

# Analytical study of a gas of gluonic quasiparticles at high temperature: Effective mass, pressure, and trace anomaly

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The thermodynamical properties of a pure Yang-Mills theory  $SU(N)$  is described by a gas of gluonic quasiparticles with temperature-dependent mass  $m(T)$  and a bag function  $B(T)$ . The analytic behavior of  $m(T)$  and the pressure  $p$  in the temperature range  $2.5-5T_c$  are derived and constraints on the parameters defining  $B(T)$  are discussed. The trace anomaly  $\theta = \rho - 3p$  is evaluated in the high  $T$  domain: it is dominated by a quadratic behavior  $\theta = nKT^2$ , where  $n = 2(N^2 - 1)$  is the number of degrees of freedom and  $K$  is an integration constant which does not depend on the bag function  $B(T)$ . The quadratic rise of  $\theta$  is in good agreement with recent lattice simulations.

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## I. INTRODUCTION AND SUMMARY

The concept of quasiparticle is a valuable tool toward an effective description of complicated interactions. An important example is that of Yang-Mills (YM)  $SU(N)$  theories at nonzero temperature  $T$ , where their intrinsic nonperturbative nature renders the applicability of a perturbative treatment limited [1]. Descriptions of the YM system at  $T > 0$  via gluonic quasiparticles have been performed in a variety of approaches, e.g. Refs. [2–8] and refs. therein. Different *Ansätze* have been tested and the outgoing thermodynamical quantities, such as energy density and pressure, have been compared to lattice results [9–11]. An appropriate, but at the same time analytically tractable, description of a gas of gluons is also a necessary step toward the understanding of the quark gluon plasma, see for instance Ref. [12] for a review.

Besides the temperature-dependent gluonic mass  $m = m(T)$ , also a temperature-dependent bag energy  $B = B(T)$  has been widely used to describe nonperturbative properties of YM theories, such as the trace anomaly and the gluon condensate. With these two basic ingredients the energy density and the pressure read (see Refs. [3,4] and references therein)

$$\rho = \rho_p + B(T), \quad p = p_p - B(T), \quad (1)$$

where the suffix ‘ $p$ ’ denotes the quasiparticle part

$$\rho_p = n \int_k \frac{\sqrt{k^2 + m^2(T)}}{\exp\left[\frac{\sqrt{k^2 + m^2(T)}}{T}\right] - 1},$$

$$p_p = -Tn \int_k \log\left[1 - \exp\left[-\frac{\sqrt{k^2 + m^2(T)}}{T}\right]\right], \quad (2)$$

where  $\int_k = \int \frac{d^3k}{(2\pi)^3} = \int_0^\infty \frac{k^2 dk}{2\pi^2}$  and  $n$  represents the degeneracy of the particle spectrum. Restricting to perturbative degrees of freedom only, one has  $n = 2(N^2 - 1)$  in the case of a  $SU(N)$  YM theory.

In this article, we study the high  $T$  properties of the gluonic gas expressed in Eqs. (1) and (2). On the practical side, we are interested in the behavior in the range between, say,  $2.5T_c$  and  $5T_c$ , where  $T_c$  is the critical temperature for deconfinement, above which gluonic quasiparticles are the relevant degrees of freedom. In this energy range, lattice data for the energy density and pressure show a plateau [9–11], which is compatible with a linear increase of the quasiparticle mass with the temperature  $T$ ,  $m \propto T$ . Because of this linear increase, the Stefan-Boltzmann limit is not reached, but a saturation at lower values is obtained. This situation persists as long as logarithmic corrections due to the running coupling of QCD are neglected. In fact, the correct perturbative behavior is  $m \propto T/\sqrt{\log(T/\Lambda_{\text{QCD}})}$ , which implies a slow approach of the pressure and energy density to the expected Stefan-Boltzmann values, see the lattice simulation in Ref. [13]. In the present work, we do not include these logarithmic corrections. This simplification is applicable in the temperature range  $2.5-5T_c$ , which is high when compared to the critical temperature, but not high enough for the logarithmic terms to become relevant.

Strong constraints on the gas of quasiparticles can be derived by imposing that the system fulfills thermodynamical self-consistency [2–6], which is a consequence of the first principle of thermodynamics:

$$\rho = T \frac{dp}{dT} - p. \quad (3)$$

The bag constant  $B(T)$  is assumed to have the following behavior:

$$B(T) = B_{\text{NP}}(T) + B_P(T) \quad (4)$$

with  $B_{\text{NP}}(T) = ncT^\alpha$  and  $B_P(T) = n\delta T^4$

for  $2.5T_c \lesssim T \lesssim 5T_c$ . (5)

The term  $B_P(T) = n\delta T^4$  is included in order to recover the expected results of perturbation theory in the high

temperature regime, where the effective gluon mass grows linearly with  $T$  (up to the previously mentioned logarithmic corrections, which are not considered here.)

The term  $B_{\text{NP}}(T) = ncT^\alpha$ , where  $\alpha$  is a real number smaller than 4 and  $c$  is a constant with the dimension of  $[\text{Energy}]^{4-\alpha}$ , describes the ‘‘nonperturbative’’ bag contribution relevant above the phase transition [14]. At the present stage, the only and general requirement about the function  $B_{\text{NP}}(T)$  is that it is dominated by a powerlike term  $T^\alpha$  in the high  $T$  region. It is the aim of this work to constrain the value of  $\alpha$  and  $c$  by using mathematical considerations and lattice results.

