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No hair theorems for stationary axisymmetric black holes

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We present a nonperturbative proof of the no-hair theorems corresponding to scalar and Proca fields for stationary axisymmetric de Sitter black hole spacetimes. Our method also applies to asymptotically flat and under a reasonable assumption, to asymptotically anti-de Sitter spacetimes.

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I. INTRODUCTION

The classical no-hair conjecture for black holes states that any gravitational collapse reaches a final stationary state characterized only by a small number of parameters. A part of this conjecture has been proven rigorously by taking different matter fields, known as the no-hair theorem (see e.g. [1–3]) and deals with the uniqueness of stationary black hole solutions characterized only by mass, angular momentum, and charges corresponding to long range gauge fields such as the electromagnetic field. Any nontrivial field configuration other than the long range gauge fields present at the exterior of a stationary black hole is known as "hair." In particular, it has been shown that static, spherically symmetric black holes do not support hair corresponding to scalars in convex potentials, Proca-massive vector fields [4], or even gauge fields corresponding to the Abelian Higgs model [5,6].

All the above proofs assume the spacetime to be asymptotically flat, i.e., one can reach spacelike infinity so that sufficiently rapid fall-off conditions on the matter fields can be imposed there. But recent observations [7,8] suggest that there is a strong possibility that our universe is endowed with a small but positive cosmological constant Λ. It is generally expected that, in that case, the spacetime in its stationary state should have an outer or cosmological Killing horizon [9]. The cosmological Killing horizon acts in general as a causal boundary (see e.g. [10]) so that no physical observer can communicate beyond this horizon along a future directed path. If there is a black hole, the black hole event horizon will be located inside the cosmological horizon and the spacetime is then known as a de Sitter black hole spacetime. The observed value of the Λ is very small, of the order of $10^{-52}\ \mathrm{m}^{-2}$, and the known exact solutions [11] for a small Λ suggest that the cosmological horizon has a length scale $\sim\mathcal{O}(\frac{1}{\sqrt{\Lambda}})$ which is of course large, but not infinite. Since no physical observer can communicate beyond the cosmological horizon, in a de Sitter black hole spacetime the cosmological horizon acts as a natural boundary. So, in the most general case, one cannot assume any precise asymptotic form in the vicinity of the cosmological horizon, and hence, one cannot set $T_{ab} = 0$ over that horizon. Therefore, the extension of the no-hair theorems for de Sitter black holes are expected to be different from the $\Lambda \leq 0$ cases.

In particular, a lot of progress has been made in this topic for static de Sitter black holes. Price's theorem, which can be regarded as a perturbative no-hair theorem [12] was proved in [13] for a Schwarzschild-de Sitter background by taking massless perturbations. In [14], all the known black hole no-hair theorems were extended for a general static de Sitter black hole spacetime. The exception was that a charged solution corresponding to the false vacuum of the complex scalar of the Abelian Higgs model was obtained, which has no $\Lambda \leq 0$ analogue. In fact, this charged solution suggests that even though Λ is very small, the existence of the cosmological horizon, because of the nontrivial boundary conditions, may change local physics considerably.

It is thus an interesting task to generalize the no-hair theorems for a stationary de Sitter black hole. For an asymptotically flat spacetime, the no-hair proofs for a rotating black hole for scalar and Proca fields were first given in [15], assuming time reversal symmetry of the matter equations. For a discussion on the $2 + 1$ dimensional no-hair theorem, see [16]. See also [17] for a scalar no-hair theorem in stationary asymptotically flat spacetimes with nonminimally coupled scalar fields. In the following we shall give a proof of the no-hair theorems for scalar and Proca-massive vector fields for a de Sitter black hole spacetime. Our method will be considerably different from that of [15].

This paper is organized as follows. In the next section, we outline all the necessary assumptions and the geometrical set up we work in. In Sec. III, we give the proof of the no-hair theorems for the scalar and Proca fields. Finally, we discuss our results. We set $c = 1 = G$ throughout.

II. THE GEOMETRICAL SET UP

In this section, we outline the particular geometrical set up we need to describe our spacetime. More details can be found in [9].

