

Group theory and dynamics of neutrino mixing

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There is a direct group-theoretical connection between neutrino mixing and horizontal symmetry that can be established without any dynamical input. Such a connection is reviewed and expanded in this article. For certain symmetry groups \mathcal{G} including A_4 and S_4 , it is shown that a generic $U(1) \times \mathcal{G}$ Higgs potential of a valon yields exactly the alignments dictated by the group-theoretical approach, but energy can now be used to discriminate different alignments. This mechanism possibly explains why starting from an A_4 group, the tribimaximal mixing matrix with an enhanced S_4 symmetry is more preferable than the one without it.

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I. INTRODUCTION

After Chadwick discovered the neutron in 1932, Heisenberg came up with the “isotopic spin” to distinguish it from the proton. This new quantum number corresponds to a global symmetry $SU(2)$ which explains the similarity of neutrons and protons in strong interactions. Now that we have three generations of quarks and leptons, should there not be a new “family quantum number” to distinguish them, and a corresponding “horizontal (or family) symmetry” to explain the similarity of their interactions?

Unlike neutrons and protons which have nearly the same mass, the masses of fermions of different generations are vastly different. Moreover, they mix. Thus if horizontal symmetry is present, it has to be badly broken, a fact which makes the symmetry difficult to recognize unless enough remnants survive from the breaking to tell the story. We suggest that such remnants are indeed present and they are hidden in the mixing matrix.

Neutrino mixing is successfully described by the tribimaximal mixing (TBM) matrix [1]. This mixing is most frequently explained by models with an A_4 symmetry, but models based on S_4 as well as many other groups \mathcal{G} [2] can also do the job. With so many successful models, it is hard to know which of them is the correct horizontal symmetry and, indeed, whether horizontal symmetry really exists or not.

To shed some light on this question from a different angle, we have previously developed a purely group-theoretical method to connect neutrino mixing with horizontal symmetry [3]. This method differs from the conventional one in that only symmetry considerations are involved. The presence of Higgs fields is never assumed, nor the details of a model Lagrangian. In this approach, the broken symmetry (which we call *residual symmetry*) and the minimal unbroken horizontal symmetry of the *left-handed leptons* are derived from the mixing

matrix U . In particular, if neutrino mixing is given by the TBM, then the smallest unbroken horizontal symmetry is S_4 , not A_4 . Models based on A_4 can explain TBM because in those models there is an accidental symmetry which elevates the A_4 symmetry to an S_4 symmetry. These results will be reviewed in Sec. II.

The same formalism can in principle be applied to fermion mixing as well [3], but unless we make suitable approximations to the Cabibbo-Kobayashi-Maskawa (CKM) mixing, all that we get is the group $SU(3)$, nothing smaller. For that reason we will not discuss quark mixing any further in this paper. This difference, however, brings up a very important question, as to whether there is a common origin between neutrino mixing and quark mixing that is smaller than $SU(3)$. This question will be taken up in a forthcoming paper [4].

The group-theoretical procedure can be reversed to obtain mixing matrices U from a symmetry group \mathcal{G} . The detail will be discussed in Sec. III, but let us outline now how that differs from the usual dynamical method. In the dynamical approach, U is to some extent determined by the vacuum alignments, and the alignments are in principle obtained by minimizing a Higgs potential V . There are however frequently too many \mathcal{G} -invariant terms available to construct V , so that just about any vacuum alignment can be obtained by a suitable choice of V . What one does then is to use the experimental results to decide what U one wants, and what alignments that can give rise to such a U , then proceeds to design a V that does the job. To keep it robust, this is often accomplished by imposing another symmetry in addition to \mathcal{G} to throw away the unwanted terms in V .

In the group-theoretical approach, U is determined by the residual symmetries F (for the charged-lepton sector) and G_i ($i = 1, 2, 3$) (for the neutrino sector) which are members of \mathcal{G} . No peeking of data is needed so one is not prejudiced by the experimental outcome. There is no dynamics to tune, and the existence of Higgs is not assumed. For a given \mathcal{G} , there is usually more than one

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possible choice of residual symmetries and thus more than one U , but within equivalence only very few survive. By equivalence I mean the following. Since Majorana phases are not measurable in neutrino oscillation experiments, and since there is no way for group theory to distinguish which flavor neutrino is which, and which neutrino mass eigenstates are to be labeled 1, 2, or 3, we will consider two mixing matrices equivalent if they differ only by row and column phases, and by row and column permutations. With a convenient choice of permutation, the mixing can be specified by three mixing angles and one CP phase, contained in the triplet $(\sin\theta_{12}, \sin\theta_{23}, \sin\theta_{13}e^{-i\delta})$. Two U 's with the same triplet parameters are considered to be equivalent.

Normally we need an F and three mutually commuting G_i to specify a U . A mixing matrix determined this way will be called a *full-mixing* matrix. These are the mixing matrices obtained from \mathcal{G} by symmetry alone. In order to allow TBM to occur under A_4 , we will also discuss *partial-mixing* matrices, which are obtained from a full-mixing matrix by replacing two of its columns with arbitrary parameters, subject however to the constraint of unitarity. Equivalently, they are mixing matrices determined by F and one of the three G_i 's, plus additional parameters needed to specify U . For each full-mixing there are always associated three types of partial-mixings, which will normally not be mentioned separately.

However, the mixing matrix obtained from a partial-mixing of \mathcal{G} with the other two columns filled has its own symmetry group $\bar{\mathcal{G}}$, which contains \mathcal{G} as a subgroup. Unless the entries in the other two columns are carefully chosen, most likely $\bar{\mathcal{G}}$ would end up to be an infinite group, perhaps as large as $SU(3)$. Thus, the entries of the other two columns are severely restricted if we stipulate $\bar{\mathcal{G}}$ to be a finite group. In particular, if the entries are those given by the full-mixing matrix, then $\bar{\mathcal{G}} = \mathcal{G}$. In the case of A_4 , the smallest finite group containing it is S_4 , and there is one partial-mixing mode of A_4 whose $\bar{\mathcal{G}}$ is S_4 . That is how TBM is produced from A_4 .

In the case of S_4 , there are *two* full-mixing patterns. One is the familiar TBM specified by the physical parameters $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, 0)$, and the other is one specified by $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, which is seldom mentioned in the literature. The full-mixing of A_4 is the Cabibbo-Wolfenstein mixing matrix, specified by $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{e^{-\pi i/2}}{\sqrt{3}})$, though TBM is allowed in a partial-mixing mode. There are two possible full-mixings in A_5 , but we will work out only one whose physical parameters are $(\frac{1}{\sqrt{3\varphi}}, \frac{1}{\sqrt{2}}, 0)$, where $\varphi = (1 + \sqrt{5})/2 = 1.618$ is the golden ratio. Once again, these results are obtained in a purely group-theoretical way, without the help of any dynamics.

Sections IV, V, and VI are devoted to the connection between group-theoretical and dynamical approaches. Defining an *invariant eigenvector* to be an eigenvector

with eigenvalue $+1$, the basic connection is that the vacuum alignment in every irreducible representation (IR) has to be the invariant eigenvector of the corresponding residual-symmetry operator in that IR [3]. That connection is reviewed in Sec. IV. In case such an eigenvector does not exist, the alignment would be forced to be zero and the coupling with such a valon cannot contribute. This in principle would reduce our ability to make models because we need at least three independent Yukawa couplings in each sector to fit the three charged-lepton and the three neutrino masses. Nevertheless, in the case of S_4 and A_4 which we discuss in some detail, there remain just enough parameters to fit all the data in type-II seesaw models. For A_4 we also discuss the partial-mixing results, both because the phenomenologically interesting TBM belongs to that category, and because the result of the discussion would be useful in Sec. V. We see from these discussions that a large number of models can be constructed for each group and, indeed, many of them have already appeared in the literature. Similar discussions can be carried out for other groups.

With the basic connection, one important question to ask is whether a generic Higgs potential can be constructed to yield just these vacuum alignments. As remarked before, in order to avoid the appearance of arbitrary alignments, an additional symmetry would have to be imposed to throw away unwanted \mathcal{G} -invariant terms. In the case of $\mathcal{G} = A_4$, that additional symmetry turns out to be $U(1)$ (an appropriate Z_n would do as well) as we shall show in Sec. V. The calculation involves writing down a generic $U(1) \times A_4$ potential in terms of Clebsch-Gordan (CG) coefficients in a convenient basis, then minimizing it.

Can this result be generalized to other groups? On the one hand, this calculation requires intimate knowledge of the group A_4 , such as the explicit CG coefficients and the details of all the $(U(1) \times A_4)$ -invariant terms, so it is difficult to see how to generalize it from A_4 to other groups \mathcal{G} . On the other hand, the result of minimization can be expressed in general group-theoretical terms, as the invariant eigenvectors of members of A_4 , and that gives hope that one might find a way to bypass the details of A_4 to generalize the result to other groups \mathcal{G} . This is indeed true as will be discussed in Sec. VI. In the first part of that section, an alternative proof for the A_4 result is presented that is quite independent of the specific A_4 details. This proof is then generalized to other groups \mathcal{G} in the rest of that section.

Besides demonstrating the consistency between the group-theoretical and the dynamical approaches, the introduction of the Higgs potential also serves another purpose. There is no way for group theory to choose between different residual symmetries, but Higgs potentials might, as different vacuum alignments might give rise to different Higgs energy. In that case the one with the lowest Higgs energy wins out.

This mechanism can, for example, be used to solve a puzzle. TBM is a full-mixing of S_4 but only a partial-mixing of A_4 . The full-mixing of A_4 is given by the Cabibbo-Wolfenstein matrix which gives too large a solar angle and too large a reactor angle. If A_4 is the correct symmetry group, as many models seem to prefer, one has to answer two questions. (1) Why does nature choose a partial-mixing rather than a full-mixing more appropriate to A_4 from the symmetry (group-theoretical) point of view? (2) Why should nature choose a partial-mixing of A_4 and then promote it to a larger S_4 to give TBM, rather than choosing S_4 directly as the symmetry group from the very start?

With the result of Sec. V, we may have an answer to question 1. If the Higgs self-couplings are all positive, then the full-mixing pattern has a higher energy than the partial-mixing pattern, which is presumably why nature prefers partial-mixing. Once in a partial-mixing mode, economy would like to make the mixing matrix TBM because $\bar{G} = S_4$ is the smallest group that contains A_4 . Question 2 is harder to answer, but the answer might come from dynamics as well. In a forthcoming publication [4], it will be shown that if we ignore the much smaller quark mixing with its third generation, then there is something common between quark mixing and leptonic mixing described by A_4 , but there is no such commonality between quark mixing and leptonic mixing described by S_4 .

II. FROM MIXING TO HORIZONTAL GROUP

We review in this section how the horizontal symmetry group \mathcal{G}_L for left-handed leptons can be derived from the neutrino mixing matrix U [3]. We will assume \mathcal{G}_L to be a subgroup of $SU(3)$, and we are particularly interested in small \mathcal{G}_L to minimize the amount of breaking necessary to obtain the desired mixing. For that reason we will concentrate on finite subgroups of $SU(3)$.

Let M_e be the 3×3 charged-lepton mass matrix, and $\bar{M}_e := M_e^\dagger M_e$ the effective mass (squared) matrix connecting left-handed to left-handed charged leptons. This matrix is clearly Hermitian. Similarly, let $\bar{M}_\nu = \bar{M}_\nu^T$ be the Majorana mass matrix for the left-handed neutrinos. This matrix is symmetric on account of the Majorana nature of the neutrinos. In the basis where \bar{M}_e is diagonal, the neutrino mixing matrix U is the matrix that renders $U^T \bar{M}_\nu U$ diagonal.

A *residual-symmetry operator* in the charged-lepton sector is a unitary matrix F which transforms \bar{M}_e into itself: $F^\dagger \bar{M}_e F = \bar{M}_e$. Similarly, a residual-symmetry operator in the neutrino sector is a unitary matrix G which transforms \bar{M}_ν into itself: $G^T \bar{M}_\nu G = \bar{M}_\nu$. G^T rather than G^\dagger has to be used to maintain $\bar{M}_\nu = \bar{M}_\nu^T$. An important consequence that follows is $G^2 = 1$. As a result, the eigenvalues of G are $+1$ or -1 . Since $\det G = 1$, G has one $+1$ and two -1 eigenvalues. There are exactly three such operators, given by

$$\begin{aligned} G_1 &= u_1 u_1^\dagger - u_2 u_2^\dagger - u_3 u_3^\dagger, \\ G_2 &= -u_1 u_1^\dagger + u_2 u_2^\dagger - u_3 u_3^\dagger, \\ G_3 &= -u_1 u_1^\dagger - u_2 u_2^\dagger + u_3 u_3^\dagger, \end{aligned} \quad (1)$$

where u_i is the i th column vector of the mixing matrix U . It is easy to see that the three G_i 's mutually commute, and that the product of any two is equal to the third. Moreover, if we define an *invariant eigenvector* to be an eigenvector with eigenvalue $+1$, then u_i is the invariant eigenvector of G_i .

Since \bar{M}_e is diagonal, F may be taken to be any diagonal unitary matrix, with determinant 1. If F is to be a member of a finite group, then there is an integer n so that $F^n = 1$. We divide F into two categories, the degenerate ones, and the *nondegenerate* ones. The latter consists of F 's whose three diagonal entries are all different, which necessitates $n \geq 3$. This category is specially important because in that case \bar{M}_e must be diagonal whenever F is. This will be used in the next section to recover U from \mathcal{G}_L .

F and G_i are symmetry operators of the left-handed mass matrices, so they are the remaining symmetry operators of the *left-handed* leptons after the horizontal group \mathcal{G}_L for left-handed leptons is broken. Conversely, the smallest \mathcal{G}_L must be the group generated by F and G_i , and this is how the group \mathcal{G}_L can be obtained from the mixing matrix U . We shall use the notation $\mathcal{G}_L = \{F, G_1, G_2, G_3\}$ to denote this. Since the product of two distinct G_i is equal to the third, we can write $\mathcal{G}_L = \{F, G_i, G_j\}$, as long as $i \neq j$. We shall call the group generated this way the *full group*.