As a last remark, we stress that the bag function  $B(T) = B_{\text{NP}}(T) + B_P(T)$  is proportional to the number of degrees of freedom  $n = 2(N_c^2 - 1)$ , in agreement with general large  $N$  scaling arguments [15].

The temperature-dependent mass  $m(T)$  can be analytically evaluated at high  $T$  (details are in Sec. II) and takes the form ( $\alpha \neq 2$ )

$$m(T) = T \sqrt{\frac{4\pi^2}{D(a_0)} \frac{\alpha c}{2 - \alpha} T^{\alpha-4} + k \frac{\Lambda^2}{T^2} - \frac{8\pi^2 \delta}{D(a_0)}}, \quad (6)$$

where  $D(a_0)$  is a positive real number which shall be specified later on,  $\alpha$ ,  $c$ ,  $\delta$  are the already-introduced parameters defining  $B(T)$ ,  $n$  is the number of degrees of freedom,  $\Lambda$  is the Yang-Mills scale and finally  $k$  is an integration constant related to the differential Eq. (3);  $k$  is not determined by the choice of  $B(T)$  but is a further parameter entering in the model.

Mathematically- and physically-based considerations about Eq. (6) will lead us to establish nontrivial relations between the parameters:

- The parameter  $\delta$  parametrizes the linear increase of  $m$  with  $T$ . As evident from Eq. (6),  $\delta \leq 0$  in order to avoid an imaginary mass at high  $T$ . Moreover, a careful study of the equations will lead us to establish also an upper limit on its absolute value:  $|\delta| \leq 0.0151$ .
- Many approaches show that the nonperturbative bag function  $B_{\text{NP}}$  is a positive number [15,16]. In the present framework, it means that  $c > 0$ . Moreover, the contribution of nonperturbative physics to the effective gluon mass should be positive, a property which also assures that no instability emerges at low  $T$ . Thus, it is a general physical requirement that the nonperturbative contribution to  $m^2(T)$  given by  $\frac{4\pi^2}{D(a_0)} \frac{\alpha c}{2 - \alpha} T^{\alpha-2}$  in Eq. (6) is also positive. This, in turn, implies a consistent limitation on the choice of the parameter  $\alpha$ :  $0 \leq \alpha < 2$ . Note, for the very same requirement we are led to conclude that the integration constant  $k$  is positive.
- The case  $\alpha = 2$  is somewhat peculiar because of the emergence of logarithms in the solution, see details in Sec. II D. However, it is also unfavored because of similar arguments.

As a next step of this work, we turn to the explicit expression for the pressure  $p$  in the high  $T$  limit, see Sec. III for details. We shall find that  $p$  is expressed by the sum of three terms ( $\alpha \neq 2$ ):

$$p = -nc \frac{2}{2 - \alpha} T^\alpha - \frac{n}{4\pi^2} D(a_0) k \Lambda^2 T^2 + (\bar{p}_p(a_0) - n\delta) T^4. \quad (7)$$

The first, negative term scales as  $T^\alpha$  (just as  $B_{\text{NP}}$ ); the second, also negative, term scales as  $T^2$  (in agreement with the phenomenological argument of Ref. [17]) and is proportional to the constant  $k$ ; the third, positive term goes as  $T^4$ , but the coefficient is slightly smaller than the Stefan-Boltzmann limit, see later on for details. The overall pressure scale as  $n \propto N^2$  in agreement with large  $N$  scaling arguments.

The final and main subject of the present work is the study of the trace anomaly at high  $T$  (Sec. IV). The trace anomaly  $\theta$ , defined as

$$\theta = \rho - 3p = 4B + \rho_p - 3p_p, \quad (8)$$

has been in the center of a vivid debate in the last years (see Refs. [17–22] and references therein). Strict dilatation invariance would imply that  $\theta$  vanishes in a dilatation invariant theory, such as a gas of photons. In a YM theory, this symmetry is broken by quantum effects and  $\theta$  does not vanish: this is the so-called trace anomaly. We aim to show that, in the context of a gas of quasiparticle with the general form of the bag constant  $B(T)$  given in Eq. (4), the following high  $T$  behavior holds:

$$\theta = \rho - 3p = nCT^\alpha + nKT^2 \quad \text{for } T \geq 2T_c, \quad (9)$$

where  $C = 2c \frac{4-\alpha}{2-\alpha}$  is a constant determined by the nonperturbative parameters of the model  $\alpha$ ,  $c$  (i.e., those parameters which define the nonperturbative bag function  $B_{\text{NP}}(T) = ncT^\alpha$ ), and  $K = \frac{k\Lambda^2 D(a_0)}{2\pi^2}$  is proportional to the previously introduced integration constant  $k$ . The main result is that the trace anomaly  $\theta$  can be decomposed in a term which behaves as the nonperturbative contribution to the bag constant  $B_{\text{NP}}(T) = ncT^\alpha$ , and a term which goes as  $T^2$ . Restricting to the favored interval  $0 \leq \alpha < 2$ , one is led to conclude that the quadratic rise dominates at  $T$  large enough. Remarkably, the  $T^2$  rise of  $\theta$  is a general property, which is independent on the nonperturbative bag constant  $B_{\text{NP}}$ .

