Magic spinor product methods in loop integrals

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We present an approach to higher point loop integrals using Chinese magic in the virtual loop integration variable. We show, using the five point function in the important $e^+e^- \rightarrow f\bar{f} + \gamma$ process for initial state radiation as a pedagogical vehicle, that we get an expression for it directly reduced to one scalar 5-point function and 4-, 3-, and 2- point integrals, thereby avoiding the computation of the usual three tensor 5-pt Passarino-Veltman reduction. We argue that this offers potential for greater numerical stability.

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With the advent of the LHC, we enter the era of precision QCD, by which we mean predictions for QCD processes at the total precision tag of 1% or better. This is analogous to the per mille level era of EW corrections at LEP energies. Radiative effects at the level of $\mathcal{O}(\alpha_s^2)$ have to be controlled on the QCD side, and those at the level of $\mathcal{O}(\alpha L \alpha_s)$, $\mathcal{O}(\alpha^2 L^2)$ on the QED $\otimes$ QCD and QED sides have to be controlled systematically, both from the physical precision standpoint and from the technical precision standpoint, in order to optimize physics discovery at the LHC.[1] In Ref. [1], we have developed a platform for the realization of such corrections ultimately on an event-by-event basis based on exact, amplitude-based resummation of QED and QCD together, wherein residuals for hard photons and hard gluons are simultaneously calculated order-by-order in perturbation theory in powers of α and α_s. These residuals, which are infrared finite and, for hadron-hadron applications, collinearly finite require then exact evaluation of higher point and (higher) loop Feynman diagrams in an appropriate reduction scheme for any attendant tensor properties as first developed systematically in Ref. [2], for example. Recently, alternative approaches have been developed in Refs. [3,4] to deal with the growing complexity of the method in Ref. [2] as the number of legs beyond four and/or loops beyond one increases. Here, we focus on higher point one-loop functions.[2]

It has been demonstrated that n-point functions, for $n = 1, \cdots, 4$, at one-loop, reduced to scalar functions using the method of Ref. [2], are tractable for fast MC event generator implementation for arbitrary masses and kinematics for high energy scattering processes [6–22]. It has also been demonstrated [23–26] that, at one-loop, higher point scalar functions can be reduced to sums of four-point scalar functions. In Refs. [24,27–30] representations of the scalar four-point function that cover arbitrary masses and the momenta relevant to most high energy collider applications have been given and these are suitable for fast MC implementation. Thus, when one is discussing higher point functions at one-loop, we can consider, at least for most collider physics applications, that the 1, 2, 3, and 4 point functions at one-loop are known in a practical way so that the main issue can be considered to be the representation of the higher point functions in terms of these known functions.

When we consider any higher point function, two of the most important aspects of any reduction procedure for recasting it in terms of the "known", lower point functions are its numerical stability and its usefulness for Monte Carlo event generator realization, as we have in mind for our residuals $\hat{\bar{\bar{\beta}}}_{n,m}$ in Ref. [1] for example. Given the simplification that has been shown for the "Chinese magic" polarization scheme [31–33] for real emission of massless gauge particles in such functions, it is natural to seek further simplification and numerical stability in the virtual emission and reabsorption processes as well by exploiting the same scheme. It is this that we pursue in what follows.

For the reader unfamiliar with the Chinese magic polarization scheme for massless gauge bosons, which is historically associated to the preprint in Ref. [31], the key observation is that the gauge invariance of the attendant massless gauge theory allows one to use an attendant set of polarization vectors which, when the chiral forms of the respective spin $\frac{1}{2}$ charged particles' wave functions are used, eliminate radiation from one entire side of a charged line and, simultaneously, simplify considerably the calculation of the part of the amplitude that remains, almost like "magic," hence the name. This is possible because of a representation of the respective polarization vector for helicity λ_γ and 4-momentum k_γ, $\epsilon^\mu_{\lambda_\gamma}$, as a matrix element of the Dirac gamma matrix, γ^μ, between the spinor of helicity λ_γ and four-momentum k_γ, $|k_\gamma \lambda_\gamma\rangle$, and the massless spinor state $\langle \rho \lambda_\gamma|$, $\rho^2 = 0$, up to a normalization factor, so that the Chisholm identity [see Eq. (12) below] reduces the Feynman rule factor $\epsilon^{*\mu}_{\lambda_\gamma}\gamma_\mu$ at the

[1] Here, L denotes the typical big log for the process under discussion.

[2] See Ref. [5] for some recent progress on the higher loop functions with an eye toward their use in the MC realization of the approach in Ref. [1]

respective interaction vertex to the simple expression $2[|k_\gamma - \lambda_\gamma\rangle\langle\rho - \lambda_\gamma| + |\rho\lambda_\gamma\rangle\langle k_\gamma\lambda_\gamma|]$, up to the same normalization factor, which causes one side of a line of the real radiation terms to vanish if ρ is set equal to the external 4-momentum entering(leaving) that side of the respective line. The remaining terms are then expressed in terms of simple spinor products which lend themselves to easy evaluation [31–33]. This gives a 'magically' shortened expression compared the usual Cartesian representation of the polarization vector with the squared amplitude modulus evaluated using traces over the fermion lines. We illustrate this below here as well.

