Regular black holes: Electrically charged solutions, Reissner-Nordström outside a de Sitter core

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To have the correct picture of a black hole as a whole, it is of crucial importance to understand its interior. The singularities that lurk inside the horizon of the usual Kerr-Newman family of black hole solutions signal an endpoint to the physical laws and, as such, should be substituted in one way or another. A proposal that has been around for sometime is to replace the singular region of the spacetime by a region containing some form of matter or false vacuum configuration that can also cohabit with the black hole interior. Black holes without singularities are called regular black holes. In the present work, regular black hole solutions are found within general relativity coupled to Maxwell's electromagnetism and charged matter. We show that there are objects which correspond to regular charged black holes, whose interior region is de Sitter, whose exterior region is Reissner-Nordström, and the boundary between both regions is made of an electrically charged spherically symmetric coat. There are several types of solutions: regular nonextremal black holes with a null matter boundary, regular nonextremal black holes with a timelike matter boundary, regular extremal black holes with a timelike matter boundary, and regular overcharged stars with a timelike matter boundary. The main physical and geometrical properties of such charged regular solutions are analyzed.

I. INTRODUCTION

A. Solutions of Einstein's equation and black holes

Finding solutions of Einstein's equation is an important branch of general relativity. The equation takes the form, $G_{\mu\nu} = 8\pi\tau_{\mu\nu}$, where $G_{\mu\nu}$ is the Einstein tensor, constructed from second derivatives of the metric tensor $g_{\mu\nu}$, $\tau_{\mu\nu}$ is the stress-energy tensor of the matter and fields, μ, ν are spacetime indices μ, $\nu = 0, 1, 2, 3$, and $G = 1$, $c = 1$. Arbitrarily chosen spacetimes with the corresponding metrics $g_{\mu\nu}$ usually give rise to an Einstein tensor, which in turn corresponds to an unphysical stress tensor, i.e., to matter which is of no interest. Thus, finding solutions of Einstein's equation is in general a nontrivial task. The efforts to overtake the difficulties are displayed in the exact solutions book [1], where several methods of solving Einstein's equation are given. To facilitate the work, instead of having a continuous solution throughout spacetime, one can find solutions for two regions, an interior region and an exterior region, and then match these regions through a smooth junction, a boundary surface [2]. One can also opt for a more drastic junction between both regions where a surface layer, i.e., a thin shell, is needed [2]. Usually, the junctions, or solderings, are through timelike surfaces, as in a surface of a star. One can also extend the formalism of [2] to spacelike surfaces. For a lightlike surface (also called a null surface), a special formalism can be developed, as in [3].

A well-known vacuum ($\tau_{\mu\nu} = 0$) solution of Einstein's equation is the Schwarzschild black hole. This spherically symmetric solution has an event horizon at a coordinate radius $r_{\rm h} = 2m$, where m is the mass of the black hole (see, e.g., [4]). In its full form, the solution represents a wormhole, with its two phases, the white hole and the black hole, both phases harboring singularities and connecting two asymptotically flat universes [4]. In its amputated form, the solution can represent a black hole shielding a singularity and with one asymptotically flat region, the black hole being formed from the collapse of a star or some other lump of matter (see, e.g., [4]). When, besides a mass m, one includes electrical charge q, the Reissner-Nordström solution is obtained (see, e.g., [4]). The inclusion of angular momentum J gives the Kerr solution, and the inclusion of the three parameters (m, J, q) is the Kerr-Newman solution, yielding the Kerr-Newman family of black holes within general relativity (see [4] and also [5]).

The outside of a black hole is visible. Nowadays, potent telescopes and detectors watch with ease what is going on in jets and phenomena powered out by black holes. Moreover, the outside of a black hole is well known classically (see, e.g., [6]). Thus, from the outside, there is

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astrophysical evidence and theoretical consistency, within the classical framework of general relativity, for the existence of black holes. Quantically, however, black holes still pose problems for the outside. These problems are related to the Hawking radiation and the Bekenstein-Hawking entropy (see, e.g., [7]). Although their solution is not in hand, the quantum outside problems are well posed and delineated.

The inside of a black hole, on the other hand, is another story; it is not known at all. By definition, the black hole inside is hidden; it encloses a mysterious unknown. The understanding of the inside of a black hole is one of the outstanding problems in gravitational theory. The Schwarzschild solution describes the black hole inside as an ever moving spacetime that ends on an all encompassing spacelike singularity. The Reissner-Nordström solution also has an ever moving inward spacetime that, instead, ends on a Cauchy horizon, which can then be cruised into a region where a singularity can be seen but avoided. The Kerr and the Kerr-Newman solutions have analogous properties to the Reissner-Nordström solution. The event horizon thus, for this class of black hole solutions, harbors a singularity. What is a singularity? The singularity theorems [8–11] do not tell what a singularity is. Through the imposition of some precise physical conditions, the theorems only prove generically that singularities are inevitable. But, those precise physical conditions might not be upheld in the situations they are to be used, turning the theorems useless in such circumstances. The existence of a singularity, by its very definition, means spacetime ceases to exist signaling a failure of the physical laws. So, if physical laws do exist at those extreme conditions, singularities should be substituted by some other object in a more encompassing theory. The extreme conditions, in one form or another, that may exist at a singularity, imply that one should resort to quantum gravity. Singularities are certainly objects to be resolved in the realm of quantum gravity [12].

Since there is no definite quantum gravity yet, a line of work to understand the inside of a black hole and resolve its singularity is to study classical or semiclassical black holes, with regular, i.e., nonsingular properties. These type of black holes can be motivated by quantum arguments. In this way, there has been a trend to invent regular black hole solutions with special matter cores that would substitute the true singularities of the Schwarzschild, Reissner-Nordström, Kerr, and Kerr-Newman black holes.

B. Early considerations

Indeed, Sakharov [13] and Gliner [14], in 1966, proposed that singularities, such as cosmological singularities, could be avoided by matter at superhigh densities with an inflationary equation of state, i.e., with a de Sitter core, in which the equation of state between the pressure p and the energy density ρ_{matter} is $p = -\rho_{\text{matter}}$, or, equivalently, the

matter stress-energy tensor $T_{\mu\nu}$ takes a lambda term or false vacuum form $T_{\mu\nu} = \Lambda g_{\mu\nu}$, Λ being the cosmological constant. This led to the idea that a spacetime filled with false vacuum inside a black hole horizon could provide a description of the final state of gravitational collapse. Furthermore, Zel'dovich [15] proposed afterwards that such a $T_{\mu\nu}$ could arise naturally as a result of vacuum polarization processes in gravitational interactions. These considerations may indicate that an unlimited increase of spacetime curvature during a collapse process can lead to the halt of the collapse itself if quantum fluctuations dominate the process, putting ultimately an upper bound to the value of the curvature and obliging the formation of a central core.

C. The Bardeen regular black hole

Bardeen, in 1968 [16], realized concretely the idea of a central matter core, by proposing a solution of Einstein's equation in which there is a black hole with horizons but without a singularity, the first regular black hole. The matter field content was a kind of magnetic matter field, yielding a modification of the Reissner-Nordström metric. But near the center, the solution tended to a de Sitter core solution. All the subsequent regular black hole solutions are based on Bardeen's proposal, although there has been a tremendous development on the implementation and on the analysis of the properties of regular black hole solutions.

D. Other regular black holes

A useful way to classify the regular black hole solutions is through the type of junctions needed. If there is no junction, the solution is a continuous solution throughout spacetime. If there are two simple regions, the solutions have boundary surfaces joining the two regions. In more drastic cases, the solutions have surface layers, i.e., thin shells joining the two regions.

