Generalization of the extended Lagrangian formalism on a field theory and applications

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Formalism of extended Lagrangian represents a systematic procedure to look for the local symmetries of a given Lagrangian action. In this work, the formalism is discussed and applied to a field theory. We describe it in detail for a field theory with first-class constraints present in the Hamiltonian formulation. The method is illustrated on examples of electrodynamics, Yang-Mills field, and nonlinear sigma model.

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I. INTRODUCTION

In the field theory with local symmetries, the number of variables used in the description is greater than the number of degrees of freedom. It is important to keep all the variables used to guarantee, for instance, the manifest Lorentz covariance. On the other hand, one needs to characterize, in one way or another, the physical sector of a given theory. This can be achieved using the manifest form of the local symmetry: among all variables, the physical ones turn out to be invariant under the action of local symmetries. So, knowledge the local symmetries in many cases is crucial in analysis of physical content of a theory.

The locally invariant theories are described by singular Lagrangians, so their analysis is carried out in accordance with the Dirac method for constrained systems [1]. The presence of constraints in the Hamiltonian formulation reflects the fact that the dynamics of part of the variables is dependent on the remaining ones. The constraints are divided into two groups: the first class and the second class. It is well known that the first-class constraints are closely related to local symmetries [2–4]. So, an interesting problem under investigation by various groups [5–22] is whence there is a relatively simple and practical procedure for restoration the symmetries from the known constraints. In the Hamiltonian formalism, the problem has been solved for the case of a mechanical system with first-class constraints along the following line [3]. The initial Hamiltonian action (which by construction contains the primary constraints only) can be replaced on the extended Hamiltonian action, with all the higher-stage constraints with their own Lagrangian multipliers added to the action. It leads to the equivalent formulation [2]. Local symmetries of the extended Hamiltonian action have been found in the closed form [3]. Moreover, in absence of second-class constraints, local symmetries of the initial Hamiltonian action can be restored in the algebraic way [3].

Search for the local symmetries of the initial Lagrangian action represents a separate issue. for the mechanical constrained system with first- and second-class constraints, one possible way to solve the problem has been developed in the works [5,6]. Given a singular Lagrangian L, the theory can be reformulated in terms of an extended one, $\tilde{L}$, equivalent to L. Because of special structure of $\tilde{L}$, its gauge symmetries can be found in a closed form. All the first-class constraints of L turn out to be the gauge generators of the symmetries of $\tilde{L}$. The extended Hamiltonian of initial theory turns out to be the Hamiltonian for the extended Lagrangian [6]. For a theory with first-class constraints, it is also possible to find the symmetries of the initial Lagrangian L [3,5,6]. The aim of this work is to discuss the method described above to the case of a field theory, showing explicitly the differences that arise when we move on from mechanical to a field theory, and apply it to particular models.

In [5,6], we consider an action invariant modulo the total derivative term. It should be mentioned that by appropriately extending an action, one can make it exactly invariant [7–11]. The modified action contains a surface term which is, in general, different from zero [7,8] (the generalizations for the field theory and to arbitrary or noncanonical symplectic structures may be found in [9,10]). In this case, analysis of the Hamiltonian action shows that the Hamiltonian generators acquire the surface term [11]. The method turns out to be useful in the path-integral quantization framework of a generally covariant theories in time-independent gauges [11].

This paper is divided as follows. Section II is devoted to discussing the method of finding local symmetries for singular mechanical models. In Sec. III, we generalize this method for constrained field models. The method is illustrated on the examples of electrodynamics, Yang-Mills field, and nonlinear sigma model in Sec. IV. Section V is left for conclusions.

II. SEARCH FOR SYMMETRIES—EXTENDED LAGRANGIAN APPROACH

This section is devoted to review the method of finding local symmetries of a singular mechanical system [4–6].

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It is done by deforming the initial Lagrangian in such a way that all its symmetries can easily be found in closed form. As it will be shown, all the first-class constraints of the initial Lagrangian turn out to be the gauge generators of local symmetries of the deformed Lagrangian. The symmetries of the initial Lagrangian are also found.

