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On modular properties of the AdS_3 CFT

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We study modular properties of the AdS_3 Wess-Zumino-Novikov-Witten model. Although the Euclidean partition function is modular invariant, the characters on the Euclidean torus diverge and the regularization proposed in the literature removes information on the spectrum and the usual one to one map between characters and representations of rational models is lost. Reconsidering the characters defined on the Lorentzian torus and focusing on their structure as distributions, we obtain expressions that recover those properties. We study their modular transformations and find a generalized S matrix, depending on the sign of the real modular parameters, which has two diagonal blocks and one off-diagonal block, mixing discrete and continuous representations, that we fully determine. We then explore the relations among the modular transformations, the fusion algebra and the boundary states. We explicitly construct Ishibashi states for the maximally symmetric D-branes and show that the generalized S matrix defines the one-point functions associated to pointlike and H_2-branes as well as the fusion rules of the degenerate representations of $sl(2,\mathbb{R})$ appearing in the open string spectrum of the pointlike D-branes, through a generalized Verlinde theorem.

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I. INTRODUCTION

The formulation of a consistent string theory on AdS_3 is an active area of research since more than two decades ago. Besides allowing to understand important aspects of strings propagating on nontrivial backgrounds (see [1–3] and references therein), this theory offers a controlled setting where it is possible to verify the AdS/CFT correspondence beyond the supergravity approximation as well as to grasp several features of a non rational conformal field theory (RCFT) with Lie algebra symmetry.

The world-sheet theory describing strings on Lorentzian AdS_3 is a Wess-Zumino-Novikov-Witten (WZNW) model on the universal cover of the $SL(2,\mathbb{R})$ group manifold. The spectrum proposed in [1] was verified in [2] through the computation of the one-loop partition function on a Euclidean AdS_3 background at finite temperature. Some correlation functions were determined in [3] and the fusion rules establishing the closure of the Hilbert space and the unitarity of the full interacting string theory were obtained in our previous work [4]. We showed that the spectral flow symmetry of the model requires a truncation of the operator algebra whose physical origin has not been elucidated yet. Although they satisfy several essential properties, the full consistency of the fusion rules should follow from a proof of factorization and crossing symmetry of the four-point functions, still unavailable. The correlators that have been analyzed in the literature so far are based on the analytic continuation from those of the better understood Euclidean version of the theory, the $H_3^+ \equiv \frac{SL(2,\mathbb{C})}{SU(2)}$ WZNW model [5,6]. But there are many subtleties in the relation between the Euclidean and Lorentzian models [7] and further work is necessary to put the fusion rules on a firmer ground.

In RCFT, a practical derivation of the fusion rules can be performed through the Verlinde theorem [8], often formulated as the statement that the S matrix of modular transformations diagonalizes the fusion rules. Moreover, besides leading to a Verlinde formula, the S matrix allows a classification of modular invariants and a systematic study of boundary states. It would be interesting to explore whether analogues of these properties can be found in the AdS_3 WZNW model. However, the relations among fusion algebra, boundary states and modular transformations are difficult to identify and have not been very convenient in noncompact models [9]. In general, the characters have an intricate behavior under the modular group [10–12] and, as is often the case in theories with discrete and continuous representations, these mix under S transformations.

In this paper we study the modular properties of the AdS_3 model. We start considering the characters of the relevant representations. Since the standard Euclidean characters diverge and lack good modular properties, extended characters were originally introduced in [13] (see also [14]).[1] A different approach was followed in [1] where the standard characters were computed on the Lorentzian torus and it was shown that the modular invariant partition function of the H_3^+ model obtained in [18] is recovered after performing analytic continuation and discarding contact terms. However, this trivial regularization removes information on the spectrum and the usual one to one map between characters and representations of rational

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[1]Similar problems in noncompact coset models have also been considered in [15–17]

models is lost. With the partial aim of overcoming these problems, in Sec. II we review (and redefine) the characters on the Lorentzian torus, focusing on their structure as distributions. We also consider the characters of degenerate representations of $sl(2,\mathbb{R})$, because they appear in the boundary spectrum of pointlike D-brane solutions.