It is indeed remarkable that a quadratic rise of the trace anomaly,  $\theta \simeq aT^2$ , is found in Ref. [17], where an analysis of the lattice data of Ref. [10] has been performed. Later on, this quadratic rise has been confirmed in recent lattice works [11,23]. In particular, in the lattice study of Ref. [23] the trace anomaly has been investigated for various pure Yang-Mills theories  $SU(N)$ ,  $N = 3, \dots, 8$ . The behavior  $\theta \propto nT^2$  (with a direct proportionality to the degeneracy number  $n = 2(N^2 - 1)$ ) is indeed found for  $N = 3, \dots, 8$

in a range between 2 and  $5T_c$ . Thus, the result of Eq. (9) may explain in a natural way the emergence of such quadratic behavior of  $\theta$  at high  $T$ .

In the following sections, we present the detailed derivations of the outlined results: In Secs. II and III, we derive the expressions for  $m(T)$  in Eq. (6) and for the pressure  $p$  in Eq. (7). In Sec. IV, we present the calculation leading to the expression of the trace anomaly  $\theta(T)$  in Eq. (9). Finally, in Sec. V we briefly outline our conclusions.

## II. TEMPERATURE-DEPENDENT QUASIPARTICLE MASS $m(T)$

### A. Differential equation for $m(T)$

In order to obtain the differential equation for the quasiparticle mass  $m = m(T)$ , we plug the expressions of Eqs. (1) and (2) into the thermodynamical self-consistency relation  $\rho = T \frac{d\rho}{dT} - p$ :

$$\frac{dB}{dT} = -nI(m) \frac{dm^2}{dT},$$

$$I(m) = \int_k \frac{1}{2\sqrt{k^2 + m^2}} \frac{1}{\exp[\frac{\sqrt{k^2 + m^2(T)}}{T}] - 1}, \quad (10)$$

where the bag function  $B(T)$  is given in Eq. (4).

### B. Use of dimensionless functions

It is convenient to rewrite the equations by using dimensionless quantities. To this end, we introduce the dimensionless temperature

$$\lambda = \frac{T}{\Lambda}, \quad (11)$$

where  $\Lambda$  is the Yang-Mills scale, which is of the same order of the critical temperature  $T_c$ ,  $\Lambda \sim T_c$ .

The dimensionless particle contribution to the energy density and pressure

$$\bar{\rho}_p = \frac{\rho_p}{T^4}, \quad \bar{p}_p = \frac{p_p}{T^4} \quad (12)$$

read

$$\bar{\rho}_p = \frac{n}{2\pi^2} \int_0^\infty dx \frac{x^2 \sqrt{x^2 + a^2}}{e^{\sqrt{x^2 + a^2}} - 1}, \quad (13)$$

$$\bar{p}_p = -\frac{n}{2\pi^2} \int_0^\infty dx (x^2 \ln(1 - e^{-\sqrt{x^2 + a^2}})). \quad (14)$$

The function  $a = a(\lambda)$  is the ‘‘dimensionless mass’’:

$$a = a(\lambda) = \frac{m(T)}{T} = \frac{m(\lambda\Lambda)}{\lambda\Lambda}. \quad (15)$$

We also define the dimensionless constant  $\gamma$  as

$$\gamma = c\Lambda^{\alpha-4}. \quad (16)$$

In this way, the ground state dimensionless energy density  $\bar{\rho}_{\text{gs}}$  and pressure  $\bar{p}_{\text{gs}}$

$$\bar{\rho}_{\text{gs}} = -\bar{p}_{\text{gs}} = \frac{\rho_{\text{gs}}}{T^4} = \frac{B(T)}{T^4} \quad (17)$$

read explicitly (see Eq. (4))

$$\bar{\rho}_{\text{gs}} = -\bar{p}_{\text{gs}} = n\gamma\lambda^{\alpha-4} + n\delta. \quad (18)$$

The full dimensionless energy density and pressure read

$$\bar{\rho} = \bar{\rho}_p + \bar{\rho}_{\text{gs}}, \quad \bar{p} = \bar{p}_p + \bar{p}_{\text{gs}}. \quad (19)$$

The thermodynamical self-consistency of Eq. (3) can be rewritten in terms of the reduced energy density and pressure as

$$\bar{\rho} - 3\bar{p} = \lambda \frac{d\bar{p}}{d\lambda}. \quad (20)$$

In terms of the dimensionless quantities, the differential Eq. (10) takes the form

$$n\alpha\gamma\lambda^{\alpha-4} + 4n\delta = -\frac{n}{2\pi^2} \frac{d(a^2\lambda^2)}{d\lambda} \frac{D(a)}{2\lambda}, \quad (21)$$

where the integral  $D(a)$  is given by

$$D(a) = \int_0^\infty dx \frac{x^2}{\sqrt{x^2 + a^2}} \frac{1}{e^{\sqrt{x^2 + a^2}} - 1}. \quad (22)$$

Note, the dependence on the degeneracy number  $n = 2(N^2 - 1)$  factorizes, so that the equation for  $a(\lambda)$  is independent on the number of colors  $N$ . This is in agreement with the general expectation of large  $N$  scaling, according to which the effective gluon mass  $m(T)$  scales as  $N^0$ .

### C. Constraints on the parameter $\delta$

In the limit of large  $\lambda$  (i.e., large  $T$ ), one has  $a(\lambda \gg \lambda_c) \rightarrow a_0$ . In this way, besides the logarithmic corrections, the effective mass exhibits a linear growth  $m = a_0T$ , in agreement with the expectation of perturbation theory [24] and with high  $T$  effective approaches, e.g. Ref. [25]. In the present phenomenological approach, the numerical value of  $a_0$  is related to the parameter  $\delta$  by studying the asymptotic behavior of Eq. (21):

$$\delta = -\frac{1}{8\pi^2} a_0^2 D(a_0). \quad (23)$$

In Fig. 1, left panel, the quantity  $\delta$  is plotted as function of  $a_0$ . The two properties mentioned in the Introduction can be easily proven:  $\delta \leq 0$  and  $|\delta| \leq \max(\frac{1}{8\pi^2} a_0^2 D(a_0)) = 0.0151$ .