The role of a nondegenerate F is to ensure that \bar{M}_e is diagonal when F is, and the role of G_i is to use their invariant eigenvectors to construct the columns of the mixing matrix U . Since there is no need for more than one F to nail down \bar{M}_e to its diagonal form, and since we want a minimal \mathcal{G}_L , we include only one F in the generators of \mathcal{G}_L . However, different choices of F will generally lead to different groups \mathcal{G}_L .

Sometimes we may be more sure of one column of U to be correct than another column. In that case we simply fill the other two columns of U with adjustable parameters, subject to the unitarity condition of U , of course. The full group \bar{G} associated with U then contains a subgroup \mathcal{G}_L generated by $\{F, G_i\}$, where i denotes that particular column of U that we have confidence in. The group \mathcal{G}_L will be called a *partial group*. There are clearly three possible partial groups, associated with $i = 1, 2$, and 3 , respectively. If for the sake of economy we demand \bar{G} to be a finite group, then the entries of the other two columns are severely restricted. In particular, if $\bar{G} = \mathcal{G}_L$, then the partial group is a full group.

Note that since each G_i is of order 2, a partial group must have an even order, and a full group must have an order divisible by 4. Groups of odd order, such as $\Delta(27)$, would

violate the assumption that at least one G_i is a member of the unbroken horizontal group.

The symmetry of the right-handed leptons cannot be determined by this procedure, because mixing has nothing to do with right-handed fermions. Whenever it calls for a knowledge of those, we shall make the minimal assumption that they are invariant under \mathcal{G}_L as well. In other words, if necessary, for simplicity we will implicitly assume $\mathcal{G} = \mathcal{G}_L$ in this paper.

For tribimaximal mixing [1],

$$U = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ -1 & \sqrt{2} & -\sqrt{3} \end{pmatrix} := U_0, \quad (2)$$

the explicit form of G_i using (1) works out to be

$$G_{10} = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix}, \quad G_{20} = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad (3)$$

$$G_{30} = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

To obtain a minimal group, we shall choose for F the simplest nondegenerate matrix, $F_0 = \text{diag}(1, \omega, \omega^2)$, where $\omega = e^{2\pi i/3}$. In that case, straightforward calculation using (3) shows that the full group is $\mathcal{G}_L = \{F_0, G_{10}, G_{20}, G_{30}\} = S_4$, and the three partial groups are $\{F_0, G_{10}\} = S_4$, $\{F_0, G_{20}\} = A_4$, $\{F_0, G_{30}\} = S_3$.

Furthermore, it can be shown that as long as F is nondegenerate, no matter what n is ($F^n = 1$), the full group \mathcal{G}_L always contain S_4 as a subgroup [3].

III. FROM HORIZONTAL GROUP TO MIXING

Given a finite group \mathcal{G}_L containing a nondegenerate element F , we can reverse the argument of the last section to obtain all possible mixings associated with this group. Everything can be carried out in a purely group-theoretical manner, without the presence of Higgs fields, nor the help of a Lagrangian.

In doing so, we may assume \mathcal{G}_L to be a full group, or a partial group. We will refer to the mixing matrices obtained with the former as *full-mixing matrices* of \mathcal{G}_L , and those with the latter as *partial-mixing matrices*. Since partial-mixing matrices are obtained from full-mixing matrices by replacing two columns with arbitrary entries, subject only to the unitarity constraint, there is no need to discuss them separately.

Another reason not to discuss them separately is that a partial-mixing matrix of \mathcal{G}_L is a full-mixing matrix of a larger group \mathcal{G} , as mentioned in the introduction. Hence we need to study only full-mixing matrices.

The general procedure of constructing U from \mathcal{G}_L is as follows. First, separate the order-2 elements in \mathcal{G}_L from those of order ≥ 3 . The former are candidates of G , and the

latter are candidates of F . Next, identify other order-2 elements G' that commute with a G . Then $G'' = GG'$ is an order-2 element that commutes with both G and G' , so the normalized invariant eigenvectors of G , G' , G'' in the F -diagonal basis constitute the three columns of the full-mixing matrix U . Which eigenvector occupies which column is a matter of convention, corresponding to different ways of labeling the three neutrino mass eigenstates, or equivalently different ways of assigning G , G' , G'' to be G_1 , G_2 , and G_3 . Moreover, the entries of the diagonal F may be permuted, resulting in permutations of the rows of U . In other words, the full-mixing matrix U can be determined only up to permutation of rows and columns, because group theory has no way of knowing which flavor neutrino is which, nor which mass neutrino eigenstate should be called 1, 2, or 3. When we present a U below, it is understood that such an ambiguity always exists.

We must check that F, G, G', G'' generate the group \mathcal{G}_L , and that the three normalized invariant eigenvectors in the F -diagonal basis are mutually orthogonal so that U is unitary.

If we cannot find two order-2 elements in \mathcal{G}_L that mutually commute, which would be the case, for example, when the order of \mathcal{G}_L is not divisible by 4, then \mathcal{G}_L contains only partial-mixing matrices. The actual symmetry group $\tilde{\mathcal{G}}$ of the resulting mixing matrix must be larger, with an order divisible by four.

Since Majorana phases are not measurable in neutrino oscillation experiments, we will regard two U 's differing only by Majorana phases to be equivalent. With that we may classify neutrino mixing also by the familiar Chau-Keung parametrization used in quark-mixing, which by convention leaves the (11), (12), (23), and (33) matrix elements of U real, through a suitable choice of the phases of the three columns and the three rows. A mixing matrix parametrized this way can be summarized by the triplet $\langle \sin\theta_{12}, \sin\theta_{23}, \sin\theta_{13}e^{-i\delta} \rangle$, exhibiting the mixing angles θ_{ij} and the CP -phase δ . For TBM, this physical triplet is $\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, 0 \rangle$.

Although straightforward, the task of obtaining all the full-mixing matrices from a given group might seem rather daunting because of the large number of combinatorial choices of F and G 's. In reality it is not as bad because mixings produced by members in the same conjugacy classes are often *equivalent*, i.e., they differ only by row and column phases, or the reshuffling of rows and columns, so at least for relatively small groups \mathcal{G}_L , there are only very few inequivalent mixings.

With such a foresight, we should arrange group elements into conjugacy classes. One way to do so is to make use of the character table, although characters other than those in the defining three-dimensional representation will not be used in this section. To illustrate the procedure, we shall discuss in the rest of this section how to obtain the full-mixing matrices from the groups $\mathcal{G}_L = S_4, A_4$, and A_5 .

These are the rotational symmetry groups of the regular polyhedrons.

The elements of each of these three groups can all be obtained by the repeated multiplication of two members, a and b , which are known as the *generators*. In each case, a obeys $a^2 = 1$ and b obeys $b^3 = 1$. These three groups differ from one another in the behavior of ab , which is of order 3, 4, 5, respectively, for A_4 , S_4 , and A_5 . In short, the *presentation* of these three groups is

$$\begin{aligned} A_4 &= \langle a, b | a^2 = b^3 = (ab)^3 = 1 \rangle, \\ S_4 &= \langle a, b | a^2 = b^3 = (ab)^4 = 1 \rangle, \\ A_5 &= \langle a, b | a^2 = b^3 = (ab)^5 = 1 \rangle. \end{aligned} \quad (4)$$

A. S_4

S_4 is the group of permutation of four objects, whose character table is given in Table I. It contains five classes \mathcal{C}_i ($i = 1, 2, \dots, 5$) with $|\mathcal{C}_i|$ elements, defined by the cycle structure of permutations shown in the second row. The order of the elements in each class can be read off from the cycle structure to be 1, 2, 2, 3, 4, respectively.

There are five irreducible representations, χ^α , with dimensions 1, 1, 2, 3, 3. The two three-dimensional irreducible representations IR4 and IR5 are very closely related. If $g^{3'}$ and g^3 are the representations of $g \in S_4$, respectively, in IR4 and IR5, then $g^3 = g^{3'}$ for $g \in \mathcal{C}_3$ or \mathcal{C}_4 , and $g^3 = -g^{3'}$ for $g \in \mathcal{C}_2$ or \mathcal{C}_5 . The defining three-dimensional representation (the matrices defining F and G_i) is IR5, not IR4, because $\chi^5 = -1$ but $\chi^4 = +1$ for the class \mathcal{C}_2 . Since G is of order 2, it belongs to \mathcal{C}_2 . It also has eigenvalues $+1, -1, -1$, hence the trace of G , which is the character of \mathcal{C}_2 , is -1 and not $+1$, so G must belong to IR5.

The last row lists where the generators a and b belong. From the order of the classes, clearly b has to belong to \mathcal{C}_3 , but *a priori* a could be in \mathcal{C}_2 or \mathcal{C}_3 . \mathcal{C}_3 is excluded because then both a and b are even permutations, so the group they generate is the subgroup A_4 , not the whole S_4 . Hence a must be in \mathcal{C}_2 .

Since the naming of the four objects under permutation is arbitrary, without loss of generality we may take $b = (134)$. To be a generator of S_4 together with b , a must then contain the number 2, so there are three possibilities: $a_1 = (12)$, $a_2 = (23) = ba_1b^2$, and $a_3 = (24) = b^2a_1b$. For each a_i , one can find an a'_i and an a''_i so that members of the triplet $A_i := [a_i, a'_i, a''_i]$ mutually commute and the product of two is the third. The explicit expressions for the three triplets are

$$\begin{aligned} A_1 &= [(12), (34), (12)(34)], \\ A_2 &= [(23), (14), (23)(14)], \\ A_3 &= [(24), (13), (24)(13)]. \end{aligned} \quad (5)$$

Note that $a_i, a'_i \in \mathcal{C}_2$, but $a''_i \in \mathcal{C}_3$.

Next, let us see how to represent the generators as $SU(3)$ matrices, in the basis where $b = (134)$ is diagonal. Let us start off with b . Since $b^3 = 1$ and its character $\text{Tr}(b) = \text{Tr}(C_4)$ vanishes, its diagonal form must be $b = \text{diag}(1, \omega, \omega^2)$ up to possible permutation of the entries, where $\omega = e^{2\pi i/3}$. This is the same as F_0 in TBM. From (3) and the discussion below it, we know that G_{10} and F_0 together generate S_4 , so we may take G_{10} to represent any one of the generators a_i . For definiteness we let it be $a_1 = (12)$.

Please note that notations such as a, b , may stand for the abstract elements defined by the permutation cycles, or matrices in various irreducible representations.

To find the representation of the triplets A_i in the b -diagonal basis, let $B := (a_1b)^2 = (14)(23)$. Then $Ba_1B^{-1} = a'_1$, which works out to be $a'_1 = G_{30}$ of (3). Furthermore, $a''_1 = a_1a'_1 = G_{20}$ of (3). Hence $A_1 = [G_{10}, G_{30}, G_{20}]$. Moreover, it is easy to check that $A_2 = bA_1b^2$ and $A_3 = b^2A_1b$. Hence $A_2 = F_0[G_{10}, G_{30}, G_{20}]F_0^2$ and $A_3 = F_0^2[G_{10}, G_{30}, G_{20}]F_0$.

Any one of the triplets A_i is a possible choice for $\tilde{G} := [G, G', G'']$. As to F , since F is of order ≥ 3 , it is either in \mathcal{C}_4 or \mathcal{C}_5 . Let us consider these two cases separately.

- (1) $F \in \mathcal{C}_4$. In cycle notation, we can write $F = (xyz)$ for some numbers x, y, z chosen among 1, 2, 3, 4. The permutation

$$g = \begin{pmatrix} x & y & z \\ 1 & 3 & 4 \end{pmatrix}$$

brings F into diagonal form because $gFg^{-1} = (134) = b = F_0$. Since F_0 together with any A_i generate S_4 , so does F and $A_i^g := g^{-1}A_i g$. The choice of $\tilde{G} = (G, G', G'')$ would then be any of the three A_i^g .

To compute U , we need to go to the basis where F is diagonal, in which case the representation of \tilde{G} is A_i in the b -diagonal basis. We may therefore assume from the outset that $F = b$ and $\tilde{G} = A_i$ for some i . A similar step should be carried out for the cases to be considered later, but we will often skip that step and assume F directly to be an appropriate generator of the group.

- (a) $\tilde{G} = A_1$. If we let $\tilde{G} = [G_1, G_3, G_2]$, then $G_i = G_{i0}$ and the full-mixing matrix is the TBM matrix U_0 in (2).
- (b) $\tilde{G} = A_2$. If we let $\tilde{G} = [G_1, G_3, G_2]$, then $G_i = F_0 G_{i0} F_0^2$ and the full-mixing matrix is just the TBM matrix U_0 , multiplied, respectively, by 1, ω, ω^2 in the first, second, and third rows. Since row phases are adjustable, this is equivalent to the TBM in (2).
- (c) $\tilde{G} = A_3$. If we let $\tilde{G} = [G_1, G_3, G_2]$, then $G_i = F_0^2 G_{i0} F_0$ and the full-mixing matrix is just the TBM, multiplied, respectively, by 1, ω^2, ω in the first, second, and third rows. Since row phases are adjustable, this is equivalent to the TBM in (2).

TABLE I. Character table of S_4 .

	C_1	C_2	C_3	C_4	C_5
	(1)	(12)	(12)(34)	(123)	(1234)
$ C_i $	1	6	3	8	6
χ^1	1	1	1	1	1
χ^2	1'	-1	1	1	-1
χ^3	2	0	2	-1	0
χ^4	3'	1	-1	0	-1
χ^5	3	-1	-1	0	1
		a		b	ab

(d) In conclusion, as long as $F \in C_4$, the full-mixing matrix is equivalent to the TBM in (2), characterized by the physical parameters $\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, 0 \rangle$.