We consider a (3 + 1)-dimensional stationary axisymmetric spacetime with two commuting Killing fields $\{\xi^a, \phi^a\}$,

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$$\nabla_{(a}\xi_{b)} = 0 = \nabla_{(a}\phi_{b)}, \tag{1}$$

$$[\xi, \phi]^a = 0. \tag{2}$$

ξ^a is locally timelike with norm $\xi^a\xi_a = -\lambda^2$ and generates the stationarity, whereas ϕ^a is locally spacelike with closed orbits and norm $\phi^a\phi_a = f^2$ and generates the axisymmetry. We assume that the spacetime satisfies Einstein's equations. We take the connection ∇_a to be torsion-free, i.e., for any differentiable function $g(X)$, we have $\nabla_{[a}\nabla_{b]}g(X) = 0$.

We can specify a basis $\{\xi^a, \phi^a, \mu^a, \nu^a\}$ for this spacetime, where $\{\mu^a, \nu^a\}$ are spacelike basis vectors orthogonal to both ξ^a and ϕ^a. We assume that the 2 surfaces spanned by $\{\mu^a, \nu^a\}$ form integral submanifolds. In other words, $\{\mu^a, \nu^a\}$ form the basis of a Lie algebra. We note that this assumption is valid for known stationary axisymmetric spacetimes.

A stationary axisymmetric spacetime with a black hole is, in general, rotating. In that case, ξ^a is not orthogonal to ϕ^a, and the basis $\{\xi^a, \phi^a, \mu^a, \nu^a\}$ is not orthogonal. So, in particular, there is no family of spacelike hypersurfaces which is both tangent to ϕ^a and orthogonal to ξ^a. Let us first construct a family of spacelike hypersurfaces tangent to ϕ^a. We define χ_a as

$$\chi_a = \xi_a - \frac{1}{f^2}(\xi_b\phi^b)\phi_a \equiv \xi_a + \alpha\phi_a, \tag{3}$$

so that we have $\chi_a\phi^a = 0$ everywhere. We note that

$$\chi_a\chi^a = -(\lambda^2 + \alpha^2 f^2), \tag{4}$$

so that χ_a is timelike when $\beta^2 = (\lambda^2 + \alpha^2 f^2) > 0$. The basis $\{\chi^a, \phi^a, \mu^a, \nu^a\}$ is now an orthogonal basis for the spacetime. We also have

$$\nabla_{(a}\chi_{b)} = \phi_a\nabla_b\alpha + \phi_b\nabla_a\alpha. \tag{5}$$

Our assumption that $\{\mu^a, \nu^a\}$ span an integral 2-submanifold implies that χ^a satisfies the Frobenius condition of hypersurface orthogonality [9]

$$\chi_{[a}\nabla_b\chi_{c]} = 0. \tag{6}$$

Thus, χ^a is orthogonal to the spacelike $\{\phi^a, \mu^a, \nu^a\}$ hypersurfaces, say Σ.

How do we define the horizons of our spacetime? It is known that in a rotating black hole spacetime, ξ^a becomes spacelike within the ergosphere [18], so for such spacetimes $\lambda^2 = 0$ does not in general define a horizon. It was shown in [9], by considering the null geodesic congruence over a $\beta^2 = 0$ surface, that the vector field χ^a coincides with a null Killing field over that surface. Thus, a $\beta^2 = 0$ surface is essentially a Killing or true horizon. Accordingly, we define the black hole event horizon and the cosmological event horizon to be the two $\beta^2 = 0$ surfaces. An example of this is the Kerr-Newman-de Sitter spacetime [10].

We assume that no naked curvature singularity exists anywhere in our region of interest, i.e., anywhere between the two horizons. The Einstein equation $G_{ab} + \Lambda g_{ab} = T_{ab}$ then implies that the invariants constructed from the energy-momentum tensor T_{ab} are bounded over or everywhere in the region between the two horizons. Apart from this regularity, we also assume that the horizons are "closed" surfaces.

The usual projector or the induced metric over the spacelike hypersurfaces Σ is defined as

$$h_a{}^b = \delta_a{}^b + \beta^{-2}\chi_a\chi^b. \tag{7}$$

Let D_a be the induced connection over Σ defined via the projector as $D_a := h_a{}^b\nabla_b$. Then we can project the derivative of a tensor $T_{a_1a_2\cdots}{}^{b_1b_2\cdots}$ over Σ as

$$D_a\tilde{T}_{a_1a_2\ldots}{}^{b_1b_2\ldots} := h_a{}^b h_{a_1}{}^{c_1}\ldots h^{b_1}{}_{d_1}\ldots\nabla_b T_{c_1c_2\ldots}{}^{d_1d_2\ldots}, \tag{8}$$

where $\tilde{T}$ is the projection of T over Σ, given by $\tilde{T}_{a_1a_2\ldots}{}^{b_1b_2\cdots} := h_{a_1}{}^{c_1}\cdots h^{b_1}{}_{d_1}\cdots T_{c_1c_2\ldots}{}^{d_1d_2\cdots}$. It is easy to verify that the induced connection D_a over Σ defined in Eq. (8) satisfies the Leibniz rule and is compatible with the induced metric h_{ab}.