Various lattice simulations of Yang-Mills system [10,11] show that the Stefan-Boltzmann limit of the energy density and the pressure

$$\bar{\rho}_{\text{SB}} = n \frac{\pi^2}{30}, \quad \bar{p}_{\text{SB}} = n \frac{\pi^2}{90} \quad (24)$$

is not reached at  $5T_c$ . On the contrary, a saturation at a lower value of about 90% of the Stefan-Boltzmann limit is observed. Such a saturation is obtained in the present quasiparticle approach by a nonzero value of  $a_0$  (i.e., a

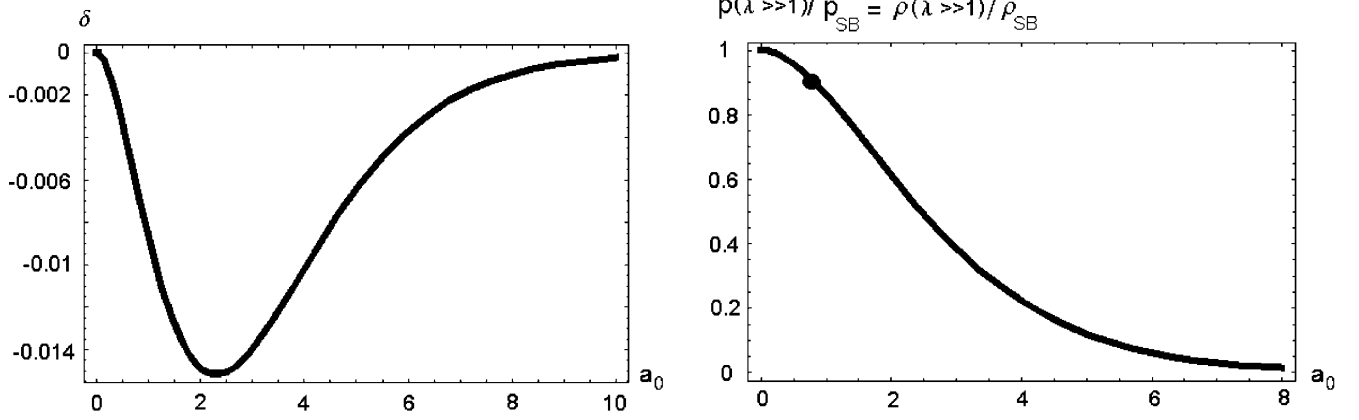


FIG. 1. Left panel: the parameter  $\delta$  is plotted as function of the asymptotic value  $a(\lambda \rightarrow \infty) = a_0$ . Right panel: the ratio  $(\bar{p}/\bar{p}_{\text{SB}})_{\lambda \gg 1} = (\bar{\rho}/\bar{\rho}_{\text{SB}})_{\lambda \gg 1}$  is plotted as function of  $a_0$ . The dot corresponds to  $\bar{\rho}/\bar{\rho}_{\text{SB}} = 0.9$  and  $a_0 = 0.83$ ; this is the saturation value obtained in lattice simulations [23].

nonzero value of  $\delta$ ). For high  $\lambda$ , the function  $\bar{p}(\lambda)/\bar{p}_{\text{SB}}$  approaches the asymptotic value  $(\bar{p}_p(a_0) - n\delta)/\bar{p}_{\text{SB}}$  (see Sec. III). In Fig. 1, right panel, the quantity  $(\bar{p}(\lambda)/\bar{p}_{\text{SB}})_{\lambda \gg 1} = (\bar{\rho}(\lambda)/\bar{\rho}_{\text{SB}})_{\lambda \gg 1}$  is plotted as a function of  $a_0$ . In order that at high  $\lambda$  the ratio  $\bar{\rho}(\lambda)/\bar{\rho}_{\text{SB}} \simeq 0.9$  holds, the value  $a_0 \simeq 0.83$  is required. This, in turn, implies that  $\delta \simeq -0.0070$ . Note, similar values for  $a_0$  have been obtained in Refs. [4,5].

As explained in the Introduction, the Stefan-Boltzmann limit for the energy density and the pressure is reached at much higher temperatures [13], at which the logarithmic decrease of  $a(\lambda)$  becomes relevant.

#### D. Analytical solution $a(\lambda)$ in the large $\lambda$ domain (i.e., $m(T)$ in the large $T$ domain)

An analytical solution of Eq. (21) can be obtained in the limit of large  $\lambda$  by approximating the function  $D(a)$  by its asymptotic values  $D(a_0)$ . In this limit, Eq. (21) can be easily solved and one obtains for  $a^2(\lambda)$  ( $\alpha \neq 2$ ):

$$a^2(\lambda) = \frac{4\pi^2}{D(a_0)} \frac{\alpha\gamma}{2-\alpha} \lambda^{\alpha-4} + \frac{k}{\lambda^2} + a_0^2, \quad (25)$$

In Fig. 2, the analytical expression (25) is compared to the numerical result of the differential Eq. (21) for a particular numerical choice; the numerical solution takes into account the full dependence on the function  $D(a)$ , which is ignored in the analytical one. However, the qualitative agreement, which becomes better and better by increasing  $\lambda$ , is visible.