Then we study their modular transformations. One could wonder about the meaning of modular transformations on a Lorentzian world sheet. Of course there is no reason to expect modular invariance of the Lorentzian partition function. However, as mentioned above, modular transformations in RCFT are intimately related to microscopic data (fusion rules, one-point functions, etc.) and these powerful relations are difficult to establish in non rational models. Given that the Euclidean characters and their modular transformations are ill defined, in Sec. III we examine the modular properties of the Lorentzian expressions with the purpose of determining the scope of those connections in the AdS_3 model. We find generalized modular maps which play an important role in the microscopic description of the theory. Real modular parameters are crucial to obtain an S matrix which, unlike those of the Euclidean models, depends on the sign of the modulus. We completely determine this generalized S matrix, which has two diagonal blocks and one off-diagonal block mixing the characters of discrete and continuous representations.

In order to explore the properties of this modular matrix, in Sec. IV we consider the maximally symmetric D-branes of the model. We explicitly construct the Ishibashi states and show that the coefficients of the boundary states turn out to be determined from the generalized S matrix, suggesting that a Verlinde-like formula could give some information on the spectrum of open strings attached to certain D-branes. Furthermore, we show in Appendix C that a generalized Verlinde formula reproduces the fusion rules of the finite dimensional degenerate representations of $sl(2,\mathbb{R})$ appearing in the boundary spectrum of the pointlike D-branes.

Conclusions are offered in Sec. V, where we compare our results with previous ones in the literature and we also draw some directions for future work.

For the benefit of the reader, we include four appendices. In Appendix A we discuss the properties of the moduli space of the Lorentzian torus. Some details of the calculations leading to the generalized S matrix are presented in Appendix B. A generalized Verlinde formula giving the fusion rules of the degenerate representations is worked out in Appendix C. Finally, in Appendix D we review the results of the one-point functions for maximally symmetric D-branes obtained in [19] and translate them to our conventions, in order to compare with the expressions obtained in the main body of the text.

II. CHARACTERS ON THE LORENTZIAN TORUS

The partition function of the AdS_3 WZNW model was computed on the Lorentzian torus in [1] because it diverges on the Euclidean signature torus, and it was shown that a modular invariant expression is obtained after analytic continuation of the modular parameters.[2] In this section we rederive the characters of the relevant representations and stress some important issues related to the regions of convergence of the expressions involved, focussing on their structure as distributions.

The spectrum of the AdS_3 WZNW model was determined in [1]. It decomposes into direct products of the normalizable continuous and lowest weight discrete representations of the left- and right-moving current algebras of $sl(2,\mathbb{R})$ generated by

$$J^a(z) = \sum_{n=-\infty}^{\infty} J^a_n z^{-n}, \qquad \bar{J}^a(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{J}^a_n \bar{z}^{-n}, \tag{2.1}$$

with $a = 3, \pm$, obeying the following commutation relations:

$$[J^3_n, J^3_m] = -\frac{k}{2} n\delta_{n+m,0}, \qquad [J^3_n, J^\pm_m] = \pm J^\pm_{n+m},$$
$$[J^+_n, J^-_m] = -2J^3_{n+m} + kn\delta_{n+m,0}, \tag{2.2}$$

with level $k \in \mathbb{R}_{>2}$. The lowest principal discrete representations $\hat{\mathcal{D}}^+_j \times \hat{\mathcal{D}}^+_j$ contain the states $|j, m, \bar{m}>$ with $-\frac{k-1}{2} < j < -\frac{1}{2}$, $m, \bar{m} \in -j + \mathbb{Z}_{\geq 0}$ and their affine descendants. The principal continuous representations $\hat{\mathcal{C}}^\alpha_j \times \hat{\mathcal{C}}^\alpha_j$ contain the states $|j, \alpha, m, \bar{m}>$ with $j \in -\frac{1}{2} + i\mathbb{R}^+$, $\alpha \in [0,1)$, $m, \bar{m} \in \alpha + \mathbb{Z}$, and their affine descendants. The spectrum also includes the spectral flow images of these representations, which can be constructed with the spectral flow operators U_w, $\bar{U}_{\bar{w}}$, defined by their action on the $sl(2,\mathbb{R})$ currents J^3, $J^\pm$ as