It will be useful to note here that if a function ψ has a vanishing Lie derivative with respect to χ, that is if $\mathcal{L}_\chi\psi = 0$, we can use the torsion-free condition to write

$$\beta\nabla_a\nabla^a\psi = D_a(\beta D^a\psi). \tag{9}$$

Next, we note that the subspace spanned by $\{\chi^a, \mu^a, \nu^a\}$ do not form a hypersurface. This is because the necessary and sufficient condition that an arbitrary subspace of a manifold forms an integral submanifold or a hypersurface is that the basis vectors of that subspace span a Lie algebra (see, e.g., [18] and references therein). It is easy to verify using the definition of χ^a in Eq. (3) that the basis vectors $\{\chi^a, \mu^a, \nu^a\}$ do not span a Lie algebra. This implies that we cannot write a condition like $\phi_{[a}\nabla_b\phi_{c]} = 0$ [9].

However, according to our assumptions, there are integral spacelike 2-submanifolds orthogonal to both χ^a and ϕ^a, and spanned by $\{\mu^a, \nu^a\}$. Then over these 2-manifolds Σ, we must have

$$\phi_{[a}D_b\phi_{c]} = 0. \tag{10}$$

Using the projector defined in Eq. (7), we write the Killing equation for ϕ_a over Σ as

$$D_{(a}\phi_{b)} = 0. \tag{11}$$

We now solve Eqs. (10) and (11) to find the expression for $D_a\phi_b$ and using the projector (7) rewrite it in terms of the full spacetime connection ∇_a

$$\nabla_a \phi_b = \frac{1}{f}[\phi_b \nabla_a f - \phi_a \nabla_b f] + \frac{f^2}{2\beta^2}[\chi_b \nabla_a \alpha - \chi_a \nabla_b \alpha]. \quad (12)$$

Also, we note that since $\{\mu^a, \nu^a\}$ span integral 2-surfaces $\bar{\Sigma}$, and χ^a and ϕ^a are orthogonal, we can project spacetime tensors over $\bar{\Sigma}$ via the projector

$$\Pi_a{}^b = \delta_a{}^b + \beta^{-2}\chi_a \chi^b - f^{-2}\phi_a \phi^b. \quad (13)$$

We can also define the induced connection $\bar{D}_a$ on $\bar{\Sigma}$ using the projector $\Pi_a{}^b$.

We will assume that any matter field also obeys the symmetry of the spacetime. In other words, if X is a matter field, or a component of a matter field, we must have

$$\mathcal{L}_\xi X = 0 = \mathcal{L}_\phi X. \quad (14)$$

We note that Eq. (14) need not hold if X is a gauge field.

We are now ready to prove the no-hair theorems.

III. NO-HAIR THEOREMS FOR SCALAR AND PROCA FIELDS

We start with the simplest case, that of a scalar field ψ moving in a potential $V(\psi)$ satisfying the equation of motion

$$\nabla_a \nabla^a \psi = V'(\psi), \quad (15)$$

where the "prime" denotes differentiation with respect to ψ and any mass term is included in $V(\psi)$. Since we are assuming stationarity and axisymmetry, we must have $\mathcal{L}_\xi \psi = 0 = \mathcal{L}_\phi \psi$, as we mentioned earlier. Since $\chi_a = \xi_a + \alpha \phi_a$, it follows that $\mathcal{L}_\chi \psi = 0$. Then, using Eq. (9), we find that Eq. (15) takes the following form over the χ-orthogonal hypersurface Σ,

$$D_a(\beta D^a \psi) = \beta V'(\psi). \quad (16)$$

We now multiply Eq. (16) by $V'(\psi)$ and integrate by parts to have

$$\int_{\partial\Sigma} \beta V'(\psi) n^a D_a \psi + \int_\Sigma \beta[V''(\psi)(D^a \psi)(D_a \psi) + V'^2(\psi)] = 0, \quad (17)$$

where $\partial\Sigma$ are spacelike closed 2-surfaces located at the boundaries of Σ, i.e., the horizons and n^a is a unit spacelike vector normal to these 2-surfaces.