1. Solutions with continuous fields

Based on a previous work on how to avoid cosmological singularities [17], Dymnikova proposed, in 1992, [18] a black hole model in which the core is de Sitter and gives way in a smooth manner into a Schwarzschild solution, with Cauchy and event horizons somewhere in between. Several subsequent works developing this idea followed [19–25]; see also [26]. Next, Ayón-Beato and García invoked nonlinear fields and sources to generate from first principles the Bardeen model as a nonlinear magnetic monopole [27], and also found a four parameter solution [28]. In addition, they also attempted to derive regular black holes from nonlinear electric fields [29]; see also [30,31], and see [32] for the extremal limit of the solutions. Bronnikov and collaborators also produced several regular black holes in which the source are fields permeating the whole spacetime, the core is an expanding universe with

de Sitter asymptotics and the exterior outer region tends to Schwarzschild [33–35]. In [36], a development along the same lines can be found. Regular black holes in quadratic gravity have also been discussed [37].

2. Solutions with boundary surfaces

Another useful way to construct regular black holes is by filling the inner space with matter up to a certain surface and then make a smooth junction, through a boundary surface, to the Schwarzschild solution as was done in [38–40], and in a more general setting in [41]. In the junction to the Schwarzschild case, this junction is made through a spacelike surface, rather than an usual timelike surface. This means that, for some parametrization, the junction exists at a single instant of time. Regular black holes in which the boundary surface is lightlike or timelike have not been found in the literature.

3. Solutions with boundary layers, i.e., thin shells

Finally, it is possible and also of interest to make the transition from an inner de Sitter core to an outer Schwarzschild, Reissner-Nordström, or other chosen spacetime, through surface layers, or thin shells. Regular black holes with thin shells of spacelike, lightlike, and timelike character have been found.

Spacelike thin shells.—Following Zel'dovich's idea [15], Markov [42] suggested a concrete upper bound for the curvature, of the order of the Planck curvature. After the Planck curvature bound is achieved, it is suggested that the matter turns into a de Sitter phase. The transition is made through a spacelike thin shell. This was developed in [43,44]; see also [45–47].

In addition, in [48] the fitting of closed and open sections of de Sitter space into a Schwarzschild solution was considered, where special care was taken in the analysis of the intrinsic properties of the spacelike surface layer of constant curvature joining the two spacetimes; see, also, [49] for a general discussion including the Kerr-Newman metric.

Lightlike thin shells.—Even before Dymnikova [18] developed her regular black hole with smooth features, Gonzalez-Diaz, in 1981, [50] took interest in finding a regular black hole. He tried a solution by direct matching of de Sitter spacetime with the Schwarzschild solution on the horizon, a null surface. Shen and Zhu reanalyzed later this soldering of de Sitter spacetime with the Schwarzschild solution [51], while Shen and Tan, in 1989, [52] generalized the Gonzalez-Diaz idea to d dimensions. It was later argued in [53] that a Schwarzschild type matching can also be achieved within a more general parametrization of the static metric by two different functions due to the jump of the product $g_{tt}g_{rr}$. However, Grøn and Soleng [54,55] showed that the direct matching onto Schwarzschild at the horizon contained in [50] is incorrect. Poisson and Israel [56] reinforced once again that no direct matching, of the type done in [50–52], is possible; de Sitter

spacetime cannot be soldered directly to an exterior Schwarzschild vacuum, since the junction conditions would be violated. It is necessary to put a thin shell of noninflationary material at a junction outside the event horizon. In [57], it was shown that the more general tentative matching proposed in [53] is also not possible (see in addition [58] for no-go theorems).

Additional tries of the same type of matching, now extending to the Reissner-Nordström spacetime, were performed in 1985 by Shen and Zhu in [59,60]. By including charge, the matching problems occurring in a Schwarzschild matching may be avoided. In 1991, Barrabès and Israel [61] gave an example where there is the possibility of joining correctly at a null surface and gave interesting examples of a lightlike thin shell matching at the Cauchy horizon (see, also, [3] for null matching); see also [44].

Timelike thin shells.—In the context of regular black holes with boundary layers, timelike matching is not found in the literature. So, it is of interest to study regular black hole solutions in such a case. Regular black holes either with a charged (usually magnetic) core or with a de Sitter core are known, but with electric charge and a de Sitter core together, as found here, seem to have not been explored. We put the electric charge on a thin shell at the surface of the object, in one instance at the inner horizon, in all the other instances below the inner horizon.

E. General results on regular black holes

General results on regular black holes, like those related to the topology and causality of these solutions, were put forward by Borde in an important development [62,63]. Also, energy conditions and other properties have been studied [64–66]. The quasilocal energy of regular black holes has been analyzed in [67]. Entropy and thermodynamics of regular black holes have been studied in [68,69].

F. Reviews on regular black holes

For a general review on regular black holes, including black holes with Gaussian sources, see [70], and for a motivation of these sources as well as a review on noncommutative black holes, see [71].

G. Connections to other works

An issue connected to regular black holes is quasiblack holes. Quasiblack holes are objects whose boundary is as near a horizon as one wants. For the outside, they act as black holes, although the inside properties are completely different [72]. Based on a worked by Guilfoyle [73], solutions of quasiblack holes with pressure, i.e., of relativistic charged spheres as frozen stars, have been found in [74]. These solutions also contain, unexpectedly, regular black holes, a particular branch of those is referred to below.

There are some interesting investigations on the dynamics of time-dependent bubbles, in which an observer in the outer region describes the system as having a horizon and a black hole, and an observer in the inner region, made of false vacuum, sees a de Sitter universe [75–77].

Related to the inside of a black hole, it has been shown that, under perturbations, an instability of the internal Cauchy horizon occurs, and a spacelike or null true singularity emerges inside a charged Reissner-Nordström black hole. This phenomenon is called mass inflation, as it shows an exponential growth of the local mass function [78].

Black holes, and, in particular, charged black holes, singular or regular, as elementary charged particles is an issue in itself that we will delve into in another work.

H. Layout of the paper

The present paper is organized as follows. In Sec. II, we lay out some properties of cold charged fluids in general relativity. In Sec. II A, the basic equations describing a charged fluid are written. Spherically symmetric equations are then set up in Sec. II B. Then, in Sec. III, regular charged black hole solutions are presented. In Sec. III A, we give the ansätze used, such as the one that describes the matter fluid by a false vacuum term. Section III B is devoted to present the regular charged black hole solutions, to the study of the main properties of these black holes, and to comment on the relation of the present solutions to other regular black hole solutions found in the literature and to some of the solutions of Guilfoyle. In Sec. IV, we conclude.