Specifically, we will use the conventions of Refs. [10,34] for spinors and polarization vectors, which are derived from the work of [31,33]. The 5-pt function which we want to analyze in these conventions as our prototypical example is shown in diagram (c) in Fig. 1. It has many applications in collider precision physics. When combined with diagrams 1(a) and 1(b) it generates a gauge invariant contribution to the ISR for $e^+e^- \to f\bar{f} + \gamma$, $f \neq e$,[3] for example, and it is a part of such a contribution to $u\bar{u} \to \mu\bar{\mu} + G$ (in an appropriate color basis), etc. Such applications and their attendant phenomenology will be taken up elsewhere. [36]. Here, we focus on the use of Chinese magic in the loop integral in Fig. 1(c) to illustrate what simplifications are possible.

More precisely, by the standard methods, we need the following Feynman integral representation of Fig. 1(c)

$$\mathcal{M}^{(1c)}_{\lambda_1\lambda_2\lambda'_1\lambda'_2\lambda_\gamma} = (2\pi)^4\delta(p_1 + p_2 - p'_1 - p'_2 - k)\mathcal{C}\frac{\int d^4q}{(2\pi)^4}\frac{\bar{v}_{\lambda_2}\gamma^\beta(\not{q} + \not{p}_1 - \not{k} + m_1)\not{\epsilon}^*_{\lambda_\gamma}(\not{q} + \not{p}_1 + m_1)\gamma^\alpha u_{\lambda_1}}{((q + p_1 - k)^2 - m_1^2 + i\epsilon)((q + p_1)^2 - m_1^2 + i\epsilon)}$$
$$\times\frac{\bar{u}'_{\lambda'_1}\gamma_\alpha(\not{q} + \not{p}'_1 + m_2)\gamma_\beta v'_{\lambda'_2}}{((q + p_1 + p_2 - k)^2 - M^2_{V_2} + i\epsilon)((q + p'_1)^2 - m_2^2 + i\epsilon)(q^2 - M^2_{V_1} + i\epsilon)} + \ldots, \tag{1}$$

where we have defined massless limit coupling factor

$$\begin{aligned}\mathcal{C} &= \mathcal{C}(\{\lambda_i\},\{\lambda'_j\})\\ &= Q_1 e G^2 G'^2(v'_1 + a'_1\lambda_2)(v_1 - a_1\lambda_1)\\ &\quad\times(v'_2 + a'_2\lambda'_2)(v_2 - a_2\lambda'_1)\end{aligned} \tag{2}$$

with the couplings Q_1e, G, and G' for the γ, V_1, and V_2, respectively. In the usual Glashow-Salam-Weinberg-'t Hooft-Veltman [37] notation, v(a) represents vector (axial-vector) coupling. The ellipsis in (2) represent the mass corrections needed to correct the massless limit used for $\mathcal{C}(\{\lambda_i\},\{\lambda'_j\})$. They are not necessary to illustrate our method and they will be restored elsewhere [36]. To get the loop integral in terms of Chinese magic, we take the following kinematics as shown in Fig. 1:

$$\begin{aligned}p_1 &= (E, p\hat{z})\\ p_2 &= (E, -p\hat{z})\\ -p_4 &= (E', p'(\cos\theta'_1\hat{z} + \sin\theta'_1\hat{x})) \equiv p'_1\\ k &= (k^0, k(\cos\theta_\gamma\hat{z} + \sin\theta_\gamma(\cos\phi_\gamma\hat{x} + \sin\phi_\gamma\hat{y})))\\ -p_4 - p_3 + k &= p_1 + p_2 = (\sqrt{s}, \vec{0})\\ -p_3 &\equiv p'_2,\end{aligned} \tag{3}$$

with $k^0 = k$, $\sqrt{s} = 2E$. Here, we introduce the alternate notations $p'_1 = -p_4$, $p'_2 = -p_3$ for cosmetic use entirely. We now introduce the two sets of magic polarization vectors associated to the two incoming lines:

$$\begin{aligned}(\epsilon^\mu_\sigma(\beta))^* &= \frac{\bar{u}_\sigma(k)\gamma^\mu u_\sigma(\beta)}{\sqrt{2}\bar{u}_{-\sigma}(k)u_\sigma(\beta)},\\ (\epsilon^\mu_\sigma(\zeta))^* &= \frac{\bar{u}_\sigma(k)\gamma^\mu \mathfrak{u}_\sigma(\zeta)}{\sqrt{2}\bar{u}_{-\sigma}(k)\mathfrak{u}_\sigma(\zeta)},\end{aligned} \tag{4}$$

with $\beta^2 = 0$ and ζ defined in Refs. [10,34], so that all phase information is strictly known in our amplitudes: the two choices for β are such that its spacelike components have the directions of the two incoming beams in the initial state. We take the basis of the four-dimensional momentum space as follows:

$$\begin{aligned}\ell_1 &= (E, E\hat{z})\\ \ell_2 &= (E, -E\hat{z})\\ \ell_3 &= E\frac{\langle\ell_2 +|\gamma^\mu|\ell_1+\rangle}{\sqrt{2}\langle\ell_2 -|\ell_1+\rangle}\\ &= \frac{-E}{\sqrt{2}}(\hat{x} + i\hat{y})\\ \ell_4 &= E\frac{\langle\ell_2 -|\gamma^\mu|\ell_1-\rangle}{\sqrt{2}\langle\ell_2 +|\ell_1-\rangle}\\ &= \frac{E}{\sqrt{2}}(\hat{x} - i\hat{y}),\end{aligned} \tag{5}$$

where we use the obvious equivalence $|\ell\sigma\rangle = u(\ell)_\sigma$ in the notation of Refs. [31–34]. The important point is that all four of these basis four-vectors are lightlike with $\ell_i^2 = 0$, $i = 1, \cdots, 4$. They therefore can participate in Chinese magic.