According to our assumption, there is no naked curvature singularity anywhere between the horizons, including the horizons. This implies that the invariants of the energy-momentum tensor is bounded on the horizons. Since $\nabla_a \psi \nabla^a \psi$ appears in the trace of the energy-momentum tensor, it follows that this quantity is bounded on the horizons. On the other hand, $\mathcal{L}_\chi \psi = 0$ implies that $\nabla_a \psi = D_a \psi$, while the inequality $(D_a \psi - n_a(n^b D_b \psi))^2 \geq 0$ implies $|n^a D_a \psi|^2 \leq (D_a \psi)(D^a \psi)$. Therefore, the quantity $n^a D_a \psi$ also remains bounded over the horizons. Then, since $\beta = 0$ over the horizons, the surface integrals in Eq. (17) vanish.

Since the inner product in the Σ integral of Eq. (17) is spacelike, it immediately follows that no nontrivial solution exists for ψ over Σ for a convex potential, i.e., if $V''(\psi) > 0$ for all values of ψ. So, for a convex $V(\psi)$, the scalar field ψ is a constant located at the minimum of the potential $V(\psi)$. Then, $\mathcal{L}_\chi \psi = 0$ ensures that we have the same trivial solution throughout the spacetime. This is the standard no-hair result for a scalar field.

For $V(\psi) = 0$, we multiply Eq. (16) by ψ and integrate by parts over Σ to get an equation similar to Eq. (17), Assuming that ψ is measurable, i.e. bounded, over the horizon [4,15], gives the no-hair result.

The no-hair statement need not hold in other kinds of potentials. For static spherically symmetric spacetimes, scalar hair may be present for nonconvex potentials, such as the double well potential $V(\psi) = \frac{\lambda}{4}(\psi^2 - v^2)^2$, which gives an unstable solution [19]. Another example is that of a conformal scalar ψ coupled to gravity by a term $V(\psi) = \frac{1}{12}R\psi^2$. The scalar field action is invariant under a conformal transformation in this theory. So, by appropriately choosing the conformal factor of the transformation, we can make ψ or $n^a D_a \psi$ diverge at $\partial\Sigma$ without causing a curvature singularity. Then, the $\partial\Sigma$ integral can be nonzero, which allows a nontrivial configuration of ψ on Σ. In fact, static spherically symmetric solutions with conformal scalar hair with $\Lambda > 0$ are known [20]. It is likely that these exceptions will also be present for stationary axisymmetric spacetimes.

Next, we consider the Proca-massive Lagrangian for the vector field

$$\mathcal{L} = -\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}m^2 A_b A^b, \quad (18)$$

where $F_{ab} = \nabla_a A_b - \nabla_b A_a$. We shall see below that proving a no-hair statement in this case is quite a bit more complicated than in the case of a scalar field. The equation of motion for A^b is

$$\nabla_a F^{ab} - m^2 A^b = 0. \quad (19)$$

The procedure, as for the scalar field, will be to construct a positive definite quadratic with a vanishing integral on Σ. Let us start by defining the potential ψ and the "electric" field e^a

$$\psi = \beta^{-1}\chi_a A^a; \qquad e^a = \beta^{-1}\chi_b F^{ab}. \quad (20)$$

The vanishing of the Lie derivatives of ψ and e_a, along the Killing fields ξ^a and ϕ^a, imply

$$\mathcal{L}_\chi \psi = 0; \qquad \mathcal{L}_\chi e^a = -\phi^a e^b \nabla_b \alpha. \quad (21)$$

Then, using Eqs. (6)–(8), it is easy to obtain the following projected equations over Σ

$$D_a(\beta\psi) = \beta e_a + \mathcal{L}_\chi A_a; \qquad D_a e^a = m^2\psi. \tag{22}$$

We now multiply the second of the Eqs. (22) with $\beta\psi$, use the first of the Eqs. (22) and integrate by parts over Σ to get

$$\int_{\partial\Sigma}\beta\psi n^a e_a + \int_\Sigma[\beta(e_a e^a + m^2\psi^2) + e^a(\mathcal{L}_\chi A_a)] = 0. \tag{23}$$