- (2) $F \in C_5$. Any F in this class obeys $F^4 = 1$, hence its possible eigenvalues are 1, $+i$, $-i$, -1 . Since we want it to be nondegenerate, has determinant $+1$, and since its character according to Table I is real, we conclude that its allowed eigenvalues are 1, i , $-i$, each occurring once and only once. For definiteness, we will let $F = a_1 b = (1342) := c_1$. This is completely general because the four permutation objects in S_4 can be assigned any label we wish to. Once again, G must be chosen so that $\{F, G\}$ generate S_4 . To see how to find such a G , write the presentation of S_4 in (4) in a slightly different form, $S_4 = \langle a, c | a^2 = c^4 = (ac)^3 = 1 \rangle$, which shows that the generator a is of order 2, the generator c of order 4, and ac is of order 3. Trying all 9 possibilities in C_2 and C_3 with $c = c_1 = (1342)$, we see that only the following four a 's, $a_1 = (12)$, $a'_1 = (34)$, $a'_3 = (13)$, and $a_3 = (24)$, can yield an order-3 ac . Thus G is confined to one of these four cases. As a consequence, the allowed triplets are $\tilde{G} = A_1$ and A_3 .

In order to find out what these matrices are in the F -diagonal basis, we first write them in the b -diagonal basis, then convert them by a similarity transformation to the F -diagonal bases.

In the b -diagonal basis,

$$F = a_1 b = G_{10} F_0 = \frac{1}{3} \begin{pmatrix} 1 & -2\omega & -2\omega^2 \\ -2 & -2\omega & \omega^2 \\ -2 & \omega & -2\omega^2 \end{pmatrix}. \quad (6)$$

This F can be diagonalized by the unitary matrix

$$V = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega & -\omega & \omega \\ \omega^2 & -\omega^2/(\sqrt{3}+1) & -\omega^2/(\sqrt{3}-1) \\ 1 & 1/(\sqrt{3}-1) & 1/(\sqrt{3}+1) \end{pmatrix} \quad (7)$$

to yield $V^\dagger F V = \text{diag}(1, i, -i) := F'$.

- (a) $\tilde{G} = A_1$. In the F -diagonal representation, $[G_1, G_2, G_3] := [G, G', G''] = V^\dagger [G_{10}, G_{30}, G_{20}] V$. The full-mixing matrix is then given by

$$U = V^\dagger U'_0 = \begin{pmatrix} -e^{\pi i/3}/\sqrt{2} & -e^{-\pi i/6}/\sqrt{2} & 0 \\ -e^{-5\pi i/12}/2 & -e^{\pi i/12}/2 & e^{\pi i/12}/\sqrt{2} \\ -e^{\pi i/12}/2 & e^{-5\pi i/12}/2 & e^{-5\pi i/12}/\sqrt{2} \end{pmatrix}, \quad (8)$$

where U'_0 is the TBM mixing matrix given in (2) with the second and third columns switched. We do that switching to put U into a more familiar form with a vanishing reactor angle.

Multiplying the first, second, and third columns by $e^{\pi i/6}$, $e^{-\pi i/12}$, $e^{5\pi i/12}$, and the first and second columns by i , -1 , we can turn U into the equivalent form

$$U \rightarrow \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 1/\sqrt{2} \end{pmatrix}, \quad (9)$$

whose mixing angles are given by the physical triplet $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$. In this case, the reactor angle remains to be zero, but both the solar and the atmospheric mixings are maximal.

- (b) $\tilde{G} = A_3$. First, note that $A_3 = [(24), (13), (24)(13)] = c_1 [(34), (12), (34)(12)] c_1^{-1}$, where $c_1 = (1342)$. In the b -diagonal representation, $[a'_3, a_3, a''_3] = c_1 [a_1, a'_1, a''_1] c_1^{-1} = F [G_{10}, G_{30}, G_{20}] F^{-1}$. In the F -diagonal basis, $[G_1, G_2, G_3] := [G', G, G''] = V^\dagger F [G_{10}, G_{30}, G_{20}] F^{-1} V = F' V^\dagger [G_{10}, G_{30}, G_{20}] V F'^{-1}$. The full-mixing matrix is therefore $F' V^\dagger U'_{\text{TBM}} = F' U_5$, with U_5 being the U in (8). Since F' is a diagonal phase matrix, the full-mixing matrix here is equivalent to (8) and (9).

- (c) In conclusion, as long as $F \in C_5$, the full-mixing matrix is equivalent to U of (9), characterized by the physical parameters $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$.

To summarize, up to equivalence, the full-mixing matrix of S_4 is either given by the TBM in (2), characterized by the physical parameters $\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, 0 \rangle$, or the matrix (9), characterized by $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$.

B. A_4

The character table of A_4 is given in Table II. In this case the order-3 elements are divided into two classes, C_3 and C_4 , with the elements in C_4 being the square, or the inverse, of the elements in C_3 . Since the generator b is of order 3, it

TABLE II. Character table of A_4 .

	C_1	C_2	C_3	C_4
	(1)	(12)(34)	(123)	(132)
$ C_i $	1	3	4	4
χ^1	1	1	1	1
χ^2	1'	1	ω	ω^2
χ^3	1''	1	ω^2	ω
χ^4	3	-1	0	0
		a	b	b^2

is either in C_3 or C_4 , which differ only by the arbitrary naming of the four objects. For definiteness we will choose it to be in C_3 . Since its three-dimensional character χ^4 vanishes, up to permutation of the entries, the diagonal form of b is again $b = \text{diag}(1, \omega, \omega^2)$. For definiteness, once again we will let $b = (134)$.

The generator a is of order 2, so it must be in C_2 . To be a generator, it must contain the permutation of the object 2, but then all of them do. Hence there are three possibilities: $a_1 = (12)(34)$, $a_2 = (13)(24)$, $a_3 = (14)(23)$. These three mutually commute, and the product of any two is the third, so they can be taken to be G, G', G'' respectively.

In the diagonal form, a nondegenerate F must be $F = b = \text{diag}(1, \omega, \omega^2) = F_0$, up to permutation of its entries. We must now find out the representation of $A = (a_1, a_2, a_3)$ in this basis.

We know from (3) and the discussions below that equation that F_0 and G_{20} generate A_4 , hence we may take $a_2 = G_{20}$. Then $a_1 = (12)(34) = (134)(13)(24)(143) = ba_1b^2 := G'_1$, and $a_3 = (14)(23) = b^2a_1b := G'_3$. Since the invariant eigenvector of G_2 is proportional to $(1, 1, 1)^T$, and invariant eigenvectors of G'_1 and G'_3 are respectively $F_0(1, 1, 1)^T = (1, \omega, \omega^2)^T$ and $F_0^2(1, 1, 1)^T = (1, \omega^2, \omega)^T$. Consequently, the full-mixing matrix of A_4 is the Cabibbo-Wolfenstein mixing matrix

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix}, \quad (10)$$

whose triplet of physical parameters can be shown to be $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{e^{-i\pi/2}}{\sqrt{3}} \rangle$.

 TABLE III. Character table of A_5 .

	C_1	C_2	C_3	C_4	C_5
	(1)	(12)(34)	(123)	(13524)	(12345)
$ C_i $	1	15	20	12	12
χ^1	1	1	1	1	1
χ^2	4	0	1	-1	-1
χ^3	5	1	-1	0	0
χ^4	3'	-1	0	$-1/\varphi$	φ
χ^5	3	-1	0	φ	$-1/\varphi$
		a	b	ab	(ab^2)

C. A_5

The character table of A_5 is given in Table III, where φ is the golden ratio $(1 + \sqrt{5})/2 = -2 \cos(4\pi/5) = 1.618$, and $1/\varphi = (-1 + \sqrt{5})/2 = \varphi - 1 = 2 \cos(2\pi/5) = 0.618$.

The elements in class C_5 are the square of the elements in C_4 . From the presentation of the group given in (4), we see that the generators a and b must in classes C_2 and C_3 respectively, and ab must in C_4 or C_5 . Which of the two it belongs to simply depends on how we label the five objects in A_5 .

For definiteness, let $b = (134)$ as before. To be a generator of A_5 , a must contain the numbers 2 and 5 in separate cycles, such as $a_{13} = (12)(35)$ or $a'_{13} = (15)(23)$. These two commute and produce a product $a''_{13} = (13)(25)$, which incidentally is not a generator because 2 and 5 are in the same cycle. In addition to $A_{13} = [a_{13}, a'_{13}, a''_{13}]$, there are two other commuting triplets, $A_{14} = [a_{14}, a'_{14}, a''_{14}]$ and $A_{34} = [a_{34}, a'_{34}, a''_{34}]$, with $a_{14} = (12)(45)$, $a'_{14} = (15)(24)$, $a''_{14} = (14)(25)$, and $a_{34} = (32)(45)$, $a'_{34} = (35)(24)$, $a''_{34} = (34)(25)$. In each case, a or a' is a possible generator of A_5 , but not a'' . Any of these three triplets is a possible choice for $\tilde{G} = [G, G', G'']$.

F may be in C_3 or C_4 , but we will only consider the case $F \in C_3$ here.¹ As usual, without loss of generality we may choose $F = b = (134) = \text{diag}(1, \omega, \omega^2) = F_0$ as before. The question is what is the three-dimensional representation for the generator a .

Representation of A_5 can be found in [6], but unfortunately it is not in the b -diagonal basis that we need. Nevertheless, from that representation, one can work out the defining representation a in the b -diagonal basis to be

$$a = \begin{pmatrix} \frac{\sqrt{5}}{3} & -\frac{1}{4} + \frac{1}{12}(\sqrt{5} + i\sqrt{3} + i\sqrt{5}\sqrt{3}) & -\frac{1}{4} + \frac{1}{12}(\sqrt{5} - i\sqrt{3} - i\sqrt{5}\sqrt{3}) \\ -\frac{1}{4} + \frac{1}{12}(\sqrt{5} - i\sqrt{3} - i\sqrt{5}\sqrt{3}) & -\frac{1}{2} - \frac{\sqrt{5}}{6} & \frac{1}{4} + \frac{1}{12}(-2\sqrt{5} - 2i\sqrt{3} + i\sqrt{5}\sqrt{3}) \\ -\frac{1}{4} + \frac{1}{12}(\sqrt{5} + i\sqrt{3} + i\sqrt{5}\sqrt{3}) & \frac{1}{4} + \frac{1}{12}(-2\sqrt{5} + 2i\sqrt{3} - i\sqrt{5}\sqrt{3}) & -\frac{1}{2} - \frac{\sqrt{5}}{6} \end{pmatrix}. \quad (11)$$

One can check that $b^3 = a^2 = 1$ and $(ab)^5 = 1$, so these a and b are indeed generators of A_5 .

¹There have been some model calculations of A_5 [5], which agree with the group-theoretical result given below. I am grateful to Prof. Feruglio for informing me that the result of the case not considered here is also contained in their paper.

This a can be equated to any one of the generators in C_2 . For definiteness, let it be $a = a_{13} = (12)(35)$. We must also find the matrix representation of its commuting partners $a'_{13} := a'$ and $a''_{13} := a''$. They are in the same class as a , so each of them can be written in the form gag^{-1} , for

$$a'' = \frac{1}{8} \begin{pmatrix} -8 & 0 & 0 \\ 0 & 0 & 1 + 3\sqrt{5} + \sqrt{3}i(1 - \sqrt{5}) \\ 0 & 1 + 3\sqrt{5} - \sqrt{3}i(1 - \sqrt{5}) & 0 \end{pmatrix}, \quad (12)$$

and then from $a' = aa''$ that

$$a' = \begin{pmatrix} -\frac{\sqrt{5}}{3} & \frac{1}{4} - \frac{1}{12}(\sqrt{5} + i\sqrt{3} + i\sqrt{5}\sqrt{3}) & \frac{1}{4} - \frac{1}{12}(\sqrt{5} - i\sqrt{3} - i\sqrt{5}\sqrt{3}) \\ \frac{1}{4} - \frac{1}{12}(\sqrt{5} - i\sqrt{3} - i\sqrt{5}\sqrt{3}) & -\frac{1}{2} + \frac{\sqrt{5}}{6} & \frac{1}{4} + \frac{1}{12}(-2\sqrt{5} - 2i\sqrt{3} + i\sqrt{5}\sqrt{3}) \\ \frac{1}{4} - \frac{1}{12}(\sqrt{5} + i\sqrt{3} + i\sqrt{5}\sqrt{3}) & \frac{1}{4} + \frac{1}{12}(-2\sqrt{5} - 2i\sqrt{3} - i\sqrt{5}\sqrt{3}) & -\frac{1}{2} + \frac{\sqrt{5}}{6} \end{pmatrix}. \quad (13)$$

The unitary matrix to diagonalize simultaneously a , a' , a'' so that $U^\dagger a U = \text{diag}(+1, -1, -1)$, $U^\dagger a' U = \text{diag}(-1, +1, -1)$, $U^\dagger a'' U = \text{diag}(-1, -1, +1)$ is

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} \varphi & \varphi^{-1} & 0 \\ -\frac{1}{2}(1 + \omega\varphi)\varphi^{-1} & \frac{1}{2}(1 + \omega\varphi)\varphi & \frac{\sqrt{3}}{2}i(1 + \omega\varphi) \\ \frac{1}{2}(1 + \omega^2\varphi)\varphi^{-1} & -\frac{1}{2}(1 + \omega^2\varphi)\varphi & \frac{\sqrt{3}}{2}i(1 + \omega^2\varphi) \end{pmatrix}. \quad (14)$$

Writing $(1 + \omega\varphi) = (1 + \omega^2\varphi)^* = \sqrt{2}e^{i\xi}$, then multiply the second and the third rows of U by $e^{-i\xi}$ and $-e^{i\xi}$ respectively, and the third column by $-i$, we can turn U into the equivalent form

$$U \rightarrow \frac{1}{\sqrt{3}} \begin{pmatrix} \varphi & \varphi^{-1} & 0 \\ -\frac{1}{\sqrt{2}\varphi} & \frac{\varphi}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}\varphi} & -\frac{\varphi}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \end{pmatrix}, \quad (15)$$

whose physical parameters are given by the triplet $\langle \frac{1}{\sqrt{3}\varphi}, \frac{1}{\sqrt{2}}, 0 \rangle$. This is the full-mixing matrix of A_5

$$X = (ba)^3 = \frac{1}{12} \begin{pmatrix} 4 & 1 - i\sqrt{3} + \sqrt{5}(3 + i\sqrt{3}) & 1 + i\sqrt{3} + \sqrt{5}(3 - i\sqrt{3}) \\ 4(1 - i\sqrt{3}) & 1 + i\sqrt{3} - \sqrt{5}(3 - i\sqrt{3}) & 1 - i\sqrt{5}\sqrt{3} \\ 4(1 + i\sqrt{3}) & 1 + i\sqrt{5}\sqrt{3} & 1 - i\sqrt{3} - \sqrt{5}(3 + i\sqrt{3}) \end{pmatrix}. \quad (16)$$

The result is

$$U = X\tilde{U} = \frac{1}{\sqrt{3}} \begin{pmatrix} \varphi^{-1} & -\varphi & 0 \\ \frac{1}{\sqrt{2}}\varphi e^{-i\eta} & \frac{1}{\sqrt{2}}\varphi^{-1}e^{-i\eta} & -i\frac{\sqrt{3}}{\sqrt{2}}e^{-i\eta} \\ \frac{1}{\sqrt{2}}\varphi e^{i\eta} & \frac{1}{\sqrt{2}}\varphi^{-1}e^{i\eta} & i\frac{\sqrt{3}}{\sqrt{2}}e^{-i\eta} \end{pmatrix}, \quad (17)$$

where η is defined by $\frac{\sqrt{15} \pm i}{12} = \frac{1}{6}e^{\pm i\eta}$.