[3] A numerical realization of the amplitude in Fig. 1 as it relates to bhabha scattering can be found in Ref. [35].

To illustrate explicitly this latter point, consider the definite case $\lambda_1, \lambda_2, \lambda_1', \lambda_2', \lambda_\gamma = +, -, +, -, +$, as all other choices for the helicities behave similarly. We write the loop momentum as

$$q = \alpha_i \ell_i \tag{6}$$

with summation over repeated indices understood. The coefficients α_i are readily determined as

$$\begin{aligned}
\alpha_1 &= \frac{q\ell_2}{2E^2} = (\mathfrak{D}_3 - \mathfrak{D}_2 - s + 2p_2k + M_{V_2}^2)/s \\
\alpha_2 &= \frac{q\ell_1}{2E^2} = (\mathfrak{D}_1 - \mathfrak{D}_0 - M_{V_1}^2)/s \\
\alpha_3 &= \frac{q\ell_4}{E^2} = -\frac{q\ell_3^*}{E^2} = -\alpha_4^* \\
\alpha_4 &= -\frac{i}{\sqrt{2s}}[c_j\mathfrak{D}_j + c_5 M_{V_1}^2 + c_6(M_{V_2}^2 + 2p_2k - s) + c_7(2kp_1)],
\end{aligned} \tag{7}$$

where we define the denominators as

$$\begin{aligned}
\mathfrak{D}_0 &= q^2 - M_{V_1}^2 + i\epsilon \\
\mathfrak{D}_1 &= (q + p_1)^2 - m_1^2 + i\epsilon \\
\mathfrak{D}_2 &= (q + p_1 - k)^2 - m_1^2 + i\epsilon \\
\mathfrak{D}_3 &= (q + p_1 + p_2 - k)^2 - M_{V_2}^2 + i\epsilon \\
\mathfrak{D}_4 &= (q - p_4)^2 - m_2^2 + i\epsilon
\end{aligned} \tag{8}$$

so that the expansion coefficients $\{c_j\}$ are (here, $\beta_1 = \frac{p}{E}$, $\beta_1' = \frac{p'}{E'}$)

$$\begin{aligned}
c_0 &= \csc\phi_\gamma\left(\frac{\csc\theta_1' e^{i\phi_\gamma}}{\beta_1' E_1'} - \frac{\csc\theta_1' e^{i\phi_\gamma}}{\beta_1'\sqrt{s}} + \frac{\csc\theta_\gamma}{\sqrt{s}} - \frac{\cot\theta_1' e^{i\phi_\gamma} - \cot\theta_\gamma}{\beta_1\sqrt{s}}\right) \\
c_1 &= \csc\phi_\gamma\left(\frac{\csc\theta_1' e^{i\phi_\gamma}}{\beta_1'\sqrt{s}} - \frac{\csc\theta_1'}{\sqrt{s}} + \frac{\cot\theta_1' e^{i\phi_\gamma} - \cot\theta_\gamma}{\beta_1\sqrt{s}} + \frac{\csc\theta_\gamma}{k^0}\right) \\
c_2 &= \csc\phi_\gamma\left(\frac{-\csc\theta_1' e^{i\phi_\gamma}}{\beta_1'\sqrt{s}} + \frac{\csc\theta_\gamma}{\sqrt{s}} + \frac{\cot\theta_1' e^{i\phi_\gamma} - \cot\theta_\gamma}{\beta_1\sqrt{s}} - \frac{\csc\theta_\gamma}{k^0}\right) \\
c_3 &= \csc\phi_\gamma\left(\frac{\csc\theta_1' e^{i\phi_\gamma}}{\beta_1'\sqrt{s}} - \frac{\csc\theta_\gamma}{\sqrt{s}} - \frac{\cot\theta_1' e^{i\phi_\gamma} - \cot\theta_\gamma}{\beta_1\sqrt{s}}\right) \\
c_4 &= -\csc\phi_\gamma\frac{\csc\theta_1' e^{i\phi_\gamma}}{\beta_1' E_1'} \\
c_5 &= \csc\phi_\gamma\left(\frac{\csc\theta_1' e^{i\phi_\gamma}}{\beta_1' E_1'} - \frac{\csc\theta_1' e^{i\phi_\gamma}}{\beta_1'\sqrt{s}} + \frac{\csc\theta_\gamma}{\sqrt{s}} - \frac{\cot\theta_1' e^{i\phi_\gamma} - \cot\theta_\gamma}{\beta_1\sqrt{s}}\right) \\
c_6 &= \csc\phi_\gamma\left(\frac{\csc\theta_1' e^{i\phi_\gamma}}{\beta_1'\sqrt{s}} - \frac{\csc\theta_\gamma}{\sqrt{s}} - \frac{\cot\theta_1' e^{i\phi_\gamma} - \cot\theta_\gamma}{\beta_1\sqrt{s}}\right) \\
c_7 &= -\csc\phi_\gamma\frac{\csc\theta_\gamma}{k^0}.
\end{aligned} \tag{9}$$