Using the fact that $\mathcal{L}_\xi A_a = 0 = \mathcal{L}_\phi A_a$, we have $\mathcal{L}_\chi A_a = (A_b\phi^b)\nabla_a\alpha$. The terms ψ^2 and e_a^2 appear in the invariants of the energy-momentum tensor which are bounded over the horizons. This implies that the surface integrals vanish, giving us the following Σ integral

$$\int_\Sigma[\beta(e_a e^a + m^2\psi^2) + (A_b\phi^b)e^a\nabla_a\alpha] = 0. \tag{24}$$

We note that for $m = 0$ the Lagrangian (18) is invariant under a local gauge symmetry $A \to A + dg$, where g is any differentiable function. Then, for $m = 0$, the components of A are not physical and need not be bounded on the horizon. Then we can always choose ψ such that the surface integrand in Eq. (23) becomes unbounded and hence the surface integral becomes nonzero.

By Eq. (20), $e^a\chi_a = 0$ and hence e^a is a spacelike vector field. Also, $\beta > 0$ between the two horizons and vanishes on the horizons. So all but the last term in Eq. (24) are positive definite. Nor can we set the last term to zero, since χ^a is not a Killing field. Thus the no-hair conjecture for the Proca field cannot be proven from Eq. (24) alone, and we need to take a more careful look at the rest of the equations of motion.

Let us first project Eq. (19) over Σ. Let a_b and f_{ab} be the Σ projections of A_b and F_{ab} defined via the projector as $a_b := h_b{}^a A_a$; $f_{ab} := h_a{}^c h_b{}^d F_{cd}$. It is easy to see that

$$h_a{}^c h_b{}^d F_{cd} = D_a a_b - D_b a_a. \tag{25}$$

We now multiply Eq. (19) by the projector to write

$$\beta h^b{}_c \nabla_a F^{ac} = m^2\beta a^b. \tag{26}$$

To relate Eq. (26) to the induced connection D_a and the projected tensor f_{ab}, we consider the expression $D_a(\beta f^{ab})$. Using the definition of the projector, we can write

$$\begin{aligned} D_a(\beta f^{ab}) &= h^b{}_e h^f{}_a \nabla_f(\beta F^{ae}) \\ &= h^b{}_e\nabla_a(\beta F^{ae}) + \beta^{-2}h^b{}_e\chi_a\chi^f\nabla_f(\beta F^{ae}). \end{aligned} \tag{27}$$

The orthogonality of χ_a and ϕ_a and Eq. (5) imply $\mathcal{L}_\chi\beta = 0$. Also, since ξ^a and ϕ^a are Killing fields, we have $\mathcal{L}_\xi F^{ab} = 0 = \mathcal{L}_\phi F^{ab}$. Then, Eq. (27) becomes

$$\begin{aligned} D_a(\beta f^{ab}) = \beta h^b{}_e\nabla_a F^{ae} + \beta^{-1}\chi_a h^b{}_e \\ \times[F^{ce}\nabla_c\chi^a + F^{ac}\nabla_c\chi^e - (F^{ce}\nabla_c\alpha)\phi^a \\ - (F^{ac}\nabla_c\alpha)\phi^e] + h^b{}_e F^{ae}\nabla_a\beta. \end{aligned} \tag{28}$$

On the other hand, from Eqs. (5) and (6) we have

$$\begin{aligned} \nabla_a\chi_b = \beta^{-1}(\chi_b\nabla_a\beta - \chi_a\nabla_b\beta) \\ + \frac{1}{2}(\phi_a\nabla_b\alpha + \phi_b\nabla_a\alpha). \end{aligned} \tag{29}$$

We substitute this expression into Eq. (28). Then, using $\chi^a\phi_a = 0$ and the definition for the electric field e^a, we find that Eq. (28) reduces to

$$D_a(\beta f^{ab}) = \beta h^b{}_e\nabla_a F^{ae} + \frac{1}{2}(e^c\nabla_c\alpha)\phi^b. \tag{30}$$

Thus, Eq. (26) becomes

$$D_a(\beta f^{ab}) = m^2\beta a^b + \frac{1}{2}(e^c\nabla_c\alpha)\phi^b. \tag{31}$$

If we multiply both sides of Eq. (31) by a_b and integrate it over Σ, we again end up with an integral which, like Eq. (24), is not guaranteed to be positive definite.

