the evolution in some good time coordinate into account which does not know about nonglobal clocks and provides an ordering to the patches. With respect to this time coordinate we can solve a well-defined initial-value problem [IVP] (as long as a time direction is given). One can even encode this relational evolution entirely with physical correlations without referring to any gauge parameter, if one keeps not only the relational configuration observables but also the relational momentum observables in mind to determine an orientation in which to evolve even at a turning point of a nonglobal clock. If a time direction is provided, one can also impose relational initial data to completely specify a classical solution. The classical solution may then be obtained by choosing a physical Hamiltonian which moves the surfaces of constant T in phase space. In the case of a nonglobal clock, this reconstruction is complicated by the fact that a given trajectory may intersect a constant time hypersurface more than once or not at all. In this case one will have to choose more than one Hamiltonian but this is merely a technical difficulty, not a fundamental problem. We will come back to this point in the main body of this article.

Because of the quantum uncertainties and the lack of a classical gauge parameter, performing a "patching" as above will no longer be possible in the full quantum theory and we are forced to employ purely relational information which will require the switching of nonglobal clocks. If relational time is defined for only a finite range, a unitary relational state evolution can not be accomplished and, as we will see, will break down earlier than the corresponding Hamiltonian evolution in the classical theory.[2] While classical evolution in nonglobal clocks is, in principle, unproblematic, nonunitary quantum evolution can lead to meaningless results long before the end of a local time is reached and it is not clear how to define relational quantum observables in this case.

Even though coordinate time may not exist in full quantum gravity at the Planck scale, one would heuristically expect that on the way to larger scales—in a semiclassical regime which ought to provide the connection to the classical solutions of general relativity—one can reconstruct a (certainly nonunique) coordinate time (for a discussion of this within loop quantum cosmology see [16]). Indeed, the notion of a time coordinate and evolution trajectory should become meaningful for coherent states whose expectation values follow the classical trajectory at least for a certain range. In a semiclassical regime, the notion of coordinate time should, therefore, make sense and we should be able to follow a similar strategy here as in the classical situation.

For most applications of quantum gravity related to potential observable effects, semiclassical evolution is sufficient, or, at least provides a large amount of information. One may then hope that such a situation makes dealing with the problem of time more feasible since this problem does not play a handicapping role classically; at the very least a dedicated analysis of semiclassical evolution should provide insights which may help in attacking the problem in full generality.

This article complements the conceptual discussion in [17] with concrete examples and a concrete discussion of the general features they exhibit. We use the effective approach to quantum constraints developed in [18,19] in the context of the problem of time; truncation at semiclassical order reintroduces some notion of classical gauge parameters. It is the aim of the present article to sidestep a number of technical issues associated to an explicit Dirac type approach and to specifically cope with the *global time problem*, while the other technical problems alluded to above will automatically be addressed in the course of the discussion. It is our goal to make physical predictions based on some set of (relational) input data, also in nondeparametrizable systems.

[2]The finite range of a clock and the resulting apparent nonunitarity are what one could call a ‘‘classical symptom’’ and a ‘‘quantum illness’’ which prevent an acceptable quantum dynamical solution in a conventional sense [15]. The point is, however, that this nonunitarity in internal time is only the result of a local dynamical interpretation of an *a priori* timeless system which, in itself is not nonunitary. These considerations are relevant for quantum gravity, since, from a certain point of view, there might not exist a fundamental notion of time at the Planck scale which would allow for a meaningful, conventional unitary evolution [4,7].

We will make use of (local) deparametrizations in order to locally scan through an *a priori* timeless physical state, thereby introducing a notion of quantum evolution. We propose a practical solution employing local, rather than global internal times and adopt and emphasize the viewpoint that the relational interpretation is, generally, only of local and semiclassical meaning, as was argued in [17]. For explicit calculations, our methods will lend themselves easily to gauge-fixing techniques, avoiding complicated derivations of complete observables. In analogy to local coordinates on a manifold, we cover the evolution trajectories by patches of local time and translate between them in order to evolve through pathologies of local clocks. The choice of time is best described and interpreted in a corresponding choice of gauge at the effective level and translating between different local clocks, therefore, requires nothing more than a gauge transformation. In addition, we find that nonunitarity at the state level translates into complex internal time. To begin with, we will focus on simple mechanical toy models which we will treat in the classical, effective and for comparison, where feasible, in a Hilbert-space approach. The first model is deparametrizable, even though we employ a nonglobal clock for the relational evolution, while the second model is a true example of a ‘‘timeless,’’ nondeparametrizable system which has previously been discussed by Rovelli [4,7].

The rest of the article is organized as follows. Section II reviews the effective treatment of a quantum Hamiltonian constraint and summarizes features of the example of the ‘‘relativistic’’ harmonic oscillator. In Sec. III we study the first of the two models, discussing its classical and quantum behavior before going through the full effective treatment truncated using the semiclassical approximation. In this model we opt to use a time variable which is nonmonotonic along every classical trajectory. We find that a consistent effective treatment of this model requires assigning a complex expectation value to the kinematical time operator. We find an explicit gauge transformation which allows us to evolve the model of Sec. III through the turning point of the nonglobal clock. A detailed discussion of general features of such transformations, as well as of the close relationship between the choice of an internal time variable and suitable gauge fixing follows in Secs. IV C and IV D. The second model is studied in Sec. V, where the effective treatment is performed following the footsteps of Sec. III. Effective evolution relative to a local time is compared to the (Hilbert space) dynamics obtained using a locally deparametrized version of the constraint, demonstrating good agreement. This model does not possess a global clock and transformations between local internal times are necessary for full dynamical evolution. At the effective level these are once again performed using gauge transformations allowing ‘‘patched-up’’ global evolution. Section VI contains several concluding remarks.

II. EFFECTIVE CONSTRAINTS

All examples in this article are quantum systems with a single constraint operator $\hat{C}$ playing a role analogous to that of the Hamiltonian constraint in general relativity. According to the Dirac quantization procedure, physical states $|\psi\rangle$ satisfy the condition $\hat{C}|\psi\rangle = 0$. When one solves for specific states represented in a Hilbert space and attempts to equip the solution space with a physical inner product, spectral properties of the zero eigenvalue of $\hat{C}$ are important: if zero is in the discrete part of the spectrum, physical states form a subspace of the kinematical Hilbert space in which the quantum constraint equation is formulated; for zero in the continuous part, on the other hand, a new physical Hilbert space must be constructed for which some methods exist [20]. These methods in practical applications, however, have a rather limited range of applicability, and so finding physical Hilbert spaces remains a challenge. For our effective procedures, assumptions about the spectrum of $\hat{C}$ need not be made; effective techniques work equally well for zero in the discrete as well as the continuous part of the spectrum of constraint operators.

Effective descriptions for canonical quantum theories [18,19] are based on a description of states not in terms of wave functions (or density matrices) but by using expectation values $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ and moments

$$\Delta(q^a p^b) := \langle(\hat{q} - \langle\hat{q}\rangle)^a(\hat{p} - \langle\hat{p}\rangle)^b\rangle_{\text{Weyl}}$$

(ordered totally symmetrically and defined for $a + b \geq 2$). (For instance, $\Delta(q^2) = (\Delta q)^2$ is the position fluctuation with only a slight change of the standard notation.)

The state space is equipped with a Poisson structure defined by

$$\{\langle\hat{A}\rangle, \langle\hat{B}\rangle\} = \frac{\langle[\hat{A}, \hat{B}]\rangle}{i\hbar} \tag{1}$$

for any pair of operators $\hat{A}$ and $\hat{B}$, extended to the moments using the Leibnitz rule and linearity. In the case of dynamics given by a true Hamiltonian, the Schrödinger evolution of states is equivalent to the evolution of expectation values and moments generated by the quantum Hamiltonian $H_Q(\langle\hat{q}\rangle, \langle\hat{p}\rangle, \Delta(\cdots)) = \langle\hat{H}\rangle$ through the Poisson bracket defined above.

For physical states parameterized by their expectation values and moments, the equation $\langle\hat{C}\rangle(\langle\hat{q}\rangle, \langle\hat{p}\rangle, \Delta(\cdots)) = 0$ defines a constraint function on the quantum phase space. In this way, classical techniques for the reduction of constrained systems can be applied even in the quantum case, one of the key features exploited in this article to address the problem of time. The quantum nature of the problem is manifest in moment-dependent correction terms in the function $\langle\hat{C}\rangle$ as opposed to the classical constraint, as well as the infinite dimensionality of the quantum phase space even for a system with finitely many classical degrees of freedom. Moreover, since the moments are *a priori* independent degrees of freedom, they are restricted by further constraints

$$C_{\text{pol}}(\langle\hat{q}\rangle, \langle\hat{p}\rangle, \Delta(\cdots)) := \langle(\widehat{\text{pol}} - \langle\widehat{\text{pol}}\rangle)\hat{C}\rangle = 0$$

for all polynomials $\widehat{\text{pol}}$ in basic operators.[3] This set of functions contains infinitely many first-class constraints for infinitely many variables; the quantum constraint functions, therefore, generate gauge transformations and solving the constraints does not directly lead to gauge invariance. The latter is only achieved after constructing Dirac observables on the quantum phase space, which provide the correct number of physical degrees of freedom. In this aspect, the effective formalism differs from standard Dirac quantization where the physical Hilbert space is devoid of gauge flows. This may be understood from noting that states in the physical Hilbert space only assign expectation values to Dirac observables, while in the effective formalism expectation values are *a priori* assigned to all kinematical variables, which even at the classical level are not gauge invariant.

For the first-class nature, the ordering of operators in the products $\widehat{\text{pol}}\,\hat{C}$ is important, which, as shown explicitly in the form written above, is not ordered symmetrically. Some of the quantum constraints then take complex values, which does not cause problems as already shown for deparameterizable systems. This complex nature of the constrained system is also rooted in the fact that the effective expectation values are assigned to all kinematical variables. It is not surprising that only some kinematical moments satisfy reality conditions after the constraints are implemented. Reality will be imposed on the physical expectation values and moments—the Dirac observables of the constrained system—and contact with the physical Hilbert space is made. We will provide further examples in this article.

Regarding the construction of Dirac observables for the constrained system defined here, we note that observables which commute with the quantum constraints translate into Dirac observables for the effective system, Poisson commuting with all the quantum constraint functions:

$$\begin{aligned}\delta\langle\hat{O}\rangle &= \{\langle\hat{O}\rangle, \langle(\widehat{\text{pol}} - \langle\widehat{\text{pol}}\rangle)\hat{C}\rangle\} \\ &= \frac{1}{i\hbar}(\langle(\widehat{\text{pol}} - \langle\widehat{\text{pol}}\rangle)[\hat{O}, \hat{C}]\rangle + \langle[\hat{O}, \widehat{\text{pol}}](\hat{C} - \langle\hat{C}\rangle)\rangle),\end{aligned} \tag{2}$$

vanishes weakly if $\hat{O}$ is a Dirac observable. By the same token, moments computed for Dirac observables are Dirac observables in the effective approach.

The set of infinitely many constraints for infinitely many variables is directly tractable by exact means only if the constraints decouple into finite sets, a situation realized

[3] The condition $\langle\hat{C}\rangle = 0$ cannot be sufficient to determine the physical state, since the mean value of $\hat{C}$ may vanish even if $\hat{C}|\psi\rangle \neq 0$.

only for constraints linear in canonical variables. More interesting systems can be dealt with by approximations which reduce the system to finite size when subdominant terms are ignored. The prime example for such an approximation is the semiclassical expansion, in which moments of high orders are suppressed compared to expectation values and lower-order moments. Semiclassicality in a very general form is implemented by the condition $\Delta(q^a p^b) = O(\hbar^{(a+b)/2})$; considering only finite orders in $\hbar$ thus allows one to restrict the infinite set of constraints to a finite one, and physical moments up to the order considered can be found more easily. When the system of all quantum constraints is reduced to finite size, we call the resulting constraints "effective," motivated by the fact that an analogous reduction in quantum-mechanical systems (combined with an adiabatic approximation) reproduces equations of motion that follow from the low-energy effective action [21].

Despite the fact that the moments can be varied independently at the effective level, they must, in general, satisfy an infinite tower of inequalities in order to represent a true quantum state. Namely, in ordinary quantum mechanics, the values assigned by a state to the various quantum moments are subject to inequalities that follow directly from the Schwarz inequality of the Hilbert space. In particular, for any two observables represented by Hermitian operators $\hat{A}$ and $\hat{B}$, we have

$$\langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle\langle(\hat{B} - \langle\hat{B}\rangle)^2\rangle \geq \tfrac{1}{4}|\langle -i[\hat{A}, \hat{B}]\rangle|^2 + \tfrac{1}{4}|\langle[(\hat{A} - \langle\hat{A}\rangle), (\hat{B} - \langle\hat{B}\rangle)]_+\rangle|^2,$$

where $[,]_+$ denotes the anticommutator. The well-known (generalized) uncertainty relation follows immediately by setting $\hat{A} = \hat{q}$ and $\hat{B} = \hat{p}$. In the present work we will *not* assume that all *kinematical* moments satisfy these inequalities, or even that their values are real. We will instead impose (order by order in the semiclassical expansion) these inequalities and reality on the relational observables *after* the constraint is solved. This is discussed in greater detail in Sec. III C 4 and in Appendix B. Notice that the generalized uncertainty relation is then the only remaining inequality at order $\hbar$.

The effective formalism provides approximation techniques for the evaluation of quantum dynamics. While it is motivated by the operator algebras of standard quantum theory, it is not necessarily equivalent to the standard theory. For instance, an expression such as $\langle\hat{q}\rangle$ need not and cannot necessarily be interpreted literally as the expectation value of a well-defined operator in a Hilbert space with a specifically defined inner product. Especially in the context of the problem of time, a crucial new feature arises—local internal time and the corresponding local relational observables, or fashionables [17]—which at present do not have a known analog at the Hilbert-space level. Changing one's local time in practice additionally amounts to a gauge transformation (see Sec. IV C), and we shall see later that different choices of gauge in the effective theory correspond to different, and in general inequivalent, choices of a Hilbert space for the quantum theory. Eventually, these new notions may be used to arrive at a generalization of quantum mechanics for situations in which time is not idealized as a monotonic parameter without turning points. If so, the generalization cannot be fully specified in the current effective framework which makes use of semiclassicality for explicit evaluations of its equations. But the examples provided in this article should play a key role in exploring these issues.

A. Example: "Relativistic" harmonic oscillator

To illustrate the procedure, we consider two copies of the canonical algebra $[\hat{t}, \hat{p}_t] = i\hbar = [\hat{\alpha}, \hat{p}_\alpha]$, subject to the constraint $\hat{C} = \hat{p}_t^2 - \hat{p}_\alpha^2 - \hat{\alpha}^2$. This system[4] has been treated in a fair amount of detail in [19,22], so here we only provide an outline. We truncate the system at order $\hbar$ of the semiclassical expansion. Specifically, this means that in addition to the terms explicitly proportional to $\hbar^{(3/2)}$, we discard all moments of third order and above, products of two or more second order moments, as well as products between a second order moment and $\hbar$. In particular, of the infinite number of degrees of freedom at this order, we only need to consider 14: four expectation values $\langle\hat{a}\rangle$, four spreads $(\Delta a)^2$ and six covariances $\Delta(ab)$, where a, b can be any of the four basic kinematical variables.

In this model, for example, one of the constraint conditions to be enforced is $C_\alpha := \langle(\hat{\alpha} - \langle\hat{\alpha}\rangle)\hat{C}\rangle = 0$. Here we are dealing with low order polynomials and the corresponding condition on expectation values and moments is straightforward to derive explicitly:

$$C_\alpha = \langle(\hat{\alpha} - \langle\hat{\alpha}\rangle)(\hat{p}_t^2 - \hat{p}_\alpha^2 - \hat{\alpha}^2)\rangle = \langle(\hat{\alpha} - \langle\hat{\alpha}\rangle)\hat{p}_t^2\rangle - \langle(\hat{\alpha} - \langle\hat{\alpha}\rangle)\hat{p}_\alpha^2\rangle - \langle(\hat{\alpha} - \langle\hat{\alpha}\rangle)\hat{\alpha}^2\rangle.$$

This quantity should be expressed in terms of the expectation values and moments, our phase-space coordinates. In each of the terms in the last expression one needs to replace powers of kinematical operators with corresponding powers of $(\hat{O} - \langle\hat{O}\rangle)$. For example, the middle term can be rewritten as

$$\langle(\hat{\alpha} - \langle\hat{\alpha}\rangle)\hat{p}_\alpha^2\rangle = \langle(\hat{\alpha} - \langle\hat{\alpha}\rangle)(\hat{p}_\alpha - \langle\hat{p}_\alpha\rangle)^2\rangle + 2\langle\hat{p}_\alpha\rangle\langle(\hat{\alpha} - \langle\hat{\alpha}\rangle)(\hat{p}_\alpha - \langle\hat{p}_\alpha\rangle)\rangle + \langle\hat{p}_\alpha\rangle^2\langle\hat{\alpha} - \langle\hat{\alpha}\rangle\rangle,$$

[4] This toy model is clearly not relativistic in the standard sense. However, here (and in the remaining models of this work) we are not interested in the precise physical interpretation of this system (of which there exist both relativistic and nonrelativistic ones), but rather in its structural properties. The constraints considered in the present article, similarly to Hamiltonian constraints in relativistic cosmology, are all quadratic in momenta.

where the last term vanishes as $\langle(\hat\alpha - \langle\hat\alpha\rangle)\rangle = \langle\hat\alpha\rangle - \langle\hat\alpha\rangle = 0$. The remaining terms need to be ordered symmetrically in order to write them in terms of moments, which can be accomplished with the use of the canonical commutation relations. Continuing with the example, the above term becomes

$$\langle(\hat\alpha-\langle\hat\alpha\rangle)\hat p_\alpha^2\rangle = \langle(\hat\alpha-\langle\hat\alpha\rangle)(\hat p_\alpha-\langle\hat p_\alpha\rangle)^2\rangle_{\rm Weyl} + \langle\hat p_\alpha\rangle(2\langle(\hat\alpha-\langle\hat\alpha\rangle)(\hat p_\alpha-\langle\hat p_\alpha\rangle)\rangle_{\rm Weyl}+i\hbar),$$

with

$$\begin{aligned}\langle(\hat\alpha-\langle\hat\alpha\rangle)(\hat p_\alpha-\langle\hat p_\alpha\rangle)^2\rangle_{\rm Weyl} &= \tfrac{1}{3}\langle(\hat\alpha-\langle\hat\alpha\rangle)(\hat p_\alpha-\langle\hat p_\alpha\rangle)^2 \\ &\quad + (\hat p_\alpha-\langle\hat p_\alpha\rangle)(\hat\alpha-\langle\hat\alpha\rangle)(\hat p_\alpha-\langle\hat p_\alpha\rangle) \\ &\quad + (\hat p_\alpha-\langle\hat p_\alpha\rangle)^2(\hat\alpha-\langle\hat\alpha\rangle)\rangle.\end{aligned}$$

Proceeding in this way, one can write the constraint condition using moments as

$$\begin{aligned}C_\alpha &= 2\langle\hat p_\alpha\rangle\Delta(p_t\alpha) - 2\langle\hat p_\alpha\rangle\Delta(\alpha p_\alpha) - i\hbar\langle\hat p_\alpha\rangle \\ &\quad - 2\langle\hat\alpha\rangle(\Delta\alpha)^2 + \Delta(\alpha p_t^2) - \Delta(\alpha p_\alpha^2) + \Delta(\alpha^3).\end{aligned}$$

Evaluating other constraints in this manner and truncating the system at order $\hbar$, the infinite set of constraint functions reduces to just five:

$$\begin{aligned}C &= \langle\hat p_t\rangle^2 - \langle\hat p_\alpha\rangle^2 - \langle\hat\alpha\rangle^2 + (\Delta p_t)^2 - (\Delta p_\alpha)^2 - (\Delta\alpha)^2 \\ C_t &= 2\langle\hat p_t\rangle\Delta(tp_t) + i\hbar\langle\hat p_t\rangle - 2\langle\hat p_\alpha\rangle\Delta(tp_\alpha) - 2\langle\hat\alpha\rangle\Delta(t\alpha) \\ C_{p_t} &= 2\langle\hat p_t\rangle(\Delta p_t)^2 - 2\langle\hat p_\alpha\rangle\Delta(p_tp_\alpha) - 2\langle\hat\alpha\rangle\Delta(p_t\alpha) \\ C_\alpha &= 2\langle\hat p_t\rangle\Delta(p_t\alpha) - 2\langle\hat p_\alpha\rangle\Delta(\alpha p_\alpha) - i\hbar\langle\hat p_\alpha\rangle - 2\langle\hat\alpha\rangle(\Delta\alpha)^2 \\ C_{p_\alpha} &= 2\langle\hat p_t\rangle\Delta(p_tp_\alpha) - 2\langle\hat p_\alpha\rangle(\Delta p_\alpha)^2 - 2\langle\hat\alpha\rangle\Delta(\alpha p_\alpha) + i\hbar\langle\hat\alpha\rangle.\end{aligned} \tag{3}$$

The constraint functions are first-class to order $\hbar$ and, therefore, generate gauge transformations through their Poisson brackets with the expectation values and moments.[5] Following [18,19], we fix the gauge that corresponds to the evolution of $\hat\alpha$ and $\hat p_\alpha$ in $\hat t$, by setting fluctuations of the latter to zero

$$(\Delta t)^2 = \Delta(t\alpha) = \Delta(tp_\alpha) = 0. \tag{4}$$

Through reorderings, imaginary contributions in the constraints have arisen, which require some of the moments to take complex values. For instance, with our gauge choice $\Delta(tp_t) = -\frac{1}{2}i\hbar$. All these moments refer to t which, when chosen as (internal) time in this deparametrizable system, is not represented as an operator and does not appear in physical moments. The gauge dependence or complex valuedness of these moments thus is no problem.

[5]The Poisson brackets between the expectation values and moments generated by two canonical pairs of operators is tabulated in Appendix A.

Moments not involving time or its momentum, on the other hand, should have a physical analog taking strictly real values. This is, indeed, the case. With the gauge fixed as above, a single gauge flow remains on the expectation values and moments evolving in t. (We need just three gauge-fixing conditions for four $o(\hbar)$ constraints because the Poisson tensor for the moments is degenerate.) It is generated by the constraint function $C_H = \langle\hat p_t\rangle \mp H_Q$ with the quantum Hamiltonian

$$H_Q = \sqrt{\langle\hat p_\alpha\rangle^2+\langle\hat\alpha\rangle^2} \times\left(1+\frac{\langle\hat\alpha\rangle^2(\Delta p_\alpha)^2 - 2\langle\hat\alpha\rangle\langle\hat p_\alpha\rangle\Delta(\alpha p_\alpha) + \langle\hat p_\alpha\rangle^2(\Delta\alpha)^2}{2(\langle\hat p_\alpha\rangle^2+\langle\hat\alpha\rangle^2)^2}\right). \tag{5}$$

Solving the Hamiltonian equations of motion for $\langle\hat\alpha\rangle(t)$, $\langle\hat p_\alpha\rangle(t)$, $\Delta(\alpha p_\alpha)(t)$, $(\Delta\alpha)^2(t)$, $(\Delta p_\alpha)^2(t)$ yields the Dirac observables of the constrained system in relational form, on which reality can easily be imposed just by requiring real initial values at some t. At this stage, we have arrived at the usual results for a deparametrized system with time t, in which evolving variables such as $\langle\hat\alpha\rangle(t)$ solving equations of motion with respect to (5) would be considered physical while no physical operator for time itself exists.

In our framework, it is gauge fixing that distinguishes one of the original variables as time without an operator analog: Time moments $\langle\hat p_t\rangle$, $(\Delta p_t)^2$, $\Delta(p_tp)$, $\Delta(p_t\alpha)$, $\Delta(tp_t)$ are eliminated using the constraints (3), while $(\Delta t)^2$, $\Delta(t\alpha)$, $\Delta(tp_\alpha)$ are fixed by the gauge condition (4). Generally, there may be several ways to interpret a given quantum constraint dynamically with respect to different choices of (internal) time. Collectively, the choice of a time variable, the associated gauge conditions and the selection of evolving variables within that gauge will be referred to, following [17], as a *Zeitgeist*. Usually, the selection of which variable to choose as clock function in which other variables may evolve relationally does not constitute a gauge choice. The effective formalism as developed here, however, provides a relationship between (the interpretation of a quantum variable as) time and gauge: we are free to fix the independent gauge flows in a way that describes and interprets relational evolution in the most convenient way. We will come back to this issue in detail in Sec. IV C; for now, we warn the reader about an inherent weakness of evolving observables, which underlies the comparison problem of time: If transformations of internal time variables are allowed, and if they are essentially implemented by gauge changes, the physical nature of some variables may appear (but is not) gauge dependent. To avoid apparently contradictory language, we use the term *fashionables* for local relational observables, as introduced in [17].

III. A MODEL OF A BAD INTERNAL CLOCK

In this section, through the use of a toy model, we showcase an effective semiclassical solution to the problem of defining quantum dynamics with respect to a time variable which is nonmonotonic along a (classical) trajectory.