Multiply the second row by $e^{i\eta}$, the third row by $-e^{-i\eta}$, the second column by -1 , the third column by i , and interchange the first and second columns, we get back to the mixing in (15).

Finally, similar calculation shows that if $(G, G', G'') = (a_{34}, a'_{34}, a''_{34})$, then its full-mixing matrix is also equivalent to (14).

In summary, as long as $F \in C_3$, the full-mixing matrix of A_5 is equivalent to (15), a mixing which is characterized by the triplet physical mixing parameters $\langle \frac{1}{\sqrt{3}\varphi}, \frac{1}{\sqrt{2}}, 0 \rangle$.

some g , which itself is a product of the generators a and b . It can be shown that $g = (ba)^2b^2 = (125)$ and $g^{-1} = b(ba)^3 = (152)$ convert $a = (12)(35)$ into $a'' = (13)(25)$. Using the representation of b and a , one can calculate from (11) to obtain

if $(G, G', G'') = (a_{13}, a'_{13}, a''_{13}) = (a, a', a'') = ((12)(35), (15)(23), (13)(25))$.

Next, consider the other triplet $(G, G', G'') = (a_{14}, a'_{14}, a''_{14}) = ((12)(45), (15)(24), (14)(25))$. Let $g = (ba)^3 = ((134)(12)(35))^3 = (15243)$. Then $(a_{14}, a'_{14}, a''_{14}) = g(a, a'', a')g^{-1}$. Note that the positions of a'' and a' are reversed. The full-mixing matrix for this triplet choice is therefore $U = X\tilde{U}$, where \tilde{U} is the matrix in (14) with the second and third columns interchanged, and

IV. CONNECTION BETWEEN GROUP THEORY AND DYNAMICS

Charged-lepton masses come from Yukawa interactions in the standard model. If neutrino masses arise from the type-I seesaw mechanism, then the Lagrangian responsible for flavor structure of mass matrices can be symbolically written as

$$\mathcal{L} = -c_\gamma \bar{e}_R e_L \phi^\gamma - h'_\gamma \bar{N}_R \nu_L \chi'^\gamma - \frac{1}{2} h_\gamma N_R^T N_R \chi^\gamma + \text{H.c.}, \quad (18)$$

where e_L, e_R are the left-handed and right-handed charged leptons, ν_L, N_R are the active and heavy Majorana neutrinos, and $\phi^\gamma, \chi^\gamma, \chi'^\gamma$ are the flavons. Isotopic spin and spacetime details are omitted, so this Lagrangian should only be used to discuss the flavor structure of mass matrices. In that regard every term is assumed to be invariant under a horizontal symmetry group \mathcal{G} , with $c_\gamma, h_\gamma, h'_\gamma$ being the Yukawa coupling constants. The index γ labels the irreducible representations of \mathcal{G} , and a sum over all γ is understood. The IR of the fermions are not explicitly specified.

For type-II seesaw, we replace the last two terms with a single term, hence

$$\mathcal{L} = -c_\gamma \bar{e}_R e_L \phi^\gamma - h_\gamma \nu_L^T \nu_L \chi^\gamma + \text{H.c.} \quad (19)$$

To obtain the mass matrices from (18) or (19), vacuum expectation values of the flavons are introduced to break the \mathcal{G} -symmetry. In order for the residual symmetries to be preserved after the breaking, F in the charged-lepton sector and G_i in the neutrino sector, the vacuum alignments have to be invariant eigenvectors (eigenvector with eigenvalue +1) of the corresponding residual operators [3],

$$F^{(\gamma)} \langle \phi^\gamma \rangle = \langle \phi^\gamma \rangle, \quad G_i^{(\gamma)} \langle \chi^\gamma \rangle = \langle \chi^\gamma \rangle, \quad G_i^{(\gamma)} \langle \chi'^\gamma \rangle = \langle \chi'^\gamma \rangle, \quad (20)$$

where $F^{(\gamma)}$ and $G_i^{(\gamma)}$ are the representations of F, G_i in $\text{IR}\gamma$. If we want to obtain a full-mixing matrix, then (20) is required to be obeyed for $i = 1, 2, 3$, but if we want to construct a partial-mixing matrix, then the equation has to be satisfied only for that i .

Assuming the invariant eigenvectors to be unique, it follows from (20) that $\langle \chi^\gamma \rangle = \langle \chi'^\gamma \rangle$ up to normalization, so we will set $\chi'^\gamma = \chi^\gamma$ in (18) from now on.

Note that (20) does not rely on the specific form of (18) or (19). All that it requires is the neutrino mass terms to be distinct from the charged-lepton mass terms, and that each has its own valon. (20) would be equally valid if valon fields appear quadratically in the Lagrangian, for example.

Our main goal is to keep the effective left-handed mass matrices \bar{M}_e and \bar{M}_ν invariant under the residual-symmetry. This can be accomplished by keeping the broken Lagrangian invariant, as is done above, but in the case of type-I seesaw, since the Dirac mass matrix M_ν enters quadratically in $\bar{M}_\nu = M_\nu^T M_N^{-1} M_\nu$, where M_N is the Majorana mass matrix of N_R , invariance of \bar{M}_ν can also be preserved *indirectly* [7] if the last term in (20) is replaced by $G_i^{(\beta)} \langle \chi'^\beta \rangle = -\langle \chi'^\beta \rangle$.

Once we know the IR of the residual-symmetry operators, vacuum alignments can be calculated. With those alignments, one can construct many models that automatically give rise to the desired neutrino mixing. Yukawa coupling constants are used to fit the fermion masses, and in the case of partial-mixing, also the remaining mixing parameters. Different models are obtained by choosing

different IR assignments of $e_L, e_R, \nu_L, N_R, \phi$, and χ , though for a successful model we must make sure that enough terms are present in each sector to fit all the masses and the remaining mixing parameters, in spite of the zero vacuum alignments forced upon us in those IR without an invariant eigenvector.

Let us see how this works explicitly. Let $\langle \alpha a, \beta b | \gamma c \rangle$ be the Clebsch-Gordan coefficient of the group \mathcal{G} , coupling orthonormal states $|\alpha a\rangle$ in $\text{IR}\alpha$ with the orthonormal states $|\beta b\rangle$ in $\text{IR}\beta$ to get an orthonormal state $|\gamma c\rangle$ in $\text{IR}\gamma$,

$$|\gamma c\rangle = \sum_{a,b} |\alpha a\rangle |\beta b\rangle \langle \alpha a, \beta b | \gamma c \rangle, \quad (21)$$

$$|\alpha a\rangle |\beta b\rangle = \sum_{\gamma,c} |\gamma c\rangle \langle \gamma c | \alpha a, \beta b \rangle.$$

The CG coefficients must obey the group-invariant condition

$$\begin{aligned} \sum_{c'} \langle \alpha a, \beta b | \gamma c' \rangle g_{c'c}^{(\gamma)} &= \sum_{a',b'} g_{aa'}^{(\alpha)} g_{bb'}^{(\beta)} \langle \alpha a', \beta b' | \gamma g \rangle, \\ \sum_{c'} g_{cc'}^{(\gamma)} \langle \gamma c' | \alpha a, \beta b \rangle &= \sum_{a',b'} \langle \gamma c | \alpha a', \beta b' \rangle g_{a'a}^{(\alpha)} g_{b'b}^{(\beta)}, \end{aligned} \quad (22)$$

for every $g \in \mathcal{G}$, or at least for every generator of \mathcal{G} . It is important to note that if we change the IR bases of g , the CG coefficients may have to be changed as well. In the case of S_4 , the CG coefficients to be used will be those that correspond to the IR given in Table IV.

Suppose e_R and e_L belong to $\text{IR}\alpha$ and $\text{IR}\beta$, respectively, and the flavons belong to $\text{IR}\gamma$. Then the charged-lepton mass matrix can be read out from either (18) or (19) to be

$$(M_e)_{ab} = \sum_{\gamma,c} c_\gamma \langle \alpha^* a, \beta b | \gamma c \rangle \langle \phi_c^\gamma \rangle. \quad (23)$$

The invariance of the charged-lepton mass matrices M_e and $\bar{M}_e := M_e^\dagger M_e$ under the residual-symmetry F can be seen from (20) and (22)

$$F^{(\alpha)\dagger} M_e F^{(\beta)} = M_e, \quad F^{(\beta)\dagger} \bar{M}_e F^{(\beta)} = \bar{M}_e. \quad (24)$$

Similarly, if N_R, ν_L belong to $\text{IR}\alpha$ and $\text{IR}\beta$, respectively, then the neutrino mass matrices are

TABLE IV. Irreducible representations and invariant eigenvectors of the first set of S_4 residual-symmetry operators.

IR	1	1'	2	3'	3
F	1	1	$\text{diag}(\omega, \omega^2)$	$\text{diag}(1, \omega, \omega^2)$	$\text{diag}(1, \omega, \omega^2)$
$\langle \phi \rangle$	1	1	(0, 0)	(1, 0, 0)	(1, 0, 0)
G_1	1	-1	σ_1	$-G_{10}$	G_{10}
G_2	1	1	$\text{diag}(1, 1)$	G_{20}	G_{20}
G_3	1	-1	σ_1	$-G_{30}$	G_{30}
$\langle \chi \rangle$	1	0	(1, 1)	(1, 1, 1)	(0, 0, 0)

$$\begin{aligned}
(M_\nu)_{ab} &= \sum_{\gamma,c} h'_\gamma \langle \alpha a, \beta b | \gamma c \rangle \langle \chi_c^\gamma \rangle, \\
(M_N)_{aa'} &= \sum_{\gamma,c} h_\gamma \langle \alpha a, \alpha a' | \gamma c \rangle_S \langle \chi_c^\gamma \rangle, \\
\bar{M}_\nu &= M_\nu^T M_N^{-1} M_\nu,
\end{aligned} \tag{25}$$

where $\langle \alpha a, \alpha a' | \gamma c \rangle_S$ is the CG coefficient symmetric in a and a' . Once again the invariance relations

$$\begin{aligned}
G_i^{(\alpha)\dagger} M_\nu G_i^{(\beta)} &= M_\nu, & G_i^{(\alpha)\dagger} M_N G_i^{(\alpha)*} &= M_N, \\
G_i^{(\beta)T} \bar{M}_\nu G_i^{(\beta)} &= \bar{M}_\nu
\end{aligned} \tag{26}$$

follow. Note that because of the Majorana nature of the neutrino, we have assumed $G_i^{(\alpha,\beta)}$ to be real. For indirect models, the first equation in (26) should have a minus sign.

In the rest of this section, we will illustrate these operations by considering the direct full-mixing models of the groups $\mathcal{G} = A_4$ and S_4 . In the case of A_4 , partial-mixing models are also discussed because they can give rise to the phenomenologically interesting tribimaximal mixing.

The first task in each case is to find out the IR of the residual-symmetry operators.

A. S_4

Remember from Sec. IV that there are two full-mixing modes, one giving rise to the TBM, and the other yielding a mixing with too large a solar angle. We shall consider how dynamical models can be constructed in each case, starting with those giving rise to the TBM.

- (1) $F = F_0, [G_1, G_2, G_3] = [G_{10}, G_{20}, G_{30}]$

Table IV shows the IR and invariant eigenvectors of the residual-symmetry operators in the basis where F is diagonal.

The one-dimensional IR $\mathbf{1}, \mathbf{1}'$ in Table IV are obtained directly from Table I because in that case the group character is the value of the IR. The three-dimensional representations $\mathbf{3}, \mathbf{3}'$ are copied from those given in the previous section. For $\mathbf{2}$, since $F^3 = 1$ and its character from Table I is $\text{Tr}(\mathcal{C}_4) = -1$, its diagonal form must be $\text{diag}(\omega, \omega^2)$. The other generator G_1 must not be diagonal, or else the two-dimensional representation is not irreducible. Since $G_1^2 = 1$, within equivalence we may choose it to be the Pauli matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

With these IR of the residual-symmetry operators, it is straightforward to work out the invariant eigenvector $\langle \phi \rangle$ of F , and the simultaneous invariant eigenvector $\langle \chi \rangle$ of G_1, G_2 , and G_3 . The column vectors are displayed in the table as row vectors for printing convenience. Some of the vacuum alignments are zero either because F^α does not have $+1$ eigenvalue, or because there is no common eigenvector of $G_1^\alpha, G_2^\alpha, G_3^\alpha$ all with eigenvalue $+1$.

With the help of Tables I and IV, many direct models possessing TBM can be constructed. These models differ from one another in their IR assignments for the right-handed fermions and flavons.