II. CHARGED FLUIDS

A. Basic equations

The cold charged fluids considered in the present work are described by the Einstein-Maxwell equations with matter, which can be written as

$$G_{\mu\nu} = 8\pi\tau_{\mu\nu}, \qquad (1)$$

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu, \qquad (2)$$

where Greek indices μ, ν, etc., run from 0 to 3, 0 corresponding to a timelike coordinate t. $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor, with $R_{\mu\nu}$ being the Ricci tensor, $g_{\mu\nu}$ the metric, and R the Ricci scalar. We put both the speed of light c and the gravitational constant G equal to unity throughout. The tensor $\tau_{\mu\nu}$ is the energy-momentum tensor, which here can be decomposed into two parts, $E_{\mu\nu}$ and $T_{\mu\nu}$,

$$\tau_{\mu\nu} = (T_{\mu\nu} + E_{\mu\nu}). \qquad (3)$$

$E_{\mu\nu}$ is the electromagnetic energy-momentum tensor, which can be written in the form

$$E_{\mu\nu} = \frac{1}{4\pi}\left(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \qquad (4)$$

where the Maxwell tensor is

$$F_{\mu\nu} = \nabla_\mu\mathcal{A}_\nu - \nabla_\nu\mathcal{A}_\mu, \qquad (5)$$

∇_μ representing the covariant derivative, and $\mathcal{A}_\mu$ the electromagnetic gauge field. In addition,

$$J_\mu = \rho_e U_\mu \qquad (6)$$

is the current density, with ρ_e and U_μ being, respectively, the electric charge density and the fluid four velocity. $T_{\mu\nu}$ is the material energy-momentum tensor, which, for the purpose of the present work, is taken in the form of an isotropic fluid

$$T_{\mu\nu} = (\rho_m + p)U_\mu U_\nu + pg_{\mu\nu}, \qquad (7)$$

where ρ_m is the fluid matter-energy density, and p is the isotropic fluid pressure.

B. Spherical equations: general analysis

We particularize here the study to spherically symmetric systems, where the charged fluid distribution is bounded by a spherical surface S, whose exterior region can be described by the electrovacuum Reissner-Nordström metric. We then assume that the spacetime inside S is static and spherically symmetric, so that the metric is conveniently written in a Schwarzschild-like form, namely,

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2), \quad (8)$$

where r is the usual radial coordinate, A and B are functions which depend upon r only, and $d\theta^2 + \sin^2 d\phi^2$ is the metric of the unit sphere, with θ and ϕ being the spherical angles. The gauge field $\mathcal{A}_\mu$ assumes the form

$$\mathcal{A}_\mu = -\phi(r)\delta_\mu^t, \qquad (9)$$

where $\phi(r)$ is the electric potential, and here δ is the Kronecker delta. The four velocity U_μ in turn is

$$U_\mu = -\sqrt{B(r)}\delta_\mu^t. \qquad (10)$$

Define r_0 as the radius of the spherical surface S. Consider the region $r < r_0$. This region is filled with matter, i.e., filled with a static charged perfect fluid distribution with spherical symmetry. Then, the relevant Einstein-Maxwell field equations for the metric (8) can be found. To find the set of basic equations, we note that the tt and rr components of the Einstein equation (1) furnish the following relations:

$$\frac{B'(r)}{B(r)} + \frac{A'(r)}{A(r)} = 8\pi r A(r)[\rho_m(r) + p(r)], \qquad (11)$$

$$(rA^{-1}(r))' = 1 - 8\pi r^2\left(\rho_m(r) + \frac{Q^2(r)}{8\pi r^4}\right), \qquad (12)$$

where a prime denotes the derivative with respect to the radial coordinate r. The only nonzero component of the

Maxwell equation (2) furnishes

$$Q(r) = 4\pi \int_0^r \rho_e(r)\sqrt{A(r)}r^2 dr = \frac{r^2 \phi'(r)}{\sqrt{B(r)A(r)}}, \quad (13)$$

where an integration constant was set to zero. $Q(r)$ is the total electric charge inside the surface of radius r. The conservation equations $\nabla_\nu T^{\mu\nu} = 0$, together with the Maxwell equation, give

$$2p'(r) + \frac{B'(r)}{B(r)}[\rho_m(r) + p(r)] - 2\frac{\phi'(r)\rho_e(r)}{\sqrt{B(r)}} = 0, \quad (14)$$

which is the only nonidentically zero component of the conservation equations. The other nonzero component of the Einstein-Maxwell equations give an additional relation, which is not independent of the above set of equations. Hence, considering that Eq. (13) gives $\phi(r)$ in terms of $\rho_e(r)$, we are left with a system of three differential equations, Eqs. (11), (12), and (14), for the five unknowns $A(r)$, $B(r)$, $\rho_m(r)$, $p(r)$, and $\rho_e(r)$. Below we study a particular choice of relations to fix the 2 degrees of freedom and show that with an appropriately chosen equation of state $p = f(\rho_m)$ and further assumptions on the functions $\rho_m(r)$ and $\rho_e(r)$, one can find regular black hole solutions. For $r \geq r_0$, we assume an electrovacuum. Then the metric and the electric potential for $r > r_0$ are given by the Reissner-Nordström solution, $B(r) = A^{-1}(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$, $\phi(r) = \frac{q}{r} + \phi_0$, ϕ_0 being an arbitrary constant which defines the zero of the electric potential, and that, in the exterior Reissner-Nordström region of the spacetimes as we consider here, can be set to zero. The parameters m and q are the mass and charge of the Reissner-Nordström solution, respectively. The matching of the interior and exterior solutions at r_0 will then have to be made.

III. ELECTRIFIED FALSE VACUUM SOLUTIONS: REGULAR CHARGED BLACK HOLES

A. Simplifying assumptions

In order for the energy-momentum tensor to satisfy the false vacuum condition, the energy density and pressure of the fluid (7) must obey the following relation

$$\rho_m(r) + p(r) = 0, \quad (15)$$

valid for all r. Indeed, in the true vacuum, for $r \geq r_0$, the condition is trivially obeyed, since $\rho_m(r) = 0$ and $p(r) = 0$. In the matter, $r < r_0$, the above equation, Eq. (15), is equivalent to a Λ term, and can be interpreted as representing a false vacuum condition. Indeed, a false vacuum is usually given in the form $T_t^t = T_r^r$ and $T_\theta^\theta = T_\phi^\phi$, the same as Eq. (15). The ansatz (15) can be interpreted as an equation of state, and supplies a constraint among the five unknown quantities of the problem.

However, since we have just three equations relating the five unknowns, we need another input. Such a degree of freedom is related to the charge density and in order to close the system usually the function $\rho_e(r)$, or $Q(r)$ if one prefers, is furnished by making an additional hypothesis. In the present case, one of the simplest ansatz one can make is through the following equation:

$$8\pi\rho_m(r) + \frac{Q^2(r)}{r^4} = \frac{3}{R^2}, \quad (16)$$

for $r \leq r_0$, and where R is a constant to be determined. This resembles the Schwarzschild assumption of constant energy density for the first interior solution found within general relativity (see, e.g., [1]). In the present case, the energy density includes not only the energy density due to the mass distribution, but it also bears the electromagnetic energy density carried by the electric field and represented by the term $\frac{Q^2(r)}{8\pi r^4}$. Now, after the assumption (15), Eqs. (14) and (13) give

$$p'(r) = \frac{\phi'(r)\rho_e(r)}{\sqrt{B(r)}} = \frac{QQ'}{4\pi r^4}. \quad (17)$$

This means that the pressure gradient $p'(r)$ is proportional to the charge density. Finally, with condition (15), it is obtained from Eq. (11) that the metric coefficients are related by $A(r)B(r) = \text{constant}$. This constant can be set to unity by a reparametrization of the coordinate t. Then,

$$B(r) = A^{-1}(r), \quad (18)$$

and we can now complete the process of integration of the full set of equations.