We introduce the model together with its classical properties in Sec. III A; its Dirac quantization is briefly discussed in Sec. III B. In Sec. III C we apply the effective scheme of [18,19] for solving constraints to define approximate dynamics; among the many viable choices for internal time, we elect to study the dynamics relative to a variable that cannot be used for a global deparametrization. Evolution with respect to such a clock variable breaks down near its turning points and translation to a new clock variable is required. Within the effective approach, the choice of a clock is practically incorporated by selecting a gauge as in (4) and, therefore, switching a clock is achieved by a gauge transformation. Another novelty is that the expectation value of the time variable acquires an imaginary contribution, a feature further discussed in Sec. IV and the second model in Sec. V. The end result of the present section is an internally consistent approximate method for evolving initial data in a nonglobal clock variable through its extremal point on the trajectory, by temporarily switching to a different variable used as internal time.

A. Classical discussion

The model we are interested in possesses a "time potential" λt and is classically determined by the constraint

$$C_{\rm class} = p_t^2 - p^2 - m^2 + \lambda t. \qquad (6)$$

We assume $\lambda \geq 0$ for concreteness. This model has been briefly discussed in [19] and structurally resembles a perturbed free relativistic particle.[6] Of particular interest to us is the fact that t exhibits a specific trait of a bad clock, namely, it is not monotonic along a classical trajectory. As regards the parametrization of the flow generated by $C_{\rm class}$, we infer from

$$\{t, C_{\rm class}\} = 2p_t \quad \text{and} \quad \{p_t, C_{\rm class}\} = -\lambda < 0, \qquad (7)$$

that

$$t(s) = -\lambda s^2 + 2p_{t0}s + t_0 \quad \text{and} \quad p_t(s) = -\lambda s + p_{t0}, \qquad (8)$$

where s is the parameter along the flow $\alpha^s_{C_{\rm class}}(x)$ generated by $C_{\rm class}$. We see that t has an extremum and runs twice through each value it assumes; therefore globally it is not a good clock function for the gauge orbits generated by $C_{\rm class}$. Note that both p_t and q provide good parametrizations of the gauge orbit and p is an obvious Dirac observable. Although this model is deparametrizable in either q or p_t, we would like to interpret the relational evolution of the configuration variable q with respect to the nonglobal clock function t.

[6] Although, again, the system is clearly not relativistic in the standard sense.

For completeness, we also note that the Dirac observables of this system are easy to find and they themselves form a canonical Poisson algebra,

$$\mathcal{Q} := q - \frac{2}{\lambda} p p_t \ \text{and} \ \mathcal{P} := p, \ \text{satisfy} \ \{\mathcal{Q}, \mathcal{P}\} = 1. \qquad (9)$$

B. Dirac quantization

Following Dirac's algorithm for a constraint quantization, one would first quantize the kinematical system in the usual way, by representing canonical operators on the space $L^2(\mathbb{R}^2, dtdq)$ as

$$\hat{t} = t, \qquad \hat{p}_t = \frac{\hbar}{i}\frac{\partial}{\partial t}, \qquad \hat{q} = q, \qquad \hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial q}.$$

The constraint function (6) can be straightforwardly quantized as $\hat{C} = \hat{p}_t^2 - \hat{p}^2 - m^2 + \lambda\hat{t}$ and the physical state condition $\hat{C}\psi_{\rm phys} = 0$ becomes a partial differential equation

$$\left(-\hbar^2\frac{\partial^2}{\partial t^2} + \lambda t - m^2 + \hbar^2\frac{\partial^2}{\partial q^2}\right)\psi(t,q) = 0. \qquad (10)$$

The operators $\hat{p}^2$ and $\hat{p}_t^2 + \lambda\hat{t}$ commute and thus can be simultaneously diagonalized. The solution to the constraint equation can be constructed from their simultaneous eigenstates. The general solution has the form

$$\psi_{\rm phys}(t,q) = \int dk f(k) {\rm Ai}\left[\left(\frac{\lambda}{\hbar}\right)^{(2/3)}(\lambda t - k^2 - m^2)\right] e^{((-ikq)/(\hbar))}, \qquad (11)$$

where ${\rm Ai}[x]$ is the bounded and integrable Airy-function. As it often happens, none of the solutions are normalizable with respect to the kinematical inner product and a separate *physical* inner product must be defined on the solutions. A common way to proceed in the context of quantum cosmology is to deparametrize the system with respect to a suitable time variable. The simplest option is to formulate the constraint equation as a Schrödinger equation giving evolution of wave functions of q in the time-parameter p_t

$$i\hbar\frac{\partial}{\partial p_t}\tilde{\psi}(p_t,q) = \frac{1}{\lambda}\left(-\hbar^2\frac{\partial^2}{\partial q^2} - p_t^2 + m^2\right)\tilde{\psi}(p_t,q), \qquad (12)$$

where $\tilde{\psi}(p_t,q) := \int dt \psi(t,q) e^{-itp_t/\hbar}$. We then define the physical inner product by integrating over q at a fixed value of p_t

$$\langle\psi,\phi\rangle_{\rm phys} := \int_{p_t=p_{t0}} dq \bar{\tilde{\psi}}(p_t,q)\tilde{\phi}(p_t,q) \quad . \qquad (13)$$

For solutions to (10), the result is independent of the value of p_{t0} and finite. A similar construction, one that is more complicated due to taking square roots of operators, can be performed if one chooses q to act as time. However, to our knowledge, there is no exact way to deparametrize this constraint using t. Here we are specifically interested in the

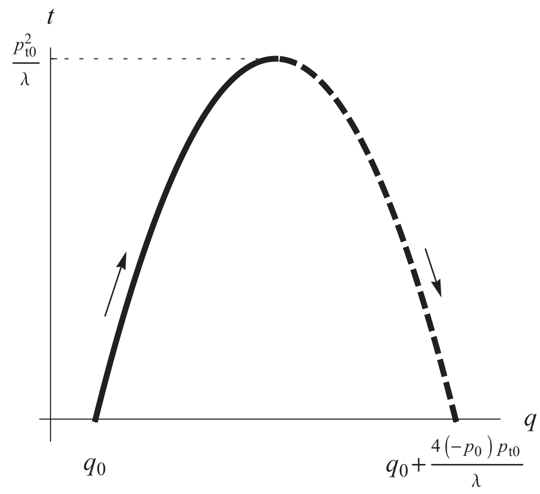

FIG. 1. A typical classical configuration space trajectory is a parabola with the peak value of t dependent on p_{t0} and the separation of branches dependent on p_0. The orientation of evolution, indicated by the arrows, is consistent with $p_0 < 0$ and p_{t0}. We refer to the left branch (solid) as "incoming" or "evolving forward in t," the right branch (dashed) as "outgoing" or "evolving backward in t."

situations where there is no obvious time variable available to perform deparametrization. For that purpose, in this toy model we choose a time variable which we know to be bad in a particular way and construct an effective initial-value formulation with respect to that variable.

Specifically, we would like to evolve initial data given at a fixed value of t on the incoming branch onto the outgoing branch (see Fig. 1). In order to do that, one inevitably has to find a way to evolve data through the extremum of t. Such an evolution can be easily performed in the classical limit and, therefore, should also be well posed at least semiclassically.

C. Effective treatment

Following the procedure outlined in Sec. II, we write the constraint functions $C_{\rm pol} = 0$ in terms of moments and truncate the system by discarding terms of order $\hbar^{(3/2)}$ and higher in the semiclassical approximation. As for the "relativistic harmonic oscillator," we have 14 kinematical degrees of freedom to this order, subject to the five effective constraints

$$
\begin{aligned}
C &= p_t^2 - p^2 - m^2 + (\Delta p_t)^2 - (\Delta p)^2 + \lambda t = 0\\
C_t &= 2p_t\Delta(tp_t) + i\hbar p_t - 2p\Delta(tp) + \lambda(\Delta t)^2 = 0\\
C_{p_t} &= 2p_t(\Delta p_t)^2 - 2p\Delta(p_t p) + \lambda\Delta(tp_t) - \frac{1}{2}i\lambda\hbar = 0 \qquad (14)\\
C_q &= 2p_t\Delta(p_t q) - 2p\Delta(qp) - i\hbar p + \lambda\Delta(qt) = 0\\
C_p &= 2p_t\Delta(p_t p) - 2p(\Delta p)^2 + \lambda\Delta(tp) = 0\ .
\end{aligned}
$$

The five effective constraints generate only four linearly independent flows due to a degenerate Poisson structure to order $\hbar$. Consequently, the 14-dimensional Poisson manifold may be reduced to a five-dimensional surface describing the five physical degrees of freedom to semiclassical order. Note that both p and, as a result of (2), $(\Delta p)^2$ commute with all five constraints and are, therefore, two obvious constants of motion of this effective system. We want to find the remaining three physical degrees of freedom as relational Dirac observables.

1. Evolution in complex t and breakdown of the corresponding gauge

Choosing t as our clock function, it is helpful to fix three out of the four independent gauge flows in order to facilitate explicit calculations and avoid keeping track of three further order $\hbar$ clocks.[7] The system, certainly, does not single out a particular gauge for us; nevertheless, with our choice of clock we can motivate certain gauges. Once a choice of time has been implemented, the clock function should not correspond to an operator and, hence, should not appear in evolving moments; it should be "as classical as possible," implying that the gauge conditions

$$\phi_1 = (\Delta t)^2 = 0 \quad \phi_2 = \Delta(tq) = 0 \quad \phi_3 = \Delta(tp) = 0 \quad (15)$$

seem reasonable. We will refer to these conditions as the t *gauge* or the Zeitgeist associated to t. At the state level, this would be closest in spirit to an inner product evaluated on $t =$ const slices in some kinematical representation. Since t is not a global time, this would lead to an apparent nonunitarity in the quantum theory, which by analogy suggests that this gauge should not be globally valid, simply because t is not a global clock. We will come back to this issue below.

Imposing the gauge conditions renders the combined system of (14) and (15) a mixture of first and second class constraints. Since there were originally four independent gauge flows, we expect at least one first-class constraint among the eight conditions given by (14) and (15). One additional independent first-class constraint may arise, but this constraint must generate a vanishing flow on the variables which we choose after solving the constraints and gauge conditions. It is easily verified that the first-class constraint with the vanishing flow on the variables q, p, t, p_t, $(\Delta q)^2$, $(\Delta p)^2$, $\Delta(qp)$ must be directly proportional to C_t in this gauge. Solving this constraint

$$C_t \approx 2p_t\Delta(tp_t) + i\hbar p_t = 0 \Rightarrow \Delta(tp_t) = -\frac{i\hbar}{2}, \qquad (16)$$

implies a saturation of the (generalized) uncertainty relation for t and p_t in this system. Here and throughout the rest of the present work $\approx$ denotes equality restricted to the region where both constraint functions and the gauge conditions of the relevant Zeitgeist are satisfied.

[7] Note that this gauge fixing occurs after quantization.

TABLE I. Poisson algebra of gauge conditions (15) with the constraints (14). First terms in the bracket are labeled by rows, second terms are labeled by columns. Note that these results only hold on the gauge surface defined in (15).

	ϕ_1	ϕ_2	ϕ_3
C	$2i\hbar$	$-2\Delta(qp_t)$	$-2\Delta(p_t p)$
C_{p_t}	$4i\hbar p_t$	$-2p_t\Delta(qp_t) - 2i\hbar p$	$-2p_t\Delta(p_t p)$
C_q	0	$-2p_t(\Delta q)^2$	$-2p_t\Delta(qp) - i\hbar p_t$
C_p	0	$i\hbar p_t - 2p_t\Delta(qp)$	$-2p_t(\Delta p)^2$

The remaining first-class constraint with nonvanishing flow on the chosen variables will generate our relational evolution in t; therefore, we refer to it as the "Hamiltonian constraint" in the t gauge. It has the form $C_H \propto C_e V^e$, where V^e is the solution to $\{\phi_i, C_e\}V^e = 0$ and $i = 1, 2, 3$ and the C_e denote the constraints of (14), except C_t. The matrix $\{\phi_i, C_e\}$ is generically of rank 3 from which we infer that there is only one independent C_H. The coefficients of this matrix are given in Table I, and, up to an overall factor, we find

$$C_H = C + \alpha C_{p_t} + \beta C_q + \gamma C_p, \tag{17}$$

where, on the constraint surface, the coefficients read

$$\alpha = -\frac{1}{2p_t}, \qquad \beta = 0 \quad \text{and} \quad \gamma = -\frac{p}{2p_t^2}. \tag{18}$$

Four nonphysical moments in this gauge may be solved for via C_t, C_{p_t}, C_q, and C_p. Equation (16) gives $\Delta(tp_t)$, the rest are given by

$$(\Delta p_t)^2 = \frac{2p^2(\Delta p)^2 + i\hbar\lambda p_t}{2p_t^2},$$
$$\Delta(p_t p) = \frac{p(\Delta p)^2}{p_t} \text{ and } \Delta(qp_t) = \frac{i\hbar p + 2p\Delta(qp)}{2p_t}. \tag{19}$$

When these relations are used together with the t gauge conditions (15), the equations of motion generated by C_H on the remaining variables read (recall that p and $(\Delta p)^2$ are constants of motion)

$$\begin{aligned}
\dot{t} &= \{t, C_H\} = 2p_t - \frac{2p^2(\Delta p)^2}{p_t^3} - \frac{i\hbar\lambda}{2p_t^2},\\
\dot{p}_t &= \{p_t, C_H\} = -\lambda,\\
\dot{q} &= \{q, C_H\} = -2p\left(1 - \frac{(\Delta p)^2}{p_t^2}\right),\\
(\dot{\Delta q})^2 &= \{(\Delta q)^2, C_H\} = -4\Delta(qp)\left(1 - \frac{p^2}{p_t^2}\right),\\
\dot{\Delta}(qp) &= \{\Delta(qp), C_H\} = -2(\Delta p)^2\left(1 - \frac{p^2}{p_t^2}\right).
\end{aligned} \tag{20}$$

These can be solved analytically by

$$\begin{aligned}
t(s) &= -\frac{p_t(s)^2}{\lambda} - \frac{p^2(\Delta p)^2}{\lambda p_t(s)^2} - \frac{i\hbar}{2p_t(s)} + c,\\
p_t(s) &= -\lambda s + p_{t0},\\
q(s) &= 2\frac{pp_t(s)}{\lambda}\left(1 + \frac{(\Delta p)^2}{p_t(s)^2}\right) + c_1,\\
(\Delta q)^2(s) &= 4(\Delta p)^2\frac{(p^2 + p_t(s)^2)^2}{\lambda^2 p_t(s)^2} + \frac{4(p^2 + p_t(s)^2)}{\lambda p_t(s)}c_2 + c_3,\\
\Delta(qp)(s) &= 2(\Delta p)^2\frac{p^2 + p_t(s)^2}{\lambda p_t(s)} + c_2,
\end{aligned} \tag{21}$$

where c, p_{t0}, and $\{c_i\}_{i=1,2,3}$ are integration constants related to the initial conditions. (These solutions, expressed via p_t, provide relational observables of the system. A comparison with (9) shows that the classical observables receive quantum corrections via the moments.) In particular, we note that to this order p_t experiences no quantum backreaction and evolves entirely classically, which is due to the fact that the only constraint function that has nontrivial bracket with p_t is C.

Neither p_t, nor t is a Dirac observable and one of them can be eliminated by using C. Combining relations (19) and the gauge conditions (15) with $C = 0$, we obtain

$$0 = p_t^4 - (p^2 + m^2 - \lambda t + (\Delta p)^2)p_t^2 + \frac{i\hbar\lambda}{2}p_t + p^2(\Delta p)^2 . \tag{22}$$

It is not difficult to see that, if we want to keep the variables q, p, $(\Delta q)^2$, $(\Delta p)^2$, $\Delta(qp)$ real (see Sec. III C 4), the above relation necessarily forces either t or p_t to be complex. When we look at the equations of motion (20) and their solutions (21), the choice is almost obvious. The equation of motion for p_t has no imaginary component and hence equipping it with a constant imaginary part appears somewhat artificial. More importantly, p_t features prominently in the solutions for q, p, $(\Delta q)^2$, $(\Delta p)^2$, $\Delta(qp)$, in order to keep all these real, we are forced to keep p_t real and, consequently, t must be complex-valued.

Let us quantify the imaginary contribution to t. We determine c by substituting both $p_t(s)$ and $t(s)$ from (21) into the constraint (22) which yields the real-valued result

$$c = \frac{p^2 + m^2 + (\Delta p)^2}{\lambda} . \tag{23}$$

The imaginary contribution to the clock t is, therefore, a quantum effect of order $\hbar$ and given by

$$\Im[t(s)] = -\frac{\hbar}{2p_t(s)} . \tag{24}$$

A more thorough analysis of the complex nature of the effective nonglobal clocks will be explored in Sec. IV and its general features have been discussed in [17].

We have previously stated that the gauge defined by the conditions (15) is related to choosing t as time. However,

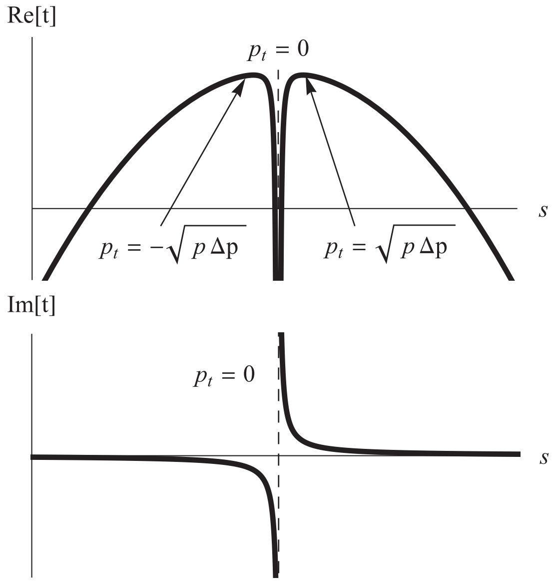

FIG. 2. Schematic plots of the real part of t (top) and the imaginary part of t (bottom) against the flow parameter s.

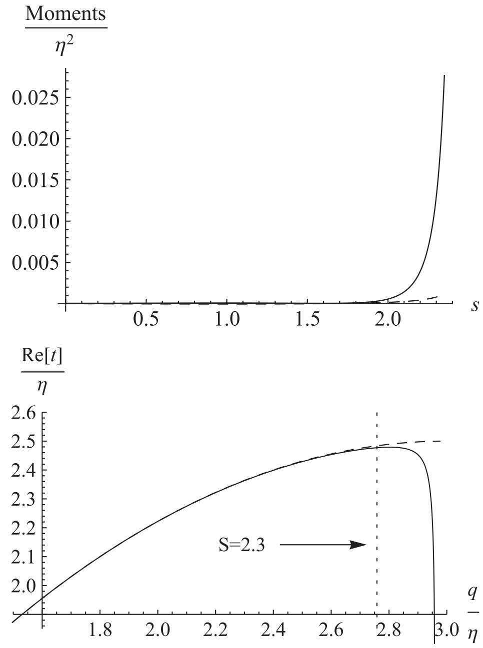

FIG. 3. Top: evolution of moments $(\Delta q)^2$ (solid) and $\Delta(qp)$ (dashed) in the t gauge ($(\Delta p)^2 = \text{const}$). Somewhere after $s = 2.3$ the spread $\Delta q := \sqrt{(\Delta q)^2}$ becomes comparable to the expectation values, as $\Delta q/\eta > .1$, and the semiclassical approximation breaks down in the t gauge. Bottom: corresponding effective trajectory (solid) and the related classical trajectory (dashed); the effective trajectory quickly diverges after $s = 2.3$.

the equations of motion, as well as their solutions are written in terms of the gauge parameter s that parametrizes the flow generated by C_H. Since t is a complex variable we can relate s to its real and imaginary parts separately. In Fig. 2, we plot the real and imaginary parts of $t(s)$, deduced directly from (21) and (23).

From the plot we see that away from $p_t = 0$, $\Re[t]$ is monotonic in s on each of the two branches and, asymptotically far away from $p_t = 0$, they become proportional. On the forward moving branch, $\Re[t]$ is increasing with s; on the backwards moving branch $\Re[t]$ is decreasing with s. From the plot we can also see that $\Re[t]$ reaches its peak value at $p_t = \pm\sqrt{p\Delta p} \neq 0$. However, at this point we can no longer trust the semiclassical approximation as the small value of p_t in the denominators in the equations of motion (21) will result in values of the moments that no longer satisfy the assumed drop-off.

Figure 2 also shows that $\Im[t]$ is monotonic in s in the same regimes. Thus, when it comes to parametrizing dynamics using t, we have the option of using either $\Im[t]$ or $\Re[t]$. We opt to refer to the real part of t as "time," for several reasons: (1) in the classical limit the imaginary part vanishes and it is, indeed, the real part of t that matches the classical internal time; (2) for large p_t or small λ when the time-dependent term in the constraint becomes insignificant, the imaginary part of t is small and approximately constant; (3) finally, as we will see later, the expectation value that reproduces $\Im[t]$ in the case of a free relativistic particle is based on integrating at a fixed value of (parameter) t equal to precisely the real part of the expectation value.

As one would expect from the classical behavior of t, this gauge is not valid for the whole "quantum trajectory." In particular, we noted that p_t evolves entirely classically, so that its solution is simply given by (8). As a result p_t passes through zero for a finite value of the evolution parameter s, which immediately implies the breakdown of the t gauge: the coefficients in (18) and in (21) become singular, the magnitudes of the moments $(\Delta q)^2$ and $\Delta(qp)$ blow up, thereby violating semiclassicality. An example of this divergence is shown in Fig. 3. Here $\eta := \sqrt{p^2 + m^2}$ provides us with a classical length-scale on the phase space, and the quantum length-scale is set to $\sqrt{\hbar} = .01\eta$. Classical quantities such as p, m, λ are all of order η, while the values of second order moments are initially of order $\hbar$. Qualitative features of the plot are insensitive to the precise values chosen so long as the relative scales are preserved.

Because of the nonglobal nature of the relational clock t, this breakdown does not come unexpected. In order to evolve a semiclassical state through the turning point of the clock, we, therefore, need to switch the gauge and—unlike in the classical case—the clock (see also Sec. IV C on this issue). A more complete discussion of the breakdown of the gauge and its counterpart on the exact side of the quantum theory will be discussed in the second model in Sec. V, while the transformation to the q *gauge* and the evolution through the turning point will be discussed in Secs. III C 2 and III C 3 below.

2. Evolution through the extremal point of $\Re[t]$ in a new gauge

Based on the evidence that the t gauge (15) fails globally due to the fact that t is a nonglobal time function, we can, instead, make use of the fact that, e.g., q *is* a good clock variable for the entire trajectory. For the evolution through the t-turning point we could, therefore, simply choose the following q *gauge* ("as if we chose q as time")

$$\tilde{\phi}_1=(\Delta q)^2=0 \quad \tilde{\phi}_2=\Delta(tq)=0 \quad \tilde{\phi}_3=\Delta(qp_t)=0\ . \tag{25}$$

This gauge is closest in spirit to choosing a $q=\text{const}$ slicing in an analogous treatment of the model at the Hilbert space level and since q is a good clock, in this gauge we expect to be able to evolve through the extremum in $\Re[t]$ without difficulty. Such a procedure of adapting the gauge to a good local clock should work in general even if no global clock functions exist, since generically we expect the existence of some degree of freedom which may serve as a good local clock where other clock degrees of freedom fail. To evolve through the whole trajectory one would in general need to switch gauges, which we discuss in Sec. III C 3 below.