We saw in the last section that the residual-symmetry operators for S_4 and A_4 are in IR $\mathbf{3}$, hence the left-handed fermions must transform as $\mathbf{3}$. One can then work out from the character table the allowed IR of the flavons for every assignment of the right-handed fermion. The result is listed in row 2 of Table V. Since $\langle \phi \rangle = 0$ for IR $\mathbf{2}$ and $\langle \chi \rangle = 0$ for IR $\mathbf{3}$, neither of them can be used to construct models. Taking that into account, the final number of free parameters available in model construction is listed in the third row for the charged-lepton sector, the fourth row for the neutrino sector if it is given by a type-II seesaw, and the fifth row for the neutrino sector given by a type-I seesaw. Those in the neutrino sector require a little bit of explanation.

For type-II seesaw, ν_L is coupled to itself so only $\mathbf{3} \times \mathbf{3}$ coupling is relevant. For type-I seesaw, the heavy Majorana neutrino N_R can be assigned to any IR. The first of the two numbers in the last row refers to the number of free parameters in the Dirac neutrino mass term. Once the assignment α of N_R is determined, the number of parameters in the Majorana mass terms is determined by $\alpha \times \alpha$, and that appears as the second number in the last row. If we consider indirect models [7], then the number of parameters is even larger than that.

Let us see what Table V is telling us. First look at the charged-lepton sector, in row 3. Since the number of free parameters always happens to match the dimension of the representation, there are always just enough of them to fit the three charged-lepton masses.

For type-II seesaw, there is again exactly three free parameters to fit the three active neutrino masses. For type-I seesaw, the number of free parameters shown is the sum of the M_ν and M_N parameters, which is generally more than enough to fit the active neutrino masses. The remaining ones can be used, for example, to tune the properties of the heavy neutrinos.

As a concrete illustration, we will write down the mass matrices of a type-I model where e_L, e_R, ν_L, N_R

TABLE V. Number of free parameters in S_4 models with a TBM mixing.

RHF	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{2}$	$\mathbf{3}'$	$\mathbf{3}$
Flavons	$\mathbf{3}$	$\mathbf{3}'$	$\mathbf{3}', \mathbf{3}$	$\mathbf{1}', \mathbf{2}, \mathbf{3}', \mathbf{3}$	$\mathbf{1}, \mathbf{2}, \mathbf{3}', \mathbf{3}$
e parameter	1	1	2	3	3
Type-II ν	-	-	-	-	3
Type-I ν	0 + 1	1 + 1	1 + 3	3 + 3	3 + 3

all belong to **3**. It is the kind of assignment that a $SO(10)$ grand unified theory would want. In that case, according to Tables IV and V, M_e , M_ν , and M_N should contain exactly three parameters each. Using the CG coefficients in the F_0 -diagonal basis, these mass matrices can be worked out to be $M_e = \text{diag}(\bar{c}_1 + 2\bar{c}_{3'}, \bar{c}_1 - \bar{c}_{3'} + \bar{c}_3, \bar{c}_1 - \bar{c}_{3'} - \bar{c}_3)$, where $\bar{c}_1 = c_1/\sqrt{3}$, $\bar{c}_{3'} = c_{3'}/3\sqrt{2}$, and $\bar{c}_3 = c_3/\sqrt{6}$. Furthermore,

$$M_N = \begin{pmatrix} \bar{h}_1 + 2\bar{h}_{3'} & \bar{h}_2 - \bar{h}_{3'} & \bar{h}_2 - \bar{h}_{3'} \\ \bar{h}_2 - \bar{h}_{3'} & \bar{h}_2 - \bar{h}_{3'} & \bar{h}_1 + \bar{h}_{3'} \\ \bar{h}_2 - \bar{h}_{3'} & \bar{h}_1 + \bar{h}_{3'} & \bar{h}_2 - \bar{h}_{3'} \end{pmatrix}, \quad (27)$$

where $\bar{h}_1 = h_1/\sqrt{3}$, $\bar{h}_{3'} = h_{3'}/3\sqrt{2}$, and $\bar{h}_2 = h_2/\sqrt{6}$. The expression for M_ν is identical except that h_α is to be replaced by h'_α .

This example illustrates some general points discussed above:

- (a) Each of M_e , M_ν , M_N is specified by three independent parameters, as shown in Table V above.
- (b) The three parameters in M_e can be fitted by the three charged-lepton masses.
- (c) There are six parameters to specify $\bar{M}_\nu = M_\nu^T M_N^{-1} M_\nu$. In addition to fitting the three active neutrino masses, we can use the remaining three to specify the heavy N_R masses.
- (d) M_e is automatically diagonal. This is because we are using CG coefficients in the F_0 -diagonal (hence \bar{M}_e -diagonal) basis. As remarked below Eq. (22), if we use some other basis, then the CG coefficient will be rotated and the resulting M_e may no longer be diagonal.
- (e) Whatever the parameters are, both M_ν and M_N turn out to be magic (all row sums are equal) and 2–3 symmetric (invariant under a simultaneous exchange of the second and third columns, and second and third rows). Hence $\bar{M}_\nu = M_\nu^T M_N^{-1} M_\nu$ is also magic and 2–3 symmetric, and the mixing matrix U is automatically TBM [8].

- (2) $F = G_{10}F_0$, $[G_1, G_2, G_3] = [G_{10}, G_{20}, G_{30}]$

This is the other allowed full-mixing of S_4 , given in (6)–(9). Unlike the TBM mixing, whose models abound in the literature, this one is seldom discussed presumably because its phenomenology is less attractive. Another reason might be that the appropriate vacuum alignments in this case are a bit strange, as we shall see. Nevertheless, we can follow the general procedure to construct models that lead to this kind of mixing.

There are two different ways to proceed. Either we go directly to the F -diagonal basis, as prescribed before, or we stay in the F_0 -diagonal basis that we are familiar with. The former has the advantage of having \bar{M}_e automatically diagonal, but the CG

TABLE VI. Irreducible representations and invariant eigenvectors of the second set of S_4 residual-symmetry operators.

IR	1	1'	2	3'	3
F	1	-1	$\sigma_1 \text{diag}(\omega, \omega^2)$	$-G_{10}F_0$	$G_{10}F_0$
$\langle\phi\rangle$	1	0	(1, ω)	(0, 0, 0)	($\omega, \omega^2, 1$)
G_1	1	-1	σ_1	$-G_{10}$	G_{10}
G_2	1	1	$\text{diag}(1, 1)$	G_{20}	G_{20}
G_3	1	-1	σ_1	$-G_{30}$	G_{30}
$\langle\chi\rangle$	1	0	(1, 1)	(1, 1, 1)	(0, 0, 0)

coefficients in the F -diagonal basis must be worked out anew. If we travel the latter route, the CG coefficients are the same ones used in (27), but we have to work out the new alignments, and have to perform the diagonalization of \bar{M}_e afterward. To illustrate better the connection between the group-theoretical and the dynamical approaches, we choose to work with the latter.

In the b -diagonal basis, the IR of G_i is the same as in Table IV, and the IR of F can be obtained from Table IV by calculating G_1F . The resulting IR as well as the invariant eigenvectors are shown in Table VI below. Since the IR for G_i remains unchanged, that part of the table is identical to Table VI. The only difference comes in the upper part of the table in which F is replaced by G_1F in every IR.

The parameter count for this kind of model is shown in Table VII. The only variation from Table V comes from the different vanishing $\langle\phi\rangle$'s, now occurring in **1'** and **3'** instead of **2**.

As a result, in order to have enough parameters to fit the charged-lepton masses, e_R must now be in **3**, not **3'**, nor a combination of **1** and **2**. Other than that, things are pretty much the same as before except that M_e is now no longer diagonal. To see the difference between this case and TBM, let us illustrate again with the model in which e_L , e_R , ν_L , N_R all belong to **3**. In that case, using the F_0 -diagonal CG coefficient as before, but now a different $\langle\phi\rangle$, we get

$$M_e = \begin{pmatrix} \bar{c}_1 & \omega\bar{c}_2 + \bar{c}_3/\sqrt{3} & \bar{c}_2 - \omega^2\bar{c}_3/\sqrt{3} \\ \bar{c}_2 + \omega^2\bar{c}_3/\sqrt{3} & \bar{c}_1 - \omega\bar{c}_3/\sqrt{3} & \omega\bar{c}_2 \\ \omega\bar{c}_2 - \bar{c}_3/\sqrt{3} & \bar{c}_2 & \bar{c}_1 + \omega\bar{c}_3/\sqrt{3} \end{pmatrix}, \quad (28)$$

where $\bar{c}_1 = c_1/\sqrt{3}$, $\bar{c}_2 = c_2/\sqrt{6}$, and $\bar{c}_3 = c_3/\sqrt{2}$. Since the $\langle\chi\rangle$'s are the same as in Table IV, the neutrino mass

TABLE VII. Number of free parameters for the other S_4 model.

RHF	1	1'	2	3'	3
Flavons	3	3'	3', 3	1', 2, 3', 3	1, 2, 3', 3
e parameter	1	0	1	2	3
Type-II ν					3
Type-I ν	0 + 1	1 + 1	1 + 3	3 + 3	3 + 3

matrices M_ν and M_N are identical to those given in (27), which are diagonalized by the TBM matrix U_0 of (2). If for convenience we switch the second and the third columns, then it is U'_0 as explained below (8). This U'_0 however is not the mixing matrix because M_e is not diagonal.

Since M_e commutes with F , which is nondegenerate, the matrix V in (7) that diagonalized F must also diagonalize M_e . Indeed,

$$V^\dagger M_e V = \begin{pmatrix} \bar{c}_1 + 2\omega^2 \bar{c}_2 & & & \\ & \bar{c}_1 - \omega^2 \bar{c}_2 + \omega \bar{c}_5 & & \\ & & & \\ & & & \bar{c}_1 - \omega^2 \bar{c}_2 - \omega \bar{c}_5 \end{pmatrix}, \quad (29)$$

which gives rise to the charged-lepton mass-squared matrix $V^\dagger \bar{M}_e V = \text{diag}(m_e^2, m_\mu^2, m_\tau^2)$, with

$$m_e^2 = (\bar{c}_1 - \bar{c}_2)^2 + 3\bar{c}_2^2, \quad (30)$$

$$m_{\mu,\tau}^2 = \bar{c}_1^2 + \bar{c}_2^2 + \bar{c}_5^2 + \bar{c}_1 \bar{c}_2 \mp (\bar{c}_1 - \bar{c}_2) \bar{c}_5.$$

The neutrino mixing matrix is now given by $U = V^\dagger U'_0$, whose explicit result is already given in (8) and (9).

B. A_4

We shall discuss for this group both the full-mixing and the partial-mixing models. For full-mixing, $\langle \chi \rangle$ is the simultaneous invariant eigenvector of G_1 , G_2 , and G_3 , and is denoted by $\langle \chi_{all} \rangle$. For partial-mixing defined by G_i , $\langle \chi \rangle$ is the invariant eigenvector of that G_i , denoted as $\langle \chi_i \rangle$. Table VIII exhibits the IR of the residual operators and the vacuum alignments.

The parameter count for these models is given in Table IX.

We see once again that no matter how e_R is assigned, there are always just enough parameters to fit the charged-lepton masses. For type-II neutrinos, after using three to fit the neutrino masses, there is one left over to fit the remaining mixing. It is this extra parameter that can turn a partial-mixing mode of A_4 into a TBM, thereby promoting

TABLE VIII. Irreducible representations and invariant eigenvectors of A_4 .

IR	1	1'	1''	3
F	1	ω	ω^2	$\text{diag}(1, \omega, \omega^2 = F_0)$
$\langle \phi \rangle$	1	0	0	(1, 0, 0)
G_1	1	1	1	$F_0 G_{20} F_0^2 := G'_{10}$
$\langle \chi_1 \rangle$	1	1	1	(1, ω , ω^2)
G_2	1	1	1	G_{20}
$\langle \chi_2 \rangle$	1	1	1	(1, 1, 1)
G_3	1	1	1	$F_0^2 G_{20} F_0 := G'_{30}$
$\langle \chi_3 \rangle$	1	1	1	(1, ω^2 , ω)
$\langle \chi_{all} \rangle$	1	1	1	(0, 0, 0)

TABLE IX. Number of free parameters for the full-mixing and partial-mixing A_4 model.

RHF	1	1'	1''	3
Flavons	3	3	3	1, 1', 1'', 3_S, 3_A
e parameter	1	1	1	3
Type-II ν				
G_1				4
G_2				4
G_3				4
<i>all</i>				3
Type-I ν				
G_1	1 + 1	1 + 1	1 + 1	5 + 4
G_2	1 + 1	1 + 1	1 + 1	5 + 4
G_3	1 + 1	1 + 1	1 + 1	5 + 4
<i>all</i>	0 + 0	0 + 0	0 + 0	3 + 3

the A_4 symmetry to a S_4 symmetry. In the case of full-mixing, listed under “all”, the mixing matrix is completely determined to be the Cabibbo-Wolfenstein matrix, so the number of parameters is just enough to fit the neutrino masses. For type-I mixing, as before, there are more parameters than necessary to fit the active neutrino masses and the remaining mixing if in the partial-mixing mode, so the extra ones can be applied to tune the properties in the heavy neutrino sector.

V. HIGGS POTENTIAL OF A_4

We established in Secs. II and III a connection between the mixing matrix and horizontal symmetry, purely on symmetry grounds, without the presence of Higgs fields nor a Lagrangian. We also saw in Sec. IV that if flavons are introduced in a Lagrangian, then their vacuum alignments must be invariant eigenvectors of the residual-symmetry operators for the two formalisms to agree. In a dynamical model, vacuum alignments are obtained by minimizing a Higgs potential, so for consistency we must find a suitable Higgs potential to yield just those alignments.

A completely generic Higgs potential contains so many \mathcal{G} -invariant terms that just about any vacuum alignment can be obtained by adjusting the self-coupling parameters. In order to confine the solutions to invariant eigenvectors, a rationale must be found to chop down the number of terms in the potential. The usual way to accomplish that is to impose an additional symmetry on top of \mathcal{G} .