A. Construction of extended Lagrangian and Hamiltonian

Starting from a singular Lagrangian $L(q^A, \dot{q}^A)$, one applies the Dirac procedure, obtaining Hamiltonian and complete Hamiltonian given by H_0 and H. The system of constraints is given by $\{G_I\} = \{\phi^\alpha, T_a\}$, where ϕ^α are primary constraints and we denote T_a all the further stage constraints. We suppose that all of them are first class (they obey the algebra $\{G_I, G_J\} = c_{IJ}{}^K G_K$, $\{G_I, H\} = b_I{}^J G_J$) and that the procedure stops at N-th stage. It is equivalent to the existence of local symmetries for L of the type [2,3,12],

$$\delta q^A = \varepsilon R_0^A + \dot{\varepsilon} R_1^A + \ldots + \frac{d^{N-1}\varepsilon}{d\tau^{N-1}} R_N^A. \quad (1)$$

We construct the following function, defined on phase space parameterized by q^A, $\tilde{p}_A$, s^a, π_a, v^α, v^a,

$$\tilde{H}(q^A, \tilde{p}_A, s^a, \pi_a, v^\alpha, v^a) = \tilde{H}_0(q^A, \tilde{p}_j, s^a) + v^\alpha \phi_\alpha(q^A, \tilde{p}_B) + v^a \pi_a, \quad (2)$$

where

$$\tilde{H}_0 = H_0(q^A, \tilde{p}_j) + s^a T_a(q^A, \tilde{p}_j). \quad (3)$$

The functions ϕ_α, H_0 and T_a were taken from the initial formulation.

We affirm that $\tilde{H}$ is the complete Hamiltonian for a Lagrangian $\tilde{L}(q^A, \dot{q}^A, s^a)$ (to be determined), $\tilde{H}_0$ is the Hamiltonian for $\tilde{L}$ as well as $\phi_\alpha = 0$ and $\pi_a = 0$ are primary constraints (π_a are conjugate momenta for s^a variables). Furthermore, L and $\tilde{L}$ are equivalent. To show all these facts, first we write the following equation of motion:

$$\dot{q}^i = \frac{\partial \tilde{H}}{\partial \tilde{p}_i} = \frac{\partial H_0}{\partial \tilde{p}_i} - v^\alpha \frac{\partial f_\alpha}{\partial \tilde{p}_i} + s^a \frac{\partial T_a}{\partial \tilde{p}_i}. \quad (4)$$

This equation can be inverted with respect to $\tilde{p}_i$ in a neighborhood of the point $s^a = 0$ (for details, see [5]). Let us denote the solution as

$$\tilde{p}_i = \omega_i(q^A, \dot{q}^i, v^\alpha, s^a). \quad (5)$$

Now, on space q^A,s^a we define

$$\tilde{L}(q^A, \dot{q}^A, s^a) = (\omega_i \dot{q}^i + f_\alpha(q^A, \omega_j)\dot{q}^\alpha - H_0(q^A, \omega_j) - s^a T_a(q^A, \omega_j))|_{\omega_i(q,\dot{q},s)}. \quad (6)$$

In the definition above, we have used the notation

$$\omega_i(q^A, \dot{q}^i, v^\alpha, s^a)|_{v^\alpha \to \dot{q}^\alpha} \equiv \omega_i(q, \dot{q}, s). \quad (7)$$

If we now suppose that $\tilde{L}$ is some singular Lagrangian, then a direct calculation shows that $\tilde{H}_0$ and $\tilde{H}$ are its corresponding Hamiltonian and complete Hamiltonian, respectively. The Dirac method applied to $\tilde{H}$ shows that all the higher-stage constraints of the initial theory are now, at most, secondary ones. It implies, in particular, that the local symmetry of $\tilde{L}$ is of $\dot{\epsilon}$-type, and hence has simple structure as compared to $\overset{(N-1)}{\epsilon}$-type symmetry of initial formulation (see Eq. (1)). If one now fixes the gauge $s^a = 0$ for the constraints $\pi_a = 0$, the sector (s^a, π_a) disappears of the extended formulation. Then one is faced again with the initial formulation. Since L is one of the gauges of $\tilde{L}$, the equivalence between the two formulations is proved. Hence, it is only matter of convenience to analyze the extended or the initial Lagrangian.