In order to simplify the situation, we now further project Eq. (31) over the spacelike 2-submanifolds orthogonal to both χ_a and ϕ_a, which we have assumed to exist. We use the projector $\Pi_a{}^b$ defined in Eq. (13) and follow the same procedure as before. Since ϕ^a is a Killing field, $\mathcal{L}_\phi f_{ab} = 0 = \mathcal{L}_\phi a_b$ and we simply have, after a little computation,

$$\bar{D}_a(f\beta\bar{f}^{ab}) = m^2 f\beta\bar{a}^b, \tag{32}$$

where the "bar" denotes the respective fields after projection onto these spacelike 2-submanifolds. Contracting both sides of Eq. (32) by $\bar{a}_b$, integrating by parts and, using the same boundedness arguments over the horizons as before, we have

$$\int_{\bar{\Sigma}}\beta f(\bar{f}_{ab}\bar{f}^{ab} + m^2\bar{a}^b\bar{a}_b) = 0. \tag{33}$$

Since the 2-submanifolds are spacelike, the integrand in Eq. (33) is positive definite. This yields $\bar{f}_{ab} = 0 = \bar{a}_b$ everywhere over the 2-submanifolds. Also, it is easy to check using $\mathcal{L}_\xi\bar{a}_b = 0 = \mathcal{L}_\xi\bar{f}_{ab}$ and $\mathcal{L}_\phi\bar{a}_b = 0 = \mathcal{L}_\phi\bar{f}_{ab}$ that $\mathcal{L}_\chi\bar{a}_b = 0 = \mathcal{L}_\chi\bar{f}_{ab}$. This implies that $\bar{f}_{ab} = 0 = \bar{a}_b$ throughout the manifold.

It follows that A_b is of the form

$$A_b = \Psi_1(x)\chi_b + \Psi_2(x)\phi_b. \tag{34}$$

The commutativity of the two Killing fields ξ^a and ϕ^a implies that $\mathcal{L}_\chi\alpha = 0 = \mathcal{L}_\phi\alpha$. Also, we recall that since A_a is a physical matter field, its Lie derivatives vanish along ξ^a and ϕ^a. Then it is easy to verify from Eq. (34) that $\mathcal{L}_\phi A_b = 0$ implies $\mathcal{L}_\phi\Psi_1 = 0 = \mathcal{L}_\phi\Psi_2$; and $\mathcal{L}_\chi A_b = (A_a\phi^a)\nabla_a\alpha$ implies that $\mathcal{L}_\chi\Psi_1 = 0 = \mathcal{L}_\chi\Psi_2$.

With the ansatz (34), the Proca Lagrangian (18) becomes

$$\mathcal{L} = \frac{1}{2}(\beta\nabla_a\Psi_1 + 2\Psi_1\nabla_a\beta)^2 - \frac{1}{2}(f\nabla_a\psi_2 + 2\Psi_2\nabla_a f)^2 + f^2\Psi_2(\nabla_a\psi_1)(\nabla^a\alpha) + \frac{f^4\Psi_2^2}{2\beta^2}(\nabla_a\alpha)(\nabla^a\alpha) + \frac{2f^2}{\beta}\Psi_1\Psi_2(\nabla_a\beta)(\nabla^a\alpha) + \frac{m^2}{2}(\beta^2\Psi_1^2 - f^2\Psi_2^2). \quad (35)$$

The equations of motion for the two degrees of freedom Ψ_1 and Ψ_2 are then

$$\nabla_a(\beta^2\nabla^a\Psi_1) - 2\beta(\nabla_a\beta)(\nabla^a\Psi_1) + \nabla_a(2\beta\psi_1\nabla^a\beta) - 4\Psi_1(\nabla_a\beta)(\nabla^a\beta) + \nabla_a(f^2\Psi_2\nabla^a\alpha) - \frac{2f^2}{\beta}\Psi_2(\nabla_a\beta)(\nabla^a\alpha) - m^2\beta^2\Psi_1 = 0, \quad (36)$$

and

$$\nabla_a(f^2\nabla^a\Psi_2) - 2f(\nabla_a f)(\nabla^a\Psi_2) + \nabla_a(2f\psi_2\nabla^a f) - 4\Psi_2(\nabla_a f)(\nabla^a f) + \frac{f^4\Psi_2}{\beta^2}(\nabla_a\alpha)(\nabla^a\alpha) + \frac{2f^2}{\beta}\Psi_1(\nabla_a\beta)(\nabla^a\alpha) + f^2(\nabla_a\Psi_1)(\nabla^a\alpha) - m^2 f^2\Psi_2 = 0. \quad (37)$$