B. Electrified de Sitter false vacuum solutions: Regular charged black holes

1. Finding the solutions

Bearing in mind that for $r > r_0$ it is a true electrovacuum solution and thus the Reissner-Nordström solution, and using the ansätze (15) and (16), Eqs. (11) and (12) integrate to

$$B(r) = A^{-1}(r) = \begin{cases} 1 - \frac{r^2}{R^2}, & r \leq r_0, \\ 1 - \frac{2m}{r} + \frac{q^2}{r^2}, & r \geq r_0, \end{cases} \quad (19)$$

where an integration constant was set to zero to avoid a spacetime singularity at $r = 0$. By joining the interior and the exterior metrics at $r = r_0$, i.e., by equating the metric coefficients [functions $B(r)$ and $A(r)$, see Appendix A] in Eq. (19) with the coefficients of the exterior Reissner-Nordström metric, the constant R in Eq. (19) is found in terms of the parameters r_0, m, and q, through the equation

$$\frac{1}{R^2} = \frac{1}{r_0^3}\left(2m - \frac{q^2}{r_0}\right). \quad (20)$$

From the above set of equations, we can infer several things. From Eq. (19), one can infer that

$$r_0 \leq R, \tag{21}$$

otherwise one would have three zeros, i.e., three horizons, for the metric corresponding to (19), an odd number of horizons, and this is not possible for regular black holes (as well as for the electrovacuum ones in general relativity) in asymptotically flat spacetimes. For $r_0 \leq R$, one can also infer from Eq. (20) that $1 - \frac{2m}{r_0} + \frac{q^2}{r_0^2} \geq 0$, which means that either r_0 is inside or on the inner, or Cauchy, horizon r_-, or r_0 is outside or on the outer, event, horizon r_+. Since we want black holes, we search for solutions in which $r_0 \leq r_-$, but the procedure dictates what kind of solutions there are.

Our choices (15) and (16) for $r \leq r_0$ can now be written for $\rho_m(r)$ and $p(r)$ in terms of the electric charge $Q(r)$, namely,

$$8\pi\rho_m(r) = \frac{3}{R^2} - \frac{Q^2(r)}{r^4}, \tag{22}$$

$$8\pi p(r) = -\frac{3}{R^2} + \frac{Q^2(r)}{r^4}. \tag{23}$$

Now, using Eqs. (22) and (23), as well as Eq. (17), we can determine $Q(r)$. The result is $Q(r) = 0$ in the region where there is matter, where $p \neq 0$, $r < r_0$. And $Q'(r) = 0$ in the electrovacuum region, where $p = 0$, $r \geq r_0$. With this, one can build a solution where $Q(r)$ is a step function, more precisely, it may be represented by the step Heaviside theta function

$$Q(r) = \begin{cases} 0, & r < r_0, \\ q, & r \geq r_0. \end{cases} \tag{24}$$

In such a case, the charge density is proportional to a Dirac delta function, the charge $\rho_e(r)$ is spread at the surface $r = r_0$. Furthermore, one gets $p'(r) = 0$ everywhere throughout the spacetime, except at $r = r_0$, where it is the product of the Heaviside theta, or step, function by the Dirac delta function.

The electric potential $\phi(r)$ is obtained from Eqs. (13) and (24), and may be written as

$$\phi(r) = \begin{cases} -\frac{q}{r_0}, & r \leq r_0, \\ -\frac{q}{r}, & r \geq r_0. \end{cases} \tag{25}$$

As it is known, the exterior geometry, i.e., for $r > r_0$, depends on $\phi'(r_0)$ only through the total charge $q = Q(r_0)$, i.e., in that region, the only important quantity is the value of the electric field on the surface $r = r_0$, as expected.

Now, we present other important relations between the parameters. At r_0, Eqs. (22) and (23) turn into

$$8\pi\rho_m(r_0) = \frac{3}{R^2} - \frac{q^2}{r_0^4}, \tag{26}$$

$$8\pi p(r_0) = -\frac{3}{R^2} + \frac{q^2}{r_0^4}. \tag{27}$$

Using the appropriate soldering conditions [2], one should have $p(r_0) = 0$ at r_0 (see also Appendix A). From Eq. (27), this means

$$Rq = \sqrt{3}r_0^2, \tag{28}$$

where we assumed $q > 0$ without loss of generality. This also means from Eq. (26) that $\rho_m(r_0) = 0$. Using Eq. (28) in Eq. (20), we find

$$mr_0 = \tfrac{2}{3}q^2. \tag{29}$$

Interestingly, in the case $R = r_0$, putting the results of Eqs. (28) and (29) into the horizon equation [i.e., the radii for which $A(r) = 0$], the boundary surface $r = r_0$ coincides with the inner horizon, i.e., the Cauchy horizon r_-. For all the other cases, r_0 is inside r_-. The event horizon r_+, if there is one, is then always outside the matter. One can also check that the second fundamental form is continuous at r_0. This is equivalent to $B'(r)$ being continuous at r_- (see Appendix A). It is easy to verify that this is true for the values given in Eqs. (28) and (29) and only for these values. In fact, as we show in Appendix A, the condition of zero pressure at the surface $r = r_0$ implies that the metric fields $g_{tt}(r) = B(r)$ and $g_{rr}(r) = 1/B(r)$ are C^1 functions at the boundary, which means that the Israel junction conditions [2] are satisfied, even in the case r_0 is a lightlike surface.

Note also that when there is no charge, there is no black hole since for $q = 0$ one has $r_0 = 0$ and $m = 0$, leaving a Minkowski vacuum spacetime. Thus, the limit of zero charge is not a regular Schwarzschild black hole, but rather a Minkowski spacetime. This is in tune with the claims of Grøn and Soleng [54,55] and Poisson and Israel [56] that one cannot have pure de Sitter joined at the event horizon r_+ to a Schwarzschild vacuum spacetime. However, as we show here, one can have pure de Sitter plus a charged shell joined to Reissner-Nordström vacuum spacetime.

2. The final solution in brief

There are four parameters: m, q, r_0, and R. From Eqs. (28) and (29), the de Sitter radius R and the black hole mass m are fixed, respectively, by

$$R = \sqrt{3}r_0^2\frac{1}{q}, \tag{30}$$

assuming positive q, and

$$m = \frac{2}{3r_0}q^2. \tag{31}$$

So there are two free parameters, r_0 and q, for example. Fixing q, one can change r_0. From Eqs. (21) and (28), one finds

$$r_0 \geq \frac{\sqrt{3}}{3} q, \tag{32}$$

as well as,

$$m \leq \frac{2}{\sqrt{3}} q. \tag{33}$$

To summarize the solution, we display it here in its entirety. Defining, as usual, the Heaviside θ function as $\theta(r - r_0) = 1$ when $r - r_0 \geq 0$, and $\theta(r - r_0) = 0$ when $r - r_0 < 0$, and the Dirac delta function as the derivative of it, $\delta(r - r_0) = [\theta(r - r_0)]'$, where a $\prime$ denotes the radial derivative, one can write all the functions in succinct form. The line element is

$$ds^2 = -B(r)dt^2 + B^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2), \tag{34}$$

where

$$B(r) = \left(1 - \frac{r^2}{R^2}\right)\theta(r_0 - r) + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)[1 - \theta(r_0 - r)], \tag{35}$$

where R and m should be thought of as being given in terms of r_0 and q by Eqs. (30) and (31). The electric potential is

$$\phi(r) = -\frac{q}{r_0}\theta(r_0 - r) - \frac{q}{r}[1 - \theta(r_0 - r)]. \tag{36}$$

Also,

$$Q(r) = q\theta(r - r_0), \tag{37}$$

$$Q'(r) = q\delta(r - r_0), \tag{38}$$

$$\rho_e(r) = \frac{q\sqrt{B(r_0)}}{4\pi r_0^2}\delta(r - r_0), \tag{39}$$

$$\rho_m(r) = \frac{q^2}{8\pi r_0^4}[1 - \theta^2(r - r_0)]\theta(r_0 - r), \tag{40}$$

$$p(r) = -\frac{q^2}{8\pi r_0^4}[1 - \theta^2(r - r_0)]\theta(r_0 - r), \tag{41}$$

$$p'(r) = \frac{q^2}{4\pi r_0^4}\delta(r - r_0); \tag{42}$$

see Appendix B for how to obtain Eq. (42) from Eq. (41). Using the fact that $\theta^2(r - r_0) = \theta(r - r_0)$ (this can only be used if derivatives are not to be taken), Eqs. (40) and (41) simplify to $8\pi\rho_m(r) = -8\pi p(r) = \frac{3}{R^2}[1 - \theta(r - r_0)]$.