B. Restoration of local symmetries

Before we obtain the local symmetries of extended and initial formulation, it is important to note two points, already cited in the Introduction. The first one is that a gauge symmetry, in Lagrangian or Hamiltonian actions, is defined modulo a total derivative. Moreover, we want to find local symmetries of the initial Lagrangian action. These two topics make our analysis different from the one considered in the papers [7,8], where the main idea is to reformulate only the Hamiltonian action, adding boundary terms, to make it fully gauge-invariant. We start with the extended Hamiltonian formulation, passing through extended Lagrangian action, and finally we arrive at the initial formulation.

We will begin with the Hamiltonian action,

$$S_{\tilde{H}\tilde{L}} = \int d\tau (\tilde{p}_A \dot{q}^A + \pi_a \dot{s}^a - \tilde{H}). \quad (8)$$

According to Dirac conjecture [3], the first-class constraints are believed to generate gauge transformations. So, one considers the transformations $\delta_I q^A = \epsilon^I \{q^A, G_I\}$, $\delta_I \tilde{p}_A = \epsilon^I \{\tilde{p}_A, G_I\}$, where $\epsilon^I = \epsilon^I(\tau)$ are arbitrary functions, that is not necessarily zero at the endpoints and I may assume any fixed value α or a. Omitting total derivative terms, it is possible to show that these transformations imply that $\delta S_{\tilde{H}\tilde{L}}$ is proportional to ϕ_α, T_a. Then, it is possible to find appropriate transformations for v^α, s^a, that leaves $S_{\tilde{H}\tilde{L}}$ invariant. In fact, direct calculations show that the transformations below,

$$\delta_I q^A = \epsilon^I \{q^A, G_I\}, \qquad \delta_I \tilde{p}_A = \epsilon^I \{\tilde{p}_A, G_I\},$$
$$\delta_I s^a = \dot{\epsilon}^a \delta_{aI} + \epsilon^I b_I{}^a - s^b \epsilon^I c_{bI}{}^a - v^\beta \epsilon^I c_{\beta I}{}^a,$$
$$\delta_I \pi_a = 0, \; \delta_I v^\alpha = \dot{\epsilon}^\alpha \delta_{\alpha I}, \qquad \delta_I v^a = (\delta_I s^a), \quad (9)$$

keep the Hamiltonian action invariant (modulo a surface term) [5]. It prompts us to find the symmetries of the extended Lagrangian action,

$$S_{\tilde{L}} = \int d\tau \tilde{L}, \tag{10}$$

in closed form. Namely, the following variations

$$\delta_I q^A = \epsilon^I \{q^A, G_I\}|_{p\to\omega(q,\dot{q},s)},$$
$$\Leftrightarrow \begin{cases} \delta_I q^\alpha &= \epsilon^\alpha \delta_{\alpha I}, \\ \delta_I q^i &= \epsilon^I \frac{\partial G_I}{\partial \tilde{p}_i}|_{p\to\omega(q,\dot{q},s)} \end{cases};$$
$$\delta_I s^a = (\dot{\epsilon}^a \delta_{aI} + \epsilon^I b_I{}^a - s^b \epsilon^I c_{bI}{}^a - \dot{q}^\beta \epsilon^I c_{\beta I}{}^a)|_{p\to\omega(q,\dot{q},s)}, \tag{11}$$

represent the local symmetries of the action. This demonstration may be found in [5].

Let us obtain the symmetries of the initial action. To do this, we must eliminate the sector s^a of the extended formulation in an appropriate way. So, consider the combination of symmetries of $\tilde{L}$,

$$\delta \equiv \sum_I \delta_I, \tag{12}$$

which obeys $\delta s^a = 0$ for all s^a. If one uses the property $\tilde{L}(q^A, \dot{q}^A, s^a = 0) = L(q^A, \dot{q}^A)$, then L is invariant under any transformation,

$$\delta q^A = \sum_I \delta_I q^A|_{s^a=0}, \tag{13}$$

which obeys $\delta s^a = 0|_{s^a=0}$, that is,

$$\dot{\epsilon}^a + \epsilon^I b_I{}^a - \dot{q}^\beta c_{\beta I}{}^a = 0. \tag{14}$$

We have $[a]$ equations for $[\alpha] + [a]$ variables ϵ^I. When there are only first-class constraints, this system can be solved iteratively [3], leading to $[\alpha]$ local symmetries of L. This cumbersome calculation is given in [5]. We observe that we are not discarding surface terms. They are absorbed in the definition of gauge transformation in both cases: Hamiltonian and Lagrangian actions.