Thus, the $\{c_j\}$ are determined explicitly by the center of momentum system (cms) kinematics that we use. The consequence to note is that the Chinese magic now carries over to the loop variable via the identity

$$
\begin{aligned}
\not{q} &= \alpha_j \ell_j \\
&= \sum_{j=1}^{2} \alpha_j (|\ell_j+\rangle\langle \ell_j+| + |\ell_j-\rangle\langle \ell_j-|) + \alpha_3 \frac{\sqrt{2}E}{\langle p_2 - | p_1 + \rangle} (|\ell_2-\rangle\langle \ell_1-| + |\ell_1+\rangle\langle \ell_2+|) \\
&\quad + \alpha_4 \frac{\sqrt{2}E}{\langle p_2 + | p_1 - \rangle} (|\ell_2+\rangle\langle \ell_1+| + |\ell_1-\rangle\langle \ell_2-|) \\
&\equiv \sum_{j=1}^{2} \alpha_j (|p_j+\rangle\langle p_j+| + |p_j-\rangle\langle p_j-|) + \alpha_3 \frac{\sqrt{2}E}{\langle p_2 - | p_1 + \rangle} (|p_2-\rangle\langle p_1-| + |p_1+\rangle\langle p_2+|) \\
&\quad + \alpha_4 \frac{\sqrt{2}E}{\langle p_2 + | p_1 - \rangle} (|p_2+\rangle\langle p_1+| + |p_1-\rangle\langle p_2-|) \\
&\equiv \sum_{j=1}^{2} \alpha_j (|p_j+\rangle\langle p_j+| + |p_j-\rangle\langle p_j-|) + \tilde{\alpha}_3 (|p_2-\rangle\langle p_1-| + |p_1+\rangle\langle p_2+|) + \tilde{\alpha}_4 (|p_2+\rangle\langle p_1+| + |p_1-\rangle\langle p_2-|),
\end{aligned} \tag{10}
$$

where we work in the massless limit for this numerator algebra so that we take $\ell_1 \equiv p_1$, $\ell_2 \equiv p_2$ in (10). Here, we defined as well

$$
\begin{aligned}
\tilde{\alpha}_3 &\equiv \alpha_3 \frac{\sqrt{2}E}{\langle p_2 - | p_1 + \rangle} = -\frac{\alpha_3}{\sqrt{2}} \\
\tilde{\alpha}_4 &\equiv \alpha_4 \frac{\sqrt{2}E}{\langle p_2 + | p_1 - \rangle} = \frac{\alpha_4}{\sqrt{2}}.
\end{aligned} \tag{11}
$$

From the standpoint of efficient and numerically stable MC event generator realization of the correction in Fig. 1, the explicit form the α_j cannot be stressed too much.

Upon introducing the representation (10) into the numerator, N, of the integrand in (1) we get, from the standard identities

$$
\begin{aligned}
\not{\epsilon}^*_{\lambda_\gamma} &= \frac{\sqrt{2}}{\langle k - \lambda_\gamma | \ell_1 \lambda_\gamma \rangle} [|\ell_1 \lambda_\gamma\rangle\langle k \lambda_\gamma| + |k - \lambda_\gamma\rangle\langle \ell_1 - \lambda_\gamma|], \\
\gamma^\rho \langle \ell_1 \lambda | \gamma_\rho | \ell_2 \lambda \rangle &= 2[|\ell_1 - \lambda\rangle\langle \ell_2 - \lambda| + |\ell_2 \lambda\rangle\langle \ell_1 \lambda|], \\
\not{\ell}_1 &= |\ell_1+\rangle\langle \ell_1+| + |\ell_1-\rangle\langle \ell_1-|,
\end{aligned} \tag{12}
$$

the reduction

$$
\begin{aligned}
N = \frac{4\sqrt{2}}{\langle k - | p_1 + \rangle} \{ &(A_1 \langle p_2 + | p'_1 - \rangle\langle p'_2 - | p_2 + \rangle \\
&+ A_2 \langle p_2 + | p'_1 - \rangle\langle p'_2 - | p_1 + \rangle)(A_3 \langle p_2 + | p'_1 - \rangle \\
&\times \langle p'_1 - | p_1 + \rangle + A_4 \langle p_1 + | p'_1 - \rangle\langle p'_1 - | p_1 + \rangle) \\
&+ \tilde{\alpha}_4 (A_1 \langle p_2 + | p_1 - \rangle\langle p'_2 - | p_2 + \rangle + A_2 \langle p_2 + | p_1 - \rangle \\
&\times \langle p'_2 - | p_1 + \rangle)(A_3 \langle p_2 + | p'_1 - \rangle\langle p_2 - | p_1 + \rangle \\
&+ A_4 \langle p_1 + | p'_1 - \rangle\langle p_2 - | p_1 + \rangle)\},
\end{aligned} \tag{13}
$$

where we defined

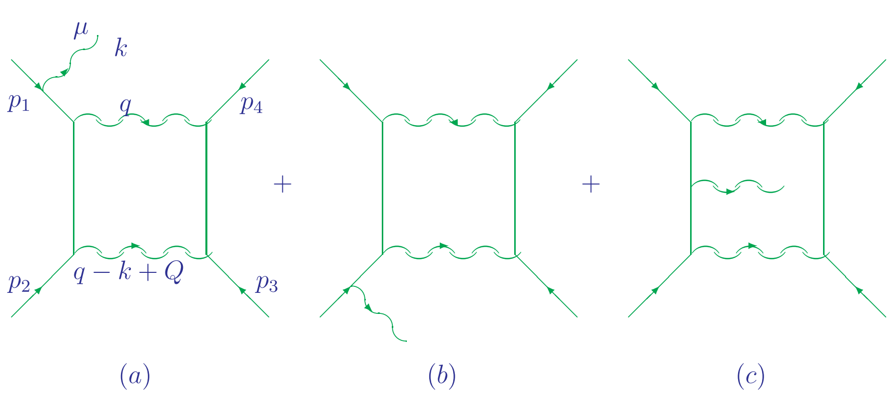

FIG. 1 (color online). ISR 5-point function contributions with fermion and vector boson masses m_f, m_B, $f = 1, 2$, $B = V_1, V_2$ and with four momenta p_i, k as shown, with $Q \equiv p_1 + p_2$. Radiation is shown from the initial state line with electric charge $Q_1 e$, where e is the electric charge of the positron—here p_1 is the incoming fermion 4-momentum, p_2 is the incoming antifermion 4-momentum. When the quantum numbers allow it, the crossed graphs for the internal vector boson exchanges must be added to what we show here.