The analytical expression of the function  $m^2(T)$  is easily obtained by making use of Eqs. (11), (15), and (25):

$$m^2(T) = \frac{4\pi^2}{D(a_0)} \frac{\alpha c}{2-\alpha} T^{\alpha-2} + k\Lambda^2 + a_0^2 T^2. \quad (26)$$

As a result of the obtained expression for  $m(T)$ , we can discuss the constraints on the parameters  $c$  and  $\alpha$ :

- (i) The squared mass is constituted of three terms. The first term in Eq. (26) depends on the parameters  $c$  and  $\alpha$ , which define the nonperturbative bag pressure  $B_{\text{NP}} = cT^\alpha$ . The second contribution in Eq. (26),  $k\Lambda^2$ , is constant and is directly proportional to the integration constant  $k$ . The last term in Eq. (26) dominates at high  $T$ , implying the linear behavior  $m(T) \simeq a_0 T$ , as predicted by perturbative calculations and effective theories of QCD. As already anticipated in Sec. I, the natural requirements that  $B_{\text{NP}} = cT^\alpha$  and the corresponding contribution to  $m^2(T)$ , given by  $\frac{4\pi^2}{D(a_0)} \frac{\alpha c}{2-\alpha} T^{\alpha-2}$ , are positive numbers implies that  $\gamma > 0$ ,  $0 \leq \alpha < 2$ . Similarly,  $k > 0$ .
- (ii) In the limit  $\alpha = 0$ , the first term in Eq. (26) vanishes and the second, constant contribution dominates: this situation corresponds to the simple approximation with a constant gluon mass. This possibility has

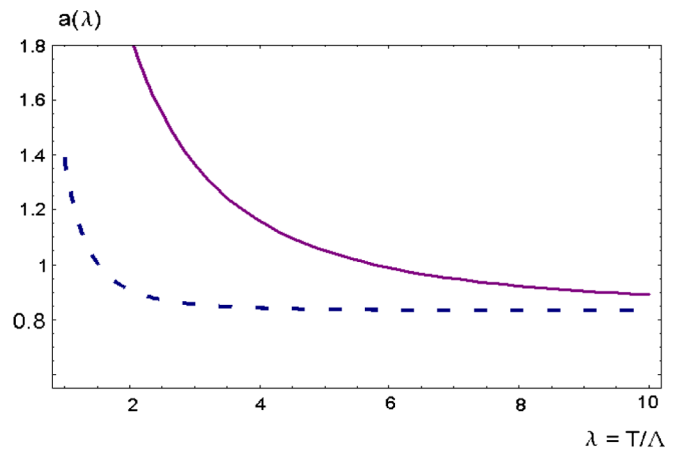


FIG. 2 (color online). Comparison of the analytic, approximate expression of Eq. (25) (solid line) with the full solution of the differential Eq. (21) (dashed line) for a particular numerical case:  $\alpha = 1$ ,  $\gamma = 0.034$ ,  $k = 10$ .

however been ruled out by a precise comparison with lattice data [4]. More generally, when  $\alpha$  is small, the first term in Eq. (26) is also negligible (unless the parameter  $c$  is anomalously large); at low  $T$ , only the second constant term survives. On the contrary, when  $\alpha \approx 2$  the first term is very large, unless the parameter  $c$  is very small. We thus conclude that  $\alpha$  should be not too close to the boundaries 0 and 2, but somewhere in between.

- (iii) In the case  $\alpha = 2$ , a slightly different solution is obtained:

$$a^2(\lambda) = -\frac{8\pi^2\gamma}{D(a_0)} \frac{\log\lambda}{\lambda^2} + \frac{k}{\lambda^2} + a_0^2, \quad (27)$$

where an additional logarithm arises. The mass contribution of the nonperturbative first term is negative for  $\gamma > 0$  (that is for the here considered choice  $c > 0$ ). Thus, also the case  $\alpha = 2$  is regarded as phenomenologically unfavored.

- (iv) In order to include the logarithmic corrections in the very high  $T$  domain, one should modify the perturbative bag pressure  $B_p(T)$  in such a way that the perturbative mass behavior  $m \propto T/\sqrt{\log T/\Lambda}$  holds. One obtains the constraint  $T^{-3}dB_p/dT \propto (2\ln^{-1}\lambda - 2\ln^{-2}\lambda)$ , thus leading to more complicated expressions involving logarithms. A detailed study of this subject represents an interesting outlook. Although the formulas will be more involved, a link with studies of Ref. [7], in which the starting point is the perturbative behavior at very large temperature, can be driven.

### III. PRESSURE IN THE HIGH $T$ DOMAIN

We turn to the explicit expression of the pressure for large  $T$ . To this end, we expand Eq. (14) around the asymptotic value  $a_0^2$

$$\bar{p}_p(a) = \bar{p}_p(a_0) + \left(\frac{d\bar{p}_p(a)}{da}\right)_{a_0} (a - a_0) + \dots \quad (28)$$

Using the equality  $\left(\frac{d\bar{p}_p(a)}{da}\right)_{a_0} = -\frac{n}{2\pi^2}a_0D(a_0)$  and approximating  $a_0(a - a_0) = a^2 - a_0^2$  (valid at the considered order), one gets

$$\bar{p}_p = \bar{p}_p(a_0) - n\frac{\alpha\gamma}{2-\alpha}\lambda^{\alpha-4} - \frac{n}{4\pi^2}D(a_0)\frac{k}{\lambda^2}. \quad (29)$$