For a range of parameters, the solution represents regular black holes with a de Sitter core, and an electric energy-less matter coat at r_0, and Reissner-Nordström all the way up. When there is no charge, there is no regular black hole since for $q = 0$ one has $r_0 = 0$ and $m = 0$, leaving a Minkowski vacuum spacetime.

3. The solutions and their properties

Since we are not interested in the case $q = 0$, as it gives the trivial Minkowski solution, we can parametrize all quantities in terms of q. Equation (32) tell us that r_0 obeys, $r_0 \geq \frac{q}{\sqrt{3}}$. Thus let us put

$$r_0 = \alpha q, \tag{43}$$

for some α, with $\alpha \geq 1/\sqrt{3}$. Then, Eq. (30) gives

$$R = \sqrt{3}\alpha^2 q, \tag{44}$$

and Eq. (31) furnishes

$$m = \frac{2}{3\alpha} q. \tag{45}$$

Now, we have to compare r_0 with r_- and r_+. These latter radii are given by

$$r_\pm = m \pm \sqrt{m^2 - q^2}. \tag{46}$$

Putting Eq. (45) into Eq. (46) yields r_- as

$$r_- = \left(\frac{2}{3\alpha} - \sqrt{\frac{4}{9\alpha^2} - 1}\right)q, \tag{47}$$

and r_+ as

$$r_+ = \left(\frac{2}{3\alpha} + \sqrt{\frac{4}{9\alpha^2} - 1}\right)q. \tag{48}$$

A plot of the several radii as a function of $\alpha = r_0/q$ is given in Fig. 1.

The solutions can be divided into four classes, which are listed and briefly discussed in the following. Figure 2 plots the function $B(r)$ and Fig. 3 draws the Carter-Penrose diagrams for each class. Figures 1–3 help in the explanation of the classes.

Regular nonextremal black holes with a lightlike matter boundary at the inner horizon.—When $\alpha = r_0/q$ has its minimum value $r_0/q = \sqrt{3}/3$, then the radius r_0 of the matter coincides with the Cauchy horizon r_-, so that $r_-/q = \sqrt{3}/3$. There is matter up to r_-. The event horizon at r_+ is at a larger radius. From Eq. (48) with $\alpha = \sqrt{3}/3$, one finds $r_+/q = 3r_-/q = \sqrt{3}$. This is a solution for a perfectly regular black hole, Reissner-Nordström outside the matter. One can also find a relation between the surface charge density and the horizon radii. From Eq. (39) defining a proper surface electric charge density as $\rho_e(r) = \sigma_e\sqrt{B(r)}\delta(r - r_0)$, one finds $\sigma_e = \frac{q}{4\pi r_0^2} = \frac{1}{4\pi}\frac{3}{q}$. This can also be put in the form $\sigma_e = \frac{1}{4\pi}\frac{\sqrt{3}}{r_-}$. Using $r_+/q = 3r_-/q = \sqrt{3}$ and squaring the result, one obtains $\sigma_e^2 = \frac{1}{16\pi^2}\frac{r_+}{r_-^2}$. For this solution, in which $R^2 = q^2/3$, one can show that the surface gravities of the de Sitter horizon and of the Cauchy Reissner-Nordström horizon are equal.
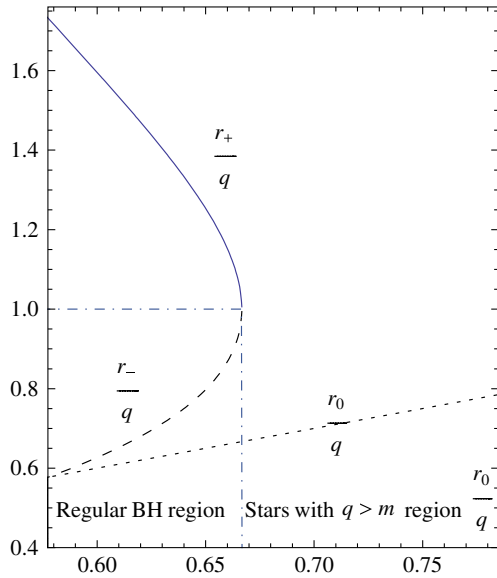
FIG. 1 (color online). A plot of the several radii as a function of $\alpha = r_0/q$. The minimum value of α is $\sqrt{3}/3$, and the vertical dashed-dotted line is at $\alpha = 2/3$.
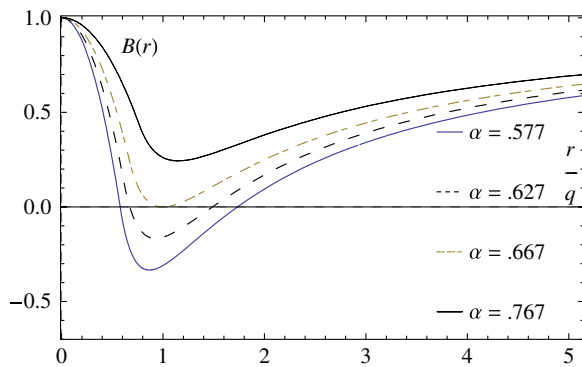

FIG. 2 (color online). A plot of $B(r)$ for four different $\alpha \equiv \frac{r_0}{q}$: $\alpha = \sqrt{3}/3$ (solid line), $\alpha = \sqrt{3}/3 + .1$ (dashed line), $\alpha = 2/3$ (dash-dotted line), and $\alpha = 2/3 + .1$ (dotted line).
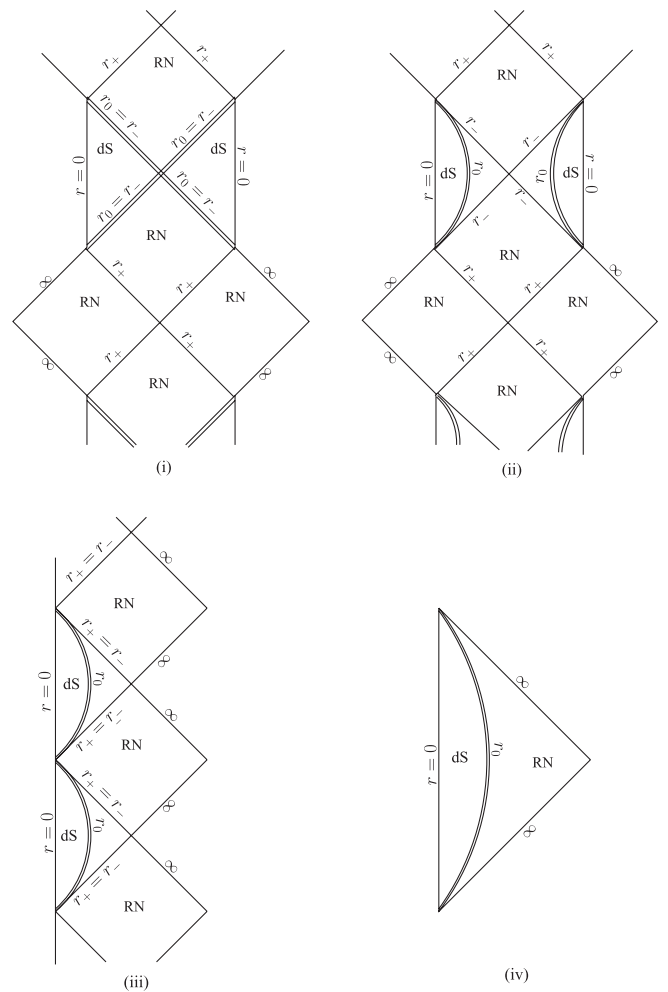