$$\begin{aligned} A_1 &= \tilde{\alpha}_4\langle p_1 + |k-\rangle + \alpha_2\langle p_2 + |k-\rangle, \\ A_2 &= (1+\alpha_1)\langle p_1 + |k-\rangle + \tilde{\alpha}_3\langle p_2 + |k-\rangle \\ A_3 &= \alpha_2\langle p_1 - |p_2+\rangle, \\ A_4 &= \tilde{\alpha}_4\langle p_1 - |p_2+\rangle \end{aligned} \tag{14}$$

for the magic choice $\beta = p_1$. Note that the magic has killed all but one set of the terms with three factors of the virtual momentum expansion coefficients and that, in the numerator of the propagator (before) after the real emission vertex, it has eliminated the terms associated with $(\not p_1)\, \not k$ as well as half of the terms in the respective virtual momentum expansion in former case. While we have eliminated a large fraction of the possible terms on the right-hand side (rhs)of (13), one can ask how it compares in length with what one would get from the usual approaches of taking traces on the fermion lines. To be specific, in the traditional method that leads to traces on fermion lines, one needs to compare the length of $2\Re\mathcal{M}_B^*\mathcal{M}^{(1c)}$, where $\mathcal{M}_B$ is the respective Born amplitude that would interfere with the one-loop amplitude to create the one-loop correction to the respective cross section. In the Chinese magic representation, we get immediately that only radiation from the antiparticle (p_2) incoming line contributes with the simple result (repeated indices are summed with $G_1 = G,\ G_2 = G'$ and $s' = (p_1 + p_2 - k)^2$ as usual)

$$\begin{aligned} &\mathcal{M}_{B+-+-+} \\ &= (2\pi)^4\delta(p_1 + p_2 - p_1' - p_2' - k) \\ &\quad\times \frac{2\sqrt{2}ieQ_1G_j^2(v_j' - a_j')(v_j - a_j)\langle p_2' - |p_1+\rangle}{\langle k - |p_1+\rangle\langle k - |p_2+\rangle(s' - M_{V_j}^2 + i\epsilon)} \\ &\quad\times[\langle p_1 - |p_2+\rangle\langle p_2 + |p_1'-\rangle - \langle p_1 - |k+\rangle\langle k + |p_1'-\rangle] \end{aligned} \tag{15}$$

so that computing $2\Re\mathcal{M}_B^*\mathcal{M}^{(1c)}$ just involves multiplying N in (13) by the complex conjugate of this simple expression and taking twice the real part. If we proceed with the usual trace on the fermion lines method, one needs the trace of two sets of terms with 10 Dirac gamma matrices multiplied by a factor with the trace of 6 Dirac gamma matrices: this means one has $2\cdot 9\cdot 7\cdot 5\cdot 4\times 5\cdot 4 = 2520\times 20 = 50,\ 400$ terms, each of which requires Passarino-Veltman reduction of 3, 2, and 1 5-point tensor integrals. In Ref. [38], another approach that leads as well to traces over fermions is used in which one first expands the amplitude under study in a gauge invariant tensor basis with scalar coefficients and uses Chinese magic-type [31–33] representations of the helicity states to express the attendant helicity amplitudes in terms of these invariant scalar coefficient functions. The key step is the use of projection operators, $\mathcal{P}(X)$ in the notation of Ref. [38], which project out the scalar coefficient X. To compare with our approach, we observe the following: the Born amplitude tensor structure is one of the tensor structures in the respective expansion basis and to project its coefficient the respective projection operator evaluates a linear combination of the trace on the fermion lines of the Hermitian conjugate of this Born level tensor structure in product with the Feynman amplitude and the traces on the fermion lines of the Hermitian conjugates of the other tensor structures in product with the same amplitude. Thus, our counting of terms given for the evaluation of $2\Re\mathcal{M}_B^*\mathcal{M}^{(1c)}$ using the traditional traces on fermion lines gives a lower limit to the number of terms that would be generated by the methods of Ref. [38] for our calculation.[4] Looked at this way, we can appreciate better the great simplification that (13) represents. It follows that this form of N in (13) has efficiently reduced the problem of reduction of the 5-point function with three, two, and one tensor indices(index) in the Passarino-Veltman formalism to the problem of a single scalar 5-point function and lower 4, 3, and 2 point functions with the coefficients already explicitly expressed in terms of the cms kinematic variables that are so crucial to efficient MC event generation. Efficient MC event generator realization of the latter functions is known [6–22], where it is understood that one uses the results in Refs. [23–26] to express the scalar 5-point function in terms of scalar 4-point functions using our explicit kinematics above. These last remarks are made more manifest when one notes the introduction of the result for N in (13) into the integral in (1) leads to the integrals

$$\int\frac{d^4q}{(2\pi)^4}\frac{\mathfrak{D}_i\mathfrak{D}_j\mathfrak{D}_k;\mathfrak{D}_i\mathfrak{D}_j;\mathfrak{D}_j;1}{\mathfrak{D}_0\mathfrak{D}_1\mathfrak{G}_2\mathfrak{D}_3\mathfrak{D}_4}, \qquad i,j,k = 0,\cdots,4 \tag{16}$$

all of which are known from the lower point functions we advertised when the results for the representation of the scalar 5-point function in terms of 4-point functions in Refs. [24–26] are used.[5] We get a bonus: *no evaluation*