The full dimensionless pressure  $\bar{p} = \bar{p}_p + \bar{p}_{gs}$  reads at high  $\lambda$

$$\bar{p} = -n\gamma\frac{2}{2-\alpha}\lambda^{\alpha-4} - \frac{n}{4\pi^2}D(a_0)\frac{k}{\lambda^2} + (\bar{p}_p(a_0) - n\delta). \quad (30)$$

By multiplying by  $T^4$ , we find the pressure  $p$  for large  $T$ :

$$p = -nc\frac{2}{2-\alpha}T^\alpha - \frac{n}{4\pi^2}D(a_0)k\Lambda^2T^2 + (\bar{p}_p(a_0) - n\delta)T^4. \quad (31)$$

We thus have also decomposed the pressure into three contributions: the first term in Eq. (31) scales exactly as the bag function  $B_{NP}$ . The second, negative term in Eq. (31) scales as  $T^2$  and is proportional to the integration constant  $k$ . Note, a similar quadratic contribution to the pressure has been postulated in Ref. [17]. There is, however, an important point to stress: here we have shown that there is no need to introduce at hand a quadratic contribution to the

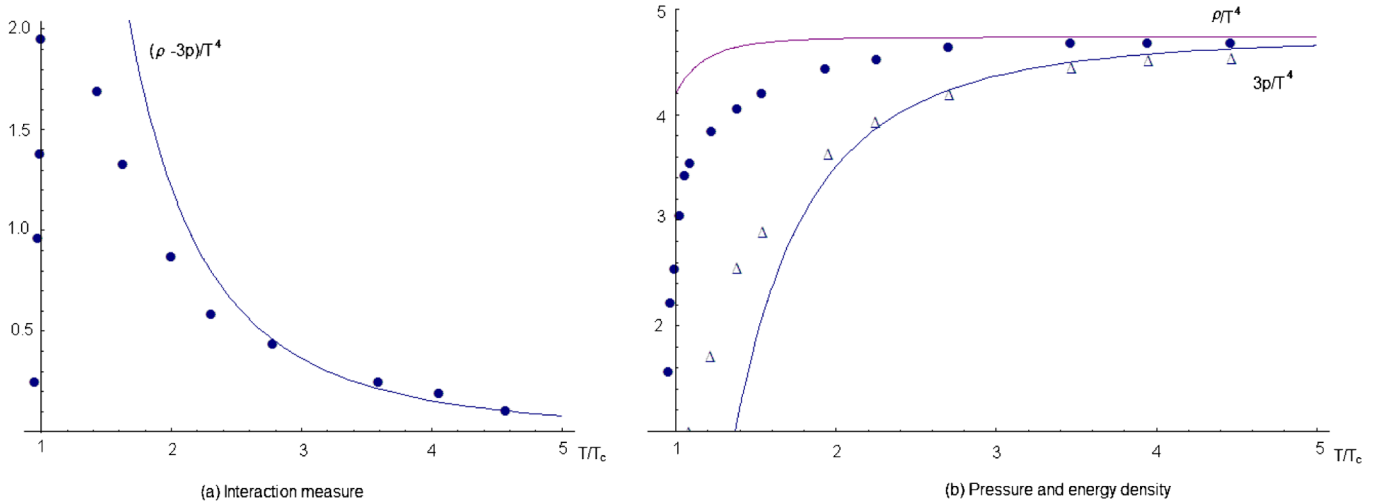


FIG. 3 (color online). Comparison with the lattice results of Ref. [10]. The parameters  $\alpha = 1$ ,  $\gamma = 0.034$ ,  $k = 10$  are used. Left panel: the interaction measure is plotted. The lattice point for  $T/T_c = 4.58$  has been used to determine  $\Lambda = 1.55T_c$ . Right panel: the quantities  $\rho$  (the upper curve is the theoretical result, the dots the lattice points) and  $3p$  (the lower curve is the theoretical result, the triangles are the lattice points) are shown.

pressure from the very beginning. The quadratic contribution naturally emerges as the result of the equation, independently on the choice of the bag function  $B_{\text{NP}}(T)$ . Finally, the last term in Eq. (31) describes the high  $T$  asymptotic limit, which differs from the Stefan-Boltzmann value as depicted in the right panel of Fig. 1. For other works on the pressure in the high  $T$  domain, see Refs. [26,27] and references therein. For a direct comparison with lattice data, see Fig. 3.

#### IV. TRACE ANOMALY

The interaction measure

$$\Delta = \frac{\theta}{T^4} = \bar{\rho} - 3\bar{p} \quad (32)$$

is evaluated by making use of the thermodynamical self-consistency of Eq. (20):

$$\Delta = \bar{\rho} - 3\bar{p} = \lambda \frac{d\bar{p}}{d\lambda} = \lambda \frac{d\bar{p}_p}{d\lambda} + \lambda \frac{d\bar{p}_{\text{gs}}}{d\lambda}, \quad (33)$$

where in the last step the dimensionless pressure has been decomposed into its particle and ground state contributions.