Take $\mathcal{G} = A_4$ for example. It contains four irreducible representations, **1**, **1'**, **1''**, **3**. Since the only nontrivial alignment occurs in **3**, let us consider a renormalizable Higgs potential $V(\psi, \psi^*)$ of the complex triplet flavon field $\psi = (a_1, a_2, a_3)^T$. It turns out that if V contains only $\psi^* \psi$ and $\psi^* \psi^* \psi \psi$ terms, without terms like $\psi \psi$, $\psi^* \psi \psi$, $\psi^* \psi \psi \psi$, etc., then its minimization would yield just the invariant eigenvectors of A_4 . Such a potential can be arrived at by assigning ψ a nontrivial $U(1)$ quantum number, ψ^* the opposite quantum number, and imposing an additional $U(1)$ symmetry on V . A suitable $Z(N)$ would do as well.

In addition to showing that the solution to its equations of motion are invariant eigenvectors, we shall also show that alignments for A_4 members of the same class have the same Higgs energy. These general properties remain valid if A_4 is replaced by any simply reducible group \mathcal{G} , as we will see in the next section.

Such a potential has been considered before [9]. For the quartic terms, if we choose to couple the two ψ 's first, and separately the two ψ^* 's, then we obtain four terms depending on whether $\psi\psi$ are coupled to $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, or $\mathbf{3}$. Using \mathcal{P}^α to denote the projection operator for irreducible representation α , the most general V is

$$V = \langle \psi\psi | f_1 \mathcal{P}^1 + f_{1'} \mathcal{P}^{1'} + f_{1''} \mathcal{P}^{1''} + f_3 \mathcal{P}^3 | \psi\psi \rangle - \mu^2 \langle \psi | \psi \rangle, \quad (31)$$

where f_α are the coupling constants.

With the help of explicit A_4 CG coefficients given in Appendix , V can be written as

$$V = \frac{1}{3}(f_1 |y_1|^2 + f_{1'} |y_{1'}|^2 + f_{1''} |y_{1''}|^2 + 2f_3(k^2 - |y_3|^2)) - \mu^2 k, \quad (32)$$

where

$$\begin{aligned} y_1 &= a_1^2 + 2a_2 a_3, & y_{1'} &= a_3^2 + 2a_1 a_2, \\ y_{1''} &= a_2^2 + 2a_1 a_3, & y_3 &= a_1 a_2^* + a_2 a_3^* + a_3 a_1^*, \\ k &= |a_1|^2 + |a_2|^2 + |a_3|^2. \end{aligned} \quad (33)$$

The equations of motion are obtained by setting $\partial V / \partial a_i = 0$ and $\partial V / \partial a_i^* = 0$. In this way, we obtain the following equation and its complex conjugate:

$$\begin{aligned} 0 &= f_1 y_1 \begin{pmatrix} a_1^* \\ a_3^* \\ a_2^* \end{pmatrix} + f_{1'} y_{1'} \begin{pmatrix} a_2^* \\ a_1^* \\ a_3^* \end{pmatrix} + f_{1''} y_{1''} \begin{pmatrix} a_3^* \\ a_2^* \\ a_1^* \end{pmatrix} \\ &+ f_3 \left[2k \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - y_3 \begin{pmatrix} a_2 \\ a_3 \\ a_1 \end{pmatrix} - y_3^* \begin{pmatrix} a_3 \\ a_1 \\ a_2 \end{pmatrix} \right] \\ &- \frac{3}{2} \mu^2 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}. \end{aligned} \quad (34)$$

It is much easier to solve this cubic equation of a_i by first converting it to a quadratic equation in y_a and k . To accomplish that, we multiply (34) on the left, respectively, by (a_1^*, a_2^*, a_3^*) , (a_3^*, a_1^*, a_2^*) , (a_2^*, a_3^*, a_1^*) to get

$$f_1 |y_1|^2 + f_{1'} |y_{1'}|^2 + f_{1''} |y_{1''}|^2 + 2f_3(k^2 - |y_3|^2) - (3/2)\mu^2 k = 0, \quad (35)$$

$$f_1 y_1 y_{1''}^* + f_{1'} y_{1'} y_1^* + f_{1''} y_{1''} y_{1'}^* + f_3(k y_3^* - y_3^2) - (3/2)\mu^2 y_3^* = 0, \quad (36)$$

$$f_1 y_1 y_{1'}^* + f_{1'} y_{1'} y_1^* + f_{1''} y_{1''} y_1^* + f_3(k y_3 - y_3^{*2}) - (3/2)\mu^2 y_3 = 0. \quad (37)$$

Substituting (35) into (32), we can simplify the Higgs energy expression to

$$V = -\frac{1}{2}\mu^2 k. \quad (38)$$

Subtracting the complex conjugate of (37) from (36), and asking the equality to be true for all f_α , we obtain

$$y_1 y_{1''}^* = y_1^* y_{1'} = y_{1'}^* y_{1''}. \quad (39)$$

Let us now proceed to solve (39).

- (1) If $y_1 = 0$, then $y_{1'}$ or $y_{1''}$ must vanish. More generally, if one of the y_s ($s = 1, 1', 1''$) vanishes, then one or both of the other two must also vanish. In this way, we obtain four possible solutions: $y := [y_1, y_{1'}, y_{1''}] \propto [0, 0, 0], [1, 0, 0], [0, 1, 0], [0, 0, 1]$.
- (2) If none of the y_s vanish, then it follows from (39) that $|y_1| = |y_{1'}| = |y_{1''}|$. Dividing (39) by $y_1 y_{1'}^*$, we get $(y_{1''}/y_1)^* = (y_{1'}/y_1) = (y_{1''}/y_1)(y_{1'}/y_1)^*$. Hence $(y_{1''}/y_1)^3 = 1$, $(y_{1'}/y_1)^3 = 1$, and $(y_{1''}/y_1) = (y_{1'}/y_1)^2$. The solutions are therefore $y = y_1[1, 1, 1], y_1[1, \omega, \omega^2]$, and $y_1[1, \omega^2, \omega]$.

Substituting these solutions of y_s into (33), we can obtain the solutions of $\psi = (a_1, a_2, a_3)$. Because of the $U(1)$ invariance, the solutions below can always be multiplied by an arbitrary phase factor of unit modulus.

- (1)
 - (a) Suppose $y_{1'} = y_{1''} = 0$.
 - (i) One obvious solution is $a_2 = a_3 = 0$. In that case, $a_1 = 0$ if and only if $y_1 = 0$. The solution $\psi = (0, 0, 0)$ gives rise to $k = y_3 = 0$, and a Higgs energy $V = 0$ from (38). Obviously it also satisfies the equation of motion (34). The solution $\psi = \sqrt{k}(1, 0, 0)$ gives rise to $y_3 = 0$ and $y_1 = k$. It satisfies (34) and (35) if $k = 3\mu^2/2(f_1 + 2f_3)$. The Higgs energy is then given by (38) to be $V = -3\mu^4/2(f_1 + 2f_3)$.
 - (ii) There is also a solution with $a_2 \neq 0$. In that case, $y_{1'} = y_{1''} = 0$ implies $a_1 = -a_2^2/2a_2$ and $a_2^2 = a_3^3/a_2$. The latter yields $a_3 = a_2 \omega^m$ for $m = 0, 1$, or 2 , and the former demands $a_1 = -a_2 \omega^{2m}/2$. From (33) we now obtain $y_1 = 9a_2^2 \omega^m/4$, $k = 9|a_2|^2/4$, and $y_3 = 0$. We can now check that (34) is satisfied provided $k = 3\mu^2/2(f_1 + 2f_3)$, so the Higgs energy $V = -3\mu^4/2(f_1 + 2f_3)$ is degenerate with that of the $\psi = \sqrt{k}(1, 0, 0)$ solution. This solution (actually three solutions corresponding to $m = 0, 1, 2$) can then be written as $\psi = (\sqrt{k}/3)(1, -2\omega^m, -2\omega^{2m})$,

or more explicitly, $\psi = (\sqrt{k}/3)(1, -2, -2)$, $(\sqrt{k}/3)(1, -2\omega, -2\omega^2)$, $(\sqrt{k}/3)(1, -2\omega^2, -2\omega)$.

- (b) The solutions for $y_1 = y_{1''} = 0$, $y_{1'} \neq 0$ can be similarly obtained. It leads to the solutions $\psi = \sqrt{k}(0, 1, 0)$, $(\sqrt{k}/3)(-2, 1, -2)$, $(\sqrt{k}/3)(-2\omega, 1, -2\omega^2)$, and $(\sqrt{k}/3)(-2\omega^2, 1, -2\omega)$. The values for k and V are respectively $k = 3\mu^2/2(f_{1'} + 2f_3)$ and $V = -3\mu^4/2(f_{1'} + 2f_3)$.
- (c) The solutions for $y_1 = y_{1'} = 0$, $y_{1''} = 0$ are $\psi = \sqrt{k}(0, 0, 1)$, $(\sqrt{k}/3)(-2, -2, 1)$, $(\sqrt{k}/3)(-2\omega, -2\omega^2, 1)$, $(\sqrt{k}/3)(-2\omega^2, -2\omega, 1)$. The values for k and V are respectively $k = 3\mu^2/2(f_{1''} + 2f_3)$ and $V = -3\mu^4/2(f_{1''} + 2f_3)$.
- (d) In short, the solutions of (b) and (c) are obtained from (a) with appropriate permutations.

(2)

- (a) $y = y_1[1, 1, 1]$. From (33), we need to have $a_1^2 + 2a_2a_3 = a_3^2 + 2a_1a_2 = a_2^2 + 2a_1a_3$. The first equality leads to $(a_1 - a_3)(a_1 + a_3 - 2a_2) = 0$, and the second equality leads to $(a_2 - a_3)(a_2 + a_3 - 2a_1) = 0$. The solution is $a_1 = a_2 = a_3$, hence $\psi = (\sqrt{k}/3)(1, 1, 1)$. Using (33), we get $y_1 = y_{1'} = y_{1''} = y_3 = k$.

We can check that (34) is satisfied provided $k = 3\mu^2/2(f_1 + f_{1'} + f_{1''})$. The Higgs energy is then $V = -3\mu^4/4(f_1 + f_{1'} + f_{1''})$.

- (b) $y = y_1[1, \omega, \omega^2]$. From (33), we need to have $a_1^2 + 2a_2a_3 = \omega^2(a_2^2 + 2a_1a_2) = \omega(a_2^2 + 2a_1a_3)$. The first equality leads to $(a_1 - \omega a_3) \times (a_1 + \omega a_3 - 2\omega^2 a_2) = 0$, and the second equality leads to $(a_2 - \omega^2 a_3)(a_2 + \omega^2 a_3 - 2\omega a_1) = 0$. The solution is $a_1 = \omega^2 a_2 = \omega a_3$, hence $\psi = (\sqrt{k}/3)(1, \omega, \omega^2)$. Using (33), we get $y_1 = \omega^2 y_{1'} = \omega y_{1''} = y_3 = k$.

We can check that (34) is satisfied provided $k = 3\mu^2/2(f_1 + f_{1'} + f_{1''})$. The Higgs energy is then $V = -3\mu^4/4(f_1 + f_{1'} + f_{1''})$.

- (c) $y = y_1[1, \omega^2, \omega]$. The solution is $\psi = (\sqrt{k}/3) \times (1, \omega^2, \omega)$, with $k = 3\mu^2/2(f_1 + f_{1'} + f_{1''})$. The Higgs energy is $t V = -3\mu^4/4(f_1 + f_{1'} + f_{1''})$.

(3) In summary, the solutions can be classified according to their energy into five categories:

- (a) $V = 0$. In this case $\psi = (0, 0, 0)$.
- (b) $V = -3\mu^4/4(f_1 + 2f_3)$. In this case $\psi = \sqrt{k}(1, 0, 0)$, $(\sqrt{k}/3)(1, -2, -2)$, $(\sqrt{k}/3) \times (1, -2\omega, -2\omega^2)$, or $(\sqrt{k}/3)(1, -2\omega^2, -2\omega)$, with $k = 3\mu^2/2(f_1 + f_3)$.
- (c) $V = -3\mu^4/4(f_{1'} + 2f_3)$. In this case $\psi = \sqrt{k}(0, 1, 0)$, $(\sqrt{k}/3)(-2, 1, -2)$, $(\sqrt{k}/3) \times (-2\omega, 1, -2\omega^2)$, or $(\sqrt{k}/3)(-2\omega^2, 1, -2\omega)$, with $k = 3\mu^2/2(f_{1'} + f_3)$.
- (d) $V = -3\mu^4/4(f_{1''} + 2f_3)$. In this case $\psi = \sqrt{k}(0, 0, 1)$, $(\sqrt{k}/3)(-2, -2, 1)$, $(\sqrt{k}/3) \times$

$(-2\omega, -2\omega^2, 1)$, or $(\sqrt{k}/3)(-2\omega^2, -2\omega, 1)$, with $k = 3\mu^2/2(f_{1''} + f_3)$.

- (e) $V = -3\mu^3/4(f_1 + f_{1'} + f_{1''})$. In this case $\psi = (\sqrt{k}/3)(1, 1, 1)$, $(\sqrt{k}/3)(1, \omega, \omega^2)$, or $(\sqrt{k}/3) \times (1, \omega^2, \omega)$, with $k = 3\mu^2/2(f_1 + f_{1'} + f_{1''})$.

In particular, if $f_1, f_{1'}, f_{1''}, f_3$ are all positive, energetics dictates that the solution $\psi = (0, 0, 0)$ is disfavored. This is the solution that leads to the A_4 full-mixing matrix of Cabibbo and Wolfenstein, rather than the experimentally correct mixing of the tribimaximal type, obtainable using the first solution of (b) in the charged-lepton sector, and the first solution of (e) in the neutrino sector.

Moreover, if $f_1 < f_{1'}$ and $f_{1''}$, then it is $(1, 0, 0)$, etc. that is energetically favored, and not $(0, 1, 0)$, $(0, 0, 1)$, etc.

In this way, we see that there is a rational why A_4 might prefer partial-mixing, which may lead to a tribimaximal matrix, rather than a full-mixing that leads to a Cabibbo-Wolfenstein matrix.

We proceed now to discuss the connection between these vacuum alignment solutions and the residual-symmetry operators. Table X shows the vacuum alignment solutions above, and the operator for which the alignment is an invariant eigenvector. F, G'_1, G_2, G'_3 are the generators of A_4 in Table VIII, but with the subscript 0 omitted.