In the presence of second-class constraints, local symmetries of L cannot be generally restored according to the procedure discussed above. The reason is that a number of equations of the system (14) can be equal or more than the number of parameters ϵ^a, see an example of this kind in the work [6].

III. GAUGE SYMMETRIES FOR CONSTRAINED FIELD MODELS

Let us discuss the method of finding local symmetries for constrained field models. It will be carried out in the same way as described in the previous section. However, we will point out some special novelties which are present when the method is applied for a singular field model.

Let we have a singular Lagrangian $L = \int d^3x \mathcal{L}(\varphi^A, \partial_\mu \varphi^A)$. The indices A may correspond to various types of fields. The conjugate momenta are defined by

$$p_A = \frac{\delta L}{\delta \dot{\varphi}^A} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^A}. \tag{15}$$

Suppose that we have carried out the corresponding Hamiltonization. The notation follows directly from the previous section. Since L may depend on spatial derivative of the fields, we observe that further stage constraints may depend on spatial derivative of the momenta. It gives rise to the first novelty when we begin the procedure of finding local symmetries. We write the equation of motion,

$$\dot{\varphi}^i = \frac{\partial \mathcal{H}_0}{\partial \tilde{p}_i} - v^\alpha \frac{\partial f_\alpha}{\partial \tilde{p}_i} + s^a \frac{\delta T_a}{\delta \tilde{p}_i}. \tag{16}$$

This equation should be inverted in terms of $\tilde{p}_i$ to construct the extended Lagrangian. Nevertheless, in general case one is faced with a partial derivative of $\tilde{p}_i$. To avoid this problem, let us suppose that the constraints are, at most, linear in spatial derivative of the momenta. In this case, Eq. (16) can be inverted. We point out that constraints with polynomial form in fields and corresponding momenta do not represent any restriction to inversion of (16), see [5]. Although restrictive, to our acknowledge, all important physical models that possess local invariance bear this particular structure in Hamiltonian formulation, i.e. with linear constraints in spatial derivative of the momenta. Indeed, electrodynamics, Yang-Mills field, standard model, string, and membrane theories are of this type. There is another novelty that must be taken into account: the coefficients of the gauge algebra may not be functions but operators, e.g., $\{G_I, G_J\} \sim \partial_j \partial^j G_K$. Finally, the gauge generators are

$$G = \int d^3x \epsilon^I(x) G_I(x). \tag{17}$$

Integration is taken over all the space. The method of finding symmetries is now carried out analogously.

At this point, it may be interesting to discuss certain special subtleties present in singular field models that do not result directly from the generalization of point mechanics to the continuous case. In classical systems, physical degrees of freedom are understood to be the minimum number of variables necessary to fully describe the model. In a field theory, physical degrees of freedom can be understood to be the minimum number of fields in each point of underlying space where the fields are defined, which completely describe the model. For instance, we say that electrodynamics has two degrees of freedom, since it is possible to eliminate two of the four components of the vector $A_\mu = A_\mu(x)$, for each point in space-time parameterized by x^μ. In fact, there are 8 field components in phase space (A_μ and its corresponding momenta) together with two first-class constraints. After the gauge is fixed, we are left with four second-class constraints. Hence, there are only $8 - 4 = 4$ independent field components. Consequently, only two components of A_μ remain independent on configuration space of fields. We must also be careful with the meaning of constraint in a field theory. In classical systems, each constraint (algebraic equation involving coordinates and momenta) allows us to eliminate one differential equation from all the equations of motion describing the model. This means that not all variables have independent dynamics. For a field theory, a constraint can also be a differential equation. Hence, elimination of nonphysical degrees of freedom does not follow directly. This point may also be exemplified using electrodynamics: p_i are the conjugate momenta for A_i and $p_0 \approx 0$ is the primary constraint. The evolution of p_0 leads to the secondary constraint $\partial_i p_i \approx 0$. The elimination of any degree of freedom using the secondary constraint is not as obvious as it is for the primary one. (For a discussion of the above points, see [13,14,23]).

IV. APPLICATIONS

We will consider some specific examples of constrained field models for applying the method presented, including electrodynamics, Yang-Mills, and nonlinear sigma model.