We immediately notice that this gauge is inconsistent with treating the moments of $\hat{p}$ and $\hat{q}$ as independent phase-space degrees of freedom, since several of them are completely fixed by the gauge conditions. We, therefore, interpret q as a clock in this gauge (see also Sec. IV C on this issue) and eliminate the remaining moments of $\hat{p}$ and $\hat{q}$ through constraints leaving the free variables t, p_t, q, p, $(\Delta t)^2$, $(\Delta p_t)^2$, $\Delta(tp_t)$. The first-class constraint with vanishing flow on these variables is now given by C_q. Solving this constraint then implies $\Delta(qp)=-\frac{i\hbar}{2}$ and, together with (25), the saturation of the uncertainty relation between $\hat{q}$ and $\hat{p}$. The "Hamiltonian constraint" of the q *gauge* reads

$$\tilde{C}_H=C+\tilde{\alpha}C_t+\tilde{\beta}C_{p_t}+\tilde{\gamma}C_p, \tag{26}$$

where the coefficients are given on the constraint surface by

$$\tilde{\alpha}=-\frac{\lambda}{4p^2},\qquad \tilde{\beta}=-\frac{p_t}{2p^2}\quad\text{and}\quad \tilde{\gamma}=-\frac{1}{2p}. \tag{27}$$

These coefficients are clearly well behaved along the entire trajectory, as long as the constant of motion $p\neq 0$. In addition to $\Delta(qp)$, we eliminate the three remaining unphysical moments through constraints

$$\begin{aligned}(\Delta p)^2&=\frac{p_t^2}{p^2}(\Delta p_t)^2+\frac{\lambda p_t}{p^2}\Delta(tp_t)+\frac{\lambda^2}{4p^2}(\Delta t)^2,\\ \Delta(p_tp)&=\frac{p_t}{p}(\Delta p_t)^2+\frac{\lambda}{2p}\left(\Delta(tp_t)-\frac{i\hbar}{2}\right),\\ \Delta(tp)&=\frac{p_t}{p}\left(\Delta(tp_t)+\frac{i\hbar}{2}\right)+\frac{\lambda}{2p}(\Delta t)^2.\end{aligned} \tag{28}$$

The dynamical equations generated by this Hamiltonian constraint on the q gauge surface are

$$\begin{aligned}\dot{t}&=2p_t-\frac{2p_t(\Delta p_t)^2+\lambda\Delta(tp_t)}{p^2},\qquad \dot{p}_t=-\lambda,\\ \dot{q}&=-2p+\frac{\lambda^2(\Delta t)^2+4p_t^2(\Delta p_t)^2+4\lambda p_t\Delta(tp_t)}{2p^3},\\ (\dot{\Delta} t)^2&=\frac{4(p^2-p_t^2)\Delta(tp_t)-2\lambda p_t(\Delta t)^2}{p^2}\\ \dot{\Delta}(tp_t)&=\frac{4(p^2-p_t^2)(\Delta p_t)^2+\lambda^2(\Delta t)^2}{2p^2},\\ (\dot{\Delta} p_t)^2&=\frac{2\lambda p_t(\Delta p_t)^2+\lambda^2\Delta(tp_t)}{p^2}.\end{aligned} \tag{29}$$

As in the t gauge before, p_t evolves classically $p_t(\tilde{s})=-\lambda\tilde{s}+p_{t_0}$. The moments evolve according to

$$\begin{aligned}(\Delta t)^2(\tilde{s})&=\frac{p_t(\tilde{s})^2}{p^2}\tilde{c}_1+\frac{4(p_t(\tilde{s})^2+p^2)^2}{\lambda^2p^2}\tilde{c}_2\\ &\quad+\frac{4p_t(\tilde{s})(p_t(\tilde{s})^2+p^2)}{\lambda p^2}\tilde{c}_3,\\ (\Delta p_t)^2(\tilde{s})&=\frac{p_t(\tilde{s})^2}{p^2}\tilde{c}_2+\frac{\lambda p_t(\tilde{s})}{p^2}\tilde{c}_3+\frac{\lambda^2}{p^2}\tilde{c}_1,\\ \Delta(tp_t)(\tilde{s})&=-\frac{2p_t(\tilde{s})^2+p^2}{p^2}\tilde{c}_3-\frac{2p_t(\tilde{s})(p_t(\tilde{s})^2+p^2)}{\lambda p^2}\tilde{c}_2\\ &\quad-\frac{\lambda p_t(\tilde{s})}{p^2}\tilde{c}_1.\end{aligned} \tag{30}$$

The above solutions can be substituted into the equations of motion for $q(\tilde{s})$ and $t(\tilde{s})$, which can then be integrated separately.

Once again, we can eliminate yet another variable. By using $C=0$ combined with (28), we obtain an equation for p,

$$\begin{aligned}&p^4-(p_t^2-m^2+(\Delta p_t)^2+\lambda t)p^2+p_t^2(\Delta p_t)^2\\ &\quad+\lambda p_t\Delta(tp_t)+\frac{\lambda^2}{4}(\Delta t)^2=0.\end{aligned} \tag{31}$$

We see that there is no need to make either p or q complex to satisfy this equation. Nor are there any explicitly imaginary terms in the equations of motion or their solutions. Nevertheless, in order to consistently switch between the t gauge and the q gauge, we will require q to carry an imaginary contribution in this gauge analogous to (24)

$$\Im[q(\tilde{s})]=-\frac{\hbar}{2p}\quad, \tag{32}$$

which in this case is constant, since p is a constant of motion.

Finally, we note that—as expected—the evolution in this gauge encounters no difficulty near the extremal point of t when $p_t = 0$. The coefficients in (27) stay finite and we can see from (30) that the moments of $\hat{p}_t$ and $\hat{t}$ remain well behaved as we go through $p_t = 0$. In the next section we describe a method for switching between the two gauges.

3. Switching gauges

The two gauges discussed in Secs. III C 1 and III C 2 describe evolution of two different sets of degrees of freedom. If we switch from one gauge to another, for example, to evolve through the turning point of a time function, we need to be able to translate between the two sets of variables. We recall that the original gauge orbit for the truncated system of constraints (14) is, in general, four-dimensional. The three gauge-fixing equations of either (15) or (25) restrict us to a one-dimensional flow on this gauge orbit generated by the remaining first-class constraint (17) or (26), respectively. In order to ensure that the two sets of variables lie on the same four-dimensional gauge orbit we need to find a gauge transformation which takes us from the surface defined by (15) to the one defined by (25) and vice versa.

In other words, to transform from the t gauge to the q gauge we need to find a combination of the constraint functions $G = \sum_i \xi_i C_i$, such that a (possibly finite) integral of its flow transforms the variables as

$$\begin{cases} (\Delta q)^2 = (\Delta q)_0^2 \\ \Delta(tq) = 0 \\ \Delta(p_t q) = \Delta(p_t q)_0 \end{cases} \rightarrow \begin{cases} (\Delta q)^2 = 0 \\ \Delta(tq) = 0 \\ \Delta(p_t q) = 0 \end{cases}, \qquad (33)$$

where the subscript $_0$ labels the value of the corresponding variable prior to the gauge transformation. In general, one would expect such a transformation to be unique up to the flows generated by C_H and $\tilde{C}_H$, since they preserve the corresponding sets of gauge conditions (see Sec. IV D for additional discussion). To get a unique answer, and to make the transformation induced on the expectation values small, we fix the multiplicative coefficient of C in G to zero.

For convenience, we only present and work with the flows generated by the constraint functions rather than displaying the generators themselves whose explicit expressions turn out to be rather complicated and less well behaved than their flows. The flow generated by a generator G will be denoted by $\alpha_G^s(x)$, $x \in \mathcal{C}$, where $\mathcal{C}$ denotes the constraint surface and s is the gauge parameter along the flow. Its (finite) action on a quantum phase-space function f can be computed via a derivative expansion

$$\alpha_G^s(f)(x) := f(\alpha_G^s(x)) = \sum_{n=0}^{\infty} \frac{s^n}{n!}\{f, G\}_n(x), \qquad (34)$$

where $\{f, G\}_n := \{\{f, G\}_{n-1}, G\}$ and $\{f, G\}_0 = f$. The Hamiltonian vector field of the generator G is denoted by X_G and we have $X_G(f) := \{f, G\}$. The required flows for the transformation may be computed explicitly with the aid of the table in Appendix A. There is still some freedom in choosing a path for the gauge transformation: as mentioned at the beginning of Sec. III C, the five constraints generate only four independent flows. Removing C still leaves us with three independent flows which we can combine. At this point we construct the gauge transformation in two steps. First we search for a flow that satisfies $X_{G_1}(\Delta(qp)) = X_{G_1}(\Delta(tq)) = 0$ on the constraint surface and rescale the flow such that $X_{G_1}((\Delta q)^2) = 1$. The second step involves finding the flow that satisfies $X_{G_2}((\Delta q)^2) = X_{G_2}(\Delta(tq)) = 0$ and rescaling this flow such that $X_{G_2}(\Delta(qp)) = 1$. The required gauge transformation will then be given by the flow[8] $\alpha_G^s(f)(x) := \alpha_{G_2}^{-(\Delta(qp)_0 + i\hbar/2)} \circ \alpha_{G_1}^{-(\Delta q)_0^2}(f)(x)$ if we can argue that the second and higher derivative terms in the respective expansion via (34) can be consistently neglected to order $\hbar$. Equation (34) implies that to linear order in the derivative expansion we also have $\alpha_{G_2}^u \circ \alpha_{G_1}^v = \alpha_{G_1}^v \circ \alpha_{G_2}^u$ for fixed values of u, v. Note that this composition of the G_1 and G_2 flows only determines α_G^s up to rescalings of G and, consequently, the value of s where the new q gauge is reached, but any such α_G^s will be suitable.

For the particular system at hand, the procedure simplifies if we impose, in addition to the constraint functions, the gauge condition $\Delta(tq) = 0$, which is shared by both the t gauge and the q gauge and is preserved by α_{G_1} and α_{G_2} by construction; we then find for the other variables

$$X_{G_1}(t) = \frac{\lambda}{4p^2}, \quad X_{G_2}(t) = -\frac{1}{p_t}, \quad X_{G_1}(q) = 0,$$
$$X_{G_2}(q) = \frac{1}{p}, \quad X_{G_1}((\Delta t)^2) = -\frac{p_t^2}{p^2}, \quad X_{G_2}((\Delta t)^2) = 0,$$
$$X_{G_1}((\Delta p_t)^2) = -\frac{\lambda^2}{4p^2}, \quad X_{G_2}((\Delta p_t)^2) = \frac{\lambda^2}{p_t},$$
$$X_{G_1}(\Delta(tp_t)) = \frac{\lambda p_t}{2p^2}, \quad X_{G_2}(\Delta(tp_t)) = -1.$$

Noting that p has a vanishing bracket with all constraints and p_t with all constraints except for C, whose flow is neither contained in α_{G_1} nor in α_{G_2}, we see that all of the derivatives are constant, and thus the gauge transformation is infinitesimal and, indeed, simply given by the terms up to linear order in the derivative expansion (34) of $\alpha_G^s(f)(x) := \alpha_{G_2}^{-(\Delta(qp)_0 + i\hbar/2)} \circ \alpha_{G_1}^{-(\Delta q)_0^2}(f)(x)$. Without this simplification,

[8] This expression might appear surprising at a first glance since gauge parameters are real valued. However, the flow of G_2 can be understood via $\alpha_{G_2}^{-(\Delta(qp)_0 + i\hbar/2)} = \alpha_{G_2}^{-\Delta(qp)_0} \circ \alpha_{iG_2}^{-\hbar/2}$ which directly follows from (34).

one may, in general, have to integrate the flows numerically.[9] The initial value for $(\Delta t)^2$ is zero as we are starting with the t gauge, initial values of $\Delta(tp_t)$ and $(\Delta p_t)^2$ can be deduced from (16) and (19), respectively. We find the complete transformation of t gauge variables into the q gauge variables to order $\hbar$ given by

$$\begin{aligned} t &= t_0 + \frac{i\hbar + 2\Delta(qp)_0}{2p_t} - \frac{(\Delta q)_0^2\lambda}{4p^2} \\ q &= q_0 - \frac{i\hbar + 2\Delta(qp)_0}{2p} \qquad (\Delta t)^2 = (\Delta q)_0^2\frac{p_t^2}{p^2} \\ (\Delta p_t)^2 &= \frac{p^2(\Delta p)_0^2 - \Delta(qp)_0\lambda p_t}{p_t^2} + \frac{\lambda^2}{4p^2}(\Delta q)_0^2 \\ \Delta(tp_t) &= \Delta(qp)_0 - \lambda\frac{p_t}{2p^2}(\Delta q)_0^2. \end{aligned} \tag{35}$$

No gauge transformations for p_t and p are listed since these variables are invariant along the flow of G. The reverse transformation can be obtained in an identical manner, or simply by inverting (35)

$$\begin{aligned} t &= t_0 - \frac{2p_t(i\hbar + 2\Delta(tp_t)_0) + (\Delta t)_0^2\lambda}{4p_t^2} \\ q &= q_0 + \frac{p_t(i\hbar + 2\Delta(tp_t)_0) + (\Delta t)_0^2\lambda}{2pp_t} \\ (\Delta q)^2 &= (\Delta t)_0^2\frac{p^2}{p_t^2} \\ (\Delta p)^2 &= \frac{4p_t^2(\Delta p_t)_0^2 + 4\lambda p_t\Delta(tp_t)_0 + \lambda^2(\Delta t)_0^2}{4p^2} \\ \Delta(qp) &= \frac{\lambda}{2p_t}(\Delta t)_0^2 + \Delta(tp_t)_0. \end{aligned} \tag{36}$$

In particular, both q and t acquire imaginary contributions during these transformations. We point out that these contributions exactly cancel out the imaginary terms (24) and (32), so that upon transformation from the t gauge to the q gauge t becomes real and q acquires the imaginary term (32) and vice versa. Observe that in the case of the global clock function q in the q gauge, its imaginary part is a constant of motion and, therefore, does not play any role for evolution, while in the case of the nonglobal clock t in the t gauge, its imaginary part is actually dynamical. We return to this characteristic in Sec. IV B. For more discussion of gauge switching and an argument for the irrelevance of the precise instant of the gauge change see Sec. IV C and IV D.

Figure 4 gives a segment of a semiclassical trajectory that has been evolved through the extremal point of t by temporarily switching to the q gauge. The initial conditions and the values of parameters used here are identical to the ones used to generate Fig. 3. We switch to the q gauge before the moments have a chance to become large (at $s = 1.8$). The evolution in the q gauge stays semiclassical through the turning point in t and sufficiently far away from the extremum ($\tilde{s}$ evolved from 0 to 1.4); the reverse gauge transformation yields a semiclassical outgoing state in the t gauge. Incoming and outgoing trajectories in the t gauge were continued into the region where the q gauge was used in order to demonstrate their divergence. We note that, although the quantities $q(\Re[t])$ in the t gauge and $t(\Re[q])$ in the q gauge refer to different pairs of objects (two examples of fashionables in the terminology of [17]) from the point of view of quantum mechanics, their classical limits correspond to the same correlations between q and t and plotting one trajectory as following the other (with jumps of $o(\hbar)$ between the trajectories as a consequence of the gauge changes above) makes sense for a semiclassical state. The resulting composite trajectory agrees extremely well with its classical counterpart, which is why the latter is not present in the plot.

4. Effective positivity conditions and physical states

In the discussion of dynamics in the t gauge, we implicitly interpreted the variables $q(s)$, $p(s)$, $(\Delta q)^2(s)$, $\Delta(qp)(s)$, $(\Delta p)^2(s)$ as expectation values and moments of a canonical pair of evolving operators, with t keeping track of the "flow of (internal) time." In order to make this interpretation consistent, these variables must have the correct Poisson algebra, which follows directly from the canonical

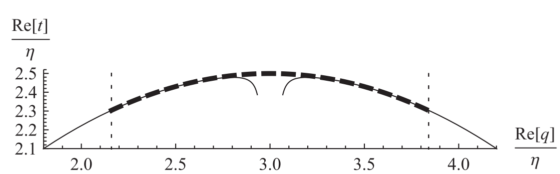

FIG. 4. Plot of the semiclassical trajectory evolved past the extremal point in the t gauge (solid part of the trajectory), by temporarily switching to the q gauge (dashed part of the trajectory). Dotted vertical lines indicate the points where gauges were switched.

[9] In general, the Poisson structure of the quantum phase space is such that the Poisson bracket of the $o(\hbar)$-quantum constraint functions with a quantum phase space function of a certain order preserves or increases the order in $\hbar$, while, for instance, Poisson brackets of ratios of moments can actually decrease the order in $\hbar$. This follows from the Poisson algebra of moments in Appendix A. Now the rescaling of the flow such that, e.g., $X_{G_1}((\Delta q)^2) = 1$ has the consequence that G_1 will be of order $\hbar^0$, consisting of ratios of moments which, in general, may lead to negative orders of $\hbar$ when taking higher derivatives of moments along the flow. It is then not consistent anymore to neglect the higher derivative terms in the expansion (34) of the flow action even if one multiplies with $o(\hbar)$ values of the flow parameter. In such situations one must numerically integrate the flow. However, in general, we expect the gauge transformation between the t and the q gauge to be infinitesimal to order $\hbar$.

commutation relation. The nontrivial brackets of this algebra are

$$\{q,p\}=1,\quad \{(\Delta q)^2,(\Delta p)^2\}=4\Delta(qp)$$
$$\{(\Delta q)^2,\Delta(qp)\}=2(\Delta q)^2,\quad \{\Delta(qp),(\Delta p)^2\}=2(\Delta p)^2. \quad (37)$$

In particular, t must have a vanishing bracket with the rest of the above variables. These relations are, of course, satisfied kinematically simply by construction. However, when we introduce gauge conditions the Poisson bracket on the gauge surface is defined with the use of the Dirac bracket [23]. It is an important feature of the gauge conditions (15) that the Dirac brackets between precisely the free variables in the t gauge are the same as their kinematical counterparts. For the details we refer the interested reader to [19].

The above result ensures that the dynamics is consistent with that of a pair of operators subject to the canonical commutation relation. However, if we are to interpret these operators as self-adjoint (which is required for well-behaved observables), we have to impose additional conditions on their expectation values and moments:

$$q,p,(\Delta q)^2,(\Delta p)^2,\Delta(qp)\in\mathbb{R}\quad (\Delta p)^2,(\Delta q)^2\geq 0$$
$$(\Delta q)^2(\Delta p)^2-(\Delta(qp))^2\geq\frac{1}{4}\hbar^2. \quad (38)$$

These conditions, in particular, guarantee similar conditions holding to order $\hbar$ for any polynomial constructed out of symmetrized products of $\hat{q}$ and $\hat{p}$ (see Appendix B). There is, of course nothing that would prevent us from imposing these conditions on the initial values of the variables. However, it is *a priori* not clear whether such conditions will be preserved by the dynamics in either gauge or by the gauge transformations. Below we list the specific results that ensure the consistency of the effective dynamics with the interpretation of the variables we have chosen as observable expectation values and moments. The details of the calculations may be found in Appendix B. We find that

(i) the conditions (38) are preserved by the dynamics of the t gauge,
(ii) the conditions on the expectation values and moments of $\hat{t}$ and $\hat{p}_t$ analogous to (38) are preserved by the dynamics in the q gauge,
(iii) if the variables in the t gauge satisfy (38), then the gauge transformed variables satisfy the q gauge analog of (38).

IV. COMPLEX INTERNAL TIME AND RELATIONAL OBSERVABLES

In this section we reflect on some of the general features of the effective analysis performed on the model of Sec. III. We focus on the interpretation of the imaginary contribution to internal time, transformations between local choices of clocks (Zeitgeist) and the status of relational observables in a system without global time. Complex internal time arising in the effective approach to local clocks and in local deparametrizations at the state level has been discussed in detail in [17], along with general issues related to relational evolution and observables and we refer the interested reader to that work. However, the results concerning complex internal time are worth summarizing in the context of the concrete examples provided within the present manuscript, which we do in Sec. IVA. Considerations of this section are general, and hence equally applicable to the second model studied in Sec. V, for which some of the general discussions of this section will be helpful.

A. Imaginary contribution to internal time

At this moment, it is useful to pause and ask how meaningful an imaginary contribution to time can be. First, we would like to acquire some intuition regarding its origin. From a certain point of view this feature is not entirely surprising—after all, there are old and well-known arguments in quantum mechanics saying that time cannot be a self-adjoint operator. Otherwise, it would be conjugate to an energy operator bounded from below for stable systems. Since a self-adjoint time operator would generate unitary shifts of energy by arbitrary values, a contradiction to the lower bound would be obtained. The result of complex expectation values for local internal times obtained here looks similar at first sight—a non-self-adjoint time operator could, certainly, lead to complex time expectation values—but it is more general. In the model of Sec. III, we are using a linear potential which does not provide a lower bound for energy. The usual arguments about time operators thus do not apply; instead our conclusions are drawn directly from the fact that we are dealing with a time-dependent potential. (For time-independent potentials, $\langle\hat{t}\rangle$ does not appear in the effective constraints and can consistently be chosen real. The time dependence is thus crucial for the present discussion.)

Rather, the imaginary contribution to internal time may be regarded in the same vein as the imaginary contributions to the various unphysical moments [see, e.g., Eq. (16)]—as an artifact of assigning expectation values to all kinematical observables, which typically do not project in any natural way to self-adjoint operators on the physical Hilbert space. We recall a simple example given in [17] of a physical inner product, which in a deparametrizable system assigns a complex "expectation value" to internal time. A free relativistic particle in 1 + 1 Minkowski spacetime,[10] is subject to the constraint

[10]In this example, t has the usual notion of proper time as experienced by inertial observers in addition to the more general notion of internal time as a phase-space degree of freedom of the cotangent bundle of Minkowski space. In this context, as in our other examples, we are interested only in the phase-space notion of internal times.

$$\left(-\hbar^2\frac{\partial^2}{\partial x_0^2}+\hbar^2\frac{\partial^2}{\partial x_1^2}-m^2\right)\psi(x_0,x_1)=0. \quad (39)$$

The standard inner product used for positive frequency solutions has the form

$$(\phi,\psi):=i\hbar\int_{-\infty}^{\infty}\left(\bar{\phi}(x_0,x_1)\frac{\partial}{\partial x_0}\psi(x_0,x_1)-\left(\frac{\partial}{\partial x_0}\bar{\phi}(x_0,x_1)\right)\psi(x_0,x_1)\right)dx_1|_{x_0=t}. \quad (40)$$

Evaluating the "expectation value" of the kinematical internal time operator, using a positive frequency solution with this inner product,[11] yields

$$\langle\hat{t}\rangle=(\phi,x_0\phi)=t-\frac{i\hbar}{2}\left\langle\frac{\hat{1}}{p_t}\right\rangle. \quad (41)$$

To order $\hbar$ the imaginary part is identical to Eq. (24), and, indeed, to the analogous result in Sec. V given in Eq. (85). The key ingredient in this result is the use of both ϕ and $\partial\phi/\partial x_0$ in the construction of the inner product, which is ultimately related to the fact that the constraint equation is second order in the time derivative, so that locally both ϕ and $\partial\phi/\partial x_0$ are independent degrees of freedom. This suggests a generalization of the form of the imaginary contribution to $\langle\hat{t}\rangle$, to all constraints where $\hat{p}_t$ appears quadratically. One may then ask whether the effective procedure supports such a generalization. It was, indeed, demonstrated in [17], that for any constraint of the form

$$\hat{C}=\hat{p}_t^2-\hat{p}^2+V(\hat{q},\hat{t}),$$

the imaginary contribution at order $\hbar$ is precisely the same in the effective framework, $\Im[t]=-\hbar/2\langle\hat{p}_t\rangle$.

One choice was made at the beginning of the effective analysis, namely, the gauge fixing of the effective constraints. We used the gauge fixing that worked well for deparametrizable systems, but it may not be suitable for nondeparametrizable ones. One could then try to change the gauge-fixing conditions and perhaps move the complex valuedness to some of the kinematical moments rather than the internal time expectation value. It is, however, unlikely that this would give a general procedure because the form of the constraints would require gauge-fixing conditions adapted to the system under consideration, and, in particular, to the potential. The gauge-fixing conditions used here, on the other hand, work for arbitrary potentials and are specifically motivated by and associated to our choice of clock and corresponding relational time (see also Sec. IV C).

[11]Strictly speaking, this is clearly not a true expectation value, since the kinematical internal time operator does not preserve the (physical) positive frequency Hilbert space. Nevertheless, we can use this inner product as a well-defined bilinear form in this case.

Finally, there is concrete evidence, that this imaginary contribution is a generic feature associated with local deparametrizations of a Dirac constraint of the form

$$(\hat{p}_t^2-\hat{H}^2(\hat{t},\hat{q},\hat{p}))\psi(q,t)=0, \quad (42)$$

where $\hat{H}^2$ is a positive operator at least on some set of states. For example, such a constraint features in the Wheeler-DeWitt (WDW) equation in homogeneous and isotropic cosmology. In general, Eq. (42) is not equivalent to a Schrödinger equation

$$(-i\hbar\partial_\tau+\hat{H}(\tau,\hat{q},\hat{p}))\psi(q,\tau)=0, \quad (43)$$

since the solutions to the latter satisfy

$$-\hbar^2\partial_\tau^2\psi=\hat{H}^2\psi+i\hbar\partial_\tau\hat{H}\psi. \quad (44)$$

The inequivalence formally appears to be of order $\hbar$ and is based in part on erroneously identifying the kinematical operator $\hat{t}$ of Eq. (42) with the time parameter τ of Eq. (43). In [17] it was shown, however, that Eq. (42) and an internal time version of Eq. (43) are both solved by the same state (in the sense that their expectation values vanish) at order $\hbar$, if one defines

$$\hat{t}=\hat{\tau}-\frac{i\hbar}{2}\hat{p}_\tau^{-1}, \quad (45)$$

(for states outside the zero-eigenspace of $\hat{p}_\tau$) where the (continuous) eigenvalues of the kinematical internal time operator $\hat{\tau}$ assume the role of the parameter τ of the Schrödinger equation. The internal time Schrödinger equation represents a local deparametrization of Eq. (42) and arises from a kinematical quantization of one of the two factors of a classical factorization of the quadratic constraint, $C=(p_\tau-H(\tau,q,p))(p_\tau+H(\tau,q,p))$, where both internal time τ and p_τ are dynamical phase-space variables. The result once again agrees with the general form of the imaginary contribution obtained effectively. This comparison of the quadratic relativistic constraint with a local (internal time) Schrödinger equation at the state level is demonstrated on a concrete example in Sec. V B 2. We also compare the corresponding semiclassical dynamics of local deparametrization to the effective evolution in Sec. V C 1.