$$U_{-w} J^3(z) U_w = J^3(z) + \frac{k}{2}\frac{w}{z}, \quad U_{-w} J^\pm(z) U_w = z^{\mp w} J^\pm(z),$$
$$\bar{U}_{-\bar{w}} \bar{J}^3(\bar{z}) \bar{U}_{\bar{w}} = \bar{J}^3(\bar{z}) + \frac{k}{2}\frac{\bar{w}}{\bar{z}}, \quad \bar{U}_{-\bar{w}} \bar{J}^\pm(\bar{z}) \bar{U}_{\bar{w}} = \bar{z}^{\mp\bar{w}} \bar{J}^\pm(\bar{z}), \tag{2.3}$$

where $U_{-w} = U_w^{-1}$, $\bar{U}_{-\bar{w}} = \bar{U}_{\bar{w}}^{-1}$ and $w = \bar{w} \in \mathbb{Z}$.[3] Using the Sugawara construction, the action of U_w, $\bar{U}_{\bar{w}}$ on the zero modes of the Virasoro generators is found to be

$$U_{-w} L_0 U_w = L_0 - wJ^3_0 - \frac{k}{4}w^2,$$
$$\bar{U}_{-\bar{w}} \bar{L}_0 \bar{U}_{\bar{w}} = \bar{L}_0 - \bar{w}\bar{J}^3_0 - \frac{k}{4}\bar{w}^2, \tag{2.4}$$

and the eigenvalues of L_0, $\bar{L}_0$ are, in general, not bounded from below. For states in the discrete series it is often

[2]The same expression was independently obtained in [20] where the Euclidean version of AdS_3 was constructed from the axial coset $SL(2,\mathbb{R})/U(1)_A$, using path integral techniques.

[3]The right and left spectral flow numbers w, $\bar{w}$ are not necessarily equal in the single cover of $SL(2,\mathbb{R})$ where $\bar{w} - w$ is the winding number around the compact closed timelike direction.

convenient to work with spectral flow images of both lowest and highest weight representations, which are related by the identification $\hat{\mathcal{D}}_j^{+,w} \equiv \hat{\mathcal{D}}_{-(k/2)-j}^{-,w+1}$.

The characters on the Lorentzian signature torus are defined from the standard expressions as

$$\chi_{\mathcal{V}_L}(\theta_-, \tau_-, u_-) = \mathrm{Tr}_{\mathcal{V}_L} e^{2\pi i\tau_-(L_0-(c/24))} e^{2\pi i\theta_- J_0^3} e^{\pi i u_- K},$$
$$\chi_{\mathcal{V}_R}(\theta_+, \tau_+, u_+) = \mathrm{Tr}_{\mathcal{V}_R} e^{2\pi i\tau_+(\bar{L}_0-(\bar{c}/24))} e^{2\pi i\theta_+ \bar{J}_0^3} e^{\pi i u_+ K}, \tag{2.5}$$

where $\tau_\pm$, $\theta_\pm$, $u_\pm$ are independent real parameters, $c = \bar{c}$ are the left- and right-moving central charges and K is the central element of the affine algebra. The traces are taken over the left and right representation modules of the Hilbert space of the theory, $\mathcal{V}_L$ and $\mathcal{V}_R$, respectively. The Euclidean version of (2.5) is obtained replacing the real parameters by complex ones. For completeness, a description of the moduli space of the Lorentzian torus is presented in Appendix A.