A. Local symmetry of electrodynamics

Let us consider the Lagrangian of electromagnetic field,

$$L = -\frac{1}{4}\int d^3x F_{\mu\nu}F^{\mu\nu} = \int d^3x\left[\frac{1}{2}(\dot{A}_i - \partial_i A_0)^2 - \frac{1}{4}F_{ij}F^{ij}\right], \tag{18}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The primary constraint and conjugate momenta are given by

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_0} = p_0 = 0 \Rightarrow \phi_1 \equiv p_0 = 0, \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_i} = p_i = \dot{A}_i - \partial_i A_0 \Rightarrow \dot{A}_i = p_i + \partial_i A_0. \tag{20}$$

The Hamiltonian H_0 and complete Hamiltonian H are

$$H_0 = \int d^3x\left[\frac{1}{2}p_i^2 + p_i\partial_i A_0 + \frac{1}{4}F_{ij}^2\right], \tag{21}$$

$$H = H_0 + \int d^3x v^0 p_0, \tag{22}$$

where v^0 is the corresponding Lagrange multiplier. The secondary constraint follows from the consistency condition $0 = \{p_0(x_1), H\}$. It leads to $T_2 \equiv \partial_i p_i = 0$. There are no further constraints.

The gauge algebra is

$$\{p_0, \partial_i p_i\} = 0, \quad \{p_0, H_0\} = \partial_i p_i, \quad \{\partial_i p_i, H_0\} = 0. \tag{23}$$

The extended Hamiltonian takes the form

$$\tilde{H} = \int d^3x\left[\frac{1}{2}\tilde{p}_i^2 + \tilde{p}_i\partial_i A_0 + \frac{1}{4}F_{ij}^2 + s^2\partial_i\tilde{p}_i + v^2\pi_2 + v^0\tilde{p}_0\right]. \tag{24}$$

Starting from

$$\dot{A}_i = \{A_i, \tilde{H}\} = \tilde{p}_i + \partial_i A_0 - \partial_i s^2 \Rightarrow \tilde{p}_i = \dot{A}_i - \partial_i A_0 + \partial_i s^2, \tag{25}$$

we find $\tilde{L}$,

$$\tilde{L} = \int d^3x\left[\frac{1}{2}(\dot{A}_i - \partial_i A_0 + \partial_i s^2)^2 - \frac{1}{4}F_{ij}^2\right]. \tag{26}$$

The symmetries of $\tilde{L}$ are given by

$$\delta_1\colon\ \delta_1 A_i = 0; \qquad \delta_1 A_0 = \delta_1 s^2 = \epsilon^1, \tag{27}$$

$$\delta_2\colon\ \delta_2 A_i = \int d^3x\epsilon^2\{A_i, \partial_l p_l\} = -\partial_i\epsilon^2, \tag{28}$$

$$\delta_2 A_0 = 0; \qquad \delta_2 s^2 = \dot{\epsilon}^2. \tag{29}$$

The symmetries of L are directed restored, see ((13) and (14)),

$$\delta_1 A_i + \delta_2 A_i = -\partial_i\epsilon^2, \tag{30}$$

$$\delta_1 A_0 + \delta_2 A_0 = \epsilon^1, \tag{31}$$

where the ϵ's obey the equation

$$\dot{\epsilon}^2 + \epsilon^1 = 0 \Rightarrow \epsilon^1 = -\dot{\epsilon}^2. \tag{32}$$

Defining $\epsilon^2 \equiv -\alpha$, we obtain the well-known gauge symmetry of electrodynamics,

$$A_\mu(x^\nu) \rightarrow A'_\mu(x^\nu) = A_\mu(x^\nu) + \partial_\mu\alpha(x^\nu), \tag{33}$$

where $\alpha = \alpha(x^\nu)$ is an arbitrary space-time scalar function.