Here are some remarks concerning Table X.

- (1) The first seven solutions in the left column are the invariant eigenvectors of the 11 operators in the right column, each of them responsible for a partial-mixing. Together with the identity operator, these 12 operators together form the A_4 group.
- (2) The last solution is the simultaneous eigenvector of G'_1, G_2 , and G'_3 relevant for full-mixing.
- (3) Solutions under 3(c) and 3(d) are not listed in the table because they are obtained from 3(b) by permutations. They are the invariant eigenvectors of the same A_4 members expressed in a similarly permuted basis.
- (4) The first four solutions 3(b) have the same energy, and the corresponding operators are all in class \mathcal{C}_3 or \mathcal{C}_4 of Table II. The next three solutions 3(e) also have the same energy, and the corresponding

TABLE X. Dynamical alignments are invariant eigenvectors of appropriate operators.

Alignment	operator
$(1, 0, 0)$	F, F^2
$(1, -2, -2)$	$FG_2F, F^2G_2F^2$
$(1, -2\omega, -2\omega^2)$	F^2G_2, G_2F
$(1, -2\omega^2, -2\omega)$	G_2F^2, FG_2
$(1, 1, 1)$	G_2
$(1, \omega, \omega^2)$	$G'_1 = F^2G_2F$
$(1, \omega^2, \omega)$	$G'_3 = FG_2F^2$
$(0, 0, 0)$	(G'_1, G_2, G'_3)

operators are all in class C_2 . In other words, vacuum alignments for operators in the same class always have the same energy.

VI. GROUP THEORY AND DYNAMICS

We saw in the last section that, barring permutations that serve only to relabel the neutrino mass eigenstates, every minimizing solution of the scalar potential V is an invariant eigenvector of some group element of A_4 . Moreover, conjugate group elements give rise to identical energy.

We shall present another derivation of these facts in the first subsection below. This alternative derivation relies more on the general property of groups, and is therefore more amenable to generalization. The generalization will be carried out in the second subsection.

The following simple fact will be used in this alternative approach.

Lemma:—Let g^α be the α irreducible representation of $g \in \mathcal{G}$. Let ϕ^α be an eigenvector of g^α with eigenvalue λ^α . Let $\Phi^\rho := (\phi^{\alpha_1} \phi^{\alpha_2} \cdots \phi^{\alpha_n})^\rho$ denote a state in the irreducible representation ρ obtained by coupling the n states ϕ^{α_j} . Then either $\Phi^\rho = 0$, or it is an eigenvector of g^ρ with eigenvalue $\Lambda = \prod_{j=1}^n \lambda^{\alpha_j}$.

Proof:—This follows immediately from the fact that $g^\rho(\prod_i \phi^{\alpha_i})^\rho = (\prod_i (g^{\alpha_i} \phi^{\alpha_i}))^\rho$.

The following is an immediate corollary of lemma 1.

A. Another derivation of the solution of $U(1) \times A_4$

An alternative derivation of the result of the last section is presented here. It has the advantage of being easily generalizable from A_4 to a general group \mathcal{G} .

The main idea is the following. Start as before from the Higgs potential $V = \sum_{\alpha, f, \bar{\alpha}} \langle \psi \psi | \mathcal{P}^{\bar{\alpha}} | \psi \psi \rangle - \mu^2 \langle \psi | \psi \rangle$. Its minimization is determined by six vectorial equations of motion, $\partial V / \partial \psi_c = 0$ and $\partial V / \partial \psi_c^* = 0$. The last three is the complex conjugate of the first three so they need not be considered separately. We will show, using the lemma proven above, that if ψ is an invariant eigenvector of an element of A_4 , then every nonzero term in the three equations of motion is proportional to ψ , with a scalar coefficient common to the three equations. ψ is then a solution of the equations of motion if we set this scalar coefficient to be zero, a condition that would be used to determine the normalization of the solution ψ . Once ψ is known, its energy can be computed from the above expression of V .

Here are the details. The idea is simple but unfortunately the notations are a bit cumbersome.

A_4 has four irreducible representations, $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, $\mathbf{3}$. Other than the antisymmetric coupling $3 \times 3 \rightarrow 3_A$ which does not appear in (31), and will therefore be forgotten from now on, each irreducible representation $\bar{\alpha}$ occurs at most once in the decomposition of $\beta \times \gamma$. The Clebsch-Gordan coefficients $\langle \bar{\alpha} \bar{\alpha} | \beta \beta, \gamma \gamma \rangle$ for $\beta \times \gamma \rightarrow \bar{\alpha}$ can be expressed in terms of the $3j$ -symbol,

$$\langle \bar{\alpha} \bar{\alpha} | \beta \beta, \gamma \gamma \rangle = \sqrt{[\alpha]} \begin{pmatrix} \alpha & \beta & \gamma \\ a & b & c \end{pmatrix},$$

where $[\alpha]$ is the dimension of the irreducible representation α , and the $3j$ -symbol is completely symmetric in its columns. The state $|\bar{\alpha} \bar{\alpha}\rangle$ is the complex-conjugated state of $|\alpha \alpha\rangle$. See Appendix for further details.

Consider a typical quartic term in the real Higgs potential $V = \sum_{\alpha, f, \bar{\alpha}} \langle \psi \psi | \mathcal{P}^{\bar{\alpha}} | \psi \psi \rangle - \mu^2 \langle \psi | \psi \rangle$,

$$\begin{aligned} Q^\alpha &:= \langle \psi \psi | \mathcal{P}^{\bar{\alpha}} | \psi \psi \rangle \\ &= [\alpha] \sum_{a, b, c, e, f} \langle \psi_e | \langle \psi_f | \begin{pmatrix} 3 & 3 & \alpha \\ e & f & a \end{pmatrix}^* \begin{pmatrix} \alpha & 3 & 3 \\ a & b & c \end{pmatrix} | \psi_b \rangle | \psi_c \rangle. \end{aligned} \quad (40)$$

Its contribution to the equation of motion is

$$\begin{aligned} \frac{\partial Q^\alpha}{\partial \psi_c} &:= Q_c^\alpha \\ &= 2[\alpha] \sum_{a, b, c, e, f} \langle \psi_e | \langle \psi_f | \begin{pmatrix} 3 & 3 & \alpha \\ e & f & a \end{pmatrix}^* \begin{pmatrix} \alpha & 3 & 3 \\ a & b & c \end{pmatrix} | \psi_b \rangle, \end{aligned} \quad (41)$$

and its complex conjugate $\partial Q^\alpha / \partial \psi_c^*$. In these formulas, $|\psi_b\rangle$ stands for the b th component of a three-dimensional complex column vector ψ , and $\langle \psi_e |$ the complex conjugate of $|\psi_e\rangle$.

We will use g^ρ to denote IR_ρ of a $g \in A_4$. Suppose g^α has an orthonormal set of eigenvectors $\phi^{\alpha m}$ ($m = 1, 2, \dots, [\alpha]$), with eigenvalues λ_m , then $\delta_{ad} = \sum_m \phi_a^{\alpha m*} \phi_d^{\alpha m}$, so (40) and (41) can be rewritten as

$$Q^\alpha = [\alpha] \sum_m Y^{\alpha m*} Y^{\alpha m}, \quad (42)$$

$$Q_c^\alpha = 2[\alpha] \sum_m Y^{\alpha m*} \Phi_c^{\alpha m} := 2[\alpha] \sum_m Q_c^{\alpha m}, \quad (43)$$

where

$$Y^{\alpha m} = \sum_{abc} \begin{pmatrix} \alpha & 3 & 3 \\ a & b & c \end{pmatrix} \phi_a^{\alpha m} \psi_b \psi_c, \quad (44)$$

$$\Phi_c^{\alpha m} = \sum_{ab} \begin{pmatrix} \alpha & 3 & 3 \\ a & b & c \end{pmatrix} \phi_a^{\alpha m} \psi_b. \quad (45)$$

It follows from the symmetry of the $3j$ -symbol that $Y^{\alpha m}$ transforms as $\mathbf{1}$, and $\Phi^{\alpha m}$ transforms as $\mathbf{3}$.

We can now start working toward the goal of showing that invariant eigenvectors of g are solutions to the equations of motion derived from V .

Let ψ be an invariant eigenvector of g , i.e., $g^3 \psi = \psi$. Since $g^\alpha \phi^{\alpha m} = \lambda_m \phi^{\alpha m}$, according to the lemma, either $Y^{\alpha m} = 0$, or $g^1 Y^{\alpha m} = \lambda_m Y^{\alpha m}$. However, since $g^1 = 1$ for every g in the identity representation $\mathbf{1}$, we must have $\lambda_m = 1$ in order for $Y^{\alpha m} \neq 0$.

The lemma also implies that either $\Phi^{\alpha m} = 0$ or $g^3 \Phi^{\alpha m} = \lambda_m \Phi^{\alpha m}$. Consequently, if $Q_c^{\alpha m} \neq 0$, meaning both $Y^{\alpha m}$ and $\Phi^{\alpha m}$ nonzero, then it is necessary for $\Phi^{\alpha m}$ to be an invariant eigenvector of g^3 . We know from Table VIII that every g^3 has one and only one invariant eigenvector, hence $\Phi^{\alpha m} = \kappa^\alpha \psi$ for some $\kappa^\alpha \neq 0$, if $Q_c^{\alpha m} \neq 0$.

The importance of this observation is the following. There are six equations of motion, namely, the following three ($c = 1, 2, 3$), and their three complex conjugates:

$$\begin{aligned} & \sum_{\alpha=11'1''3} 2[\alpha] f_\alpha \sum_m Q_c^{\alpha m} - \mu^2 \psi_c \\ &= \sum_{\alpha=11'1''3} 2[\alpha] f_\alpha \sum_m Y^{\alpha m*} \Phi_c^{\alpha m} - \mu^2 \psi_c = 0. \end{aligned} \quad (46)$$

From the observation above, we see that if ψ is an invariant eigenvector of g^3 , then every nonzero term in (46) is proportional to ψ , hence *all* the six equations of motion are satisfied provided the single scalar equation

$$\sum_{\alpha=11'1''3} 2[\alpha] f_\alpha Y^{\alpha m*} \kappa^\alpha - \mu^2 = 0 \quad (47)$$

is obeyed. Note that the single m in the equation is that one for which $\Phi^{\alpha m}$ is the invariant eigenvector of g^3 . Since every quantity in this equation is known, except the overall normalization of ψ , this then is an equation to determine the normalization of ψ . For that reason we shall sometimes refer to (47) as the *normalization equation*.

Knowing that ψ is an invariant eigenvector of g with known normalization computed in (47), we can substitute this solution into the expression of V to compute its energy.

To summarize, we have shown that the invariant eigenvector ψ of every group element g of A_4 is a dynamical vacuum alignment, satisfying the equation of motion derived from the Higgs potential V . We will now show that conjugate elements in A_4 give rise to the same energy V .

Let $g' = hgh^{-1}$ be an element conjugate to g . If $g^3 \psi = \psi$, then $g'^3 \psi' = \psi'$ for $\psi' = h^3 \psi$. Similarly, $g'^{\alpha} \phi'^{\alpha m} = \lambda_m \phi'^{\alpha m}$ if $g^\alpha \phi^{\alpha m} = \lambda_m \phi^{\alpha m}$ and $\phi'^{\alpha m} = h^\alpha \phi^{\alpha m}$. If we denote the quantities in (44) and (45) built up of the primed quantities by $Y'^{\alpha m}$ and $\Phi_c'^{\alpha m}$, respectively, then $Y'^{\alpha m} = Y^{\alpha m}$ and $\Phi_c'^{\alpha m} = h^3 \Phi_c^{\alpha m}$. Furthermore, if $\Phi^{\alpha m} = \kappa^\alpha \psi$, then $\Phi_c'^{\alpha m} = \kappa^\alpha \psi'$, with the same κ^α . For that reason, the normalization equation (47) is identical for the primed as the unprimed quantities, so ψ and ψ' have the same normalization, thus both solutions give rise to the same energy V .

The eigenvalues of G_i^3 are $+1, -1, -1$, and the eigenvalues of all other g^ρ can be obtained from Table VIII. Using this information, we can make a closer comparison between the present derivation and the explicit calculation in Sec. V.

If g is an order-3 element, then $g^\alpha \neq 1$ in the $\alpha = 1', 1''$ representations, hence $Y^{\alpha m} = 0$ for $\alpha = 1', 1''$. This corresponds to the $y_{1'} = y_{1''} = 0$ solutions in the previous

section. Namely, the first four rows of Table X, or solution 3(b) before that table.

If g is order-2, for the singlet representations $\alpha = 1, 1', 1''$, every $g^\alpha = +1$, hence according to the lemma, none of these three $Y^{\alpha m}$ (there is only one m) have to vanish, and we can take $\phi^{\alpha m} = 1$ in all cases. This corresponds to the solutions with $|y_1| = |y_{1'}| = |y_{1''}| = 0$ in the last section, given by 3(e) and the next three rows of Table X.

B. Generalization to other groups \mathcal{G}

The alternative derivation for A_4 discussed above is so general that it can be easily adapted to many other groups \mathcal{G} . For technical simplicity we shall confine ourselves here to the simply reducible groups. The validity of the theorem below certainly goes beyond the simply reducible groups, as A_4 itself is not simply reducible. For those groups, the same idea should still be applicable, though generalization and some modification in details may be necessary.

A *simply reducible group* is a group in which (i) g and g^{-1} are always in the same class, and (ii) in the decomposition of the Kronecker product $\beta \times \gamma$ of any two irreducible representations β and γ , every irreducible representation $\bar{\alpha}$ occurs at most once. A consequence of (i) is that all characters have to be real. A_4 is not simply reducible because it contains complex characters, but groups like $SU(2)$, $SO(3)$, S_3 , S_4 , and the quaternion group, are simply reducible.