In the remaining of this section we review (and redefine) the complete set of characters of the relevant representations making up the spectrum of the bulk AdS$_3$ conformal field theory and of the finite dimensional representations appearing in the open string spectrum of some brane solutions.

To lighten notation, from now on τ, θ, u will denote the real parameters τ_-, θ_-, u_- and the following compact notation will be used: $\chi_j^{\pm,w} := \chi_{\hat{\mathcal{D}}_j^{\pm,w}}$, $\chi_j^{\alpha,w} := \chi_{\hat{\mathcal{C}}_j^{\alpha,w}}$.

A. Discrete representations

The naive computation of the characters (2.5) for the discrete representations leads to θ and τ dependent divergences. This is not a problem because the characters are typically not functions but distributions. Indeed, similarly as the characters of the continuous representations, which contain a series of delta functions [1], those of the discrete representations need also be interpreted as distributions.

Let us consider the distributions constructed from the series defining the characters of the discrete representations. Shifting $\tau \to \tau + i\xi_1$ and $\theta \to \theta + i\xi_2^w$ in (2.5), where ξ_1, ξ_2^w are two real non vanishing parameters, a regular distribution can be defined. Indeed, the deformed characters of discrete representations in an arbitrary spectral flow sector w can be written in terms of those of unflowed representations as

$$\chi^{+,w}_{j,\xi_2^w,\xi_1}(\theta,\tau,u) = e^{i\pi ku}\sum_{\mathrm{n}} \epsilon_{\mathrm{n}} < \mathrm{n}| U_{-w} e^{2\pi i(\tau+i\xi_1)(L_0-(c/24))} e^{2\pi i(\theta+i\xi_2^w)J_0^3} U_w |\mathrm{n}>,$$

where $|n>$ is a complete orthonormal basis in $\hat{\mathcal{D}}_j^{+,0}$, with norm $\epsilon_n = \pm 1$ (recall that this model is not unitary). Since U_w is unitary, $U_w|n>$ defines an orthonormal basis in $\hat{\mathcal{D}}_j^{+,w}$ and from (2.3) one can rewrite

$$\chi^{+,w}_{j,\xi_2^w,\xi_1} = e^{i\pi ku} e^{-2\pi i\tau(k/4)w^2} e^{2\pi i\theta(k/2)w} \sum_{\mathrm{n}} \epsilon_{\mathrm{n}} < \mathrm{n}| e^{2\pi i(\tau+i\xi_1)(L_0-(c/24))} e^{2\pi i(\theta - w\tau + i(\xi_2^w - w\xi_1))J_0^3} |\mathrm{n}>. \tag{2.6}$$

Choosing an orthonormal basis of eigenvectors of L_0 and J_0^3, the following behavior of the sum is easy to see

$$\chi^{+,w}_{j,\xi_2^w,\xi_1} \sim \sum_{N,n=0}^{\infty} \rho(n,N) e^{2\pi i[(1+w)\tau - \theta + i((1+w)\xi_1 - \xi_2^w)]N} e^{2\pi i[\theta - w\tau + i(\xi_2^w - w\xi_1)]n},$$

where $\rho(n,N)$ gives the degeneracy of states.[4] This expression is convergent for parameters in the ranges[5]

$$\xi_1 > 0, \qquad (1+w)\xi_1 > \xi_2^w > w\xi_1, \tag{2.7}$$

and it gives

$$\chi^{+,w}_{j,\xi_2^w,\xi_1} = e^{i\pi ku} e^{-2\pi i(\tau+i\xi_1)(k/4)w^2} e^{2\pi i(\theta+i\xi_2^w)(k/2)w} \frac{e^{-(2\pi i(\tau+i\xi_1)/k-2)(j+(1/2))^2} e^{-2\pi i(\theta+i\xi_2^w - w(\tau+i\xi_1))(j+(1/2))}}{i\vartheta_{11}(\theta + i\xi_2^w - w(\tau+i\xi_1), \tau+i\xi_1)}. \tag{2.8}$$