B. Dynamics with a complex relational clock

As we saw in the previous section, the expectation value of internal time can acquire an imaginary contribution even in the standard treatments of deparametrizable systems. The difference is only that deparametrizable systems with a global internal time do not force us to include the imaginary part, while systems with local internal times do. This can also be seen from the shape of the generic imaginary contribution $\Im[t]=-\hbar/2\langle\hat{p}_t\rangle$: While in the presence of a "time potential," p_t will fail to be a constant of motion and, consequently, $\Im[t]$ will actually be dynamical, in the absence of a "time potential" in the constraint p_t is automatically a Dirac observable and, therefore, $\Im[t]$

a constant of motion. But a constant imaginary contribution, in contrast to a dynamical one, is not needed in order to avoid a violation of the constraints since it can be interpreted as an integration constant at the effective level and does not even appear in the constraints in the absence of a "time potential." Indeed, the WDW and (the internal time version of the) Schrödinger equation, Eqs. (42) and (43), are automatically equivalent in this case. The imaginary contribution to internal time may, therefore, be disregarded altogether for relational evolution in the absence of a "time potential," but it cannot be neglected otherwise.

We emphasize that a nonglobal clock necessarily implies a "time potential," while a time-dependent potential does not automatically imply a nonglobal clock.[12] The dynamical imaginary contribution is, therefore, more general than a pure consequence of nonunitarity following from nonglobal clocks. Nevertheless, the imaginary contribution becomes more prominent where the momentum of the clock variable becomes small and is, thus, especially relevant near turning points of nonglobal clocks. In fact, the dynamical imaginary contribution, being inversely proportional to the kinetic energy of the clock variable, can be interpreted as a measure for the quality of the relational clock: the higher the clock's momentum, i.e., the further away it is from a turning point where quantum effects restrict its applicability, the smaller the imaginary term and the better behaved the clock. This coincides with the intuition that, the faster the clock, the better its time resolution. The inverse kinetic energy also appears in other discussions of the qualities of clocks. A brief comparison of this and further references may be found in [17].

Facing a dynamical imaginary part, we ought to make sense out of such a "vector time" with two separate degrees of freedom. (Relational) time is commonly understood as a single (scalar) degree of freedom and, in principle, we may choose any (real) phase-space function which is reasonably well behaved. In this light, we appoint the real part of the clock function for relational time, for several reasons: (1) it gives the correct classical internal time in the classical limit; (2) for small "time potentials," or in the absence thereof, the imaginary contribution is approximately, or exactly constant, respectively; (3) the "expectation value," Eq. (41), reproducing the specific imaginary term for the free relativistic particle is based on a constant real parameter time slicing; (4) the Schrödinger regime (obtained from a local deparametrization of the relativistic constraint) which, at least locally, should give a conventional quantum time evolution, is based on a real-valued time, and (5) as we will see in an example in Fig. 8 in Sec. V C 1 below, the dynamical imaginary contribution for nonglobal clocks can fail to be monotonic where the real part serves as a suitable local clock.

[12]For instance, in a relativistic system governed by a constraint $C = p_t^2 - H^2(q, p, t)$, where $H^2 > 0\ \forall\, t$, the clock t will be global.

C. Switching clocks is equivalent to changing gauges

From the point of view of the Poisson manifold of the effective framework no variables or gauges are preferred over others and we could, in principle, choose a q gauge like (25) and still use t as our clock for relational evolution. However, as we will see in the second model in Sec. V, the effective evolution in a given τ gauge is matched by a Schrödinger type state evolution (43) in internal time τ, where the conventional Schrödinger type inner product is defined on constant-τ slicings. This Schrödinger regime analog can, thus, only be meaningfully interpreted as local evolution in τ. Moreover, when nevertheless using, e.g., t as a local clock in the q gauge in Sec. III C 2, one faces the undesirable consequence that moments involving t or p_t become evolving degrees of freedom, while the moments of our actual variables of interest, (q, p), are (at least partially) gauge fixed, essentially leaving only an evolution parameter q. The resulting moments would no longer be associated to a canonical pair, which has an impact on Dirac brackets and unnecessarily complicates the physical relational interpretation of such moments relative to t. Consequently, it is unavoidable to switch the local clock in the effective procedure when choosing a new gauge; the choice of gauge is intimately intertwined with the choice of (internal) time and changing the clock and corresponding time is practically tantamount to changing gauge and Zeitgeist. Accordingly, certain questions about (physical) correlations of variables are best described in certain gauges and in each gauge we evolve a *different* set of relational observables which is associated to the chosen relational clock.

The peculiar circumstance that the set of degrees of freedom that evolve in relational time appears to depend on the gauge has its roots in the fact that, by the choice of Zeitgeist, local relational observables considered here describe the system in partially gauge fixed form. While the physical information computed for the system is, certainly, gauge independent, its presentation in gauge fixed form depends on the gauge chosen. One can illustrate this feature also with the standard notions of partial and complete observables. Complete relational observables (invariant under all gauge flows) can be understood as gauge invariant extensions of gauge restricted quantities [8,10,23]; when restricting a complete observable to certain fixed values of some clock functions (parametrizing the full gauge orbit), it is reduced to a "partial" observable, evaluated on a gauge-fixing surface. In such a gauge not all correlations between the phase-space degrees of freedom are accessible and, hence, not all questions about correlations meaningful. (The choice of clock functions along full gauge orbits, of course, does not constitute gauge fixing.) Evolving partial observables along the (full) gauge orbits results in complete relational observables that clearly depend on the choice of the relational clock functions,[13] just as the gauge-fixing surfaces corresponding to constant values of (some of) the clock functions and the associated partial relational observables do.

In the effective framework as well one could gauge invariantly extend the local relational observables of the different Zeitgeister to complete observables by, apart from the $o(\hbar^0)$ clock t or q, taking three further $o(\hbar)$ clock functions into account to keep track of the remaining three gauge flows on quantum phase space.[14] However, for practical reasons, it is advantageous to gauge fix these three $o(\hbar)$ clocks such that the relational evolution we want to describe in the $o(\hbar^0)$ clock can be expressed and compared to Hilbert-space approaches in the most convenient way. One possibility is by using the mentioned relationship of the effective framework with a (local) deparametrization in an internal time Schrödinger regime. To define a Schrödinger type evolution, one can choose which slicing to employ [where the constant-t slicing is the most convenient one when choosing t as internal time and corresponds to the deparametrization given by (43)]. The choice of the slicing and corresponding inner product determines how the spreads of the states solving the internal time Schrödinger equation are measured. For instance, in standard constant-τ slicing for (43) (corresponding to constant-t slicing and evolution in t in the relativistic system), not all the fluctuations of $\hat{q}$ can vanish and the variable appears to be of quantum nature, while $\hat{\tau}$ is projected to the role of a classical parameter τ since the spreads related to $\hat{\tau}$ will vanish. In constant q slicing the situation is reversed. Note, however, that deparametrizations with respect to different internal time variables will, in general, yield different quantum theories with inequivalent Hilbert spaces.

Alternatively, we could use a tilted slicing that corresponds to neither configuration coordinate. For a concrete example recall the free relativistic particle, which is subject to (39). This constraint equation is Lorentz-invariant and we can construct a physical inner product on its solutions of the same form as (40) but evaluated in a different Lorentz frame on surfaces of constant x'_0, where $x'_\mu = \Lambda_\mu{}^\nu x_\nu$ are the boosted coordinates; the corresponding multiplicative kinematical operators will be denoted by $\hat{x}'_\mu$. Kinematical expectation values and moments of $\hat{t}$ and $\hat{q}$ are linear combinations of the expectation values and moments of $\hat{x}'_\mu$. For instance, by linearity of the expectation values, the correlation $\Delta(tq) = \Lambda^\mu{}_0 \Lambda^\nu{}_1 \Delta(x'_\mu x'_\nu) = \Lambda^1{}_0 \Lambda^1{}_1 (\Delta x'_1)^2$, which is nonzero unless the boost is trivial.

(Here the last equality follows as fluctuations of $\hat{x}'_0$ vanish to order $\hbar$, when evaluated in this inner product.) In this tilted slicing one can construct a local Schrödinger evolution and still use $\langle \hat{t} \rangle$ as internal time, though unfamiliar nonvanishing moments (involving $\hat{t}$) severely complicate the interpretation of $\hat{t}$ and $\hat{q}$ as a relational time reference and an evolving variable, respectively.

On the other hand, the quantum phase space of the effective framework, being representation independent, must contain information about a general class of slicings in a (local) deparametrization. This is the reason why unusual (time) moments such as $\Delta(qt)$ do not necessarily vanish in the effective formalism. The three $o(\hbar)$ clocks do not represent true internal coordinates, but parametrize the slicings and thereby the (in general inequivalent) corresponding Hilbert-space representations. Hence, the three conditions fixing the three $o(\hbar)$ flows will fix the slicing and Hilbert-space representation to which the effective relational evolution will correspond. Certainly, when choosing t as the relational $o(\hbar^0)$ clock, we could choose gauge conditions differing from the t Zeitgeist; however, these would correspond to tilted slicings and are, consequently, less convenient for calculations as well as interpretations. Furthermore, the q Zeitgeist can be interpreted in terms of slicings parallel to the t axis and is, thus, not useful for describing evolution in t.

In the light of the present discussion, one may interpret the evolution generated by the remaining first-class (Hamiltonian) constraint in a given Zeitgeist (e.g., (17) in t Zeitgeist in Sec. III C 1) which preserves this gauge and the effective positivity (see Sec. III C 4) as describing an approximate, locally unitary evolution for semiclassical states in a given (preserved) slicing in a local deparametrization. In addition, the imaginary contribution to internal time is clearly dependent on the chosen Zeitgeist at the effective level and the slicing in a local deparametrization; when employing tilted slicings or gauges differing from the Zeitgeist, the imaginary contribution to the internal clock will take a different form.

In conclusion, certain questions about correlations are best addressed in certain gauges and we are, indeed, evolving different sets of (partial) relational observables in different Zeitgeister. The presence of additional gauge flows and slicings also explains the observation that $\langle \hat{t} \rangle (\langle \hat{q} \rangle)$ and $\langle \hat{q} \rangle (\langle \hat{t} \rangle)$ are not in one-to-one correspondence, while the analogous statement (at least locally) holds in the classical system.

D. The moment of gauge and clock change

Here we argue that the precise instant of the gauge change is irrelevant, as long as the semiclassical approximation is valid before and after the gauge transformation. The instant when to perform the change of the clock then becomes a matter of convenience.

[13]Different choices of clocks parametrizing the full gauge orbits will yield different parameter families of observables, although still describing the correlations on the same gauge orbits (albeit along different flow lines).

[14]In general, global obstructions may prevent the clock functions from globally parametrizing the full gauge orbit.

Let q_1 and q_2 be two configuration variables, which we use as local clocks, and let $\mathcal{C}$ be the constraint surface, $\mathcal{G}_1$ the q_1 gauge surface and $\mathcal{G}_2$ the q_2 gauge surface (in $\mathcal{C}$). Denote by $\alpha^s_{C_{H_1}}(x)$ ($x \in \mathcal{G}_1$) the flow of the "Hamiltonian constraint" in the q_1 gauge (i.e., the $\mathcal{G}_1$-preserving first-class flow) and by $\alpha^u_{C_{H_2}}(y)$ ($y \in \mathcal{G}_2$) the flow of the "Hamiltonian constraint" in the q_2 gauge, where s, u are gauge parameters along the flows. Furthermore, denote by $\alpha^t_G(x)$ the flow of the generator G of some fixed gauge transformation which maps between the q_1 and q_2 gauge for certain values of t and which, for the sake of avoiding ordering ambiguities, we assume to be free of caustics (see Secs. III C 3 and V C 2 for explicit constructions of such transformations in the examples).

For the moment, assume that both $\mathcal{G}_1$ and $\mathcal{G}_2$ provide complete submanifolds of $\mathcal{C}$ and that there are no global obstructions to either the q_1 or the q_2 gauge. Recall that the first-class nature of a constraint algebra with n independent flows ensures that the flows are integrable to an n-dimensional submanifold in $\mathcal{C}$, the gauge orbit $\mathfrak{g}$ [23].

For simplicity, consider a classical constraint $C(q_1, q_2, p_1, p_2)$ on a four-dimensional phase space. Then the quantum phase space to semiclassical order will be 14-dimensional and governed by five quantum constraint functions which generate four independent flows [18,19]. Hence, $\dim\mathcal{C} = 9$ and $\dim\mathfrak{g} = 4$. $\mathcal{G}_1$ and $\mathcal{G}_2$ are each described by three independent conditions, thereby fixing three of the four independent flows in $\mathfrak{g}$. C_{H_1} (C_{H_2}) generates the only independent gauge flow which preserves $\mathcal{G}_1$ ($\mathcal{G}_2$), implying $\dim\mathfrak{g}\cap\mathcal{G}_1 = \dim\mathfrak{g}\cap\mathcal{G}_2 = 1$, where the sets $\mathfrak{g}\cap\mathcal{G}_1$ and $\mathfrak{g}\cap\mathcal{G}_2$ are the curves $\alpha^s_{C_{H_1}}(x)$ ($x \in \mathcal{G}_1$) and $\alpha^u_{C_{H_2}}(y)$ ($y \in \mathcal{G}_2$). Now $\alpha^t_G(x) \in \mathfrak{g}\ \forall\ t$ and $\alpha^{t=t^*}_G(x) \in \mathfrak{g}\cap\mathcal{G}_2$ for some t^* and $x \in \mathcal{G}_1$. This map obviously has an inverse, namely α_{-G}, since the flow lines of a single generator form a congruence in $\mathfrak{g}$, and, thus, no point lies on two different such flow lines. Therefore, points along $\alpha^s_{C_{H_1}}$ are mapped 1-to-1 to points along $\alpha^u_{C_{H_2}}$ via α_G, and we must have

$$\alpha^{t=t_1^*}_G \circ \alpha^s_{C_{H_1}}(x) = \alpha^u_{C_{H_2}} \circ \alpha^{t=t_2^*}_G(x), \tag{46}$$

for some $x \in \mathcal{G}_1$, some $s, u \in \mathbb{R}$ and fixed t_1^*, t_2^* determined via the conditions $\alpha^{t=t_2^*}_G(x) \in \mathcal{G}_2$ and $\alpha^{t=t_1^*}_G \circ \alpha^s_{C_{H_1}}(x) \in \mathcal{G}_2$.

Since the gauge transformation α_G maps the points along the C_{H_1}-generated trajectory in $\mathcal{G}_1$ bijectively to points along the C_{H_2}-generated trajectory in $\mathcal{G}_2$ we always map between the same two trajectories and, therefore, it does not matter when precisely the gauge and the clock are switched.

Locally, this argument also holds in systems without global clocks and which suffer from global obstructions to the q_1 and q_2 gauges, as long as one works in a regime in which the respective gauges are valid before and after the gauge transformation and are consistent with the semiclassical approximation. In this regime, it should also be irrelevant when precisely the gauge and the clock are changed. In Sec. V C 3, we numerically demonstrate this argument and its consistency with the semiclassical approximation in an example.

E. Relational observables as "fashionables"

As can be seen explicitly in the models studied in the present work, relational observables of the type $\langle\hat{q}\rangle(\langle\hat{t}\rangle)$ can be given meaning even if $\langle\hat{t}\rangle$ is not used as an internal time throughout the evolution. This feature is implemented by switching gauges for nonglobal clocks. Such gauge transformations imply shifts of the order $\hbar$ in correlations of expectation values and moments as one changes clocks. This is not surprising; it merely underlines the fact that expectation values of the same kinematical variable taken in different Zeitgeister translate into different relational observables. Semiclassically, however, the differences are only of order $\hbar$.

We see that relational observables appear to be only of local nature[15]: a Zeitgeist comes with its own set of relational observables and since a Zeitgeist is typically only temporary, one is forced to use different relational observables to describe the full evolution. Just as with local coordinates on a manifold, we cover a semiclassical evolution trajectory by patches of local internal times and translate between them. We, therefore, follow [17] and refer to the correlations of the evolving expectation values and moments with the (real part of) the expectation value of a local internal clock in its corresponding Zeitgeist as *fashionables*. An explicit examples of a fashionable is the correlation of $q(s)$ and $\Re[t(s)]$ of Eq. (21) (see Figs. 3 and 4). These quantities are only defined so long as the corresponding Zeitgeist is valid and may subsequently "fall out of fashion" when the Zeitgeist changes. By analogy, we also use the term fashionables to denote the expectation values of operators obtained via local deparametrizations (for example $\langle\hat{q}_2\rangle(q_1)$ and $\langle\hat{p}_2\rangle(q_1)$ of Eq. (71)).

It should be noted that the notion of *fashionables* is, in fact, state dependent, in contrast to usual operator versions of quantum relational Dirac observables. Fashionables are associated to a choice of Zeitgeist and different Zeitgeister are valid for ranges depending on the semiclassical states considered. A fashionable breaks down together with the corresponding Zeitgeist when it is rendered invalid, e.g., at a turning point of the corresponding clock. Fashionables, therefore, reflect the local nature of quantum relational

[15]Relational observables have perhaps been understood as a local concept in the formulations provided before, but so far they have been made sense of in a quantum setting only in the effective framework as developed in [17]. For a discussion of difficulties in the Hilbert-space picture, see the comment by Hájíček cited in [7].

evolution and are somewhat closer to a physical interpretation by being state-dependent. Thereby, they also avoid certain technical and interpretational problems of operator versions of quantum relational observables, such as non-self-adjointness issues in the presence of a purely local time (see also the general discussion concerning fashionables in [17]). In practice, the local nature of observables does not prevent one from computing physically meaningful predictions, as these typically refer to finite ranges of time. Moreover, since data is consistently transferred between local choices of a clock, one can evolve them through the turning point by temporarily switching to a new Zeitgeist and employing the old Zeitgeist before and after the turning point.

Apart from being generally of merely local nature, it appears that the standard concept of relational evolution has only semiclassical meaning and that the standard notion of (locally unitary) relational time evolution breaks down together with complex relational time in a highly quantum state of a system without a global clock. For a discussion of this issue, we again refer the interested reader to [17].

Unlike a conventional Hilbert-space representation, the effective approach in its present form does not by itself rigorously define a quantum theory, but rather provides a tool for evaluating quantum dynamics. In deparametrizable models, a close relationship between these two formulations has been found and discussed [22]. On the other hand, when going beyond deparametrizable systems, the effective method can still be used to evaluate quantum dynamics, while local internal times and fashionables have not been made sense of in the Hilbert-space picture, which indicates that the effective constructions presented here already go somewhat beyond usual formulations of quantum physics. At this stage, we are not entitled to formulate effective dynamics as a true alternative to quantum mechanics because mainly the semiclassical setting has been developed so far. Given the enormous difficulties of dealing with time at the Hilbert-space level of nondeparamterizable systems, some nontruncated form of effective equations may be a more suitable setting and eventually be independent of Hilbert-space constructions.

V. A TIMELESS MODEL: THE TWO-DIMENSIONAL ISOTROPIC HARMONIC OSCILLATOR WITH FIXED TOTAL ENERGY

The previous example in Sec. III was deparametrizable, even though one could locally employ a nonglobal clock which already revealed a number of consequences of the *global time problem*, in particular, for the effective approach. Some of these features were subsequently discussed in more generality in Sec. IV, complementing [17]. Now we explore all this in detail in a truly timeless, nondeparametrizable system comprised of the two-dimensional isotropic harmonic oscillator with prescribed total energy. This toy model, previously discussed by Rovelli in [4,7], leads to closed orbits in the classical phase space and, consequently, does not admit global clocks. The issue of changing clocks/gauges becomes inevitable. In our discussion we will compare the classical, effective and Hilbert-space approaches to this model.

A. Classical discussion

Classically, the model is governed by the constraint

$$C_{\text{class}} = p_1^2 + p_2^2 + q_1^2 + q_2^2 - M \tag{47}$$

with a constant M. The dynamical equations are given by

$$\{q_i, C_{\text{class}}\} = 2p_i \quad \text{and} \quad \{p_i, C_{\text{class}}\} = -2q_i, \tag{48}$$

$(i = 1, 2)$ and straightforwardly solved by

$$q_{1_{\text{cl}}}(s) = \sqrt{A}\sin(2s), \quad q_{2_{\text{cl}}}(s) = \sqrt{M-A}\sin(2s+\phi), \tag{49}$$

$$p_{1_{\text{cl}}}(s) = \sqrt{A}\cos(2s), \quad p_{2_{\text{cl}}}(s) = \sqrt{M-A}\cos(2s+\phi), \tag{50}$$

where s is the parameter along $\alpha^s_{C_{\text{class}}}(x)$ and $0 \le A \le M$, $0 \le \phi \le 2\pi$. The canonical pair of Dirac observables ϕ and A satisfies

$$\begin{aligned} 2A &= M + p_1^2 - p_2^2 + q_1^2 - q_2^2, \\ \tan\phi &= \frac{p_1 q_2 - p_2 q_1}{p_1 p_2 + q_1 q_2}, \end{aligned} \tag{51}$$

and completely coordinatizes the reduced phase space, which is topologically a sphere and, thus, no cotangent bundle [7]. The classical system clearly does not possess any global clock functions; indeed, if we choose one of the q_i as a clock, we see that this function will encounter a sequence of turning points along a classical trajectory. The classical trajectories are ellipses in configuration space, periodic and, therefore closed.

Because of this periodicity of the orbits, states which are related by an integer number of revolutions around such an ellipse are described by identical phase-space information. One could only distinguish these states via the gauge parameter s which, however, is not a physical degree of freedom. In order to distinguish states related by complete numbers of revolutions, one would need an extra phase-space degree of freedom. Furthermore, the group generated by this constraint is $U(1)$ which is compact. The number of revolutions around the ellipse, therefore, has no physical meaning, in spite of the fact that the gauge parameter may run over an infinite interval. We thus identify states related by complete numbers of revolution.

1. Evolving observables

For the quantization of the model it turns out to be advantageous to use the following over-complete set of Dirac observables [7]

$$L_x = \frac{1}{2}(p_1p_2 + q_2q_1), \quad L_y = \frac{1}{2}(p_2q_1 - p_1q_2), \text{ and}$$
$$L_z = \frac{1}{4}(p_1^2 - p_2^2 + q_1^2 - q_2^2), \tag{52}$$

which satisfy the constraint

$$L_x^2 + L_y^2 + L_z^2 = \frac{M^2}{16} \tag{53}$$

and the usual angular momentum (Poisson) brackets. These variables may then be quantized via group quantization. The observable L_y can be interpreted as the angular momentum of the system which also provides the orbits with an orientation.

In spite of the *a priori* timelessness of this model, one can give it a (local) evolutionary interpretation. Given the timeless initial data ϕ and A, the classical solution is completely specified and prediction of relational information is possible. Choose a local clock, say q_1, and evolve the other variables of interest, in this case q_2 and p_2, with respect to τ, where τ are the possible values of q_1. The relational Dirac observables corresponding to this evolution are, obviously, double valued, since the orbit is closed and are given by

$$\begin{aligned} q_2^{\pm}(\tau) &= \sqrt{M/A - 1}(\tau\cos\phi \pm \sqrt{A - \tau^2}\sin\phi), \\ p_2^{\pm}(\tau) &= \sqrt{M/A - 1}(-\tau\sin\phi \pm \sqrt{A - \tau^2}\cos\phi). \end{aligned} \tag{54}$$

(where τ is now a parameter). The expressions with index $+$ refer to evolution forward in q_1 time, while the expressions with index $-$ refer to backward evolution in q_1 (see Sec. VA 2 for additional discussion). The fact that these correlations are double valued does not constitute a problem, since the value of ϕ provides an orientation of the orbit. Starting at a point of the ellipse at a given value of q_1, the direction of relational evolution in q_1 is provided by the orientation and one may evolve in this manner around the ellipse without having to switch the clock at the classical level. Indeed, at the two turning points of q_1 the relational momentum observable is nonvanishing and, consequently, determines the direction of evolution. One can simply switch, for instance, from q_2^+ to q_2^- and change the direction of τ since the system moves back in q_1.[16] This way a consistent relational evolution is obtained along the trajectory which is entirely encoded within Dirac observables and no use of any gauge parameter is made. For later reference, it is useful to note that one could arrive at the same predictions of correlations by providing—instead of ϕ and A—relational initial data, e.g., $q_2^+(\tau = \tau_0)$ and $p_2^+(\tau = \tau_0)$, plus the orientation of the ellipse which is encoded in the angular momentum L_y. Notice that the orientation must be specified since, given the values of q_1, q_2, p_2, one can only solve for p_1 up to sign via Eq. (47). This is due to the relativistic/quadratic nature of the constraint and the reason why, in general, one needs to provide a time direction in which to evolve (or equivalently a Hamiltonian) apart from the initial data [13], in order to pose a well-defined IVP; purely relational information cannot coordinatize the space of solutions of systems governed by relativistic constraints.[17]

We will perform the precise analogue of this local relational evolution in the effective and quantum theory.