Let us now project Eqs. (36) and (37) over Σ and form quadratic integrals as before. Since $\mathcal{L}_\chi\Psi_1 = 0 = \mathcal{L}_\chi\Psi_2$, the fact that ∇_a is torsion-free implies that $\mathcal{L}_\chi(\nabla_a\Psi_1) = 0 = \mathcal{L}_\chi(\nabla_a\Psi_2)$. It is straightforward to calculate similarly that $\mathcal{L}_\chi(\nabla_a\alpha) = \mathcal{L}_\chi(\nabla_a f) = \mathcal{L}_\chi(\nabla_a\beta) = 0$. So the 1-forms $(\nabla_a\beta, \nabla_a\alpha, \nabla_a f)$ are spacelike. We can now project Eqs. (36) and (37) over Σ to get

$$D_a(\beta^3 D^a\Psi_1) - 2\beta^2(D_a\beta)(D^a\Psi_1) + D_a(2\beta^2\Psi_1 D^a\beta) - 4\beta\Psi_1(D_a\beta)(D^a\beta) + D_a(\beta f^2\Psi_2 D^a\alpha) - 2f^2\Psi_2(D_a\beta)(D^a\alpha) - m^2\beta^3\Psi_1 = 0, \quad (38)$$

and

$$D_a(f^2\beta D^a\Psi_2) - 2\beta f(D_a f)(D^a\Psi_2) + D_a(2\beta f\psi_2 D^a f) - 4\beta\Psi_2(D_a f)(D^a f) + \frac{f^4\Psi_2}{\beta}(D_a\alpha)(D^a\alpha) + 2f^2\Psi_1(D_a\beta)(D^a\alpha) + \beta f^2(D_a\Psi_1)(D^a\alpha) - m^2\beta f^2\Psi_2 = 0. \quad (39)$$

We now multiply Eq. (38) by Ψ_1 and Eq. (39) by Ψ_2; add them and integrate by parts. The surface integrals do not survive because Ψ_1 and Ψ_2 and their derivatives are bounded on $\partial\Sigma$, and we have

$$\int_\Sigma \beta[(\beta D_a\Psi_1 + 2\Psi_1 D_a\beta)^2 + (fD_a\Psi_2 + 2\Psi_2 D_a f)^2 - \frac{f^4\Psi_2^2}{\beta^2}(D_a\alpha)(D^a\alpha) + m^2(\beta^2\Psi_1^2 + f^2\Psi_2^2)] = 0. \quad (40)$$

This is clearly not positive definite due to the presence of the third term. We can naively interpret that term as the centrifugal effect on the field due to the rotation of the spacetime. We now investigate whether the rotation can actually be so large that the integrand in Eq. (40) becomes negative.

Let us consider the Killing identity for ϕ_b

$$\nabla_b\nabla^b\phi_a = -R_a{}^b\phi_b. \quad (41)$$

Contracting Eq. (41) by ϕ^a and using Eq. (12), we get

$$\nabla_b\nabla^b f^2 = \left[4(\nabla_a f)(\nabla^a f) - \frac{f^4}{\beta^2}(\nabla_a\alpha)(\nabla^a\alpha) - 2R_{ab}\phi^a\phi^b\right]. \quad (42)$$

We now project Eq. (42) onto Σ, multiply by Ψ_2^2 and integrate by parts to get

$$\int_\Sigma \beta[4f\Psi_2(D_a\Psi_2)(D^a f) + 4\Psi_2^2(D_a f)(D^a f) - \frac{\Psi_2^2 f^4}{\beta^2}(D_a\alpha)(D^a\alpha) - 2\Psi_2^2 R_{ab}\phi^a\phi^b] = 0. \quad (43)$$

Subtracting Eq. (43) from Eq. (40), we now have

$$\int_\Sigma \beta[(\beta D_a\Psi_1 + 2\Psi_1 D_a\beta)^2 + f^2(D_a\Psi_2)(D^a\Psi_2) + 2\Psi_2^2 R_{ab}\phi^a\phi^b + m^2(\beta^2\Psi_1^2 + f^2\Psi_2^2)] = 0. \quad (44)$$

So, the no-hair result $\Psi_1 = 0 = \Psi_2$ will follow from Eq. (44) if $R_{ab}\phi^a\phi^b \geq 0$. We have assumed that the spacetime satisfies Einstein's equations, so in particular

$$R_{ab}\phi^a\phi^b = \left(T_{ab} - \frac{1}{2}Tg_{ab}\right)\phi^a\phi^b + \Lambda f^2. \quad (45)$$