This character defines a regular distribution and, given that the series of regular distributions are continuous with respect to the weak limit, this implies

$$\chi_j^{+,w}(\theta,\tau,u) = e^{i\pi ku} \frac{e^{-(2\pi i\tau/k-2)(j+(1/2)-w(k-2)/2)^2} e^{-2\pi i\theta(j+(1/2)-w(k-2)/2)}}{i\vartheta_{11}(\theta + i\epsilon_2^w, \tau + i\epsilon_1)}, \tag{2.9}$$

[4] Notice the different letters' styles: Roman type n labels a generic basis and italic n appears in the eigenvalues of J_0^3.

[5] Because of the degeneracy, these are sufficient (and not necessary) conditions. However, an explicit calculation in this region gives the inverse ϑ_{11} function having the same poles, which then turns them into necessary conditions.

where we have used the identity

$$\vartheta_{11}(\theta + i\epsilon_2^w - w(\tau + i\epsilon_1), \tau + i\epsilon_1) = (-)^w e^{-\pi i\tau w^2 + 2\pi i\theta w}\vartheta_{11}(\theta + i\epsilon_2^w, \tau + i\epsilon_1) \tag{2.10}$$

and the $i\epsilon$'s denote the usual $i0$ prescriptions, constrained as the corresponding finite parameters in (2.7), which dictate how to avoid the poles of ϑ_{11}^{-1} at $n\tau \in \mathbb{Z}$, $m\tau + \theta \in \mathbb{Z}$, for $n \in \mathbb{N}$, $m \in \mathbb{Z}$. These poles are easily seen in the following alternative expression for the elliptic theta function

$$\frac{1}{\vartheta_{11}(\theta + i\epsilon_2^w, \tau + i\epsilon_1)} = \frac{-e^{-i(\pi/4)\tau}}{\sin[\pi(\theta + i\epsilon_2^w)]}\frac{1}{\prod_{n=1}^{\infty}[1 - e^{2\pi i n(\tau + i\epsilon_1)}]}\frac{1}{\prod_{n=1}^{\infty}[1 - e^{2\pi i(n\tau - \theta + i\epsilon_3^{n,w})}][1 - e^{2\pi i(n\tau + \theta + i\epsilon_4^{n,w})}]}, \tag{2.11}$$

with

$$\epsilon_3^{n,w} = n\epsilon_1 - \epsilon_2^w \qquad \epsilon_4^{n,w} = n\epsilon_1 + \epsilon_2^w, \tag{2.12}$$

i.e., $\epsilon_3^{n,w} > 0(<0)$ for $n \geq 1 + w(n \leq w)$ and $\epsilon_4^{n,w} > 0(<0)$ for $n \geq -w(n \leq -1 - w)$.

Notice that, in the weak limit, one can take $\epsilon_1, \epsilon_2^w = 0$ in the arguments of the exponential terms in (2.9) because they are perfectly regular.

It is useful to rewrite (2.9) using the identity (B1), which allows to change the signs of ϵ_2^w, $\epsilon_3^{n,w}$ and $\epsilon_4^{n,w}$, in order to get the following expressions in terms of only one parameter, say $\epsilon_2^{w'}$, with arbitrary w':