FIG. 3. The Carter-Penrose diagram in the possible four distinct situations, three for regular black holes and one for a star: (i) The regular nonextremal black hole with a lightlike matter boundary, where $r_0 = r_-$. (ii) The regular nonextremal black hole with a timelike matter boundary, where $r_0 < r_-$. (iii) The regular extremal black hole with a timelike matter boundary, where $r_0 < r_- = r_+$. (iv) The regular overcharged star with a timelike matter boundary, where there is no $r_\pm$. In all cases, r_0 represents the surface matching an interior de Sitter spacetime region to an exterior Reissner-Nordström region. Note the regular black hole constraint $r_0 \leq r_-$.

The surface pressure and surface energy density are both zero at r_-.

Regular nonextremal black holes with a timelike matter boundary inside the inner horizon.—For larger r_0/q, i.e., larger r_0, the Cauchy horizon r_- grows faster and remains outside the radius of the matter, $r_0 < r_-$. There is matter up to r_0 and then outside the matter stand two horizons. This is also a solution for a perfectly regular black hole, Reissner-Nordström outside the matter.

Regular extremal black holes with a timelike matter boundary inside the double horizon.—When $r_0/q = 2/3$, then the Cauchy horizon and the event horizon coincide, the solution represents an extremal regular black hole. Since $r_- = r_+ = m = q$, we have $r_0 = \frac{2}{3}r_+$. The horizon is now at the furthest coordinate distance from the surface

of the matter at r_0. In fact, the horizon is at an infinite proper distance from the surface of the matter.

Regular overcharged stars with a timelike matter boundary.—For $r_0/q > 2/3$, one gets from Eq. (31) that $q > m$, the solution has charge greater than mass. Thus, from Eq. (46), there are no horizons and therefore no black holes. An overcharged star, with no horizons, pops up. The star that was hidden behind a horizon comes into light. The horizons have disappeared. It is of remark that r_0 of the first star is still smaller than r_- of the extremal regular black hole. Perhaps this is no surprise, as we are familiar with the fact that the Reissner-Nordström solution has the

same feature: when the Reissner-Nordström parameters are such that $m < q$, the degenerate horizon $r_+ = r_-$ disappears and the singularity at $r = 0$ becomes bare for these overcharged solutions.

Several other features should be mentioned and emphasized.

(i) For a range of parameters, the solutions are thus regular electrically charged black hole solutions. They are built from false vacuum up to, but not at, r_0. The metric for $r < r_0$ is the de Sitter metric, where the isotropic pressure is constant ($p(r) = -\rho_{\mathrm{m}}(r) = 3/8\pi R^2$), and goes to zero at r_0. Furthermore, since the charge density $\rho_{\mathrm{e}}(r)$ is a Dirac delta function centered in $r = r_0$, the total charge q is distributed uniformly on the surface $r = r_0$. At r_0, there is thus a thin electrical layer of an energy-less field, and exterior to it is pure Reissner-Nordström, with two horizons at r_- and r_+. The radius to charge ratio and mass to charge ratio of the solution are well-defined quantities.

(ii) From Eq. (45), we see that as r_0 increases the mass m decreases. This is due to the fact that the pressure, and thus the density, both decrease fast as r_0 increases.

(iii) The limit of zero charge of these solutions is a Minkowski spacetime, rather than a Schwarzschild spacetime.

(iv) These regular charged black hole solutions have boundaries which are either timelike or, in one instance, lightlike. The boundaries for regular black holes found in the literature are spacelike.

(v) Note that, if the charge q is the elementary charge, i.e., the electron charge e, then Eq. (32) gives that the radius r_0 of the particle is of the order of the Planck radius and from Eq. (33) the mass m is of the order of the Planck mass. The solution could then be a model for a heavy elementary charged particle.

(vi) Because of the fact that the matter of the regular black hole is in the inside or at the Cauchy horizon, these solutions may suffer the mass inflation instability [78].

4. Comments on the regular nonextremal black hole with a lightlike matter boundary at the inner horizon r_-

Since the regular nonextremal black hole with a lightlike matter boundary at the inner horizon r_- (see Sec. III B 3) has some history, we comment on it. We start by reviewing and connecting our solution to other works that also found this particular solution. Afterwards, we comment that this particular solution belongs to the Weyl-Guilfoyle class of solutions.

Other works that also found this particular solution.— As noticed in [54–58], a direct soldering of a matter solution onto Schwarzschild through the event horizon is not possible. Indeed, a simple calculation [56] shows that the radial pressure p_r for a static spherically symmetric junction at $r = r_0$ is $p_r = -\rho_{\mathrm{m}}\theta(r_0 - r) + \frac{1}{2}\rho_{\mathrm{m}}r_0\delta(r - r_0)$.

In terms of proper radial distance s, with $ds = \sqrt{A}dr$ [where A is the metric function defined in (8)], the delta function is $\delta(r - r_0) = \sqrt{A}\delta(s)$. At the Schwarzschild event horizon $1/A(r_+) = 0$, and thus the delta function itself is a singular distribution. Thus, besides being a distribution, the surface pressure becomes singular when an event horizon is chosen as the boundary. Therefore, papers [50–53] must be incorrect. A Schwarzschild event horizon is then not the place to make a continuation to a de Sitter phase, whereas other places might give a suitable continuation. However, when one introduces charge, and the matter is joined instead to a Reissner-Nordström spacetime, the impossibility of matching at the horizon can be avoided by making $\rho_{\mathrm{m}} = 0$ and $p_r = 0$ at the event horizon, as well as at the Cauchy horizon.

(i) The two papers by Shen and Zhu of 1985 on regular black holes with Reissner-Nordström asymptotics [59,60] indeed contain, in a hidden manner, the regular nonextremal black hole with a lightlike matter boundary at the inner horizon r_-. Both papers [59,60] have the same content, but oddly the authors do not self-cite either paper. Paper [60] is more detailed.

Shen and Zhu's papers [59,60] give a generalization of the Gonzalez-Diaz [50] work (see, also, [51,52]), by considering the case with charge. In this way, they avoid the matching problem. In [59,60], it is claimed that by taking the limit of zero charge they recover the Gonzalez-Diaz [50] solution. This claim seems incorrect since Eq. (34) of [60] shows that in this limit there is a singularity at $r = 0$, and so there is no regular Schwarzschild black hole.