The ground state contribution is easily evaluated:

$$\lambda \frac{d\bar{p}_{\text{gs}}}{d\lambda} = n\gamma(4 - \alpha)\lambda^{\alpha-4}. \quad (34)$$

The calculation of the particle contribution to the interaction measure goes via two steps. First, we rewrite it by making use of Eqs. (14):

$$\lambda \frac{d\bar{p}_p}{d\lambda} = -\frac{n}{2\pi^2} \lambda D(a) a \frac{da}{d\lambda} = -\frac{n}{4\pi^2} \lambda D(a) \frac{da^2}{d\lambda}. \quad (35)$$

As a second step, in the large  $\lambda$  domain one can replace  $D(a)$  with the asymptotic value  $D(a_0)$  and then evaluate the derivative  $\frac{da^2}{d\lambda}$  by using Eq. (25):

$$\lambda \frac{d\bar{p}_p}{d\lambda} = n\alpha\gamma \frac{4 - \alpha}{2 - \alpha} \lambda^{\alpha-4} + n \frac{D(a_0)}{2\pi^2} \frac{k}{\lambda^2}. \quad (36)$$

By putting the results together, one finds for  $\lambda \gtrsim 2\lambda_c$  ( $\alpha \neq 2$ )

$$\Delta = \bar{\rho} - 3\bar{p} = 2n\gamma \frac{4 - \alpha}{2 - \alpha} \lambda^{\alpha-4} + n \frac{D(a_0)}{2\pi^2} \frac{k}{\lambda^2}, \quad (37)$$

which consists of two terms: a term which scales as the ground state contribution  $\bar{\rho}_{\text{gs}}$ , and a term which scales as  $\lambda^{-2}$ .

By multiplying  $\Delta$  by  $T^4$ , one obtains the trace anomaly  $\theta$  as a function of  $T$  ( $\alpha \neq 2$ ):

$$\theta = 2nc \frac{4 - \alpha}{2 - \alpha} T^\alpha + n \frac{D(a_0)}{2\pi^2} k \Lambda^2 T^2, \quad (38)$$

which is Eq. (9) discussed in the Introduction. The first term depends on  $c$  and  $\alpha$  (i.e., the parameters which define  $B_{\text{NP}}(T)$ ), while the second term describes a quadratic rise

of  $\theta$ , is proportional to the integration constant  $k$  and is independent on the bag function  $B$ . The very same term proportional to  $k$  was responsible for a constant contribution to the effective gluon mass, see Eq. (26). In the favored range  $0 \leq \alpha < 2$ , the rise  $T^\alpha$  is realized for small  $T$ , while the quadratic rise of  $\theta$  dominates for large enough temperature. The temperature at which this change happens depends on the particular numerical values of the parameters, and cannot be determined by analytical considerations.

It is, however, possible to use some lattice results about the trace anomaly in order to constrain the numerical values of the parameters of the model. The quantity  $\theta$  scales as  $3.3T_c^2/T^2$  for  $T \gtrsim 1.5-2T_c$  [4]. Then, from Eq. (38) and  $\Lambda \sim T_c$ , it follows that  $k \simeq 10$ . It is also possible to obtain a rough estimation of the upper limit of the parameter  $\gamma = c\Lambda^{4-\alpha} \sim cT_c^{4-\alpha}$ . In fact, the quantity  $\theta/T^2$  is, to a good approximation, constant for  $T \gtrsim 2T_c$  [17]. This, in turn, means that the first term in Eq. (38) is smaller than the second term for  $T \gtrsim 2T_c$ . Through simple algebra, one obtains the upper limit  $\gamma \lesssim \frac{2-\alpha}{4-\alpha} \frac{D(a_0)k}{2\pi^2} 2^{1-\alpha}$ .

We now turn to a direct comparison of our theoretical curves with the lattice results of Ref. [10]. Since our theoretical functions depend on the variable  $\lambda = T/\Lambda$ , where  $\Lambda \sim T_c$  but not exactly equal, care is needed: it is first necessary to determine  $\Lambda$ . To this end, we chose  $\Lambda$  in such a way that the theoretical result for the interaction measure  $\Delta$  reproduces the lattice point at the highest simulated value of  $T/T_c = 4.57$  (at which  $\Delta = 0.10$ ). One obtains the relation  $\Lambda = 1.55T_c$ . In Fig. 3, the plot of the interaction measure (left panel) and energy and pressure (right panel) are shown: it is visible that the agreement is acceptable for  $T \gtrsim 2.5T_c$  and increases for increasing  $T$ . On the contrary, the theoretical results for  $T/T_c \lesssim 2.5$  deviate from the lattice simulations. This is expected because the present version of the model cannot describe the physical properties close to the phase transition.

Further comments are in order:

- (i) For the here-considered case  $c > 0$  (i.e. for  $B_{\text{NP}} > 0$ ), a positive contribution of the first term to the trace anomaly is obtained for  $0 \leq \alpha < 2$  (see Eq. (38)), which is the same interval outlined previously. If, on the contrary,  $2 < \alpha < 4$  one would have a negative  $\theta$  for high enough  $T$ , in disagreement with all present simulations. This represents a further confirmation of the outlined range of  $\alpha$ .
- (ii) The case  $\alpha = 2$  leads to a slightly modified form:

$$\theta = 4nT^2 \left( c - c \log\left(\frac{T}{\Lambda}\right) + \frac{D(a_0)}{8\pi^2} k \Lambda^2 \right). \quad (39)$$

A negative  $\theta$  at high  $T$  is realized. This fact is at first sight a further argument against the choice  $\alpha = 2$ . However, the appearance of the logarithms in the solution implies that a full study of the present case

is only possible when the logarithmic corrections are taken into account.