$$\begin{aligned}\chi_j^{+,w<w'}(\theta, \tau, u) = {}& (-)^w e^{i\pi k u}\frac{e^{-(2\pi i\tau/k-2)(j+(1/2)-w(k-2)/2)^2}e^{-2\pi i\theta(j+(1/2)-w(k-2)/2)}}{i\vartheta_{11}(\theta + i\epsilon_2^{w'}, \tau + i\epsilon_1)} \\ & - (-)^w e^{i\pi k u}\frac{e^{-(2\pi i\tau/k-2)(j+(1/2)-w(k-2)/2)^2}e^{-2\pi i\theta(j+(1/2)-w(k-2)/2)}}{\eta^3(\tau + i\epsilon_1)} \\ & \times \sum_{n=1+w}^{w'}(-)^n e^{2i\pi\tau(n^2/2)}\sum_{m=-\infty}^{\infty}(-)^m\delta(\theta - n\tau + m)\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\chi_j^{+,w>w'}(\theta, \tau, 0) = {}& (-)^w e^{i\pi k u}\frac{e^{-(2\pi i\tau/k-2)(j+(1/2)-w(k-2)/2)^2}e^{-2\pi i\theta(j+(1/2)-w(k-2)/2)}}{i\vartheta_{11}(\theta + i\epsilon_2^{w'}, \tau + i\epsilon_1)} \\ & + (-)^w e^{i\pi k u}\frac{e^{-(2\pi i\tau/k-2)(j+(1/2)-w(k-2)/2)^2}e^{-2\pi i\theta(j+(1/2)-w(k-2)/2)}}{\eta^3(\tau + i\epsilon_1)} \\ & \times \sum_{n=1+w'}^{w}(-)^n e^{2i\pi\tau(n^2/2)}\sum_{m=-\infty}^{\infty}(-)^m\delta(\theta - n\tau + m).\end{aligned} \tag{2.14}$$

These expressions are in perfect agreement with the spectral flow symmetry, which implies $\chi_j^{+,w}(-\theta, \tau, u) = \chi_{-(k/2)-j}^{+,-w-1}(\theta, \tau, u)$. They lead to the following contribution to the partition function:

$$Z_{\mathcal{D}}^{\mathrm{AdS}_3} = \sqrt{\frac{k-2}{2i(\tau_- - \tau_+)}}\frac{e^{i\pi k(u_- - u_+)}e^{2\pi i(k-2)/4((\theta_- - \theta_+)^2/\tau_- - \tau_+)}}{\vartheta_{11}(\theta_- + i\epsilon_2^0, \tau_- + i\epsilon_1)\vartheta_{11}^*(\theta_+ - i\epsilon_2^0, \tau_+ - i\epsilon_1)} + \ldots, \tag{2.15}$$

where the ellipses stand for the contributions of the contact terms. This expression differs formally from the equivalent one in [1], where no ϵ prescription or contact terms were considered. Nevertheless, the ultimate goal in [1] was to reproduce the Euclidean partition function continuing the modular parameters away from the real axes and discarding contact terms such as those of the characters of the continuous representations.

B. Continuous representations

A similar analysis can be performed for the characters of the continuous representations. Using (2.6), one can compute these characters in terms of those of the unflowed continuous representations. The result is

$$\chi_j^{\alpha,w} = e^{i\pi ku} \frac{-2\sin[\pi(\theta - w\tau)] e^{-2\pi i\tau(k/4)w^2} e^{2\pi i\theta(k/2)w} e^{-(2\pi i\tau/k-2)(j+(1/2))^2} e^{2\pi i(\theta - w\tau)\alpha}}{\vartheta_{11}(\theta - w\tau, \tau + i\epsilon_1)} \sum_{n=-\infty}^{\infty} e^{2\pi i(\theta - w\tau)n}$$
$$= e^{i\pi ku} \frac{e^{2\pi i\tau((s^2/k-2)+(k/4)w^2)}}{\eta^3(\tau + i\epsilon_1)} \sum_{m=-\infty}^{\infty} e^{-2\pi im(\alpha+(k/2)w)} \delta(\theta - w\tau + m), \tag{2.16}$$

where the following identity was used

$$\sum_{n=-\infty}^{\infty} e^{2\pi ixn} = \sum_{m=-\infty}^{\infty} \delta(x + m). \tag{2.17}$$

One could be tempted to interpret this sum of delta functions as the infinite volume of the target space. If it were, then one should assume the volume factor is not modular invariant. To see this, let us consider the limit $\theta \approx 0$. In the $w = 0$ case, the delta factors read $\sum_m e^{-2\pi im\alpha}\delta(\theta + m) \sim \delta(\theta)$. So, after a modular transformation one finds $\delta(\frac{\theta}{\tau}) = |\tau|\delta(\theta)$. This prevents one from simply taking the limit $\theta = 0$ discarding the δ function. The modular transformation will differ from the $\theta \neq 0$ case, and so it will not give the correct modular S matrix (which must not depend on θ).