B. Local symmetry of Yang-Mills field

In the pioneer work [24], Yang and Mills (YM) have considered the idea of interact a original set of fields, invariant under a group with constant parameters, with a new field (gauge field). It was accomplished by postulating the invariance of the system under the original group but having now arbitrary functions as parameters. We will discuss this field model via the Dirac procedure and we shall find its local symmetries. Let us consider the YM Lagrangian,

$$L = \int d^3x\mathcal{L} = -\frac{1}{4}\int d^3x F^a_{\mu\nu}F^{a\mu\nu}, \tag{34}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + igf^{abc}A^b_\mu A^c_\nu$. L has global $SU(N)$ symmetry, the field A_μ assumes values on the corresponding Lie algebra with generators T^a,

$$A_\mu = A_\mu^a T^a, \tag{35}$$

and f^{abc} are the structure constants,

$$[T^a, T^b] = i f^{abc} T^c. \tag{36}$$

The primary constraints and conjugate momenta are

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_0^a} = p_0^a = 0 \Rightarrow \phi_1^a = p_0^a = 0, \tag{37}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_i^a} = p_i^a = \dot{A}_i^a - \partial_i A_0^a + igf^{abc}A_0^b A_i^c \Rightarrow \dot{A}_i^a = p_i^a + \partial_i A_0^a - igf^{abc}A_0^b A_i^c. \tag{38}$$

The Hamiltonian H_0 and complete Hamiltonian H are given by

$$H_0 = \int d^3x \left[\frac{1}{2}(p_i^a)^2 + p_i^a \partial_i A_0^a - igf^{abc}A_0^b A_i^c p_i^a + \frac{1}{4}(F_{ij}^a)^2\right] \tag{39}$$

$$H = H_0 + \int d^3x \lambda^a p_0^a, \tag{40}$$

where λ^a are the corresponding Lagrange multipliers. The secondary constraint follows from the consistency condition $0 = \{p_0^a(x_1), H\}$. One finds $T_2^a = \partial_i p_i^a - igf^{abc}p_i^b A_i^c = 0$. There are no further constraints. The gauge algebra is

$$\{\phi_1^a, \phi_1^b\} = \{\phi_1^a, T_2^b\} = 0, \tag{41}$$

$$\{T_2^a(x_1), T_2^b(x_2)\} = -igf^{abc}T_2^c(x_1)\delta(x_1 - x_2), \tag{42}$$

$$\{\phi_1^a, H_0\} = T_2^a, \tag{43}$$

$$\{T_2^a, H_0\} = igA_0^b f^{bac} T_2^c. \tag{44}$$

The extended Hamiltonian takes the form

$$\tilde{H} = \int d^3x \left[\frac{1}{2}(\tilde{p}_i^a)^2 + \tilde{p}_i^a \partial_i A_0^a - igf^{abc}A_0^b A_i^c \tilde{p}_i^a + \frac{1}{4}(F_{ij}^a)^2 + (s^2)^a(\partial_i p_i^a - igf^{abc}p_i^b A_i^c) + (v^2)^a \pi_2^a + v^a \tilde{p}_0^a\right]. \tag{45}$$

Starting from

$$\dot{A}_i^a = \{A_i^a, H\} = \tilde{p}_i^a + \partial_i A_0^a - igf^{abc}A_0^b A_i^c + -\partial_i (s^2)^a - igf^{bac}A_i^c (s^2)^b, \tag{46}$$

we find

$$\tilde{p}_i^a = \dot{A}_i^a - \partial_i(A_0^a - (s^2)^a) + igf^{abc}(A_0^b - (s^2)^b)A_i^c. \tag{47}$$

Thus, $\tilde{L}$ reads

$$\tilde{L} = \int d^3x \left\{\frac{1}{2}(\dot{A}_i^a - \partial_i(A_0^a - (s^2)^a) + igf^{abc}(A_0^b - (s^2)^b)A_i^c)^2 + -\frac{1}{4}(F_{ij}^a)^2\right\}. \tag{48}$$

The symmetries of $\tilde{L}$ are given by

$$\delta_1\colon\ \delta_1 A_i^a = 0; \delta_1 A_0^a = (\epsilon^1)^a; \qquad \delta_1 (s^2)^a = (\epsilon^1)^a; \tag{49}$$

$$\delta_2\colon\ \delta_2 A_i^a = -\partial_i(\epsilon^2)^a - igf^{abc}(\epsilon^2)^b A_i^c; \quad \delta_2 A_0^a = 0; \delta_2 (s^2)^a = (\dot{\epsilon}^2)^a. \tag{50}$$