2. Local relational evolution generated by physical Hamiltonians

If we interpret Eq. (54) as physical motion in q_1, we would like to find a physical Hamiltonian which generates this motion in the reduced phase space. Such a Hamiltonian is not the constraint, but itself a Dirac observable which moves a given transversal surface (time level) in phase space [8–10]. Given data on a transversal surface, this data will be moved onto another transversal surface in a direction determined by the Hamiltonian. More precisely, the "time direction" is provided by its Hamiltonian vector field. The trouble in the present model is, obviously, that these transversal surfaces may be intersected twice or not at all by the classical orbit. The two intersections of a trajectory with given orientation also come with two different evolution directions because the trajectory is closed. These two opposite directions can, certainly, not both be generated by one and the same physical Hamiltonian, since it moves the transversal surface in only one direction in phase space. Thus, unlike in systems with global clocks, we are required to perform a change of Hamiltonian at the turning points of the clock. In order to evolve from the surface determined by the nonglobal clock q_1, we need two Hamiltonians, one of which generates evolution for q_2^+ and p_2^+ in the positive q_1 direction until the turning point of q_1 and the second of which then generates evolution for q_2^- and p_2^- in the opposite direction, away from the turning point. Let us explore this in more detail.

Choosing q_1 as local time, we may factorize Eq. (47) classically into a pair of constraints linear in p_1,

$$C = (p_1 + H(\tau))(p_1 - H(\tau)) = \tilde{C}_+\tilde{C}_-, \quad \text{where } H(\tau) = \sqrt{M - \tau^2 - p_2^2 - q_2^2}. \tag{55}$$

The dynamical equations now read $\{\cdot, C\} = \tilde{C}_+\{\cdot, \tilde{C}_-\} + \tilde{C}_-\{\cdot, \tilde{C}_+\}$. Away from the turning points in q_1 time we have $H(\tau) > 0$ and, therefore, $C = 0$ implies that one of

[16]Continuation to larger absolute values of τ will produce meaningless complex correlations in Eq. (54) which simply indicates that the system will never reach such values of the local clock.

[17]In nonrelativistic parametrized systems, where the momentum conjugate to the time function appears linearly, the time direction is automatically given.

the following two possibilities (but not both simultaneously) is true

$$\tilde{C}_+ = 0 \Leftrightarrow \tilde{C}_- = 2p_1 < 0 \Rightarrow q_1' = \{q_1, C\} = 2p_1 < 0 \\ \text{and } \{\cdot, C\} \propto -\{\cdot, \tilde{C}_+\}, \tag{56}$$

or,

$$\tilde{C}_- = 0 \Leftrightarrow \tilde{C}_+ = 2p_1 > 0 \Rightarrow q_1' = \{q_1, C\} = 2p_1 > 0 \\ \text{and } \{\cdot, C\} \propto +\{\cdot, \tilde{C}_-\}. \tag{57}$$

Hence, on the set defined by $\tilde{C}_\pm = 0$ we may use $\tilde{C}_\pm$ as evolution generator, but notice that the flow generated by $\tilde{C}_+$ is directed opposite to the one generated by C. Furthermore, since $\{q_1, \tilde{C}_\pm\} = 1$, $\tilde{C}_\pm$ and, thus, $\pm H(\tau)$ are evolution generators for q_2 and p_2 in q_1 time. In particular, on the part of the constraint surface, where $\tilde{C}_+$ vanishes and, thus, may be used as an evolution generator (whose Hamiltonian vector field points in opposite direction to the one determined by C), we have $q_1' = 2p_1 < 0$ and, therefore, the system governed by C moves back in q_1 time. As a consequence, while $-H(\tau)$ generates evolution for q_2 and p_2 forward in q_1 time, $+H(\tau)$ does precisely the opposite. Note, moreover, that the two Hamiltonians $\pm H(\tau)$ are themselves relational Dirac observables which generate the physical equations of motion

$$\dot{q}_2 = \pm\{q_2, H(\tau)\} = \mp \frac{p_2}{H(\tau)}, \tag{58}$$

$$\dot{p}_2 = \pm\{p_2, H(\tau)\} = \pm \frac{q_2}{H(\tau)}, \tag{59}$$

where $\dot{}$ denotes a time derivative with respect to τ. As can be easily checked by using Eq. (54), the solution to the equations of motion generated by $+H(\tau)$ will reproduce classically q_2^- and p_2^-, while the solutions to the equations generated by $-H(\tau)$ will provide q_2^+ and p_2^+. Consequently, in the solutions q_2^+ and p_2^+ in (54) τ must run forward, while for q_2^- and p_2^- it must run backwards. Care must be taken at the turning point of q_1 time, where $p_1 = H = 0$. Here we have to perform the change from $-H(\tau)$ to $+H(\tau)$, or vice versa.

The situation here is quite different from the case of the free relativistic particle for two reasons. Firstly, in the constraint for the free relativistic particle the two momenta come with opposite signs and $t' = \{t, C_{\text{particle}}\} = \{t, -p_t^2 + p^2\} = -2p_t$, which entails that forward evolution in the clock t is only possible where $p_t < 0$. Secondly, p_t is a Dirac observable which implies that in this model no change of Hamiltonian needs to be performed. Neither of the two issues occurs in the nonrelativistic case, where p_t appears linearly and the time direction is automatically given.

B. The quantum theory

The constraint (47), when promoted to a quantum operator in the Dirac procedure, reads

$$\hat{C} = \hat{p}_1^2 + \hat{p}_2^2 + \hat{q}_1^2 + \hat{q}_2^2 - M. \tag{60}$$

The quantization of this model is straightforward, since zero lies in the discrete part of the spectrum of the constraint.[18] The physical Hilbert space is, therefore, a subspace of the kinematical Hilbert space $L^2(\mathbb{R}^2, dq_1 dq_2)$, where the physical inner product is identical to the kinematical inner product and simply given by

$$\langle \psi, \phi \rangle_{\text{phys}} = \int_{-\infty}^{+\infty} dq_1 dq_2 \bar{\psi}(q_1, q_2)\phi(q_1, q_2). \tag{61}$$

The general form of the physical states is

$$\psi_{\text{phys}}(q_1, q_2) = \sum_{n=0}^{M/(2\hbar)-1} c_n \psi_n(q_1)\psi_{M/(2\hbar)-n-1}(q_2), \tag{62}$$

($c_n = \text{const}$) and ψ_n denotes the n-th eigenstate of the one-dimensional harmonic oscillator. The Dirac observables in Eq. (52) are also straightforwardly quantized, since there is no factor ordering ambiguity involved. For some aspects discussed here see also [4,7].

The inner product may easily be obtained from group averaging, where $P = \int_0^{2\pi} ds e^{-i\hat{C}s/\hbar}$, in fact, is a true projector. The integration range of 2π is due to the constraint being a $U(1)$ generator and compatible with the classical identification of states on the orbit which are related by integer numbers of revolution.

1. Timelessness

A priori, there should be no time evolution and no IVP since there is no true time. Indeed, in the (q_1, q_2) representation, Eq. (60) provides an elliptic PDE; thus, there is no well-defined IVP for this quantum model, but rather a boundary value problem. The ‘‘initial data’’ characterizing the quantum solution is in a sense timeless. This is also highlighted by the inner product (61) which integrates out both configuration variables and, therefore, cannot be captured by the standard inner products based on constant time slicings. The latter are usually related to the existence of a well-posed IVP.

In spite of this *a priori* timelessness, we can give a local dynamical interpretation to the quantum theory in analogous fashion to the classical theory. (The relational evolution to be discussed here is only an emergent local evolutionary interpretation of a timeless model. Consequently, the apparent nonunitarity in the nonglobal clock evolution and possible decoherence effects related to this are an artefact of this emergent interpretation. The model itself is neither nonunitary nor decohering since there is no true time.

[18] We assume here that M is chosen to the extent that there exist n_1, n_2 such that $2\hbar(n_1 + n_2 + 1) - M = 0$ and zero actually lies in the spectrum of $\hat{C}$.

For that reason, the issue of "quantum illnesses," raised, for instance, in [15], is not directly applicable here.) The ensuing differences between the classical and quantum theory are, as usual, merely due to the quantum uncertainties; however, these have more severe implications in the absence of a global clock.

Again, we can give a meaning to orientation in the quantum theory, namely, via $\hat{L}_y$, which—being a Dirac observable—is a well-defined operator on $\mathcal{H}_{\text{phys}}$. Its positive and negative eigenspaces distinguish the orientation which also provides a direction of evolution. By superimposing the two, a superposition of evolution in both directions is, in principle, possible.

However, owing to the quantum uncertainties, the relational concept of evolution seems to be only of an essentially semiclassical and certainly local nature when dealing with nonglobal clocks and even in this regime, quantum effects have severe consequences. When asking for the value of, say, q_2 when a certain value of q_1 is realized, one faces the problem that due to the spread, parts of the state may already be "beyond their turning point" in q_1. Classically, this results in a quite meaningless complex-valued correlation between the two configuration variables (just extend $|\tau|$ beyond A in Eq. (54)) which merely indicates that the system never reaches this point. In the quantum theory, the correlation of the two variables, thus, loses meaning earlier than in the classical theory; the larger the quantum uncertainties, i.e., the larger the spread of the state, the earlier the concept of the relational correlation breaks down. At a given value of the clock q_1 part of the system is lost and an apparent nonunitarity shows up. This, certainly, also applies to semiclassical states and, therefore, one cannot fully reach the classical turning point without changing the clock beforehand. Here, one cannot simply switch between, e.g., q_2^+ and q_2^-, as one could classically, and as a consequence relational Dirac observables only have a local meaning.

By the same token, the peak of a coherent physical state may follow a classical trajectory exactly while expectation values computed in an internal time Schrödinger regime can only do so locally. Such a Schrödinger regime results from a local deparametrization and is aimed at locally approximating the timeless physical state and the information contained in it by locally scanning through it, thereby introducing a notion of quantum evolution. The Schrödinger regime for this model, is explicitly discussed in Sec. V B 2 below. For this regime we need an (emergent) inner product based on constant internal time slicings (for only the part of a coherent physical state which either corresponds to, e.g., q_2^+ or q_2^-) and such a slicing becomes troublesome near the classical turning point of the chosen clock due to the apparent nonunitarity, and eventually breaks down. Since the breakdown occurs earlier the greater the quantum uncertainties, it becomes apparent that the internal time Schrödinger evolution is only meaningful here in a semiclassical regime. And even then, an expectation value trajectory cannot

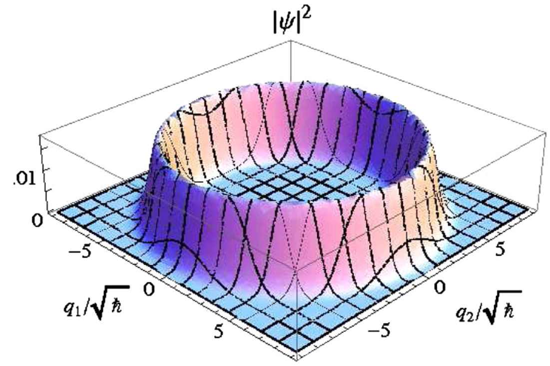

FIG. 5 (color online). Square amplitude of a coherent solution to the constraint (60), with $M = 50\hbar$, peaked about a circular configuration space trajectory.

completely reproduce the corresponding classical trajectory near the turning point, even though the peak of the coherent state may do so.

Thus, while the question for what value, say, q_2 takes when q_1 reads such and such seems to be meaningless if the state is extremely quantum, it is meaningful for a semiclassical state, where at least locally the expectation value evaluated in some "emergent" inner product based on constant q_1 slicings follows a classical trajectory until close to the q_1-turning point. For highly quantum states in systems without globally valid clock variables, however, the standard concept of (locally unitary) relational evolution seems to disappear in conjunction with the standard notion of relational time. For a more detailed general discussion of this feature we refer the interested reader to [17]. The analysis of the present toy model supplies several general statements in [17] with concrete examples.

Let us, therefore, investigate relational evolution via local deparametrizations and how to reconstruct the information of the physical state from it in the semiclassical regime. We refrain from explicitly employing elliptic coherent physical states here, but in order to visually facilitate the discussion we present an example of such a state for this model in Fig. 5 (the interested reader may find the recipe for the construction in this particular model in [24]). In the semiclassical regime it is also reasonable to consider only the solutions to Eq. (60) which consist purely of positive or negative eigenstates of $\hat{L}_y$ such that we avoid superposition of evolution in both directions and are in a position to essentially repeat the same procedure here as in the classical case.

We now have four methods for investigating the semiclassical regime: the Dirac method, the reduction method,[19] evolution in an approximate local Schrödinger

[19]Since in the reduced phase-space quantization the parameter τ survives in the quantum theory, it is the only method in which the timeless physical inner product (61) may be used in order to compute expectation values at a fixed value τ of q_1; otherwise this physical inner product does not admit a sense of evolution.

regime or in the effective approach. This issue has been partially analyzed in the reduction method (which in this simple case turns out to be equivalent to the Dirac method) via group quantization by Rovelli in [7], therefore, we will focus on the local Schrödinger regime in Sec. V B 2 and the effective approach in Sec. V C, both truncated at order $\hbar$, in this article. We will show that both yield equivalent results.

2. A local internal time Schrödinger regime

Since relational quantum evolution seems feasible for semiclassical states, we would like to locally construct an internal time Schrödinger regime which reproduces one branch of the timeless physical state. This can be achieved by simply translating the local relational motion generated by the two Hamiltonians of Sec. VA 2 into the quantum theory and may, therefore, be understood as a local deparametrization with a valid IVP. To construct this Schrödinger regime, we require q_1 (or q_2)—in analogy to the parameter τ in (55)—to appear as a parameter rather than as an operator, and the corresponding states do not exist in the Hilbert space of the previous subsection. We therefore need a new Hilbert space, with a new inner product, in which we integrate only over q_2 at a fixed value of the parameter q_1. The Schrödinger regime using q_2 as an internal clock naturally requires a further new Hilbert space, in which the roles of q_1 and q_2 are reversed. From the point of view of standard Hilbert-space quantum theory, these Schrödinger regimes thus constitute different quantizations of the classical theory: that is, they are different and, in general, inequivalent quantum theories. Even though solutions to the resulting Schrödinger equations violate the quadratic quantum constraint with self-adjoint clock operator and are not normalizable with (61), they can be considered as approximations to the original constrained problem by referring to the analysis of [17] summarized in Sec. IVA: the WDW equation (60) is, in fact, not violated if internal time in this equation allows for an imaginary contribution. Because of the apparent nonunitarity alluded to above, the local Schrödinger regime will break down on approach to the classical turning point of the clock, and we can only hope to reconstruct/approximate the full physical state by switching clocks and deparametrizations prior to the breakdown of the respective clock. The results of this section will become essential for understanding the effective approach, since the local relational evolution of expectation values, i.e., of fashionables, obtained in both approaches will prove to be indistinguishable.

Choosing $\tilde{C}_+$ (and, thus, backward evolution in q_1) in Eq. (55), standard quantization yields

$$i\hbar \frac{\partial}{\partial q_1}\psi(q_1, q_2) = \hat{H}(\hat{q}_2, \hat{p}_2; q_1)\psi(q_1, q_2) = \widehat{\sqrt{M - q_1^2 - p_2^2 - q_2^2}}\psi(q_1, q_2), \quad (63)$$

where $\hat{H}$ is defined via spectral decomposition. The eigenfunctions of the latter are the harmonic oscillator eigenfunctions ψ_n with eigenvalues $H_n(q_1) = \sqrt{M - q_1^2 - \hbar(2n+1)}$, and, consequently, the operator is positive definite on the lower energetic eigenstates, where the time-dependent energy bound is given by $M - q_1^2$.[20] In analogy with Eq. (55) and in contrast to Eq. (60), q_1 has been reduced to a parameter here (see also Sec. IVA and [17] on this issue).

We solve Eq. (63) in the standard way—noting that $[\hat{H}(\hat{q}_2, \hat{p}_2; q_1), \hat{H}(\hat{q}_2, \hat{p}_2; q_1')] = 0$—via

$$\psi(q_2; q_1) = e^{-(i/\hbar)\int_{q_{1_0}}^{q_1} dt \hat{H}(\hat{q}_2, \hat{p}_2; t)}\psi_n(q_2; q_{1_0}) = e^{-(i/\hbar)E_n(q_1)}\psi_n(q_2; q_{1_0}), \quad (64)$$

where

$$E_n(q_1) = \int_{q_{1_0}}^{q_1} dt H_n(t) = \frac{1}{2}\Big(q_1\sqrt{M - q_1^2 - \hbar(2n+1)} - q_{1_0}\sqrt{M - q_{1_0}^2 - \hbar(2n+1)} + (M - \hbar(2n+1))\Big(\arctan\Big(\frac{q_1}{\sqrt{M - q_1^2 - \hbar(2n+1)}}\Big) - \arctan\Big(\frac{q_{1_0}}{\sqrt{M - q_{1_0}^2 - \hbar(2n+1)}}\Big)\Big)\Big). \quad (65)$$

In order to better explore the semiclassical regime, let us attempt to construct coherent states. The eigenstates of $\hat{H}$ are given by harmonic oscillator eigenmodes; therefore, it seems reasonable to make the following standard ansatz for a coherent state[21]

[20] This energy bound is related to the upper limit of the sum in the physical state (62).
[21] For convenience, we shall henceforth employ bra and ket notation.

$$|z(q_{1_0})\rangle = e^{-|z|^2/2}e^{z\hat{a}^+}|0\rangle = e^{-|z|^2/2}\sum_{n\geq 0}\frac{z^n}{\sqrt{n!}}|n\rangle, \quad (66)$$

where $|n\rangle$ is the n-th eigenstate of the harmonic oscillator,

$$\hat{a} = \frac{1}{2\hbar}(\hat{q}_2 + i\hat{p}_2) \qquad \hat{a}^+ = \frac{1}{2\hbar}(\hat{q}_2 - i\hat{p}_2) \quad (67)$$

are the usual annihilation and creation operators of the harmonic oscillator, and

$$z = \frac{q_{2_0} + ip_{2_0}}{\sqrt{2\hbar}}, \quad (68)$$

where q_{2_0} and p_{2_0} are the initial positions of the coherent state in phase space.

The coherent state will be evolved with the (local) evolution generator $\hat{H}$. Thus,

$$\begin{aligned}|z(q_1)\rangle &= e^{-(i/\hbar)\int_{q_{1_0}}^{q_1} dt\hat{H}(\hat{q}_2,\hat{p}_2;t)}|z(q_{1_0})\rangle \\ &= e^{-|z|^2/2}\sum_{n\geq 0}\frac{z^n}{\sqrt{n!}}e^{-(i/\hbar)E_n(q_1)}|n\rangle. \quad (69)\end{aligned}$$

Furthermore, the states are normalized $\langle z(q_1)|z(q_1)\rangle = 1$ with respect to the standard inner product obtained by merely integrating out q_2.

The coherent states of the harmonic oscillator are dynamical coherent states when evolved with the standard Hamiltonian. Here, however, we are not evolving with the standard Hamiltonian and, therefore, these states are only initially coherent states for our local Schrödinger regime; the states are not eigenstates of $\hat{a}$ for all times, as can be seen from

$$\hat{a}|z(q_1)\rangle = e^{-|z|^2/2}\sum_{n\geq 0}\frac{z^{n+1}}{\sqrt{n!}}e^{-(i/\hbar)E_{n+1}(q_1)}|n\rangle \not\propto |z(q_1)\rangle, \quad (70)$$

and the form of Eq. (65).

Expectation values as functions of q_1, i.e., *fashionables*, are now easily calculated

$$\begin{aligned}\langle\hat{q}_2\rangle(q_1) &= \langle z(q_1)|\hat{q}_2|z(q_1)\rangle = \langle z(q_1)|\sqrt{\frac{\hbar}{2}}(\hat{a}+\hat{a}^+)|z(q_1)\rangle \\ &= e^{-|z|^2}\sum_{n\geq 0}\frac{|z|^{2n}}{n!}\bigg(q_{2_0}\cos\bigg(\frac{E_{n+1}(q_1)-E_n(q_1)}{\hbar}\bigg) \\ &\quad + p_{2_0}\sin\bigg(\frac{E_{n+1}(q_1)-E_n(q_1)}{\hbar}\bigg)\bigg), \\ \langle\hat{p}_2\rangle(q_1) &= \langle z(q_1)|\hat{p}_2|z(q_1)\rangle = \langle z(q_1)|\sqrt{\frac{\hbar}{2}}i(\hat{a}^+-\hat{a})|z(q_1)\rangle \\ &= e^{-|z|^2}\sum_{n\geq 0}\frac{|z|^{2n}}{n!}\bigg(p_{2_0}\cos\bigg(\frac{E_{n+1}(q_1)-E_n(q_1)}{\hbar}\bigg) \\ &\quad - q_{2_0}\sin\bigg(\frac{E_{n+1}(q_1)-E_n(q_1)}{\hbar}\bigg)\bigg). \quad (71)\end{aligned}$$

The explicit expressions for the fashionables of the moments $(\Delta q_2)^2$, $(\Delta p_2)^2$ and $\Delta(q_2p_2)$ as functions of q_1 are given in Appendix C. The first two equations for $\langle\hat{q}_2\rangle$ and $\langle\hat{p}_2\rangle$, certainly, reduce to the standard (classical) equations of motion for the expectation values of the harmonic oscillator if one replaces $E_n(q_1)$ with the usual eigenvalues of the harmonic oscillator. Plots of these fashionables for a specific configuration are provided in Figs. 6 and 7 in Sec. V C 1 below, combined with a comparison with the effective results.

As an explicit example of the analysis summarized in Sec. IV A, let us discuss by how much we are violating the WDW equation (60) due to the fact that q_1 is a real parameter here. To this end, we compute

$$\begin{aligned}\langle z(q_1)|\hat{C}|z(q_1)\rangle &= \langle z(q_1)|-\hbar^2\frac{\partial^2}{\partial q_1^2}-\hat{H}^2|z(q_1)\rangle = \langle z(q_1)|i\hbar(\partial_{q_1}\hat{H})|z(q_1)\rangle = \langle z(q_1)|-i\hbar q_1(\hat{H})^{-1}|z(q_1)\rangle \\ &= -i\hbar e^{-|z|^2}\sum_{n\geq 0}\frac{|z|^{2n}}{n!}\frac{q_1}{\sqrt{M-q_1^2-\hbar(2n+1)}} = i\hbar\frac{\partial}{\partial q_1}\langle z(q_1)|\hat{H}|z(q_1)\rangle. \quad (72)\end{aligned}$$

(The last line just demonstrates the Ehrenfest theorem.) Linearizing in $\hbar$, one finds a violation of the quadratic constraint

$$\langle z(q_1)|\hat{C}|z(q_1)\rangle = -\frac{i\hbar q_1}{\sqrt{M-q_1^2}} + o(\hbar^2). \quad (73)$$

To bridge this discrepancy, we interpret q_1 as the operator (45) with expectation value having an imaginary contribution $-\frac{i\hbar}{2\langle\hat{p}_1\rangle}$ to order $\hbar$. Because of $(\Delta q_1)^2 = 0$, one finds $\langle\hat{q}_1^2\rangle = \langle\hat{q}_1\rangle^2 = q_1^2 - \frac{i\hbar q_1}{\langle\hat{p}_1\rangle} + O(\hbar^{(3/2)})$ and, with a little further calculation, it turns out that the right hand side of Eq. (73) is precisely the imaginary part of $\langle\hat{q}_1^2\rangle$. It may thus be brought to the left hand side and interpreted as the imaginary contribution to the expectation value of the clock q_1 in Eq. (60). Then, the quadratic constraint is satisfied to this order and provides an explicit example for the general derivation in [17].

Similarly, to linear order in $\hbar$, Dirac observables of the quadratic constraint are, in general, constants of motion

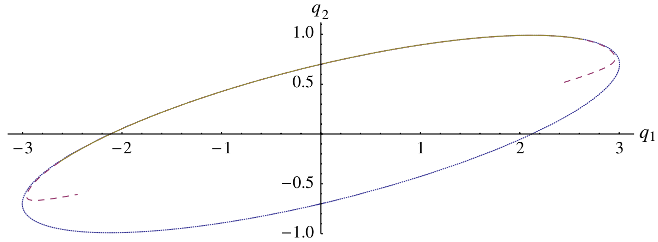

FIG. 6 (color online). Pictorial comparison of the classical relational Dirac observable $q_2(q_1)$ (full ellipse, blue curve) with the quantities $q_2(\Re[q_1])$ calculated in the effective theory using the q_1 gauge (violet dashed curve) and $\langle\hat{q}_2\rangle(q_1)$ in the Schrödinger regime (yellow solid curve). Where valid, the three curves agree perfectly. The Schrödinger regime breaks down earlier than the q_1 gauge of the effective framework. The initial data match in all three cases: we chose $q_{2_0} = 0.7$ and $p_{2_0} = -0.7$ for the Schrödinger regime, which via Eq. (C1) yields $(\Delta q_2)^2(q_1 = 0) = (\Delta p_2)^2(q_1 = 0) = \frac{\hbar}{2}$ and $\Delta(q_2 p_2)(q_1 = 0) = 0$. We have set $M = 10$ and, to amplify effects, $\hbar = 0.03$. We take these values as initial data for the effective formalism as well, and, using Eq. (84), we determine the initial value for $p_{1_0} = -2.998$ (the minus sign is necessary here, since in Eq. (63) we quantized $\tilde{C}_+$ which evolves backwards in q_1). In the effective picture, due to the imaginary contribution to q_1 in the q_1 gauge, we have set the initial value of the clock to $q_1 = -\frac{i\hbar}{2p_{1_0}}$, but employ $\Re[q_1]$ as relational clock (see also Fig. 8). The initial data for the classical curve has been chosen accordingly. As regards the axis labels: for the effective framework both q_1 and q_2 refer to the expectation values of the corresponding operators (for q_1 the real part), while for the internal time Schrödinger regime q_2 refers to the expectation value from Eq. (71) and q_1 is the real evolution parameter.

of the internal time Schrödinger regime only if the expectation value of the clock in the quadratic constraint is complex. For instance, the quantized Dirac observable A of Eq. (51) is given by $2\hat{A} = 2(M - \hat{p}_2^2 - \hat{q}_2^2) + \hat{C}$. The expectation value $\langle z(q_1)|\hat{A}|z(q_1)\rangle$ is independent of q_1 only if the expectation value of $\hat{C}$ vanishes to semiclassical order since, employing Eq. (71) and the expressions in Appendix C, one can easily convince oneself that the expectation value of $\hat{p}_2^2 + \hat{q}_2^2$ is q_1 independent.