In this case, the characters are defined as the weak limit $\epsilon_1, \epsilon_2^w \to 0$, with the constraints

$$\epsilon_1 > 0, \qquad \epsilon_2^w - w\epsilon_1 = 0, \tag{2.18}$$

and they give the following contribution to the partition function:

$$Z_{\mathcal{C}}^{AdS_3} = \sqrt{\frac{2-k}{8i(\tau_- - \tau_+)}} \frac{e^{i\pi k(u_- - u_+)}}{\eta^3(\tau_- + i\epsilon_1)\eta^{*3}(\tau_+ - i\epsilon_1)} \times \sum_{m,w=-\infty}^{\infty} e^{-2\pi i(k/4)w(\theta_- - \theta_+)} \times \delta(\theta_- - w\tau_- + m)\delta(\theta_+ - w\tau_+ + m), \tag{2.19}$$

in agreement with the expression obtained in [1].

C. Degenerate representations

Degenerate representations are not contained in the spectrum of the AdS_3 WZNW model but they play an important role in the description of the boundary CFT. Indeed, using world-sheet duality, it was argued that they make up the Hilbert space of open string excitations of S^2 branes in the H_3^+ model [21,22]. For the analysis that we shall perform in the forthcoming sections, it is useful to note the relation among their characters and those of discrete and continuous representations of the universal cover of $SL(2, \mathbb{R})$ discussed above.

The finite dimensional degenerate representations are labeled by the spin $j_{rs}^{\pm}$ defined by $1 + 2j_{rs}^{\pm} = \pm(r + s(k-2))$, with r, $s + 1 = 1, 2, 3, \ldots$ for the upper sign and r, $s = 1, 2, 3, \ldots$ for the lower one. Here we consider $J = j_{r0}^{+}$, with characters given by

$$\chi_J(\theta, \tau, u) = -\frac{2e^{i\pi ku} e^{-2\pi i\tau((2J+1)^2/4(k-2))} \sin[\pi\theta(2J+1)]}{\vartheta_{11}(\theta + i\epsilon_2, \tau + i\epsilon_1)}, \tag{2.20}$$

where the ϵ's are restricted to

$$\epsilon_1 > 0, \qquad |\epsilon_2| < \epsilon_1. \tag{2.21}$$

Extrapolating the values of the spins in the expressions obtained in the previous sections, (2.20) can be rewritten as

$$\chi_J(\theta, \tau, u) = \chi_J^{+,w=0}(\theta, \tau, u) + \chi_{-(k/2)-J}^{+,w=-1}(\theta, \tau, u) - \chi_J^{\alpha=\{J\},w=0}(\theta, \tau, u), \tag{2.22}$$

where $\{J\}$ is the sawtooth function. Actually, this relation could have been guessed from a simple inspection of the spectrum (see Fig. 1). This can be seen as a nontrivial check of the characters defined above and, simultaneously, it shows the important role played by the $i0$ prescription in the definition of the characters of discrete representations. A naive computation of these characters, ignoring the $i0'$s, would yield the (wrong) conclusion $\chi_J = \chi_J^{+,w=0} + \chi_{-(k/2)-J}^{+,w=-1}$.