The symmetries of L are easily restored,

$$\delta_1 A_i^a + \delta_2 A_i^a = -\partial_i(\epsilon^2)^a - igf^{abc}(\epsilon^2)^b A_i^c, \tag{51}$$

$$\delta_1 A_0^a + \delta_2 A_0^a = (\epsilon^1)^a, \tag{52}$$

where the ϵ's obey

$$(\dot{\epsilon}^2)^b + (\epsilon^1)^b + igA_0^a f^{abc}(\epsilon^2)^c = 0 \Rightarrow (\epsilon^1)^b = -\partial_0(\epsilon^2)^b - igf^{bca}(\epsilon^2)^c A_0^a. \tag{53}$$

Defining $(\epsilon^2)^a \equiv -\xi^a$, we obtain the expected result,

$$A_\mu^a \to A_\mu^{\prime a} = A_\mu^a + D_\mu^{ac}\xi^c, \tag{54}$$

where $D_\mu^{ac} = \delta^{ac}\partial_\mu - igf^{acb}A_\mu^b$ is the covariant derivative.

C. Local symmetry of converted nonlinear sigma model

In the work [15], a method is discussed of conversion of second-class constraints into the first class ones based on transformations that involve derivatives of the configuration-space variables. It is useful for covariant quantization of a theory and in the context of doubly special relativity [25], for example. Here, we consider the converted version of the nonlinear sigma model presented in [15]. The model is useful for the purposes of this work since, after the conversion, there are only first-class constraints. So, we look for local symmetries of the action

$$S = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi^a)^2 - 2\partial_\mu e \partial^\mu \phi^a \phi^a + \lambda((\phi^a)^2 - 1)\right]. \tag{55}$$

The primary constraints and conjugate momenta are

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}^a} = p_a = \dot{\phi}^a - 2\dot{e}\phi^a; \qquad \frac{\partial \mathcal{L}}{\partial \dot{e}} = p_e = -2\dot{\phi}^a \phi^a; \qquad \frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = p_\lambda = 0. \tag{56}$$

The expressible velocities are given by

$$\dot{e} = -\frac{1}{4\phi^2}(2\phi p + p_e);$$
$$\dot{\phi}^a = p_a - \frac{\phi^a}{2\phi^2}(2\phi p + p_e). \qquad (57)$$

We are using the notation $\phi^a\phi^a = \phi^2$. The Hamiltonian H_0 and complete Hamiltonian H are given by

$$H_0 = \int d^3x\left[\frac{1}{2}p^2 - \frac{(2\phi p + p_e)^2}{8\phi^2} + \frac{1}{2}(\partial_i\phi^a)^2 + -2\partial_i e\partial^i\phi^a\phi^a - \lambda(\phi^2 - 1)\right]; \qquad (58)$$

$$H = H_0 + \int d^3x v p_\lambda, \qquad (59)$$

where v is the corresponding Lagrange multiplier. The secondary constraint follows from the consistency condition $0 = \{p_\lambda(x_1), H\}$. One finds $G_2 = \phi^2 - 1 = 0$. We still find a tertiary constraint: $0 = \{G_2(x_1), H\} = -p_e$. $G_3 = p_e = 0$.

If we define $\{G_I\} = \{G_1 = p_\lambda, G_2 = \phi^2 - 1, G_3 = p_e\}$, then the gauge algebra is

$$\{G_I, G_J\} = 0 \Rightarrow c_{IJ}{}^K = 0 \forall I, J, K; \qquad (60)$$

$$\{G_1, H_0\} = G_2 \Rightarrow b_1{}^2 = 1, b_1{}^1 = b_1{}^3 = 0; \qquad (61)$$

$$\{G_2, H_0\} = -G_3 \Rightarrow b_2{}^3 = -1, \qquad b_2{}^1 = b_2{}^2 = 0; \qquad (62)$$

$$\{G_3, H_0\} = -\partial^i\partial_i G_2 \Rightarrow b_3{}^2 = -\partial^i\partial_i, \quad b_3{}^1 = b_3{}^3 = 0. \qquad (63)$$

Note that expressions of the form $\partial_i G_2$, $\partial^i\partial_i G_2$, etc. are consequences of the already-obtained constraints. They do not imply simplification of the dynamical equations. So we adopt the following point: spatial derivatives of constraints does not give rise to new constraints. So, the procedure stops at the third stage.