Finally, let us return to the issue of reconstructing the classical trajectory or even the full physical state from the results in this Schrödinger regime. The peak of a semiclassical state may follow a classical trajectory almost precisely. However, the expectation values can only follow the classical trajectory away from the turning point. Because of the apparent nonunitarity of evolution in q_1, the fashionables evaluated in the standard Schrödinger type inner product with $q_1 =$ const slicing must become meaningless on approach to the turning point of q_1. Heuristically, this may be understood by taking the expectation value of the unit operator which may be interpreted as the probability that the system is at some q_2 for a given value of q_1. As long as the state is sufficiently semiclassical and the peak is far enough away from the clock turning region, this expectation value should always give 1. On approach to the turning region, however, there will be parts of the state which are "beyond their turning point," precluding meaningful expectation values. Part of the system is lost which implies that the expectation value of the unit operator cannot give 1 anymore. Nonunitarity, therefore, implies that the spread in q_1 cannot vanish close to the classical turning point, since

$$(\Delta q_1)^2 = \langle q_1^2\rangle - \langle q_1\rangle^2 = q_1^2(\langle \mathbb{1}\rangle - \langle \mathbb{1}\rangle^2), \quad (74)$$

which is nonvanishing when the expectation value of the unit operator fails to be unity. This provides an analogy in the internal time Schrödinger regime for why the q_1 gauge, which among other conditions enforces $(\Delta q_1)^2 = 0$, must break down on approach to the turning point of q_1 time in the effective procedure.

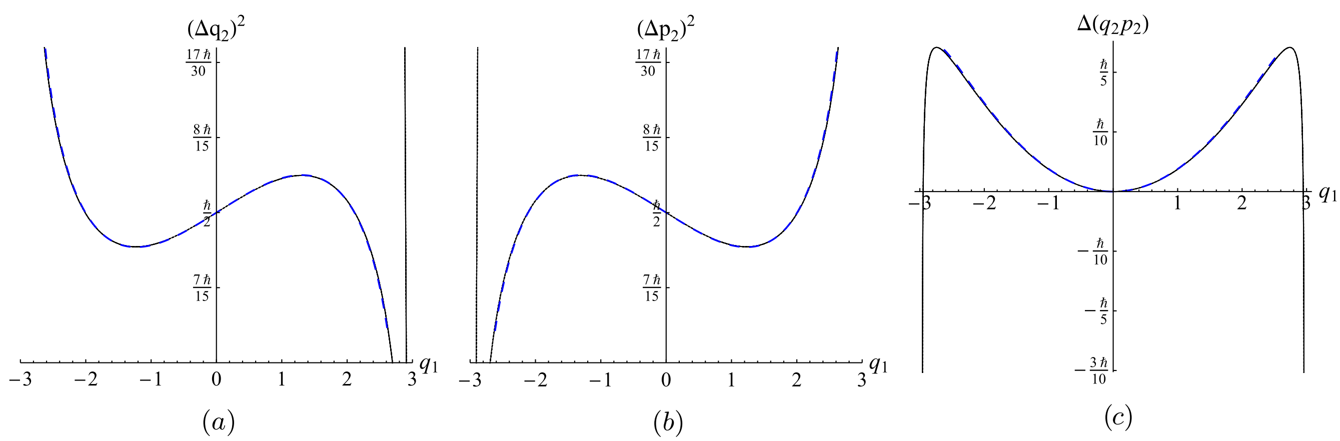

FIG. 7 (color online). Comparison of the effective (black dotted curves) and internal time Schrödinger regime results (blue dashed curves) for the fashionables in q_1 time associated to moments: (a) $(\Delta q_2)^2(q_1)$, (b) $(\Delta p_2)^2(q_1)$ and (c) $\Delta(q_2 p_2)(q_1)$. The curves agree perfectly to order $\hbar$. As explained in the main text, the Schrödinger regime breaks down earlier than the q_1 gauge of the effective framework. The breakdown of the latter is clearly demonstrated by the divergence of the effective moments near $|q_1| = 3$. The initial data is identical to the one for Fig. 6.

As a consequence, in order to reproduce information from the full physical state, we are forced to change from constant q_1 to constant q_2 slicing, and thus from q_1 to q_2 time, prior to the Schrödinger regime in q_1 time becoming invalid. Likewise, we have to switch from q_2 time back to q_1 time again, prior to the constant q_2 slicing subsequently becoming invalid and so on until we have evolved once around the classical ellipse. In order for the physical state to be reproduced, it then remains to be shown that the expectation values of the quantum Dirac observables characterizing the physical state, such as the three angular momentum operators (52), are invariant under the change of slicing. Since the two slicings used here are orthogonal to each other, one cannot smoothly translate data from one slicing to the other. In fact, one would expect jumps in the relational correlations when switching the slicing. The necessary changes in slicing here are directly analogous to the necessary changes between the q_1 and q_2 gauge in the effective approach in Sec. V C below and underline that fashionables can only locally be made sense of.

C. Effective procedure

To semiclassical order, the constraint (60) translates into the following five constraints in the effective approach

$$\begin{aligned}
C &= p_1^2 + p_2^2 + q_1^2 + q_2^2 + (\Delta p_1)^2 + (\Delta p_2)^2 + (\Delta q_1)^2 + (\Delta q_2)^2 - M = 0\\
C_{q_1} &= 2p_1\Delta(q_1p_1) + 2p_2\Delta(q_1p_2) + 2q_1(\Delta q_1)^2 + 2q_2\Delta(q_1q_2) + i\hbar p_1 = 0\\
C_{p_1} &= 2p_1(\Delta p_1)^2 + 2p_2\Delta(p_1p_2) + 2q_1\Delta(p_1q_1) + 2q_2\Delta(p_1q_2) - i\hbar q_1 = 0\\
C_{q_2} &= 2p_1\Delta(p_1q_2) + 2p_2\Delta(q_2p_2) + 2q_1\Delta(q_1q_2) + 2q_2(\Delta q_2)^2 + i\hbar p_2 = 0\\
C_{p_2} &= 2p_1\Delta(p_1p_2) + 2p_2(\Delta p_2)^2 + 2q_1\Delta(q_1p_2) + 2q_2\Delta(q_2p_2) - i\hbar q_2 = 0.
\end{aligned} \tag{75}$$

Again, there are four linearly independent flows generated by these five constraints. The 14-dimensional Poisson manifold may, therefore, be reduced to five physical degrees of freedom. Dirac observables for this system are easily obtained by translating either Eqs. (51) or (52) into the quantum theory and taking their expectation values. For instance, the over-complete set (52) now reads

$$\begin{aligned}
L_x &= \tfrac{1}{2}(p_1p_2 + q_1q_2 + \Delta(p_1p_2) + \Delta(q_1q_2)),\\
L_y &= \tfrac{1}{2}(p_2q_1 - p_1q_2 + \Delta(q_1p_2) - \Delta(p_1q_2)),\\
L_z &= \tfrac{1}{4}(p_1^2 - p_2^2 + q_1^2 - q_2^2 + (\Delta p_1)^2 - (\Delta p_2)^2\\
&\quad + (\Delta q_1)^2 - (\Delta q_2)^2).
\end{aligned} \tag{76}$$

Owing to the definition of the effective Poisson bracket (1), also these effective observables satisfy the standard angular momentum Poisson algebra. Moreover, due to Eq. (2), the moments associated to these variables, $(\Delta L_x)^2$, $(\Delta L_y)^2$, $(\Delta L_z)^2$, $\Delta(L_xL_y)$, $\Delta(L_xL_z)$ and $\Delta(L_yL_z)$, will provide the $o(\hbar)$ observables. Since classically (52) is an over-complete set, also these nine observables here are, certainly, over-complete. Indeed, to order $\hbar$, the constraint (53) can easily be translated into four relations among these effective observables, thus leaving us with the five physical degrees of freedom to this order. The explicit expressions for the moments, as well as the four relations among the full set of these observables, are rather lengthy and not particularly illuminating. We, therefore, abstain from showing them here. As regards relational evolution, the angular momentum L_y will provide an orientation to the effective trajectories.

Because of the symmetry of the model in the indices 1 and 2, we will henceforth work with indices $i, j \in \{1, 2\}$. In analogy to Eq. (15), we impose the q_i *gauge* (or the *Zeitgeist associated to* q_i)

$$\phi_1 = (\Delta q_i)^2 = 0 \quad \phi_2 = \Delta(q_iq_j) = 0 \quad \phi_3 = \Delta(q_ip_j) = 0. \tag{77}$$

The remaining first-class constraint with vanishing flow on the variables q_1, p_1, q_2, p_2, $(\Delta q_j)^2$, $(\Delta p_j)^2$, $\Delta(q_jp_j)$ is directly proportional to C_{q_i}. The solution of this constraint

$$C_{q_i} \approx 2p_i\Delta(q_ip_i) + i\hbar p_i = 0 \Rightarrow \Delta(q_ip_i) = -\frac{i\hbar}{2}, \tag{78}$$

again implies the saturation of the (generalized) uncertainty relation in (q_i, p_i).

The Hamiltonian constraint reads

$$C_H = C + \alpha C_{p_i} + \beta C_{q_j} + \gamma C_{p_j}, \tag{79}$$

where on the gauge surface (77)

$$\alpha = -\frac{1}{2p_i}, \qquad \beta = \frac{q_j}{2p_i^2} \quad \text{and} \quad \gamma = \frac{p_j}{2p_i^2}. \tag{80}$$

In addition to Eq. (78), we may solve C_{p_i}, C_{q_j} and C_{p_j} for the remaining nonphysical moments

$$(\Delta p_i)^2 = \frac{p_j^2(\Delta p_j)^2 + 2q_jp_j\Delta(q_jp_j) + q_j^2(\Delta q_j)^2 + i\hbar q_ip_i}{p_i^2}, \qquad \Delta(p_ip_j) = -\frac{2p_j(\Delta p_j)^2 + 2q_j\Delta(q_jp_j) - i\hbar q_j}{2p_i},$$
$$\Delta(q_jp_i) = -\frac{2q_j(\Delta q_j)^2 + 2p_j\Delta(q_jp_j) + i\hbar p_j}{2p_i}. \tag{81}$$

Making use of this, the relevant dynamical equations generated by C_H simplify on the gauge surface (77) and are given by

$$\dot{q}_i = \{q_i, C_H\} \approx 2p_i - \frac{i\hbar q_i}{p_i^2} - 2\frac{p_j^2(\Delta p_j)^2 + 2q_jp_j\Delta(q_jp_j) + q_j^2(\Delta q_j)^2}{p_i^3}, \quad \dot{q}_j = \{q_j, C_H\} \approx 2p_j + 2\frac{q_j\Delta(q_jp_j) + p_j(\Delta p_j)^2}{p_i^2},$$
$$\dot{p}_i = \{p_i, C_H\} \approx -2q_i - \frac{i\hbar}{p_i}, \quad \dot{p}_j = \{p_j, C_H\} \approx -2q_j - 2\frac{q_j(\Delta q_j)^2 + p_j\Delta(q_jp_j)}{p_i^2},$$
$$(\dot{\Delta q_j})^2 = \{(\Delta q_j)^2, C_H\} \approx 4\frac{q_jp_j(\Delta q_j)^2 + (p_i^2 + p_j^2)\Delta(q_jp_j)}{p_i^2}, \quad (\dot{\Delta p_j})^2 = \{(\Delta p_j)^2, C_H\} \approx -4\frac{q_jp_j(\Delta p_j)^2 + (p_i^2 + q_j^2)\Delta(q_jp_j)}{p_i^2},$$
$$\dot{\Delta}(q_jp_j) = \{\Delta(q_jp_j), C_H\} \approx 2\frac{(p_i^2 + p_j^2)(\Delta p_j)^2 - (p_i^2 + q_j^2)(\Delta q_j)^2}{p_i^2}. \tag{82}$$

This set of coupled equations is rather complicated to solve analytically, but this is not necessary for our discussion here.

Although the dynamical equation for p_i is not classical in nature, the $\hbar^0$-order part of p_i must still vanish and $p_i \to o(\hbar)$ on approach to the turning point of q_i time. In conjunction with Eq. (80), this implies that the q_i *gauge* is inconsistent with the semiclassical truncation near the q_i turning point as a result of the coefficients of the $o(\hbar)$ constraints becoming singular. In addition, we may note that due to the imaginary terms

$$C_{q_j} \underset{p_i\to o(\hbar)}{\longrightarrow} 2p_j\Delta(q_jp_j) + 2q_j(\Delta q_j)^2 + i\hbar p_j \approx 0,$$
$$C_{p_j} \underset{p_i\to o(\hbar)}{\longrightarrow} 2p_j(\Delta p_j)^2 + 2q_j(q_jp_j) - i\hbar q_j \approx 0, \tag{83}$$

combined with the assumption of real valued q_j, p_j, $(\Delta q_j)^2$, $(\Delta p_j)^2$ and $\Delta(q_jp_j)$ implies a violation of C_{q_j} and C_{p_j} to semiclassical order at the turning point. But as previously discussed, this collapse of the q_i gauge does not come unexpected, being related to a nonglobal clock.

In analogy to Eq. (22), combining C_{p_i}, C_{q_j}, C_{p_j} and C yields a further constraint proportional to C_H, which on the constraint surface in the q_i gauge reads

$$p_i^4 + (p_j^2 + q_i^2 + q_j^2 - M + (\Delta p_j)^2 + (\Delta q_j)^2)p_i^2 + i\hbar q_ip_i$$
$$+ p_j^2(\Delta p_j)^2 + 2q_jp_j\Delta(q_jp_j) + q_j^2(\Delta q_j)^2 = 0. \tag{84}$$

We may use this remaining constraint to discuss the imaginary contributions to the variables we have chosen, as a result of the $i\hbar$ term in Eq. (84). For brevity, let us only state the (expected) result here: in complete accordance with the general result of Sec. IV A and [17], it is inconsistent with the equations of motion and the constraints in the q_i gauge to keep a real-valued clock q_i and to push the imaginary contributions to its conjugate momentum p_i, while having real-valued variables associated to the pair (q_j, p_j). Instead, it is consistent to have both the variables associated to the pair (q_j, p_j) and p_i real valued, as well as a complex clock with the standard imaginary contribution, inherent to nonglobal clocks,

$$\Im[q_i] = -\frac{\hbar}{2p_i}. \tag{85}$$

A proof of this may be found in Appendix D. Note, however, that it is also possible that both q_i and p_i are complex simultaneously.

1. Local evolution and comparison to the internal time Schrödinger regime

Since we are interested in a comparison of the effective approach with the internal time Schrödinger regime, we solve the system of effective Eqs. (82) numerically in the q_1 gauge and compare the results with the ones obtained via Eq. (71) and the expressions in Appendix C. Figure 6 shows a comparison of the classical, effective and Schrödinger regime results for the configuration space ellipse for a specific configuration, whose initial data is given in the caption of the figure. These curves depict the relational Dirac observable $q_2(q_1)$ in the classical case, the relationship $q_2(\Re[q_1])$ of expectation values in the effective framework, and $\langle\hat{q}_2\rangle(q_1)$ from Eq. (71) in the Schrödinger regime where q_1 is a real parameter.[22]

The three curves are indistinguishable where valid. Notice that the Schrödinger regime breaks down somewhat

[22]Note that in the effective framework we evolve with respect to the real part of q_1, in accordance with the discussion in Sec. IV B and the one concerning Fig. 8 below. For the effective curve, the axis label q_1, therefore, actually refers to $\Re[q_1]$.

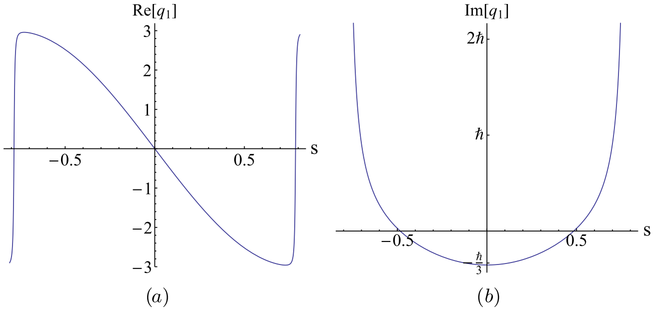

FIG. 8 (color online). Behavior of (a) the real and (b) the imaginary part of the local clock q_1 with respect to the gauge parameter s of C_H for the effective configuration with initial data as given in the caption of Fig. 6. Clearly, while $\Re[q_1]$ is monotonic along the flow of C_H (as long as the q_1 gauge is valid) and, therefore, constitutes a useful local clock, $\Im[q_1]$ does not provide a suitable clock here. The divergence of both near $|s| = 0.79$ signifies the breakdown of the q_1 gauge.

earlier than the curve of effective expectation values, due to the square roots in Eq. (65) which become imaginary for larger values of q_1 and states with higher n. The breakdown of the correlations from the effective and Schrödinger regime emphasizes the merely local nature of the fashionables. In spite of this, the plot also demonstrates that, at least locally, one can reconstruct a semiclassical orbit from the effective framework and the Schrödinger regime.

For further—nontrivial—comparison of the Schrödinger regime and the effective framework, we compare the relational evolution of their respective moments, related to the pair (q_2, p_2), in q_1 time in Fig. 7 for the same initial data as previously. The curves demonstrate that the relational evolution of the moments of both approaches agrees perfectly to this order. Since these relational moments are truly quantum in nature, this agreement provides interesting nontrivial evidence for the equivalence of these two different approaches to semiclassical order. It is also found numerically, that the discrepancies between the results of the two approaches are of $o(\hbar^2)$ or even smaller. Again, due to the square roots in Eq. (65), the Schrödinger regime in constant q_1 slicing breaks down earlier than the q_1 Zeitgeist in the effective framework. The eventual divergence of the effective moments in Fig. 7 demonstrates the breakdown of the latter.

Finally, as regards the effective evolution in q_1, Fig. 8 shows the behavior of the real and imaginary parts of q_1 with respect to the gauge parameter s of (79) for the same effective configuration. Away from the breakdown of the q_1 Zeitgeist, signified by the divergence in both the real and imaginary parts of q_1, the real part of q_1 is clearly monotonic along the flow and may thus be used as a relational clock. On the contrary, the imaginary contribution to q_1 does *not* behave monotonically and, consequently, is not a useful clock here, underlining the general argument for employing only the real part of a clock for evolution, as advocated in Sec. IV B. Note that the real part of q_1 runs backwards in the flow parameter, since we have chosen the initial data equivalently to the Schrödinger regime, where for (63) we had chosen the quantization of $\tilde{C}_+$ in Eq. (55), which generates backwards evolution in q_1.

2. Changing time and gauge transformations

Just as in the model of Sec. III we can use flows generated by the constraint functions to perform a gauge transformation from the q_i gauge to the q_j gauge. In this way, we can evolve the system through an entire closed orbit by switching the role of time back and forth between the two configuration space variables. In this section we calculate the corresponding gauge transformations; evolution through the entire orbit is explored in the following section.

Following the steps used in Sec. III C 3 to construct the gauge transformation between different Zeitgeister, we find the effect of the flows on the other variables to be given by

$$X_{G_1}(q_i) = \frac{p_i q_i - 2p_j q_j}{2p_i p_j^2}, \qquad X_{G_2}(q_i) = -\frac{1}{p_i}$$

$$X_{G_1}(p_i) = \frac{p_i}{2p_j^2}, \qquad X_{G_2}(p_i) = 0 \qquad X_{G_1}(q_j) = \frac{q_j}{2p_j^2},$$

$$X_{G_2}(q_j) = \frac{1}{p_j} \qquad X_{G_1}(p_j) = -\frac{1}{2p_j}, \qquad X_{G_2}(p_j) = 0$$

$$X_{G_1}((\Delta q_i)^2) = -\frac{p_i^2}{p_j^2}, \qquad X_{G_2}((\Delta q_i)^2) = 0$$

$$X_{G_1}((\Delta p_i)^2) = \frac{q_i(2p_j q_j - p_i q_i)}{p_i p_j^2}, \qquad X_{G_2}((\Delta p_i)^2) = \frac{2q_i}{p_i}$$

$$X_{G_1}(\Delta(q_i p_i)) = \frac{p_i q_i - p_j q_j}{p_j^2}, \qquad X_{G_2}(\Delta(q_i p_i)) = -1.$$

This time the derivatives along the flow are not constant; however, they depend only on expectation values. For the variables of interest, all of the derivatives in an expansion of the flow actions of α_{G_1} and α_{G_2} via Eq. (34) are functions of expectation values only and are thus of classical order $\hbar^0$. Second and higher derivative terms are suppressed by second and higher powers of the flow parameter, which is of order $\hbar$, since it goes from zero to $-(\Delta q_j)_0^2$ or $-(\Delta(q_j p_j)_0 + \frac{i\hbar}{2})$. Therefore, to order $\hbar$ it is sufficient to take the terms up to first order in derivatives in the flow expansion via Eq. (34) of $\alpha_G^s(f)(x_0) := \alpha_{G_2}^{-(\Delta(q_j p_j)_0 + i\hbar/2)} \circ \alpha_{G_1}^{-(\Delta q_j)_0^2}(f)(x_0)$, i.e. we have $\alpha_G^s(f)(x_0) = f_0 - (X_{G_1}(f))_0(\Delta q_j)_0^2 - (X_{G_2}(f))_0(\Delta(q_j p_j)_0 + i\hbar/2) + o(\hbar^2)$. The transformation to order $\hbar$ thus obtained has the form[23] (dropping the α's for brevity)

[23] In fact, the flows α_{G_1} and α_{G_2} have a relatively simple form and can also be integrated analytically, yielding identical results to order $\hbar$.

$$(\Delta q_i)^2 = \frac{(p_i)_0^2(\Delta q_j)_0^2}{(p_j)_0^2} \qquad (\Delta p_i)^2 = \frac{(p_j)_0^4(\Delta p_j)_0^2 + (2(p_j)_0(q_j)_0 - 2(p_i)_0(q_i)_0)\Delta(q_jp_j)_0 + (\Delta q_j)_0^2((p_i)_0(q_i)_0 - (p_j)_0(q_j)_0)^2}{(p_i)_0^2(p_j)_0^2}$$

$$\Delta(q_ip_i) = \frac{(\Delta q_j)_0^2((p_j)_0(q_j)_0 - (p_i)_0(q_i)_0)}{(p_j)_0^2} + \Delta(q_jp_j)_0$$

$$q_i = (q_i)_0 + \frac{i\hbar(p_j)_0^2 + (\Delta q_j)_0^2(2(p_j)_0(q_j)_0 - (p_i)_0(q_i)_0) + 2(p_j)_0^2\Delta(q_jp_j)_0}{2(p_i)_0(p_j)_0^2} \qquad p_i = (p_i)_0\left(1 - \frac{(\Delta q_j)_0^2}{2(p_j)_0^2}\right)$$

$$q_j = (q_j)_0 - \frac{i\hbar(p_j)_0 + 2(p_j)_0\Delta(q_jp_j)_0 + (q_j)_0(\Delta q_j)_0^2}{2(p_j)_0^2} \qquad p_j = (p_j)_0\left(1 + \frac{(\Delta q_j)_0^2}{2(p_j)_0^2}\right). \tag{86}$$

These are the explicit expressions for the free variables of the q_j gauge in terms of the free variables of the q_i gauge.[24] We note that just as in the model of Sec. III, this transformation precisely cancels out the imaginary part (85) of the time variable q_i, rendering it real in the q_j gauge, while simultaneously giving q_j precisely the correct imaginary contribution expected of a time variable, if its initial value $(q_j)_0$ is real. See Appendix B 3 for the discussion of positivity of the gauge transformed state.