Now, in a particular instance, the solution found in [59,60] reduces to the regular nonextremal black hole with a lightlike matter boundary at the inner horizon r_- found here. Let us see how. They consider two electrical coats, one at r_+, the other at r_-, such that $\rho_{1e} = \sigma_{1e}\delta(r - r_+)$ and $\rho_{2e} = \sigma_{2e}\delta(r - r_-)$, their Eqs. (17, 18), respectively. They also find that the mass density in between r_- and r_+ is given in their Eq. (28), namely $\rho_{1o} = \frac{3}{8\pi}\left[\frac{r_+ - 16\pi^2\sigma_{2e}^2 r_-^3}{r_+(r_+^2 + r_+ r_- + r_-^2)}\right]$, and the mass density between 0 and r_- is de Sitter type. There is an interesting special solution implicit in this solution. To find it, one abolishes the matter existent in between r_- and r_+. Impose $\rho_{1e} = 0$ and $\rho_{1o} = 0$. From $\rho_{1e} = 0$, nothing special comes about, but from $\rho_{1o} = 0$, one finds that $\sigma_{2e} = \frac{1}{16\pi^2}\frac{r_+}{r_-^3}$. Then, this solution is precisely our regular nonextremal black hole with a lightlike matter boundary at the inner horizon r_-, as one can check (see Sec. III B 3).

(ii) The paper by Barrabès and Israel of 1991 [61] also finds the regular nonextremal black hole with a lightlike matter boundary at the inner horizon r_-.

The direct matching conditions between an interior de Sitter region and an exterior Reissner-Nordström region having a lightlike surface as soldering surface was indeed considered by Barrabès and Israel [61] (see, also,

[3]). The authors have noticed that different matching conditions at a horizon are possible, and have given two special conditions, the static soldering and the affine soldering. In general, the soldering can be performed in anyone of the two horizons, but the surface pressure and energy density are nonzero. However, the static soldering offers a special case. When the de Sitter radius R is related to the charge of the Reissner-Nordström solution q by $R^2 = q^2/3$, then the surface pressure and density are both zero and the surface gravities of the de Sitter horizon and of the Cauchy Reissner-Nordström horizon are equal. Here, of course, the matching happens at the inner horizon. This result agrees with what we have found above, but in our case, the matching can be done at any surface $r = r_0$, timelike if $r_0 < r_-$, or lightlike if $r_0 = r_-$. The solution is the same as the hidden solution in Shen and Zhu's work [59,60], and so the same as our particular solution.

The solution is in the Weyl-Guilfoyle class of solutions.—Charged star-type solutions found by Guilfoyle [73] have a plethora of parameters from which one can choose values. These solutions contain quasiblack hole solutions as found by Lemos and Zanchin [74] and it seems they also contain many different regular black holes. Here, we indicate that in a particular limit of those charged star-type solutions [73] one obtains the regular nonextremal black holes with a lightlike matter boundary at the inner horizon mentioned in Sec. III B 3.

Indeed, taking the parameter $A \to 0$ in Eq. (25) of [73], or the limit $a \to \infty$ in [74] as we do here, we find $\lim_{a \to \infty} \frac{q^2}{m^2} = \frac{1}{2}(2 + |\frac{r_0}{m} - 2| - \frac{r_0}{m})\frac{r_0}{m}$. Because of the absolute value $|\frac{r_0}{m} - 2|$ one concludes, remarkably, there are two branches, $\frac{r_0}{m} > 2$ and $\frac{r_0}{m} < 2$. For $r_0/m > 2$, one finds $\lim_{a \to \infty} \frac{q^2}{m^2} = 0$, which gives the uncharged Schwarzschild interior solution, and it is not of our concern here. For $r_0/m < 2$, one finds $\lim_{a \to \infty} \frac{q^2}{m^2} = (2 - \frac{r_0}{m})\frac{r_0}{m}$, which is equivalent to $1 - \frac{2m}{r_0} + \frac{q^2}{r_0^2} = 0$. This branch is the one that interests us here, and we take the opportunity to make an analysis initiated in our previous paper [74] in relation to this branch. The comments here take over the comments there. Thus, r_0 is at a horizon, and consequently either at r_+ or r_-. Under closer scrutiny, one finds that there is a special regular black hole solution among a maze of regular black hole solutions, in which $r_0 = r_-$, $m = 2r_0$, and $q = \sqrt{3}r_0$. This solution, thus, belongs to Guilfoyle's class of solutions, i.e., it is a solution of Weyl-Guilfoyle type. Thus, this particular solution belongs to both Guilfoyle's class of solutions and to a particular set of solutions we have been studying here. The other regular black holes and stars we have found here are not of the Weyl-Guilfoyle type.

IV. CONCLUSIONS

Charged regular black holes and overcharged stars as solutions to certain distributions of spherically symmetric charged matter have been displayed and studied. The interior distribution of matter is constituted by a de Sitter perfect fluid with pressure p equal to the negative of the energy density ρ_m. Thus, the interior can be interpreted as a false vacuum state. The interior metric is matched into an exterior Reissner-Nordström electrovacuum region. The Einstein-Maxwell equations together with the equation of state $p = -\rho_m$, imply that the isotropic pressure and the energy density are constant throughout the spacetime, and that the electric charge must be located at the boundary of the fluid distribution. Pressure and energy density both go to zero at the boundary, and the resulting solutions represent regular charged black holes. The matter boundary is timelike, and in a limiting case is lightlike. The boundary surface is always at a radius smaller or, at most (in the lightlike case), equal to the inner Reissner-Nordström horizon. For overcharged matter, a star pops up, and the horizons evanesce.

If the charge q is the elementary charge e, then the mass of the solution is of the order of the Planck mass and the radius is of the order of the Planck radius. The solutions could then provide a model for a charged elementary Planckian particle.

Because of the fact that the matter of the regular black hole is in the inside or at the Cauchy horizon, these solutions may suffer the mass inflation instability.

In some models, such as in Dymnikova's, the experimental investigation of physical processes occurring near the external event horizon could give some information about processes occurring deeply inside the black hole. Unfortunately, in our solution near the external event horizon, there is no way to probe the inside of the black hole.

APPENDIX A: THE MATCHING CONDITIONS ON THE BOUNDARY

We will follow [2] to show that the boundary at r_0 is really a boundary surface. We need to show that the metric, or first fundamental form, $g_{\mu\nu}$ is continuous at the surface,

$g^-_{\mu\nu} = g^+_{\mu\nu}$, or what is the same thing, $ds^2_- = ds^2_+$. It is also necessary that the extrinsic curvature, or second fundamental form, $K_{\mu\nu}$ is continuous, $K^-_{\mu\nu} = K^+_{\mu\nu}$.

For the surface Σ, defined by $r = r_0$, we adopt the metric

$$ds^2_\Sigma = -d\tau^2 + r^2_0(d\theta^2 + \sin^2 d\phi^2), \qquad (A1)$$

with the intrinsic coordinates of Σ being $\xi^a = (\tau, \theta, \phi)$, while the inner and the outer metrics are given receptively by [see Eqs. (34) and (35)],

$$ds^2_- = -\left(1 - \frac{r^2}{R^2}\right)dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1}dr^2$$
$$+ r^2(d\theta^2 + \sin^2 d\phi^2), \qquad (A2)$$

and

$$ds^2_+ = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r}\right)^{-1}dr^2$$
$$+ r^2(d\theta^2 + \sin^2 d\phi^2), \qquad (A3)$$

where we have identified the coordinates (t, r, θ, φ) in both regions of the spacetime.