[4]For example, let us take the example discussed in Ref. [38], using their notation, of $q(p_2,\lambda_2)\bar{q}(p_1,\lambda_1)\to\gamma(p_3,\lambda_3)\gamma(p_4,\lambda_4)$, where we focus just on the one-loop correction from the Gross-Wilczek-Politzer [39] QCD theory with direct analysis for the respective 4-point box graph in which a gluon is exchanged between the incoming quark (q) antiquark ($\bar{q}$) pair "before" they annihilate to the two photons. For the helicities $+-++$ for the quark, antiquark, $\gamma(p_3)$, $\gamma(p_4)$, respectively, the helicity amplitude is proportional to the A_{11} scalar coefficient in Ref. [38]. Evaluation of the projection operator for A_{11} on the box graph requires the trace for a product of 12 Dirac gamma matrices, which generates $11\cdot 9\cdot 7\cdot 5\cdot 3 = 10,395$ terms, and this has to be done 5 times (there are five scalar coefficients) for a total of 51 975 terms. This is just a 4-point function. The same calculation using our methods generates a formula smaller in length than that in Eq. (20) in the text.

[5]Since the integral in (1) is manifestly UV finite, we do not need to specify what regularization is used for the two point functions because only UV finite combinations of them can occur here while the wave functions are all in four-dimensional Minkowski space. Note also that the standard trace over fermion lines would also lead to results equivalent to that in (16) but as we have seen above it would necessitate evaluation and simplification of much longer expressions in general to compute the attendant transition rate for the process.

of wave functions at complex momenta is required here. What we have done is rigorously a result of Lagrangian quantum field theory, and it therefore can serve as a cross check on methods that may not obviously so be. Evidently, the method we have illustrated can be used for any higher point function.

At this point, while we have shortened considerably the respective amplitude and have removed the Gram determinant-type factors in the tensor reductions, we are still subject to the Gram determinant-type denominator factors in the results in Refs. [24–26] for the representation of the 5-point scalar function in terms of 4-point scalar functions. We have found that these are in general still too numerically unstable for realization in the amplitude-based exact resummation MC event generators such as those in Ref. [9]. Thus, we replace the representation from Refs. [24–26] of the needed 5-point scalar function here as follows.

We start from the basic identity

$$q^2 = \mathfrak{D}_0 + M_{V_1}^2 - i\epsilon = (\alpha_i \ell_i)^2 = 2\alpha_1\alpha_2\ell_1\ell_2 + 2\alpha_3\alpha_4\ell_3\ell_4 = s\alpha_1\alpha_2 + \frac{s}{2}\alpha_3\alpha_4. \tag{17}$$

Dividing by $\mathfrak{D}_0 \cdots \mathfrak{D}_4$ and integrating over d^4q we arrive at the following representation of the required scalar 5-point function (we use the notation of Ref. [26] for E_0 itself):

$$\begin{aligned}
E_0(\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, \bar{m}_4) = \Big\{ & -D_0(0) + \frac{1+\beta_1^2}{2s\beta_1^2}[C_0(13) - C_0(12) - C_0(03) + C_0(02) + (M_{V_2}^2 - s + 2p_2k) \\
& \times (D_0(1) - D_0(0)) - M_{V_1}^2(D_0(3) - D_0(2))] - \frac{1-\beta_1^2}{4s\beta_1^2}[\Delta r_{1,0}(D_0(1) - D_0(0)) \\
& + 2\Delta\bar{p}_{1,0}(D_{11}(1)\bar{\bar{p}}(1)_1 - D_{11}(0)\bar{\bar{p}}(0)_1 + D_{12}(1)\bar{\bar{p}}(1)_2 - D_{12}(0)\bar{\bar{p}}(0)_2 \\
& + D_{13}(1)\bar{\bar{p}}(1)_3 - D_{13}(0)\bar{\bar{p}}(0)_3) - D_0(1)\bar{\bar{p}}(1)_4 + D_0(0)\bar{\bar{p}}(0)_4 - 2M_{V_1}^2(D_0(1) - D_0(0)) \\
& + \Delta r_{3,2}(D_0(3) - D_0(2)) + 2\Delta\bar{p}_{3,2}(D_{11}(3)\bar{\bar{p}}(3)_1 - D_{11}(2)\bar{\bar{p}}(2)_1 \\
& + D_{12}(3)\bar{\bar{p}}(3)_2 - D_{12}(2)\bar{\bar{p}}(2)_2 + D_{13}(3)\bar{\bar{p}}(3)_3 - D_{13}(2)\bar{\bar{p}}(2)_3 - D_0(3)\bar{\bar{p}}(3)_4 \\
& + D_0(2)\bar{\bar{p}}(2)_4) + 2(M_{V_2}^2 - s + 2p_2k)(D_0(3) - D_0(2))] - \frac{1}{4}\Big[\sum_{j=0}^{4}|c_j|^2(C_0(j,j+1) \\
& + \Delta r_{j,j+1}D_0(j) + 2\Delta\bar{p}_{j,j+1}(D_{11}(j)\bar{\bar{p}}(j)_1 + D_{12}(j)\bar{\bar{p}}(j)_2 + D_{13}(j)\bar{\bar{p}}(j)_3 \\
& - D_0(j)\bar{\bar{p}}(j)_4)) + 2\Big(\sum_{i\rangle j}^{4}\Re(c_ic_j^*)C_0(ij) + \sum_{j=0}^{4}\Re(c_j(c_5^*M_{V_1}^2 + c_6^*(M_{V_2}^2 - s + 2p_2k) \\
& + c_7^*(2kp_1)))D_0(j)\Big)\Big]\Big\}\Big/C_{E_0},
\end{aligned} \tag{18}$$