3. Evolution around the closed orbit

Finally, let us perform a sequence of gauge and clock changes until we fully evolve around the configuration space ellipse. As a result of the breakdown of the q_i Zeitgeist near the q_i turning point, the changes between the gauges and q_1- and q_2 time are required. The breakdown of the gauges and the necessity of gauge changes are precisely the effective analog of the apparent nonunitarity in the internal time Schrödinger regime in Sec. V B 2 and the ensuing breakdown of the constant q_i slicing and the resulting obligation to change the slicing and the clock. The jumps between the correlations which one would obtain when changing slicing in the Schrödinger regime translate into the jumps in correlations encountered in the gauge changes in Sec. V C 2. (As emphasized in Sec. V B, quantum relational observables valid for all classically allowed values of the chosen clock, therefore, do not exist.)

Apart from such quantum effects, the relational procedure works just as in the classical case. Because of the relativistic nature of the constraint, we are required to provide a time direction in which to evolve, since imposing only the relational initial data q_j, p_j, $(\Delta q_j)^2$, $(\Delta p_j)^2$ and $\Delta(q_jp_j)$ at a fixed value of q_i does not completely solve the IVP. As in the classical model and the Dirac approach, providing L_y, being the angular momentum, results in giving the required orientation to evolution. Using Eq. (81) and the expression for C in Eq. (75), p_i is determined up to sign when providing the relational initial data. The expression for L_y in Eq. (76) then implies that additionally providing L_y is equivalent to imposing the sign of p_i. Note that, unlike in the full quantum theory briefly described in Sec. V B and in complete accordance with semiclassicality, there cannot be a superposition of evolution in the two opposite orientations in the effective framework truncated at order $\hbar$.

Given this data, the system (82) can be solved (at least numerically) and we can relate the variables associated to (q_j, p_j) to the clock q_i and evolve forward in the q_i Zeitgeist in the given direction of evolution. Prior to the breakdown of this gauge, we translate to the q_j gauge and, thus, to a different set of fashionables. Then, just before the subsequent breakdown of the q_j Zeitgeist, we return to the q_i gauge and so forth, until fully evolving around the ellipse. In this way, the initial data is transported around the orbit independently of the gauge parameters, although employing different gauges and even different sets of fashionables in the different gauges (see also Sec. IV C).

It should be noted that, just as in Secs. V A 2 and V B 2, we could generate our physical evolution by a physical Hamiltonian, which would be obtained by simply linearizing Eq. (84) in p_i. The resulting relational evolution would, obviously, be identical to the one generated by C_H. Since the system generated by C_H is somewhat simpler to handle, we focus on Eq. (82) here. Notice also that the effective formalism reintroduces a gauge parameter even in the quantum theory (the parameter along the flow of C_H). Recall from the introduction that this gauge parameter simplifies a patching solution to the global problem of time in the classical case and that its absence in the quantum theory is one of the reasons for the difficulties occurring there. Nevertheless, the gauge parameter here is related to C_H which depends on the q_i Zeitgeist. When changing gauge, one necessarily obtains a separate gauge parameter and since the gauges break down prior to the classical turning points of the clocks, one cannot use the effective gauge parameters in the classical way to overcome the global problem of time.

As regards reconstructing the full coherent physical state from the Schrödinger regime, it was noted in Sec. V B 2 that one would need to explore whether the quantum versions of the Dirac observables (51) or (52), which

[24] Actually, not all these variables are free, as p_i can be eliminated in the q_i gauge with the use of C. We display its transformation for convenience, since we are using $(p_i)_0$ and $(p_j)_0$ within the above expressions.

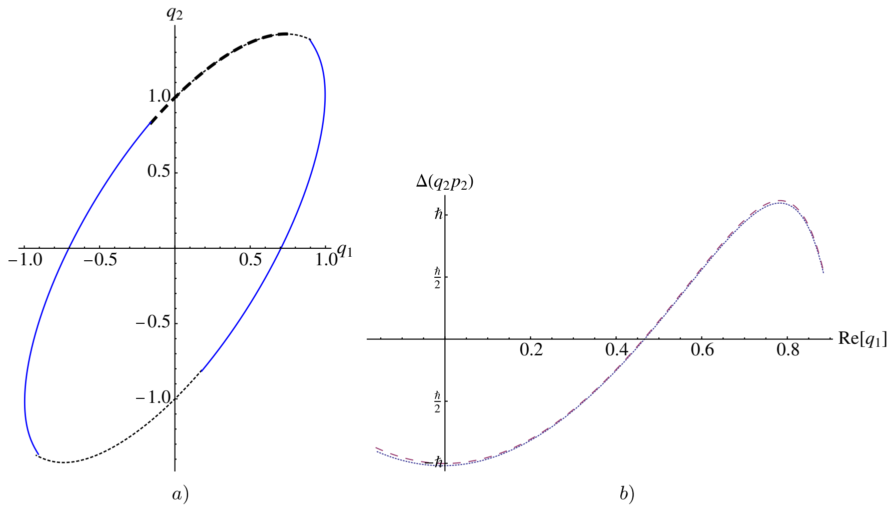

FIG. 9 (color online). (a) Reconstruction of a semiclassical physical state via gauge switching in the effective framework. The jumps between the q_1 gauge (black dotted and dashed curves) and the q_2 gauge (blue solid curves) are a consequence of the $o(\hbar)$ jumps in the gauge transformations (86). The final evolution in q_1 Zeitgeist after the fourth clock change is given by the fat black dashed curve and coincides to $o(\hbar)$ with the initial evolution in the q_1 gauge prior to the first clock change. For convenience we have labeled the axes by q_1 and q_2. It should be noted that for the curves in the q_i gauge, q_i actually refers to $\Re[q_i]$. (b) Comparison of $\Delta(q_2p_2)(\Re[q_1])$ in the q_1 gauge before (dashed curve) and after (dotted curve) the complete revolution around the ellipse. The difference between the two curves is clearly of $o(\hbar^2)$ or smaller. Initial data for both (a) and (b): $q_{1_0} = -\frac{i\hbar}{2}$, $p_{1_0} = q_{2_0} = p_{2_0} = 1$, $(\Delta q_2)_0^2 = (\Delta p_2)_0^2 = \frac{\hbar}{2}$. Furthermore, $M = 3$ and, to enhance effects, we have set $\hbar = 0.01$. The initial value for $\Delta(q_2p_2)$ follows from Eq. (84).

characterize the physical state, are constants of motion in a given constant q_i slicing and whether they are invariant under a change of slicing. In the present effective case, the answer to this problem is obvious: since the characterizing observables, for instance, (76) and their moments are complete Dirac observables of the effective system, they are invariant under the action of the constraints (75) and, therefore, also under the gauge changes of Sec. V C 2. Consequently, they are constant for the given orbit which we are analyzing and, as a result, we are always probing one and the same physical state. Since the internal time Schrödinger regime corresponds to the effective framework to this order, we conjecture that also in the Schrödinger regime, these observables remain invariant, although this is more difficult to prove explicitly.

As a specific example of an effective reconstruction of a semiclassical physical state via gauge switching, we provide a plot of the configuration space ellipse in Fig. 9(a) for a configuration whose initial data is provided in the caption of the figure. We have started in the q_1 Zeitgeist and changed gauge 4 times in the course of evolution, in order to reach the same gauge after a complete revolution around the ellipse. Since revolution numbers around the ellipse have no physical meaning in either the classical or the quantum theory, we only evolve once around the ellipse. In accordance with this, it is found that the discrepancy between the variables in the q_1 gauge before and after one complete revolution are of order $o(\hbar^2)$ or smaller. For the particular example of $\Delta(q_2p_2)(\Re[q_1])$ this is shown in Fig. 9(b); the two curves in the same gauge before and after the complete revolution agree extremely well to order $\hbar$, implying that they describe the same physical state. The jumps between the curves in the two different gauges are a consequence of the particular form of the gauge changes, as given in Sec. V C 2. In agreement with Sec. IV D, it is also found numerically that the end result does not depend on the precise instants of the intermediate gauge changes, as long as the two gauges are valid before and after the transformations. This shows consistency of the argument in Sec. IV D with the semiclassical approximation in this particular example.

Validity of the semiclassical approximation and the new and old gauge has to be checked when performing intermediate gauge changes. This is not problematic as long as the ellipse is reasonably close to a circle. For squeezed ellipses, however, when the turning points in q_1 and q_2 time may lie very close to each other, one has to be rather careful when precisely to carry out the gauge change, since in spite of a semiclassical trajectory, the spread will play a more restrictive role in this case. Nonetheless, this issue merely constitutes a practical, but not a conceptual problem.

VI. DISCUSSION AND CONCLUSIONS

In this article we have described in two simple toy models the effective approach of [17] to coping with the general problem of time in the semiclassical regime. A central additional ingredient for the interpretation of this approach is the relational concept of evolution.

By employing an effective framework, one benefits from the advantage of sidestepping many technical problems associated to the general problem of time, thereby facilitating an explicit investigation of various of its aspects, as well as their repercussions for the usual Dirac quantization.

In particular, the effective approach avoids the *Hilbert-space problem* altogether since no use of representations or physical inner products has been made at any point of the algebraic construction. The tedious problem of constructing physical states and inner products, which is often even practically impossible,[25] is replaced by evaluating an (infinite) coupled set of quantum variables which can be consistently truncated to a finite solvable system, for instance, at semiclassical order; necessary physicality conditions for observables are ultimately imposed just by reality conditions. At this stage, the effective framework can be implemented numerically and its physical properties can be studied in detail.

Although we can avoid practical problems in constructing physical Hilbert spaces, we do not intend to suggest solutions of effective constraints as full substitutes of physical states. Some questions, such as the measurement problem, can only be addressed with Hilbert-space representations. Effective techniques at present do not provide a complete description of quantum systems, but they can capture representation-independent information which is sufficient for many questions of interest.

The *multiple-choice problem*, furthermore, does not constitute a problem at the effective level, since, from the point of view of the Poisson manifold of the effective framework, all variables of a given order are treated on an equal footing. Just as in the classical case, we may choose whichever suitable (quantum) phase space clock function we desire and deparametrize in this variable. To simplify explicit calculations and interpretations, it is helpful to further impose gauge conditions on this effective constrained system, which are closely related to the choice of the clock variable and which fix all but one Hamiltonian gauge flow. Note that this gauge fixing happens *after* quantization. At this level, choosing different clocks means choosing different gauges and corresponding Zeitgeister in which to evaluate the effective system. As explicitly demonstrated in two examples, one can, moreover, translate between the different choices for internal time by means of gauge transformations. In fact, in the case of systems which admit the *global time problem* one is forced to change the local clocks in the course of relational evolution since gauges are, in general, not globally valid. It should be emphasized that deparametrizations with respect to different choices of internal time yield, in general, inequivalent Hilbert-space representations, and thus different gauges at the effective level generally correspond to different formulations of the quantum theory.

The usual *operator-ordering problem* is not entirely circumvented in this effective approach since we choose a particular ordering for the constraint operator before treating it effectively. This specific ordering, however, is not connected to the choice of a (local) time variable which happens only *after* the effective system has been constructed.

Of the technical problems briefly described in the introduction, it is only the *global time problem* and the *problem of observables* which are not automatically sidestepped by the effective approach. But by avoiding the other technical problems, the effective approach greatly facilitates the construction of a sufficient set of explicit fashionables since, although we face a larger number of degrees of freedom, the problem can be addressed in the usual classical manner which allowed for simple numerical solutions in the toy models studied in this article. The effective framework is, thus, amenable to techniques, usually aimed at a solution to the classical *problem of observables*, such as [8,10] and the perturbative expansions of [9]. Moreover, concrete evaluations of constrained systems are usually done by employing gauge fixing, for which classical methods such as those of [25] are useful.

Likewise, the effective approach enables us to perform a concrete treatment of the *global time problem* and suggests a simple patching solution. As discussed in Sec. V, the relational concept is only of a local and semiclassical nature in the absence of a global clock and, thus, the *problem of relational observables* becomes a local one. Global relational observables valid for all classical values of relational time do not exist in the quantum theory. While in the absence of global clocks it is not at all clear how to implement the relational concept and explicitly construct relational Dirac observable operators in a Dirac quantization, some simplification is offered by local deparametrization, resulting in a local internal time Schrödinger regime. In contrast to this, it is clear how to implement this scenario in a simple way within the effective semiclassical analysis, which reproduces the results of the local Schrödinger regime. An apparent nonunitarity leads to the breakdown of a constant time slicing in this procedure and to the failure of the gauge associated to the choice of local time in the effective framework. This is consistent with the related breakdown of the relational observables in the reduction and in the Dirac method on approach to a turning point [7]. To achieve a consistent evolution through turning points of a clock, we are forced to switch to a different clock and a different set of variables to be evolved, prior to reaching a turning point, which corresponds to switching to a different local Schrödinger regime and a gauge change in the effective approach. By switching to a good local clock, when another time variable approaches a turning point, we can consistently transport relational data along and thereby

[25] Reference [20] notwithstanding, for the issue of defining physical evolution in the absence of global clocks has not been addressed in these approaches.

reconstruct the entire information of a semiclassical physical state via local patches of relational evolution. To our knowledge, there is no consistent method for explicitly transferring data between different local deparametrizations of one and the same model at a Hilbert-space level. Any such method is likely to be quite involved, to lead to discontinuities in correlations and to be only applicable for states that are sufficiently semiclassical. On the other hand, the gauge changes are easily implemented on the effective side, albeit exhibiting jumps of order $\hbar$ in correlations, which underline the merely local nature of relational observables. No sharp instant for the change in time prior to a turning point has to be selected; the transformation may be performed at any point, as long as the old and new choice of time are valid before and after the clock change, respectively.

As regards relational Hamiltonian evolution, in the second model we have discussed the peculiarities associated to the IVP and the issue of time direction in the absence of a global clock. While we may classically keep one and the same relational time variable and only have to switch the sign of the physical Hamiltonian at the turning point of the clock, we are required to change the Hamiltonian operator of the internal time Schrödinger regime to a new one adapted to a new local clock *before* reaching the classical turning point. On the effective side, we could proceed similarly by linearizing the Hamiltonian constraint in the momentum conjugate to internal time in the gauge associated to the chosen clock. Such an effective physical Hamiltonian, obviously, changes together with the Hamiltonian constraint during necessary gauge changes prior to turning points of nonglobal clocks.

A final striking consequence of the *global time problem* is the inevitable appearance of a *complex internal time*. We have shown that the particular form of the imaginary contribution to the time variable is a quantum effect and a generic feature of the effective approach. Similarly, we have collected strong evidence from an expectation value calculation of the time operator in a Dirac approach to the free relativistic particle and a comparison of the quadratic Wheeler-DeWitt equation to an associated internal time Schrödinger equation that this particular imaginary contribution is also a generic feature of standard Hilbert-space quantizations. In particular, the inequivalence between the Wheeler-DeWitt and Schrödinger equation in the presence of a ‘‘time potential’’ is a result of the assumption that time is real valued in both equations. The two equations can be locally reconciled if the expectation value of internal time is allowed a particular imaginary contribution in the WDW case. By the same token, as shown in the concrete example in Sec. V B 2, Dirac observables of the system governed by the quadratic constraint are, in general, constants of motion of the associated Schrödinger regime only if internal time is complex in the Wheeler-DeWitt equation.

Despite the fact that the imaginary contribution to time also appears for globally valid clocks, the imaginary contribution can be disregarded altogether in this case, since it turns out to be a constant of motion which is not necessary for the satisfaction of the constraints. For nonglobal clocks, however, the imaginary contribution turns out to be dynamical and cannot at all be ignored. It is, therefore, rather a true nonglobal feature. When the local clock eventually needs to be exchanged together with the corresponding gauge at the effective level, the imaginary contribution is consistently removed from the old clock which subsequently turns into an evolving physical variable and pushed, accordingly, to the new clock function.

Concerning relational evolution in the presence of a dynamical imaginary contribution to internal time, we encounter the issue of a ‘‘vector time’’ with two separate degrees of freedom. In this article, however, we argue, in agreement with common sense, to only employ the real part of the internal clock as relational time, since the imaginary part causes a number of additional problems, rendering it an even worse clock than the already nonglobal real part.

In conclusion, the effective approach to the problem of time overcomes a number of technical problems and substantially facilitates the solution to various other problems, while simultaneously providing further insight into standard Hilbert-space quantizations. In particular, it is possible to master the *global time problem* at the semiclassical level and to consistently evolve data through turning points of nonglobal clocks. In this article and in [17], we have, furthermore, argued that the standard notion of relational time and the concept of relational evolution are, in general, of merely local and semiclassical nature, which disappear (together with complex relational time) for highly quantum states of systems without global clock variables.

We emphasize that these results and conclusions are based on a semiclassical analysis in simple toy models. It is, certainly, dangerous to draw any general conclusions for full quantum gravity from procedures which so far are only proven to work in simple scenarios. Moreover, further technical problems, specifically related to gravity, such as, e.g., the *spacetime reconstruction problem*, require significant advances in the effective formalism before they may be tackled. Nevertheless, we believe that the present approach is worth pursuing and promises some headway in evaluating quantum gravity theories and models in a practical way. In this light, we expect certain features, such as complex internal time, to be of a generic nature in more general models, especially in quantum cosmology.

Owing to the advantage that the effective approach simultaneously avoids many facets of the problem of time, it may be viewed as one step in the quest to ‘‘defeat the Ice Dragon'' of [3], symbolizing the conjunction of the apparently many faces of the problem of time in quantum gravity.

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APPENDIX A: POISSON ALGEBRA

Expectation values satisfy the classical Poisson algebra and have vanishing Poisson brackets with the moments of all orders. Table II lists the Poisson brackets between second order moments generated by two canonical pairs of observables. The table has originally appeared in the appendix of [19] and is reproduced here for convenience.

APPENDIX B: DISCUSSION OF POSITIVITY

1. Algebraic positivity

Positivity is understood in the algebraic sense as the condition $\langle \mathbf{A}\mathbf{A}^* \rangle \geq 0$, $\forall \mathbf{A} \in \mathcal{A}$, where $\mathcal{A}$ is some algebra. It relates directly to the Gelfand-Naimark-Segal construction of unitary representations for $*$ algebras, and is also necessary for the measurement theory and probabilistic interpretation of the state. In this appendix we focus on the unital star algebra $\mathcal{A}$ of all finite-order polynomials generated by a single canonical pair $\hat{q}$ and $\hat{p}$ subject to

$$[\hat{q}, \hat{p}] = i\hbar \mathbb{1} \quad \text{and} \quad \hat{q}^* = \hat{q}, \qquad \hat{p}^* = \hat{p}.$$

We pose the following question:

- What are the *necessary* and *sufficient* conditions one needs to place on a state on $\mathcal{A}$ such that positivity holds to order $\hbar$?
  By ''positivity holding to order $\hbar$'' we mean that $|\Im[\langle \mathbf{A}\mathbf{A}^* \rangle]| \propto \hbar^{(3/2)}$ and $\Re[\langle \mathbf{A}\mathbf{A}^* \rangle] \geq -\hbar^{(3/2)}$. The answer turns out to be simple, in addition to normalization $\langle \mathbb{1} \rangle = 1$, we need to impose

$$q, p, (\Delta q)^2, (\Delta p)^2, \Delta(qp) \in \mathbb{R} \quad (\Delta p)^2, (\Delta q)^2 \geq 0$$
$$(\Delta q)^2 (\Delta p)^2 - (\Delta(qp))^2 \geq \frac{1}{4}\hbar^2. \tag{B1}$$

We only outline the demonstration of *necessity*, as these are standard results in ordinary quantum mechanics:

(i) We recall that positivity can be used to derive $\langle \mathbf{A}^* \rangle = \langle \bar{\mathbf{A}} \rangle$, where bar denotes the complex conjugate. This immediately implies $q, p, (\Delta q)^2, (\Delta p)^2, \Delta(qp) \in \mathbb{R}$.

(ii) $\langle (\hat{q} - \langle \hat{q} \rangle \mathbb{1})(\hat{q} - \langle \hat{q} \rangle \mathbb{1})^* \rangle \geq 0$ immediately gives $(\Delta q)^2 \geq 0$, we similarly get $(\Delta p)^2 \geq 0$.

(iii) The uncertainty relation can be obtained by first deriving the Schwarz-type inequality $|\langle \mathbf{A}\mathbf{B}^* \rangle|^2 \leq \langle \mathbf{A}\mathbf{A}^* \rangle \langle \mathbf{B}\mathbf{B}^* \rangle$, and substituting $\mathbf{A} = \hat{q} - q\mathbb{1}$ and $\mathbf{B} = \hat{p} - p\mathbb{1}$.

Before we demonstrate *sufficiency*, we derive an inequality implied by (B1), which we will use on several occasions in this section and the following ones:

$$\alpha^2 (\Delta q)^2 + \beta^2 (\Delta p)^2 + 2\alpha\beta \Delta(qp) \geq 0, \quad \forall\, \alpha, \beta \in \mathbb{R}. \tag{B2}$$

This follows as

TABLE II. Poisson algebra of second order moments. First terms in the bracket are labeled by rows, second terms are labeled by columns.

	$(\Delta t)^2$	$\Delta(tp_t)$	$(\Delta p_t)^2$	$(\Delta q)^2$	$\Delta(qp)$	$(\Delta p)^2$	$\Delta(tq)$	$\Delta(p_t p)$	$\Delta(tp)$	$\Delta(p_t q)$
$(\Delta t)^2$	0	$2(\Delta t)^2$	$4\Delta(tp_t)$	0	0	0	0	$2\Delta(tp)$	0	$2\Delta(tq)$
$\Delta(tp_t)$	$-2(\Delta t)^2$	0	$2(\Delta p_t)^2$	0	0	0	$-\Delta(tq)$	$\Delta(p_t p)$	$-\Delta(tp)$	$\Delta(p_t q)$
$(\Delta p_t)^2$	$-4\Delta(tp_t)$	$-2(\Delta p_t)^2$	0	0	0	0	$-2\Delta(p_t q)$	0	$-2\Delta(p_t p)$	0
$(\Delta q)^2$	0	0	0	0	$2(\Delta q)^2$	$4\Delta(qp)$	0	$2\Delta(p_t q)$	$2\Delta(tq)$	0
$\Delta(qp)$	0	0	0	$-2(\Delta q)^2$	0	$2(\Delta p)^2$	$-\Delta(tq)$	$\Delta(p_t p)$	$\Delta(tp)$	$-\Delta(p_t q)$
$(\Delta p)^2$	0	0	0	$-4\Delta(qp)$	$-2(\Delta p)^2$	0	$-2\Delta(tp)$	0	0	$-2\Delta(p_t p)$
$\Delta(tq)$	0	$\Delta(tq)$	$2\Delta(p_t q)$	0	$\Delta(tq)$	$2\Delta(tp)$	0	$\Delta(tp_t)$ $+\Delta(qp)$	$(\Delta t)^2$	$(\Delta q)^2$
$\Delta(p_t p)$	$-2\Delta(tp)$	$-\Delta(p_t p)$	0	$-2\Delta(p_t q)$	$-\Delta(p_t p)$	0	$-\Delta(tp_t)$ $-\Delta(qp)$	0	$-(\Delta p)^2$	$-(\Delta p_t)^2$
$\Delta(tp)$	0	$\Delta(tp)$	$2\Delta(p_t p)$	$-2\Delta(tq)$	$-\Delta(tp)$	0	$-(\Delta t)^2$	$(\Delta p)^2$	0	$\Delta(qp)$ $-\Delta(tp_t)$
$\Delta(p_t q)$	$-2\Delta(tq)$	$-\Delta(p_t q)$	0	0	$\Delta(p_t q)$	$2\Delta(p_t p)$	$-(\Delta q)^2$	$(\Delta p_t)^2$	$\Delta(tp_t)$ $-\Delta(qp)$	0

$$\begin{aligned}\alpha^2(\Delta q)^2+\beta^2(\Delta p)^2+2\alpha\beta\Delta(qp) &\geq \alpha^2(\Delta q)^2+\beta^2(\Delta p)^2-2|\alpha||\beta||\Delta(qp)|\\ &\geq |\alpha|^2(\Delta q)^2+|\beta|^2(\Delta p)^2-2|\alpha||\beta|\sqrt{(\Delta q)^2(\Delta p)^2}\geq(|\alpha|\sqrt{(\Delta q)^2}-|\beta|\sqrt{(\Delta p)^2})^2\\ &\geq 0.\end{aligned}$$

To demonstrate *sufficiency* to order $\hbar$, we adopt a rather direct approach. Any finite-order polynomial in $\hat{q}$ and $\hat{p}$ can be expanded using the symmetrized products $(\hat{q}^m\hat{p}^n)_{\mathrm{Weyl}}$

$$\hat{f}=\sum_{m,n\geq 0}\alpha_{mn}(\hat{q}^m\hat{p}^n)_{\mathrm{Weyl}}=:f(\hat{q},\hat{p}).$$