Let us assume that the boundary surface Σ is timelike, which means $1 - r^2_0/R^2 > 0$ and also $1 - 2m/r_0 + q^2/r^2_0 > 0$. Hence, we note first that, since the surface Σ is spherical and nonlightlike, the radial coordinate r can be used as the matching parameter along the generators on Σ, and so the normal n_μ to the surface has only the radial component $n_r = \sqrt{g_{rr}}$. Therefore, in the present case, the extrinsic curvature has the form

$$K^\pm_{ab} = -n^\pm_r \Gamma^{r(\pm)}{}_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b}, \qquad (A4)$$

where ξ^a are the intrinsic coordinates of the surface. Now, we analyze the junction at the outer and inner surfaces. The continuity of the first fundamental form at the boundary implies that $g^+_{tt} = g^-_{tt}$ and $g^+_{rr} = g^-_{rr}$. These two conditions are satisfied by the continuity of the line elements given in Eqs. (A2) and (A3) at $r = r_0$, namely,

$$1 - r^2_0/R^2 = 1 - 2m/r_0 + q^2/r^2_0, \qquad (A5)$$

which is Eq. (20) and defines R^{-2} in terms of the parameters r_0, m, and q. One then sees that the g^+_{tt} and g^-_{tt} match at $r = r_0$, and g^+_{rr} and g^-_{rr} also match, as well as the terms for the angular part of the metric. Now, at r_0 coming from the interior one finds $K^-_{\tau\tau} = -n^-_r \Gamma^{(-)r}_{tt} \frac{dt}{d\tau} \frac{dt}{d\tau}$. By construction, one has $g^\pm_{tt} \frac{dt^2}{d\tau^2} = -1$. One also has $n^-_r = 1/\sqrt{1 - \frac{r^2_0}{R^2}}$. Noting that $\frac{dg^-_{tt}}{dr} = -\frac{2r_0}{R^2}$, one finds $\Gamma^{r(-)}_{tt} = (1 - \frac{r^2_0}{R^2})\frac{r_0}{R^2}$. Thus, $K^-_{\tau\tau} = -\frac{r_0}{R^2}\sqrt{1 - \frac{r^2_0}{R^2}}$. Similarly, the extrinsic curvature coming from the exterior Reissner-Nordström

region can also be calculated. One finds $K^+_{\tau\tau} = (\frac{m}{r^2_0} - \frac{q^2}{r^3_0}) \times$

$$\sqrt{1 - \frac{2m}{r_0} + \frac{q^2}{r^2_0}}.$$ So, in order to match $K^+_{\tau\tau}$ and $K^-_{\tau\tau}$, the parameters R, r_0, m, and q must satisfy the relation

$$\frac{r_0}{R^2} = -\frac{m}{r^2_0} + \frac{q^2}{r^3_0}. \qquad (A6)$$

This result, together with Eq. (A5), gives

$$\frac{1}{R^2} = \frac{1}{3}\frac{q^2}{r^4_0}, \qquad (A7)$$

exactly the result found above [cf. Eq. (28)]. This also means that the surface pressure vanishes at r_0, $p(r_0) = 0$. One can easily check that $K_{\theta\theta}$ and $K_{\phi\phi}$ also match. So the metric and the extrinsic curvature are continuous as is required. The electric potential ϕ is also continuous at the boundary as it is also required.

Let us assume that the boundary surface Σ is lightlike now, which means $1 - r^2_0/R^2 = 0$ and also $1 - 2m/r_0 + q^2/r^2_0 = 0$. Then, the normal vector n^μ is lightlike. The radial coordinate r cannot be used for the matching in this case. As is proposed in [3] (see, also, [61] for the original work), an alternative is using the advanced (retarded) time u as the soldering parameter, so that we can perform what Barrabès and Israel [61] call a static soldering. The new coordinate u is defined by

$$du = dt + \epsilon \frac{dr}{\left(1 - \frac{r^2}{R^2}\right)}, \quad \text{for } r < r_0, \qquad (A8)$$

$$du = dt + \epsilon \frac{dr}{\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)}, \quad \text{for } r > r_0, \qquad (A9)$$

where $\epsilon = \pm 1$, and the plus (minus) sign is associated to the outgoing (ingoing) radial light rays on the cone $u = $ constant. The parameters on Σ may be chosen as $\xi^a = (\tau = u, \theta, \varphi)$ [3,61]. The normal to Σ is therefore $n^\mu = \frac{\partial x^\mu}{\partial u}$, and, moreover, since n^μ is lightlike, an additional transverse vector N^μ is needed. This can be chosen as $N^\mu = \epsilon\frac{\partial x^\mu}{\partial r}$. Instead of the extrinsic curvature K_{ab}, the relevant quantity for a lightlike soldering is the transverse curvature $\mathcal{K}_{ab}$ defined by

$$\mathcal{K}_{ab} = -N_\mu \Gamma^\mu_{\nu\rho} \frac{\partial x^\nu}{\partial \xi^a} \frac{\partial x^\rho}{\partial \xi^b}. \qquad (A10)$$

Then, the matching of the first fundamental form at $r = r_0$ gives $g^+_{uu} = g^-_{uu}$, which results in Eq. (A5), and is accomplished as far as that equality holds even in the limit of r_0 being a solution of the equations $1 - r^2_0/R^2 = 0$ and

$1 - 2m/r_0 + q^2/r_0^2 = 0$, which is true in the particular case we are considering here. Moreover, calculating the nonzero components of $\mathcal{K}_{ab}^+$, we get

$$\mathcal{K}_{uu}^+ = \epsilon\left(\frac{m}{r_0^2} - \frac{q^2}{r_0^3}\right), \tag{A11}$$

$$\mathcal{K}_{\theta\theta}^+ = \mathcal{K}_{\varphi\varphi}^+ \sin^{-2}\theta = \epsilon r_0, \tag{A12}$$

and

$$\mathcal{K}_{uu}^- = -\epsilon\frac{r_0}{R^2}, \tag{A13}$$

$$\mathcal{K}_{\theta\theta}^- = \mathcal{K}_{\varphi\varphi}^- \sin^{-2}\theta = \epsilon r_0, \tag{A14}$$

where r_0 coincides with one of the horizons of the Reissner-Nordström spacetime. The matching between the metric coefficients on the horizon r_0 is clear. Again, imposing the continuity of the uu component of transverse curvature, i.e., equating the relations (A11) and (A13) we find that they match only in the case $R = r_0 = r_-$, r_- being the inner Reissner-Nordström horizon. As a consequence of this matching, the pressure vanishes at the

boundary surface, $p(r_0) = 0$. Note that here the lightlike case can also be analyzed as a limit of the configurations with a timelike boundary.

APPENDIX B: HOW TO OBTAIN p'

Note that the equation $\theta^2(r - r_0) = \theta(r - r_0)$ can only be used after taking derivatives. Also, the Dirac delta function is given by $\delta(r - r_0) = [\theta(r - r_0)]'$. Thus, in Eq. (41), the term in front of $\frac{q^2}{r_0^4}$ differentiates to $[\theta^2(r - r_0)\theta(r_0 - r)]' = 2\theta(r - r_0)\delta(r - r_0)\theta(r_0 - r) - \theta^2(r - r_0)\delta(r - r_0) = 2\delta(r - r_0) - \theta(r - r_0)\delta(r - r_0) = \delta(r - r_0)$, where we have used $\theta(r - r_0)\theta(r_0 - r) = 1$ in $r = r_0$ and zero otherwise, and $\theta(r - r_0)\delta(r - r_0) = \delta(r - r_0)$. Now, the term in front of $-\frac{3}{R^2}$ has derivative $[\theta(r_0 - r)]' = -\delta(r - r_0)$. Thus, the whole term yields $+\frac{3}{R^2}\delta(r - r_0)$. Since $\frac{3}{R^2} = \frac{q^2}{r_0^4}$, we have to sum both terms to give Eq. (42), i.e., $p'(r) = \frac{q^2}{4\pi r_0^4}\delta(r - r_0)$.

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