- (iii) The behavior of  $\theta$  in the large  $T$  domain as measured on the lattice is still subject to an ongoing discussion. In the work of Ref. [18], also based on the lattice data of Ref. [10], it is found that  $\theta$  grows linearly (rather than quadratically) with  $T$ :  $\theta = aT$  for  $2.5T_c \leq T \leq 5T_c$ , where  $a \approx 1.5 \text{ GeV}^3$  in the  $SU(2)$  case and  $a \approx 1.7 \text{ GeV}^3$  in the  $SU(3)$  case. We also refer to the similar results obtained in the older works of Refs. [28,29]. In Ref. [30], the linear rise has been confirmed by studying the lattice data of Ref. [11]. Our result (9) can indeed also account for an initial nonquadratic behavior of  $\theta$  (the linear one being realized for  $\alpha = 1$ , see [19]), which persists as long as the quadratic term does not become dominant.
- (iv) The linear rise of  $\theta$  has been derived within the theoretical framework described in Refs. [31], and further investigated in Refs. [19,32]. The nonperturbative sector of  $SU(2)$  or  $SU(3)$  YM theories is described by a composite, (adjoint-) scalar field  $\phi$  in the deconfined phase ( $T > T_c$ ), which emerges as an ‘‘average’’ over calorons and anticalorons (topological objects which correspond to instantons at nonzero  $T$  [33]) with trivial holonomy, see [31] for a microscopic derivation and [32] for a macroscopic one. On a length scale  $l > |\phi|^{-1}$ , it is thermodynamically exhaustive to consider only the average field  $\phi$  and neglect the (unsolvable) microscopic dynamics of all YM-field configurations, such as calorons and monopoles. One can then build up an effective theory for YM-thermodynamics valid for  $T > T_c$ , in which the scalar field  $\phi$  acts as a background field coupled to the residual, perturbative gluons. On a phenomenological level, it contributes to the energy and pressure as a temperature-dependent bag constant  $B_{\text{NP}} = 4\pi\Lambda^3 T$ , i.e. with the parameters  $\alpha = 1$  and  $\gamma = 4\pi$ . Note, in the theoretical framework of Ref. [31] the constant  $k$  was set to be very small, thus the quadratic rise starts to dominate only at very high temperatures and for this reason does not affect the phenomenology between  $2.5T_c$  and  $5T_c$ . The linear growth with  $T$  of the stress-energy tensor in the pure  $SU(2)$  YM theory is obtained as [19]  $\theta = \rho - 3p \sim^{T > 2T_c} 24\pi\Lambda^3 T \approx (1.7 \text{ GeV}^3)T$ . The coefficient  $1.7 \text{ GeV}^3$  is similar to  $1.5 \text{ GeV}^3$  found in Ref. [18]. A similar result holds in the  $SU(3)$  case.
- (v) A linear growth of the trace anomaly  $\theta$  has also been obtained within the theoretical approach described in Ref. [20], in which a Gribov-type dispersion relation is used. On the contrary, a quadratic rise of  $\theta$  is the result of Ref. [21], in which a dimension-two gluonic condensate is studied.

- (vi) In the present work, we concentrated on the high  $T$  side. If we assume that a powerlike behavior  $B_{\text{NP}}(T) = cT^\alpha$  is valid for lower temperature, the quasiparticle gluon mass blows up at a critical temperature  $T_c$  (which depends on the numerical values of  $c$  and  $\alpha$ , e.g. in Ref. [31]). This fact may signalize a confinement/deconfinement phase transition. For  $T < T_c$ , a quasiparticle description is no longer possible and the system should be described by different degrees of freedom (such as glueballs [34] and other nonperturbative states).

## V. CONCLUSIONS

In this work, we have performed an analytical study of the high temperature properties of a gas of gluonic quasiparticles with a temperature-dependent bag function. The expression of the quasiparticle mass  $m(T)$ , pressure and trace anomaly have been derived analytically for large  $T$ .

The implications and constraints on the parameters of the bag function  $B(T) = B_{\text{NP}}(T) + B_p(T)$  have been discussed. For the nonperturbative contribution  $B_{\text{NP}}(T) = ncT^\alpha$  (with  $c > 0$ ) we have found the following constraint on the parameter  $\alpha$ :  $0 \leq \alpha < 2$ . This result follows from the requirements that the gluon mass  $m(T)$  does not become imaginary for decreasing  $T$  and that the sign of the trace anomaly  $\theta = \rho - 3p$  at large  $T$  is positive, in agreement with lattice simulations. The behavior of the trace anomaly  $\theta = \rho - 3p$  at high temperatures consists of two contributions,  $\theta = n\#T^\alpha + n\#T^2$ , with a first term which goes as  $T^\alpha$ , just as the nonperturbative bag function, and a second term which goes as  $T^2$  and dominates the high  $T$  behavior. It is remarkable that this quadratic contribution is general and does not depend on the choice of the bag function. Such a quadratic behavior, and also the expected scaling with the degeneracy number  $n$ , have been confirmed in the lattice simulations of Refs. [11,23].

A variety of improvements of the present approach represents an outlook for the future: (i) Inclusion of the logarithmic corrections in order to make contact with the very high  $T$  behavior, in which the pressure and the energy density slowly approach their Stefan-Boltzmann limit. (ii) On the low-temperature side, a fit should be performed in order to determine the behavior of the nonperturbative bag pressure  $B_{\text{NP}}(T)$  beyond the simple power-law used in this work for analytical considerations. (iii) Calculation of viscosities following Ref. [35] can be performed. (iv) Inclusion of further degrees of freedom: quarks for  $T > T_c$  and confined states (glueball and mesons) for  $T < T_c$ .

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