Here, $f(\hat{q},\hat{p})$ is understood as a map from the algebra to itself, in particular, it keeps track of the ordering, which we chose to be completely symmetric in this case. In general, $\alpha_{mn}\in\mathbb{C}$, for self-adjoint elements $\alpha_{mn}\in\mathbb{R}$. We now expand the polynomial in terms of a different set of elements $\widehat{\Delta q}:=\hat{q}-q$ and $\widehat{\Delta p}:=\hat{p}-p$. Evidently

$$\begin{aligned}\hat{f}&=f(\hat{q},\hat{p})=f(q+\widehat{\Delta q},p+\widehat{\Delta p})\\ &=f(q,p)+\frac{\partial f}{\partial q}(q,p)\widehat{\Delta q}+\frac{\partial f}{\partial p}(q,p)\widehat{\Delta p}+\frac{1}{2}\frac{\partial^2 f}{\partial q^2}(q,p)(\widehat{\Delta q})^2+\frac{1}{2}\frac{\partial^2 f}{\partial p^2}(q,p)(\widehat{\Delta p})^2+\frac{\partial^2 f}{\partial q\partial p}(q,p)(\widehat{\Delta q}\,\widehat{\Delta p})_{\mathrm{Weyl}}\\ &\quad+(\text{higher powers of}\,\widehat{\Delta q},\widehat{\Delta p}).\end{aligned}$$

q and p can be any real numbers, below we set them to the expectation values $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$, which enables us to utilize semiclassical truncation. Keeping terms of order
$\hbar$ we find the expectation value of $\hat{f}$

$$\langle\hat{f}\rangle=f(q,p)+\frac{1}{2}\frac{\partial^2 f}{\partial q^2}(q,p)(\Delta q)^2+\frac{1}{2}\frac{\partial^2 f}{\partial p^2}(q,p)(\Delta p)^2+\frac{\partial^2 f}{\partial q\partial p}(q,p)\Delta(qp)+O(\hbar^{3/2}),$$

so that, again to order $\hbar$, we have

$$\begin{aligned}|\langle\hat{f}\rangle|^2&=|f|^2+\frac{1}{2}\bigg[f\bigg(\overline{\frac{\partial^2 f}{\partial q^2}}\bigg)+\bar{f}\bigg(\frac{\partial^2 f}{\partial q^2}\bigg)\bigg](\Delta q)^2+\frac{1}{2}\bigg[f\bigg(\overline{\frac{\partial^2 f}{\partial p^2}}\bigg)+\bar{f}\bigg(\frac{\partial^2 f}{\partial p^2}\bigg)\bigg](\Delta p)^2+\bigg[f\bigg(\overline{\frac{\partial^2 f}{\partial q\partial p}}\bigg)+\bar{f}\bigg(\frac{\partial^2 f}{\partial q\partial p}\bigg)\bigg]\Delta(qp)\\ &\quad+O(\hbar^{(3/)2}).\end{aligned}$$

We note that since $|\langle\hat{f}\rangle|^2\geq 0$, the truncated expression for $|\langle\hat{f}\rangle|^2$, satisfies the inequality to order $\hbar$ in the sense discussed earlier. Now consider positivity of the state evaluated on $\hat{f}$:

$$\begin{aligned}\langle\hat{f}\hat{f}^*\rangle&=\bigg\langle\bigg(f+\frac{\partial f}{\partial q}\widehat{\Delta q}+\frac{\partial f}{\partial p}\widehat{\Delta p}+\frac{1}{2}\frac{\partial^2 f}{\partial q^2}(\widehat{\Delta q})^2+\frac{1}{2}\frac{\partial^2 f}{\partial p^2}(\widehat{\Delta p})^2+\frac{\partial^2 f}{\partial q\partial p}(\widehat{\Delta q}\,\widehat{\Delta p})_{\mathrm{Weyl}}\bigg)\\ &\quad\times\bigg(\bar{f}+\overline{\frac{\partial f}{\partial q}}\widehat{\Delta q}+\overline{\frac{\partial f}{\partial p}}\widehat{\Delta p}+\frac{1}{2}\overline{\frac{\partial^2 f}{\partial q^2}}(\widehat{\Delta q})^2+\frac{1}{2}\overline{\frac{\partial^2 f}{\partial p^2}}(\widehat{\Delta p})^2+\overline{\frac{\partial^2 f}{\partial q\partial p}}(\widehat{\Delta q}\,\widehat{\Delta p})_{\mathrm{Weyl}})\bigg\rangle+O(\hbar^{3/2})\\ &=|f|^2+\frac{1}{2}\bigg[f\bigg(\overline{\frac{\partial^2 f}{\partial q^2}}\bigg)+\bar{f}\bigg(\frac{\partial^2 f}{\partial q^2}\bigg)\bigg](\Delta q)^2+\frac{1}{2}\bigg[f\bigg(\overline{\frac{\partial^2 f}{\partial p^2}}\bigg)+\bar{f}\bigg(\frac{\partial^2 f}{\partial p^2}\bigg)\bigg](\Delta p)^2+\bigg[f\bigg(\overline{\frac{\partial^2 f}{\partial q\partial p}}\bigg)+\bar{f}\bigg(\frac{\partial^2 f}{\partial q\partial p}\bigg)\bigg]\Delta(qp)\\ &\quad+\bigg|\frac{\partial f}{\partial q}\bigg|(\Delta q)^2+\bigg|\frac{\partial f}{\partial p}\bigg|(\Delta p)^2+2\Re\bigg[\frac{\partial f}{\partial q}\overline{\frac{\partial f}{\partial p}}\bigg]\Delta(qp)+O(\hbar^{3/2})\\ &=|\langle\hat{f}\rangle|^2+\bigg|\frac{\partial f}{\partial q}\bigg|(\Delta q)^2+\bigg|\frac{\partial f}{\partial p}\bigg|(\Delta p)^2+2\Re\bigg[\frac{\partial f}{\partial q}\overline{\frac{\partial f}{\partial p}}\bigg]\Delta(qp)+O(\hbar^{3/2}).\end{aligned}$$

Now $|\langle\hat{f}\rangle|^2\geq 0$, and the next three terms are positive by inequality (B2)

$$\bigg|\frac{\partial f}{\partial q}\bigg|(\Delta q)^2+\bigg|\frac{\partial f}{\partial p}\bigg|(\Delta p)^2+2\Re\bigg[\frac{\partial f}{\partial q}\overline{\frac{\partial f}{\partial p}}\bigg]\Delta(qp)\geq\bigg|\frac{\partial f}{\partial q}\bigg|(\Delta q)^2+\bigg|\frac{\partial f}{\partial p}\bigg|(\Delta p)^2-2\bigg|\frac{\partial f}{\partial q}\bigg|\bigg|\frac{\partial f}{\partial p}\bigg||\Delta(qp)|\geq 0.$$

So that, as claimed earlier, $\langle\hat{f}\hat{f}^*\rangle\geq 0$ to order $\hbar$.

2. Positivity in the model of Sec. III

Here we use the explicit form of gauge invariant functions to prove the following statements to order $\hbar$ for the relativistic particle in a λt potential:

(i) the positivity of a state is preserved by the dynamics in the t gauge,

(ii) it is also preserved by gauge transformation between the q gauge and the t gauge,

(iii) finally it is preserved by the dynamics in the q gauge.

The constraint in this model is

$$\hat{C} = \hat{p}_t^2 - \hat{p}^2 - m^2\mathbb{1} + \lambda\hat{t}.$$

A complete set of Dirac observables may be constructed from the canonical pair:

$$\hat{\mathcal{Q}} := \hat{q} - \frac{2}{\lambda}\hat{p}\hat{p}_t \text{ and } \hat{\mathcal{P}} := \hat{p}, \text{ satisfying } [\hat{\mathcal{Q}}, \hat{\mathcal{P}}] = i\hbar\mathbb{1},$$

which commute with the constraint $[\hat{\mathcal{Q}}, \hat{C}] = 0 = [\hat{\mathcal{P}}, \hat{C}]$. Below we provide the expectation values and second order moments of these observables:

$$\mathcal{Q} = q - \frac{2}{\lambda}(pp_t + \Delta(p_t p)), \quad \mathcal{P} = p, \quad (\Delta\mathcal{P})^2 = (\Delta p)^2, \quad \Delta(\mathcal{Q}\mathcal{P}) = \Delta(qp) - \frac{2}{\lambda}(\Delta(p_t pp) + p_t(\Delta p)^2 + p\Delta(p_t p))$$

$$(\Delta\mathcal{Q})^2 = (\Delta q)^2 - \frac{4}{\lambda}(\Delta(p_t qp) + p_t\Delta(qp) + p\Delta(p_t q)) + \frac{4}{\lambda^2}[\Delta(p_t p_t pp) + 2p_t\Delta(p_t pp) + 2p\Delta(p_t p_t p) + p_t^2(\Delta p)^2 + p^2(\Delta p_t)^2 + (2p_t p - \Delta(p_t p))\Delta(p_t p)].$$

Poisson brackets of these functions with constraint functions must vanish to the given order, since the operators that generate them commute with the constraint operator [see Eq. (2)]. Additionally, we note that $p = \mathcal{P}$ is a constant of motion, while p_t evolves as $p_t(s) = -\lambda s + p_{t_0}$ and is preserved by the transformation between the gauges, therefore, the condition $p_t,\ p \in \mathbb{R}$ is preserved in all situations considered here.

a. Dynamics in the t gauge

Below are the expressions for the same invariants truncated at order $\hbar$, evaluated in the t gauge, with the moments generated by $\hat{p}_t$ eliminated through constraint functions:

$$\mathcal{Q} = q - \frac{2}{\lambda}\left(pp_t + \frac{p}{p_t}(\Delta p)^2\right), \quad \mathcal{P} = p,$$

$$(\Delta\mathcal{Q})^2 = (\Delta q)^2 - 2\theta\Delta(qp) + \theta^2(\Delta p)^2, \quad (\Delta\mathcal{P})^2 = (\Delta p)^2$$

$$\Delta(\mathcal{Q}\mathcal{P}) = \Delta(qp) - \theta(\Delta p)^2, \text{ where } \theta = \frac{2(p_t^2 + p^2)}{\lambda p_t}.$$

We now re-express the gauge dependent moments in terms of these invariants:

$$(\Delta q)^2 = (\Delta\mathcal{Q})^2 + \theta^2(\Delta\mathcal{P})^2 + 2\theta\Delta(\mathcal{Q}\mathcal{P})$$

$$(\Delta p)^2 = (\Delta\mathcal{P})^2 \qquad \Delta(qp) = \Delta(\mathcal{Q}\mathcal{P}) + \theta(\Delta\mathcal{P})^2.$$

Assuming that θ is real (which holds provided p_t and p are real), one can see that the

(i) reality of invariant moments implies reality of evolving moments,

(ii) trivially $(\Delta\mathcal{P})^2 > 0 \Rightarrow (\Delta p)^2 > 0$,

(iii) $(\Delta q)^2 > 0$ follows directly from the inequality (B2),

(iv) finally one finds

$$(\Delta q)^2(\Delta p)^2 - (\Delta(qp))^2 = (\Delta\mathcal{Q})^2(\Delta\mathcal{P})^2 - (\Delta(\mathcal{Q}\mathcal{P}))^2 \geq \frac{\hbar^2}{4}.$$

In short, positivity of the observables implies positivity of t gauge variables, provided θ is real. The converse is also true: positivity of t gauge observables (together with $p_t \in \mathbb{R}$) implies positivity of the invariants. The Dirac observables are invariant under gauge transformations and, in particular, under the t gauge dynamics, which must then preserve positivity of the invariant moments and, therefore, also of the evolving moments.

b. Dynamics in the q gauge

We now verify the equivalent statement in the q gauge. In this gauge, the invariant moments to order $\hbar$ are given by:

$$(\Delta\mathcal{Q})^2 = \frac{1}{\theta\nu - 1}((\Delta t)^2 + \theta^2(\Delta p_t)^2 + 2\theta\Delta(tp_t))$$

$$(\Delta\mathcal{P})^2 = \frac{1}{\theta\nu - 1}((\Delta p_t)^2 + 2\nu\Delta(tp_t) + \nu^2(\Delta t)^2)$$

$$\Delta(\mathcal{Q}\mathcal{P}) = \frac{-1}{\theta\nu - 1}((\theta\nu + 1)\Delta(tp_t) + \theta(\Delta p_t)^2 + \nu(\Delta t)^2),$$

where $\theta = \frac{2(p_t^2+p^2)}{\lambda p_t}$ and $\nu = \frac{\lambda}{2p_t}$, so that $\frac{1}{\theta\nu-1} = \frac{p_t^2}{p^2}$. These relations are tricky to invert by hand, but the final result is exactly symmetrical, it just so happens that the above transformation is its own inverse:

$$(\Delta t)^2 = \frac{1}{\theta\nu - 1}((\Delta\mathcal{Q})^2 + \theta^2(\Delta\mathcal{P})^2 + 2\theta\Delta(\mathcal{Q}\mathcal{P}))$$
$$(\Delta p_t)^2 = \frac{1}{\theta\nu - 1}((\Delta\mathcal{P})^2 + 2\nu\Delta(\mathcal{Q}\mathcal{P}) + \nu^2(\Delta\mathcal{Q})^2)$$
$$\Delta(tp_t) = \frac{-1}{\theta\nu - 1}((\theta\nu + 1)\Delta(\mathcal{Q}\mathcal{P}) + \theta(\Delta\mathcal{P})^2 + \nu(\Delta\mathcal{Q})^2). \tag{B3}$$

If p_t and p are real and if $p \neq 0$, then $\frac{1}{\theta\nu-1} \geq 0$, with equality only when $p_t = 0$. We can use the same arguments as before to show that positivity of the invariants implies positivity of the q gauge moments (for the $p_t = 0$ case we substitute the expressions for θ and ν in terms of p_t and p first). In particular,

$$(\Delta t)^2(\Delta p_t)^2 - (\Delta(tp_t))^2 = (\Delta\mathcal{Q})^2(\Delta\mathcal{P})^2 - (\Delta(\mathcal{Q}\mathcal{P}))^2 \geq \frac{\hbar^2}{4}.$$

We note that, once we enforce $p_t, p \in \mathbb{R}$, the reality of t in this gauge follows directly from setting $\langle\hat{C}\rangle = 0$ and the reality of the moments of $\hat{t}$ and $\hat{p}_t$. Eliminating $(\Delta p)^2$ through other constraints and imposing the q gauge conditions, $\langle\hat{C}\rangle = 0$ gives

$$t = \frac{1}{\lambda}\left[p^2 + m^2 - p_t^2 + \frac{p_t^2 - p^2}{p^2}(\Delta p_t)^2 + \frac{\lambda p_t}{p^2}\Delta(tp_t) + \frac{\lambda^2}{4p^2}(\Delta t)^2\right].$$

Reality of $\mathcal{Q}$ then provides a condition on the imaginary part of q, since in this gauge

$$\mathcal{Q} = q - \frac{2}{\lambda}pp_t - \frac{2p_t}{\lambda p}(\Delta p_t)^2 - \frac{1}{p}\Delta(tp_t) + \frac{i\hbar}{2p},$$

so that $\mathcal{Q} \in \mathbb{R}$ implies $\Im[q] = -\frac{i\hbar}{2p}$, which is compatible with the transformation between the two gauges derived in Sec. III.

We have demonstrated that the positivity of the invariant observables together with $p_t \in \mathbb{R}$ results in the positivity of the evolving q gauge observables and yields the imaginary part of q. The converse can also be demonstrated, namely, starting with the positivity of the q gauge observables and $\Im[q] = -\frac{i\hbar}{2p}$, one discovers that the invariants are positive (to demonstrate that $p \in \mathbb{R}$ one needs to select the solution to the constraint functions compatible with the semiclassical approximation). This shows that positivity is preserved by the dynamics in the q gauge.

c. Gauge transformation

The gauge transformation of the second-order moments from the t gauge to the q gauge can be written as

$$(\Delta t)^2 = (\Delta q)_0^2\frac{p_t^2}{p^2}$$
$$(\Delta p_t)^2 = \frac{p^2}{p_t^2}((\Delta p)_0^2 + \mu^2(\Delta q)_0^2 - 2\mu\Delta(qp)_0)$$
$$\Delta(tp_t) = \Delta(qp)_0 - \mu(\Delta q)_0^2, \quad \text{where } \mu = \frac{\lambda p_t}{2p^2}.$$

Assuming $p_t > 0$, and that p and λ are real (which also means that μ is real), it follows in a similar way that
(i) $(\Delta q)_0^2 > 0 \Rightarrow (\Delta t)^2 > 0$,
(ii) once again, $(\Delta p_t)^2 > 0$ follows from the inequality (B2),
(iii) one also finds

$$(\Delta t)^2(\Delta p_t)^2 - (\Delta(tp_t))^2 = (\Delta q)^2(\Delta p)^2 - (\Delta(qp))^2 \geq \frac{\hbar^2}{4}.$$

So that a positive state in the t gauge transforms to a positive state in the q gauge. The reverse gauge transformation can be analyzed identically.

3. Positivity in the timeless model of Sec. V

We will not establish the positivity-preserving properties of effective dynamics within this model, instead, we point out its close relation with a local internal time Schrödinger evolution, which by construction preserves positivity so long as it remains valid.

We briefly show that the gauge transformation (86) of Sec. V C 2 consistently transfers positivity between the two sets of physical variables to order $\hbar$. Firstly, we note that the only initial parameter that has an imaginary part is $(q_i)_0$. The imaginary contribution (85) is of order $\hbar$ and leads to the imaginary contributions to the final values of q_i, p_i, $(\Delta q_i)^2$, $(\Delta p_i)^2$, $\Delta(q_ip_i)$ only at order $\hbar^2$. Hence, to order $\hbar$ these variables are real in the q_j gauge. In addition:
(i) $(\Delta q_j)_0^2 \geq 0$ implies $(\Delta q_j)^2 \geq 0$,
(ii) $(\Delta p_i)^2 \geq 0$ follows once again from the inequality (B2),
(iii) The uncertainty relation follows after some straightforward algebraic manipulations.

APPENDIX C: EXPLICIT MOMENTS FOR THE SCHRÖDINGER REGIME OF SEC. V B 2

In Eq. (71), we provided the explicit form of the expectation values for $\hat{q}_2$ and $\hat{p}_2$ as functions of q_1, i.e., as fashionables, in the internal time Schrödinger regime. Below we also provide the explicit form of the moments associated to these two operators.

$$
\begin{aligned}
(\Delta q_2)^2(q_1) &= \langle \hat{q}_2^2\rangle(q_1) - \langle \hat{q}_2\rangle^2(q_1) = \frac{\hbar}{2}\langle z(q_1)|\hat{a}^2 + \hat{a}^{+2} + 2\hat{a}\hat{a}^+ + \hat{\mathbf{1}}|z(q_1)\rangle - \langle \hat{q}_2\rangle^2(q_1)\\
&= e^{-|z|^2}\sum_{n\geq 0}\frac{|z|^{2n}}{n!}\left(\frac{q_{2_0}^2 - p_{2_0}^2}{2}\cos\left(\frac{E_n(q_1) - E_{n+2}(q_1)}{\hbar}\right) - q_{2_0}p_{2_0}\sin\left(\frac{E_n(q_1) - E_{n+2}(q_1)}{\hbar}\right)\right) + \frac{q_{2_0}^2 + p_{2_0}^2}{2}\\
&\quad + \frac{\hbar}{2} - \langle \hat{q}_2\rangle^2(q_1),\\
(\Delta p_2)^2(q_1) &= \langle \hat{p}_2^2\rangle(q_1) - \langle \hat{p}_2\rangle^2(q_1) = \frac{\hbar}{2}\langle z(q_1)| - \hat{a}^2 - \hat{a}^{+2} + 2\hat{a}\hat{a}^+ + \hat{\mathbf{1}}|z(q_1)\rangle - \langle \hat{p}_2\rangle^2(q_1)\\
&= -e^{-|z|^2}\sum_{n\geq 0}\frac{|z|^{2n}}{n!}\left(\frac{q_{2_0}^2 - p_{2_0}^2}{2}\cos\left(\frac{E_n(q_1) - E_{n+2}(q_1)}{\hbar}\right) - q_{2_0}p_{2_0}\sin\left(\frac{E_n(q_1) - E_{n+2}(q_1)}{\hbar}\right)\right) + \frac{q_{2_0}^2 + p_{2_0}^2}{2}\\
&\quad + \frac{\hbar}{2} - \langle \hat{p}_2\rangle^2(q_1),\\
\Delta(q_2p_2)(q_1) &= \frac{1}{2}\langle(\hat{q}_2 - \langle \hat{q}_2\rangle)(\hat{p}_2 - \langle \hat{p}_2\rangle) + (\hat{p}_2 - \langle \hat{p}_2\rangle)(\hat{q}_2 - \langle \hat{q}_2\rangle)\rangle = \langle(\hat{q}_2 - \langle \hat{q}_2\rangle)(\hat{p}_2 - \langle \hat{p}_2\rangle)\rangle - \frac{i\hbar}{2}\\
&= \left\langle\sqrt{\frac{\hbar}{2}}(-\langle \hat{p}_2\rangle + i\langle \hat{q}_2\rangle)\hat{a} - \sqrt{\frac{\hbar}{2}}(\langle \hat{p}_2\rangle + i\langle \hat{q}_2\rangle)\hat{a}^+ + \langle \hat{q}_2\rangle\langle \hat{p}_2\rangle + \frac{i\hbar}{2}(\hat{a}^{+2} - \hat{a}^2)\right\rangle\\
&= e^{-|z|^2}\sum_{n\geq 0}\frac{|z|^{2n}}{n!}\left(\left(\langle \hat{q}_2\rangle(q_1)q_{2_0} - \langle \hat{p}_2\rangle(q_1)p_{2_0}\right)\sin\left(\frac{E_{n+1}(q_1) - E_n(q_1)}{\hbar}\right) - \left(\langle \hat{p}_2\rangle(q_1)q_{2_0} + \langle \hat{q}_2\rangle(q_1)p_{2_0}\right)\right.\\
&\quad\left.\times\cos\left(\frac{E_{n+1}(q_1) - E_n(q_1)}{\hbar}\right) + \frac{q_{2_0}^2 - p_{2_0}^2}{2}\sin\left(\frac{E_n(q_1) - E_{n+2}(q_1)}{\hbar}\right) + q_{2_0}p_{2_0}\cos\left(\frac{E_n(q_1) - E_{n+2}(q_1)}{\hbar}\right)\right)\\
&\quad + \langle \hat{q}_2\rangle(q_1)\langle \hat{p}_2\rangle(q_1).
\end{aligned}
\tag{C1}
$$

APPENDIX D: IMAGINARY CONTRIBUTIONS IN THE q_i GAUGE OF SEC. V C

Here we want to summarize the analysis, which leads to the standard imaginary contribution (85) to the clock q_i in q_i Zeitgeist.

Linearizing $q_i = q_{i_{\rm cl}} + \hbar^{(1)}q_i$ and $p_i = p_{i_{\rm cl}} + \hbar^{(1)}p_i$ and similarly for q_j and p_j yields to first order

$$
\begin{aligned}
\hbar\,^{(1)}p_i = -\Bigg(&\frac{(\Delta q_j)^2 + (\Delta p_j)^2}{2p_{i_{\rm cl}}}\\
&+ \hbar\frac{2p_{i_{\rm cl}}(p_{j_{\rm cl}}\,^{(1)}p_j + q_{i_{\rm cl}}\,^{(1)}q_i + q_{j_{\rm cl}}\,^{(1)}q_j)}{2p_{i{\rm cl}}^2} + \frac{i\hbar q_{i_{\rm cl}}}{2p_{i{\rm cl}}^2}\\
&+ \frac{p_{j{\rm cl}}^2(\Delta p_j)^2 + q_{j{\rm cl}}^2(\Delta q_j)^2 + 2q_{j_{\rm cl}}p_{j_{\rm cl}}\Delta(q_jp_j)}{2p_{i{\rm cl}}^3}\Bigg).
\end{aligned}
\tag{D1}
$$

Since the coefficients (80) are of zeroth order, it is consistent to replace all q_i, q_j, p_i and p_j appearing in terms of order $\hbar$ in (82) by their zero-order (or classical) parts which in (D1) we have denoted by a subscript $_{\rm cl}$, and whose solutions are given in (49). To order $\hbar$ this does not modify the equations and helps for their solutions. Furthermore, remembering that all zero-order variables are kept real valued, (82) and (D1) imply that either $^{(1)}p_i$ or $^{(1)}q_i$ or both must contain imaginary contributions while all variables associated to the canonical pair (q_j, p_j) are consistently real valued as a result of real-valued equations of motion.

Requiring p_i to be real, it is obvious that

$$
\frac{d\Im[q_i]}{ds} = -\frac{\hbar q_{i_{\rm cl}}}{p_{i{\rm cl}}^2}.
\tag{D2}
$$

Using Eq. (49) and integrating this equation, precisely yields the standard imaginary contribution (85) which is also consistent with the constraint (D1) and cancels the imaginary term in the equation of motion for p_i in Eq. (82). Requiring q_i to be real valued, however, and repeating the same analysis shows that the solution for $\Im[p_i]$ would *not* reproduce the imaginary term $-i\hbar q_{i_{\rm cl}}/(2p_{i{\rm cl}}^2)$ in Eq. (D1). It is, hence, inconsistent to keep q_i real valued and push the imaginary contribution to p_i. In accordance with the analysis in Sec. IVA and [17], we, thus, find the generic $o(\hbar)$ imaginary contribution inherent to all nonglobal clocks in the effective framework.

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