where we have the identifications

$$\bar{p}_1 = p_1, \quad \bar{p}_2 = p_1 - k, \quad \bar{p}_3 = p_1 + p_2 - k, \quad \bar{p}_4 = p_1', \quad \bar{m}_0 = M_{V_1}, \quad \bar{m}_1 = m_1, \quad \bar{m}_2 = m_1, \quad \bar{m}_3 = M_{V_2}, \quad \bar{m}_4 = m_2,$$

and where the coefficient C_{E_0} is given by

$$\begin{aligned}
C_{E_0} = {} & M_{V_1}^2 - i\epsilon + \frac{1+\beta_1^2}{2\beta_1^2 s}M_{V_1}^2(M_{V_2}^2 - s + 2p_2k) + \frac{1-\beta_1^2}{4\beta_1^2 s}(M_{V_1}^4 + (M_{V_2}^2 - s + 2p_2k)^2) \\
& + \frac{1}{2}\Re[c_5c_6^*M_{V_1}^2(M_{V_2}^2 - s + 2p_2k) + c_5c_7^*M_{V_1}^2(2kp_1) + c_6c_7^*(M_{V_2}^2 - s + 2p_2k)(2kp_1)] \\
& + \frac{1}{4}[|c_5|^2M_{V_1}^4 + |c_6|^2(M_{V_2}^2 - s + 2p_2k)^2 + |c_7|^2(2kp_1)^2].
\end{aligned} \tag{19}$$

We have here used a combination of the notation from Refs. [2,25,26] so that the definitions which follow should hold true:

$$\mathfrak{D}_j = (q + \bar{p}_j)^2 - \bar{m}_j^2 + i\epsilon = q^2 + 2q\bar{p}_j + \bar{p}_j^2 - \bar{m}_j^2 + i\epsilon \equiv q^2 + 2q\bar{p}_j + r_j,$$
$$\Delta r_{i,j} \equiv r_i - r_j,$$
$$\Delta\bar{p}_{i,j} \equiv \bar{p}_i - \bar{p}_j,$$
$$D_0(j) \equiv \text{4-point scalar function obtained from 5-point scalar function by omitting denominator } \mathfrak{D}_j,$$
$$C_0(i,j) \equiv \text{3-point scalar function obtained from 5-point scalar function by omitting denominators } \mathfrak{D}_i \quad \text{and} \quad \mathfrak{D}_j,\ i \neq j,$$

where we also follow the Passarino-Veltman [2] notation of the 4-point one-tensor integral, $D_\mu(j)$, obtained by omitting denominator $\mathfrak{D}_j$ from the corresponding 5-point one-tensor integral with

$$D_\mu(j) \equiv D_{11}(j)\bar{\bar{p}}(j)_1 + D_{12}(j)\bar{\bar{p}}(j)_2 + D_{13}(j)\bar{\bar{p}}(j)_3 - D_0(j)\bar{\bar{p}}(j)_4,$$

where the four-vectors $\{\bar{\bar{p}}(j)_i\}$ are then determined in accordance with Ref. [2], with the understanding that $\bar{\bar{p}}(j)_4$ is only nonzero if it is necessary to shift the q integration variable by it to reach the standard form of the respective Passarino-Veltman representation. This expression for E_0 does not have problems with Gram determinant-type denominators.

To further exhibit the magic in the polarization vector spinor representation under display here, we record as well the results for Fig. 1(a) and 1(b) that one needs to add to our result for Fig. 1(c) to get a gauge invariant result:

$$\mathcal{M}^{(1a)}_{+-+-+} = 0, \quad \text{by magic}$$
$$\mathcal{M}^{(1b)}_{+-+-+} = (2\pi)^4\delta(p_1 + p_2 - p_1' - p_2' - k) \times \frac{4\sqrt{2}\mathcal{C}}{\langle k - |p_1+\rangle\langle k - |p_2+\rangle} \frac{\int d^4q}{(2\pi)^4} \times \frac{N'}{\mathfrak{D}_O\mathfrak{D}_1\mathfrak{D}_3\mathfrak{D}_4}, \tag{20}$$

where the numerator N' is given by

$$N' = (\langle p_2' - |p_1+\rangle\mathfrak{a}_1 + \langle p_2' - |p_2+\rangle\mathfrak{b}_1)(\langle p_1 - |p_2+\rangle \times \langle p_2 + |p_1'-\rangle - \langle p_1 - |k+\rangle\langle k + |p_1'-\rangle) + (\langle p_2' - |p_1+\rangle\bar{\mathfrak{a}}_1 + \langle p_2' - |p_2+\rangle\bar{\mathfrak{b}}_1) \times [(-2p_1(p_2 - k))\tilde{\alpha}_4 + \alpha_2\langle p_1 - |k+\rangle\langle k + |p_2-\rangle] \tag{21}$$