The main property we need from a simply reducible group is the existence of a symmetric or antisymmetric $3j$ -symbol. As before, if $\langle \bar{\alpha} \bar{\alpha} | \beta b, \gamma c \rangle$ is the Clebsch-Gordan coefficient, then the $3j$ -symbol is defined by

$$\langle \bar{\alpha} \bar{\alpha} | \beta b, \gamma c \rangle = \sqrt{[\alpha]} \begin{pmatrix} \alpha & \beta & \gamma \\ a & b & c \end{pmatrix} \quad (48)$$

and is either symmetric or antisymmetric upon the interchange of columns. Here $(\bar{\alpha} \bar{\alpha})$ is the complex-conjugate representation of $(\alpha \alpha)$; whether the $3j$ -symbol is symmetric or antisymmetric is a detail that does not concern us.

Most of the arguments used in A_4 can now be carried through to prove this theorem.

Theorem.—Let ψ be a complex variable vector in the ρ irreducible representation of a simply reducible group \mathcal{G} , carrying a nontrivial $U(1)$ quantum number opposite to that of ψ^* . Writing the most general $(U(1) \times \mathcal{G})$ -invariant renormalizable real Higgs potential as

$$V = \sum_\alpha f_\alpha \langle \psi \psi | \mathcal{P}^\alpha | \psi \psi \rangle - \mu^2 \langle \psi | \psi \rangle, \quad (49)$$

in which the sum is carried over all irreducible representations, with \mathcal{P}^α being the projection operator and f_α the Yukawa coupling constant of the α representation. Then every *unique* invariant eigenvector ψ of g^ρ for every $g \in \mathcal{G}$ is a solution of the equation of motion $\partial V / \partial \psi_c = \partial V / \partial \psi_c^* = 0$. Moreover, g 's in the same conjugacy class have the same Higgs energy V .

Proof:—The proof is essentially a copy of the procedure discussed in the last subsection. The only thing we are not sure without knowing more details about \mathcal{G} is whether every g^ρ has one and only one eigenvector with eigenvalue $+1$. This is why the theorem is limited to those g^ρ with *unique* invariant eigenvectors. The only other difference is that the $3j$ -symbol may be antisymmetric under a column exchange rather than symmetric. The only place this symmetry is used was in (44) and (45) to show that they transform as **1** and **3**, respectively. Clearly an additional minus sign under column exchange will not affect this property. Finally, whether ψ belongs to **3** or some other multidimensional irreducible representation ρ really does not matter in the proof.

With these preliminary remarks, let us proceed with the details.

Consider a typical quartic term in the real Higgs potential V ,

$$\begin{aligned} Q^\alpha &:= \langle \psi \psi | \mathcal{P}^{\bar{\alpha}} | \psi \psi \rangle \\ &= [\alpha] \sum_{a,b,c,e,f} \langle \psi_e | \langle \psi_f | \begin{pmatrix} \rho & \rho & \alpha \\ e & f & a \end{pmatrix}^* \begin{pmatrix} \alpha & \rho & \rho \\ a & b & c \end{pmatrix} | \psi_b \rangle | \psi_c \rangle. \end{aligned} \quad (50)$$

Its contribution to the equation of motion is given by

$$\begin{aligned} \frac{\partial Q^\alpha}{\partial \psi_c} &:= Q_c^\alpha \\ &= 2[\alpha] \sum_{a,b,c,e,f} \langle \psi_e | \langle \psi_f | \begin{pmatrix} \rho & \rho & \alpha \\ e & f & a \end{pmatrix}^* \begin{pmatrix} \alpha & \rho & \rho \\ a & b & c \end{pmatrix} | \psi_b \rangle, \end{aligned} \quad (51)$$

together with its complex conjugate $\partial Q^\alpha / \partial \psi_c^*$.

Suppose g^α has an orthonormal set of eigenvectors $\phi^{\alpha m}$ ($m = 1, 2, \dots, [\alpha]$), with eigenvalues λ_m . Then $\delta_{ad} = \sum_m \phi_a^{\alpha m*} \phi_d^{\alpha m}$, so (50) and (51) can be rewritten as

$$Q^\alpha = [\alpha] \sum_m Y^{\alpha m*} Y^{\alpha m}, \quad (52)$$

$$Q_c^\alpha = 2[\alpha] \sum_m Y^{\alpha m*} \Phi_c^{\alpha m} := 2[\alpha] \sum_m Q_c^{\alpha m}, \quad (53)$$

where

$$Y^{\alpha m} = \sum_{abc} \begin{pmatrix} \alpha & \rho & \rho \\ a & b & c \end{pmatrix} \phi_a^{\alpha m} \psi_b \psi_c, \quad (54)$$

$$\Phi_c^{\alpha m} = \sum_{ab} \begin{pmatrix} \alpha & \rho & \rho \\ a & b & c \end{pmatrix} \phi_a^{\alpha m} \psi_b. \quad (55)$$

It follows from the symmetry of the $3j$ -symbol that $Y^{\alpha m}$ transforms as **1**, and $\Phi^{\alpha m}$ transforms as ρ .

Suppose ψ is an invariant eigenvector of g , i.e., $g^\rho \psi = \psi$, and suppose this is the only eigenvector of g^ρ with eigenvalue $+1$. Since $g^\alpha \phi^{\alpha m} = \lambda_m \phi^{\alpha m}$, according to the

lemma, either $Y^{\alpha m} = 0$ or $g^\rho Y^{\alpha m} = \lambda_m Y^{\alpha m}$. However, since $g^\rho = 1$ for every g in the identity representation **1**, we must have $\lambda_m = 1$ in order for $Y^{\alpha m} \neq 0$.

The lemma also implies that either $\Phi^{\alpha m} = 0$ or $g^\rho \Phi^{\alpha m} = \lambda_m \Phi^{\alpha m}$. Consequently, if $Q_c^{\alpha m} \neq 0$, it is necessary for $\Phi^{\alpha m}$ to be an invariant eigenvector of g^ρ . Since this is assumed to be unique, it follows that $\Phi^{\alpha m} = \kappa^\alpha \psi$ for some κ^α if $Q_c^{\alpha m} \neq 0$.

Let us look at the equation of motion,

$$\begin{aligned} \sum_\alpha 2[\alpha] f_\alpha \sum_m Q_c^{\alpha m} - \mu^2 \psi_c \\ = \sum_\alpha 2[\alpha] f_\alpha \sum_m Y^{\alpha m*} \Phi_c^{\alpha m} - \mu^2 \psi_c = 0. \end{aligned} \quad (56)$$

From the observation above, we see that if ψ is an invariant eigenvector of g^ρ , then every nonzero term in (56) is proportional to ψ , hence equations of motion are satisfied provided the single scalar equation

$$\sum_\alpha 2[\alpha] f_\alpha Y^{\alpha m*} \kappa^\alpha - \mu^2 = 0 \quad (57)$$

is obeyed. Note that the single m in the equation is that one for which $\Phi^{\alpha m}$ is the invariant eigenvector of g^ρ . Since every quantity in this equation is known, except the overall normalization of ψ , this then is an equation to determine the normalization of ψ . For that reason we shall sometimes refer to (57) as the *normalization equation*.

This proves the first part of the theorem. What remains is to show that conjugate elements in \mathcal{G} give rise to the same energy V .

Let $g' = hgh^{-1}$ be an element conjugate to g . If $g^\rho \psi = \psi$, then $g'^\rho \psi' = \psi'$ for $\psi' = h^\rho \psi$. Similarly, $g'^{\alpha} \phi'^{\alpha m} = \lambda_m \phi'^{\alpha m}$ if $g^\alpha \phi^{\alpha m} = \lambda_m \phi^{\alpha m}$ and $\phi'^{\alpha m} = h^\alpha \phi^{\alpha m}$. If we denote the quantities in (54) and (55) built up of the primed quantities by $Y'^{\alpha m}$ and $\Phi_c'^{\alpha m}$, respectively, then $Y'^{\alpha m} = Y^{\alpha m}$ and $\Phi_c'^{\alpha m} = h^\rho \Phi_c^{\alpha m}$. Furthermore, if $\Phi^{\alpha m} = \kappa^\alpha \psi$, then $\Phi_c'^{\alpha m} = \kappa^\alpha \psi'$, with the same κ^α . For that reason, the normalization equation (47) is identical for the primed as the unprimed quantities, so ψ and ψ' have the same normalization, thus both solutions give rise to the same energy V .

VII. CONCLUSION

We have discussed in some detail the group-theoretical connection between a neutrino mixing matrix and the horizontal symmetry of left-handed leptons. Horizontal symmetry groups can be derived from mixing matrices, and mixing matrices can be obtained from horizontal groups, all without invoking the Higgs or other dynamical mechanisms, as long as we assume some trace of the original symmetry remains to be found in the mass matrices even after symmetry breaking. We have also shown that this formulation is completely consistent with the dynamical approach, in that the mixing matrices derived from a symmetry group \mathcal{G} can be obtained from a generic Higgs

potential invariant under $U(1) \times \mathcal{G}$, for many groups \mathcal{G} . Such a potential gives rise to vacuum alignments that are invariant eigenvectors of the group elements. An alignment should be assigned to the charged-lepton sector if the corresponding group element has order ≥ 3 , but to the neutrino sector if the corresponding group element has order 2. Within this rule, alignments can be further discriminated by their energies.

Other than Sec. IV, in which some S_4 and A_4 dynamical models are constructed to illustrate the consistency of the two approaches, we have not studied in detail in this article how physical models should be constructed. The models constructed in Sec. IV may not be optimal, in that mass hierarchy is not natural in those models, and the successful Koide mass relation of the charged leptons [10] is not explained. More complicated dynamics including the presence of quadratic valon fields may be necessary. Furthermore, although the $U(1) \times \mathcal{G}$ potential can give rise to all the desired vacuum alignments, and although their assignments to the two sectors can be discriminated by the order of the associated residual-symmetry operator, we have not found a purely dynamical way to do so without invoking the residual-symmetry operators explicitly. That may call for the introduction of driver valons [11] or other dynamical mechanisms.

APPENDIX

In this appendix, we list the Clebsch-Gordan (CG) series of A_4 and its CG coefficients computed in the representation of Table VIII.

The CG series $\beta \times \gamma \rightarrow \bar{\alpha}$ can be computed from Table II to be where the first row refers to $\text{IR}\gamma$, the first column refers to $\text{IR}\beta$, and the table entries refer to $\text{IR}\bar{\alpha}$. The CG coefficients $\langle \bar{\alpha} \bar{a} | \beta b, \gamma c \rangle$ are zero if $(\bar{\alpha}, \beta, \gamma)$ is not contained in an entry of the table. It is also zero if the product of F -eigenvalues of $|\beta b\rangle$ and $|\gamma c\rangle$ is not equal to that of $|\bar{\alpha} \bar{a}\rangle$. This last condition is a consequence of Eq. (22) applied to $g = F$. If we use A, B, C to denote the three states in 1, $1', 1''$, and D, E, G to denote the three states in 3, then the F -eigenvalue of A and D is 1, that of B and E is ω , and that of C and G is ω^2 . The eigenvalue rule is not shown in Table XI, but that is incorporated in the expanded Table XII.

The complex conjugate $|\bar{\alpha} \bar{a}\rangle$ of a state $|\alpha a\rangle$ is defined to be the state with the complex-conjugated F -eigenvalue. Thus $\bar{X} = X$ if $X = A, D$, but $\bar{B} = C, \bar{C} = B, \bar{E} = G, \bar{G} = E$.

TABLE XI. Clebsch-Gordan series of A_4 .

	1	1'	1''	3
1	1	1'	1''	3
1'	1'	1''	1	3
1''	1''	1	1'	3
3	3	3	3	1, 1', 1'', 3 _S , 3 _A

TABLE XII. Expanded Clebsch-Gordan series of A_4 .

	A	B	C	D	E	G
A	A	B	C	D	E	G
B	B	C	A	E	G	D
C	C	A	B	G	D	E
D	D	E	G	A, D _S	B, E _S , E _A	C, G _S , G _A
E	E	G	D	B, E _S , E _A	C, G _S	A, D _S , D _A
G	G	D	E	C, G _S , G _A	A, D _S , D _A	B, E _S

We will also write the CG coefficient simply as $\langle \bar{X} | Y, Z \rangle$ to represent the more complicated $\langle \bar{\alpha} \bar{a} | \beta b, \gamma c \rangle$.

The CG coefficients are related to the $3j$ -symbols via the equation

$$\langle \bar{\alpha} \bar{a} | \beta b, \gamma c \rangle = \sqrt{[\alpha]} \begin{pmatrix} \alpha & \beta & \gamma \\ a & b & c \end{pmatrix}, \tag{A1}$$

$$\langle \bar{X} | Y, Z \rangle = \sqrt{[X]} (X \ Y \ Z),$$

where $[\alpha]$ is the dimension of $\text{IR}\alpha$, and $[X]$ is the dimension of the IR in which state X belongs to. The $3j$ -symbol couples three states to a singlet; it vanishes unless the product of the three F -eigenvalues is 1.

There are two $3j$ -symbols when $\alpha = \beta = \gamma = 3$. The one corresponding to $\bar{\alpha} = 3_A$ will be marked with a subscript A , and the one corresponding to $\bar{\alpha} = 3_S$ will carry no subscript.

$3j$ -symbols are more convenient because of their symmetry properties. The $3j$ -symbol

$$(X \ Y \ Z)_A$$

is antisymmetric upon the interchange of two columns, and all the other $3j$ -symbols are symmetric upon such an exchange.

The nonzero $3j$ -symbol

$$(X \ Y \ Z)$$

is equal to 1 when X, Y, Z all belong to one-dimensional IRs, is equal to $1/\sqrt{3}$ if two of them belong to the three-dimensional IR, and the third one belongs to a one-dimensional IR. Finally, when all three of them belong to the three-dimensional IR, then their values are

$$(D \ E \ G)_A = -\frac{1}{\sqrt{2}}, \quad (D \ E \ G) = -\frac{1}{\sqrt{6}}, \tag{A2}$$

$$(D \ D \ D) = (E \ E \ E) = (G \ G \ G) = \sqrt{\frac{2}{3}}.$$

It can be checked that the CG coefficients obey Eq. (22) for $g = G_2$ as well.

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