We compute the energy-momentum tensor for the Lagrangian (18),

$$T_{ab} = F_{ac}F_b{}^c + m^2 A_a A_b + \mathcal{L}g_{ab}, \quad (46)$$

which yields

$$\left(T_{ab} - \frac{1}{2}Tg_{ab}\right)\phi^a\phi^b = \left(\frac{1}{2}b_a^2 + \frac{1}{2}f^2 e_a^2 + m^2 f^4\Psi_2^2\right), \quad (47)$$

where $b_a = F_{ab}\phi^b$ and e_a is the electric field defined in Eq. (20). It is easy to see that $b_a\chi^a = 0$, i.e., b_a is spacelike. The electric field e^a is also spacelike as mentioned earlier. So, Eq. (47) shows that $(T_{ab} - \frac{1}{2}Tg_{ab})\phi^a\phi^b \geq 0$ for the Proca field. Putting in all this, we can rewrite Eq. (44) as

$$\int_\Sigma \beta\Big[(\beta D_a\Psi_1 + 2\Psi_1 D_a\beta)^2 + f^2(D_a\Psi_2)(D^a\Psi_2) + m^2\beta^2\Psi_1^2 + (m^2 + 2\Lambda)f^2\Psi_2^2 + 2\Psi_2^2\Big(\frac{1}{2}b_a^2 + \frac{1}{2}f^2e_a^2 + m^2f^4\Psi_2^2\Big)\Big] = 0, \qquad (48)$$

which gives $\Psi_1 = 0 = \Psi_2$ over Σ. Since $\mathcal{L}_\chi\Psi_1 = 0 = \mathcal{L}_\chi\Psi_2$, we have $\Psi_1 = 0 = \Psi_2$ throughout the manifold. This, combined with the previous proof $\bar{a}_b = 0$, is the desired no-hair result for a de Sitter black hole for the Proca-massive vector field.

Clearly, our proof is also valid for an asymptotically flat stationary axisymmetric spacetime, $\Lambda = 0$. We have only to replace the outer boundary or the cosmological horizon by a 2-sphere at spacelike infinity with a sufficiently rapid fall-off condition of the fields. Our proof also applies to asymptotically anti-de Sitter space-time provided we assume $m^2 \geq 2|\Lambda|$ in Eq. (48) for the asymptotically AdS case. This is not a strong assumption—it only means that the Compton wavelength of the vector field is less than the cosmological length scale or the AdS radius.

As we have mentioned earlier, the no-hair proof fails for $m = 0$, i.e. for the Einstein-Maxwell system, because the local gauge symmetry implies that A_a is not a physical field, so need not be bounded on the horizon. The Kerr-Newmann-de Sitter spacetime is a black hole solution to the Einstein-Maxwell equations [11].

IV. DISCUSSIONS

To summarize, we have proven the no-hair theorems for scalar and (Proca) massive vector fields for a stationary axisymmetric de Sitter black hole spacetime. In comparison to the proof in a static spacetime, this proof contains some additional constraints such as the commutativity of the two Killing fields ξ^a and ϕ^a and the existence of spacelike 2-submanifolds orthogonal to them. Also, in order to prove the theorem for the vector field, we had to assume in Eq. (45) that the spacetime satisfies Einstein's equations. For a static spacetime one need not assume that (see e.g. [14]).

In the static case, it is necessary to assume spherical symmetry in order to prove the no-hair theorem for the Abelian Higgs model [6,14]. In fact, if we have cylindrically symmetric matter distribution, we have a cosmic string piercing the horizons [21–23]. It seems likely that we will have a stringlike solution for a rotating axisymmetric de Sitter black hole as well.

As an aside, we note that the no-hair results proven are not black hole uniqueness theorems. It is known that for $\Lambda = 0$, the Kerr spacetime is the only asymptotically flat black hole solution of the vacuum Einstein equations in 4-dimensions (see e.g. [24,25] and references therein). For $\Lambda < 0$ in $2 + 1$ dimensions, a result analogous to Birkhoff's theorem was proven for the BTZ black hole [26]. For $\Lambda > 0$, no proof of uniqueness of black hole solutions is known [27,28]. However, our results reduce the Einstein-scalar (in convex potential) and Einstein-massive vector (with no gauge symmetry) systems to vacuum Einstein equations in the presence of a stationary axisymmetric black hole. So, any proof of uniqueness of the Kerr-de Sitter black hole, if it exists, will apply to these systems as well.

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