with the definitions

$$\mathfrak{a}_1 = (1 + \alpha_1)(2p_1p_1') + \alpha_3\langle p_2 + |p_1'-\rangle\langle p_1' - |p_1+\rangle$$
$$\mathfrak{b}_1 = \alpha_2\langle p_2 + |p_1'-\rangle\langle p_1' - |p_1+\rangle + \tilde{\alpha}_4(2p_1p_1')$$
$$\bar{\mathfrak{a}}_1 = \langle p_1 - |p_2+\rangle[(1 + \alpha_1)\langle p_1 + |p_1'-\rangle + \tilde{\alpha}_3\langle p_2 + |p_1'-\rangle]$$
$$\bar{\mathfrak{b}}_1 = \langle p_1 - |p_2+\rangle[\alpha_2\langle p_2 + |p_1'-\rangle + \tilde{\alpha}_4\langle p_1 + |p_1'-\rangle]. \tag{22}$$

Again, this gives immediate reduction to the known scalar functions with considerable reduction in the number of terms requiring evaluation compared to the usual trace over fermion lines method when one computes the respective contribution to $2\Re\mathcal{M}_B^*\mathcal{M}^{1b}$. The complete phenomenology of our results for the process in Fig. 1 will appear elsewhere [36].

It is important to explain the difference between what we have done here and what was done in Refs. [3,4,40,41]. We do this in turn in a somewhat reverse chronological order. In Ref. [40], the representation of the loop variable in a basis of lightlike four-vectors is used to construct a recursion relation between one-loop n-point tensor integrals of differing rank whereas in Ref. [41] the spinor representation of the external tensor coefficient of a massless n-point tensor one-loop integral is used to reduce the rank of that integral iteratively to allow numerical implementation, using Dirac matrix methods. In both cases, the square roots of the Gram determinants appear in the denominators of the resulting representations. In our approach, explicit kinematics allows direct Chinese magic action in the complete amplitude contribution's evaluation directly to the lower point functions without Gram determinant factors to be computed in our denominators. No iteration is necessary and Chinese magic reduces considerably the number of terms in our final result. Such action is not present in Refs. [40,41]. In Ref. [4], the representation of the n-point amplitude at one-loop starts from its integrand $N(q)/(\mathfrak{D}_0\cdots\mathfrak{D}_{n-1})$ with an expansion of the numerator $N(q)$ in powers of the denominators $\{\mathfrak{D}_j\}$ with coefficients that split into a part that is independent of q and a part that integrates to zero with the understanding that the integration measure is in general in d dimensions, whereas the function $N(q)$ is defined for q in four dimensions. We refer to this representation as the Ossola-Papadopoulos-Pittau (OPP) representation after the authors in the first paper in Ref. [4]. Various methods for adding in the so-called missing rational terms generated by the mismatch between the 4-dimensional q in N and the d-dimensional $\bar{q}$ in the $\{\mathfrak{D}_j\}$ are given in Ref. [4], including the generalized d-dimensional unitarity that treats the full d-dimensional unitarity realization of the OPP representation. In all of these works, $N(q)$ or $N(\bar{q})$ is treated as a given and no procedure for exploiting Chinese magic to simplify it at the loop momentum level is considered. Moreover, the need to add in rational terms is an essential part of the procedure, whereas, as we see in our result (13), we do not have such an issue in our approach—we get the complete answer with methods that operate entirely in four dimensions.[6] More importantly, inverse powers of Gram-type determinants appear in the coefficients in the representation of N so that issues of numerical stability obtain, whereas as we show above our approach does not lead to such factors so that it should be more stable. Finally, the procedure for determining the coefficients in the representation of N involves solving the algebraic problem for the q values at which 4, then 3, then 2, and finally 1 of the $\{\mathfrak{D}_j\}$ vanish(es). This means that, in general, complex values of q are required and this forces the evaluation of $N(q)$ at such unphysical 4-momenta. Our approach avoids this issue altogether as we carry our entire calculation out in the four-dimensional real virtual loop momentum space. We then provide a completely physical cross check on the methods in Ref. [4]. Similarly, the approach in Ref. [3] also takes the integrand as a given and constructs the respective amplitude from unitarity-based on-shell (recursion) relations, where the authors in Ref. [3] are able to get both the cut-constructable and the rational parts of the amplitudes with such methods. Again, there is no exploitation of Chinese magic to simplify the amplitude at the loop variable level, the amplitude construction uses 4-particle cuts that have in general complex 4-momenta as their solutions so that wave functions are evaluated at such unphysical momenta, and the solution of these on shell relations generally introduces troublesome kinematic factors in the denominators of the representation so that numerical stability cannot be assured. Our approach avoids all of these problems and affords again a completely physical cross check on this approach as well.

The complete analytical result for the amplitude in Fig. 1 will be presented elsewhere [36]. Here, we have shown that the use of Chinese magic in the virtual loop momentum can reduce considerably the amount of algebra required for stable, efficient, manifestly physical computation of higher point virtual corrections with general mass scales, as they are needed for exact amplitude-based resummed MC event generator realization.

[6] If one wants to apply our method to lower point amplitudes that are UV divergent, in renormalizable theories one should use the known counterterms for those divergences to render the amplitudes finite first and then apply our four-dimensional methods to the UV finite subtracted amplitudes.

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