The extended Hamiltonian takes the form

$$\tilde{H} = \int d^3x\left[\frac{1}{2}\tilde{p}^2 - \frac{(2\phi\tilde{p} + \tilde{p}_e)^2}{8\phi^2} + \frac{1}{2}(\partial_i\phi^a)^2 - 2\partial_i e\partial^i\phi^a\phi^a + -\lambda(\phi^2 - 1) + s^2(\phi^2 - 1) + s^3\tilde{p}_e + v\tilde{p}_\lambda + v^2\pi_2 + v^3\pi_3\right]. \qquad (64)$$

Starting from

$$\dot{\phi}^a = \{\phi^a, \tilde{H}\} = \tilde{p}^a - \frac{2\phi\tilde{p} + \tilde{p}_e}{2\phi^2}\phi^a \qquad (65)$$

$$\dot{e} = \{e, \tilde{H}\} = -\frac{2\phi\tilde{p} + \tilde{p}_e}{4\phi^2} + s^3, \qquad (66)$$

we find

$$\tilde{p}_a = \dot{\phi}^a - 2\phi^a(\dot{e} - s^3), \qquad \tilde{p}_e = -2\phi\dot{\phi}. \qquad (67)$$

Thus, $\tilde{L}$ reads

$$\tilde{L} = \int d^3x\left\{\frac{1}{2}(\partial_\mu\phi^a)^2 - 2\phi\dot{\phi}(\dot{e} - s^3) + 2\partial_i e\partial^i\phi^a\phi^a + +(\lambda - s^2)(\phi^2 - 1)\right\}. \qquad (68)$$

The symmetries of $\tilde{L}$ are given by

$$\delta_1\colon\ \delta_1\phi^a = 0; \qquad \delta_1\lambda = \epsilon^1; \qquad \delta_1 e = 0; \qquad \delta_1 s^2 = \epsilon^1; \qquad \delta_1 s^3 = 0; \qquad (69)$$

$$\delta_2\colon\ \delta_2\phi^a = 0; \qquad \delta_2\lambda = 0; \qquad \delta_2 e = 0; \qquad \delta_2 s^2 = \dot{\epsilon}^2; \qquad \delta_2 s^3 = -\epsilon^2; \qquad (70)$$

$$\delta_3\colon\ \delta_3\phi^a = 0; \qquad \delta_3\lambda = 0; \qquad \delta_1 e = \epsilon^3; \qquad \delta_3 s^2 = b_3{}^2\epsilon^3 = -\partial_i\partial_i\epsilon^3; \qquad \delta_3 s^3 = \dot{\epsilon}^3. \qquad (71)$$

The symmetries of L are restored,

$$\delta_1\phi^a + \delta_2\phi^a + \delta_3\phi^a = 0, \qquad (72)$$

$$\delta_1\lambda + \delta_2\lambda + \delta_3\lambda = \epsilon^1, \qquad (73)$$

$$\delta_1 e + \delta_2 e + \delta_3 e = \epsilon^3, \qquad (74)$$

where the ϵ's obey,

$$\dot{\epsilon}^2 - \partial_i\partial_i\epsilon^3 + \epsilon^1 = 0, \qquad (75)$$

$$\dot{\epsilon}^3 - \epsilon^2 = 0. \qquad (76)$$

Defining $\epsilon^3 \equiv -\epsilon$, we obtain the following local symmetry:

$$\delta\phi^a = 0; \qquad \delta\lambda = \partial_\mu\partial^\mu\epsilon; \qquad \delta e = -\epsilon, \qquad (77)$$

where $\epsilon = \epsilon(x)$ is an arbitrary function of space-time coordinates.

V. CONCLUSION

In this work, we have presented a generalization of the extended Lagrangian method of finding local symmetries to the field systems. As we have illustrated in various examples, it provides a systematic method of finding gauge symmetries of a singular Lagrangian L with first-class constraints. The initial theory is deformed in a special way such that all the symmetries of the deformed Lagrangian $\tilde{L}$ can easily be found. The symmetries of L are also obtained. According to the scheme, all the first-class constraints of the initial theory are the gauge generators of the deformed theory. We also pointed out the subtleties that must be taken into account when moving from classical systems to the continuous case. In this context, we briefly discussed some fundamental definitions that are slightly complicated and do not follow directly from point mechanics to field theories, including degrees of freedom and constraints.

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