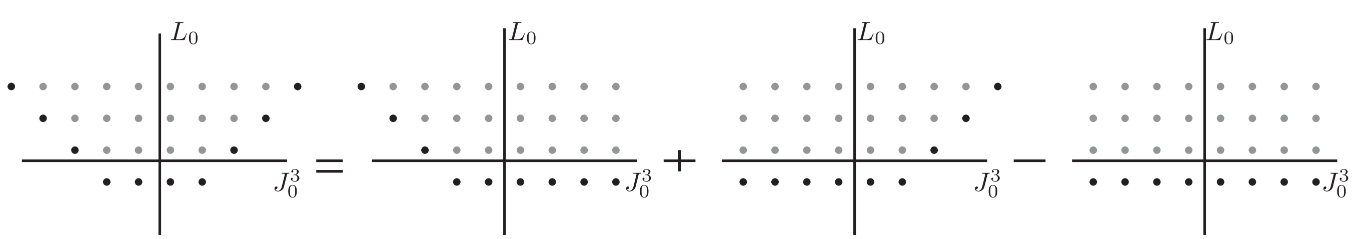

FIG. 1. The weight diagram of the degenerate representations with spin $J = j_{r0}^{+} = \frac{r-1}{2}$, $r = 1, 2, 3\ldots$ can be decomposed as the sum of the weight diagrams of the lowest and highest weight unflowed discrete representations minus that of the continuous representation of spin J.

III. MODULAR PROPERTIES

The modular transformation $\tau \longrightarrow \frac{a\tau+b}{c\tau+d}$, with integer parameters a, b, c, d such that $ad - bc = 1$, can be easily extended to include θ, u. Characters generating a representation space of the modular group transform as [23]

$$\chi_\mu\left(\frac{\theta}{c\tau+d}, \frac{a\tau+b}{c\tau+d}, u + \frac{c\theta^2}{2(c\tau+d)}\right) = \sum_\nu M_\mu{}^\nu \chi_\nu(\theta, \tau, u), \tag{3.1}$$

M being the matrix associated to the group element. Insofar as τ and u are concerned, the sign of all the parameters a, b, c, d may be simultaneously changed without affecting the transformation. The modular group $PSL(2, \mathbb{Z}) = \frac{SL(2,\mathbb{Z})}{\mathbb{Z}_2}$ is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

These transformations map $\theta \longrightarrow \theta$ and $\theta \longrightarrow \frac{\theta}{\tau}$, respectively, but inverting the signs of a, b, c, d, the mapping gives the opposite sign for θ. Therefore, the space spanned by the characters does not realize a good representation space for the modular group unless the characters are symmetric under $\theta \leftrightarrow -\theta$, e.g. for self-conjugate representations. When this is not the case, the characters form a representation of the double covering of the modular group, where S^2 is not the identity but the charge conjugation matrix. In fact, S^2 produces time and parity inversion on the torus geometry and, by CPT invariance, it transforms a character into its conjugate.

A. The S matrix

Below we will find explicit expressions for generalized S transformations of the characters introduced in the previous section, setting $u = 0$ for short, as[6]

$$\chi_\mu\left(\frac{\theta}{\tau}, -\frac{1}{\tau}, 0\right) = e^{-2\pi i(k/4)(\theta^2/\tau)} \sum_\nu S_\mu{}^\nu \chi_\mu(\theta, \tau, 0), \tag{3.2}$$

and we will show that, unlike standard expressions, they contain a sign of τ factor. This result can already be inferred from the S modular transformation of the partition function. Indeed, ignoring the ϵ's and the contact terms, one finds for the contributions from discrete representations[7]

$$\tilde{Z}_{\mathcal{D}}^{\mathrm{AdS}_3}(\tau'_-, \theta'_-, u'_-; \tau'_+, \theta'_+, u'_+) = \mathrm{sgn}(\tau_-\tau_+) \times \tilde{Z}_{\mathcal{D}}^{\mathrm{AdS}_3}(\tau_-, \theta_-, u_-; \tau_+, \theta_+, u_+), \tag{3.3}$$

while the contributions from the continuous series verify

$$Z_{\mathcal{C}}^{\mathrm{AdS}_3}(\tau'_-, \theta'_-, u'_-; \tau'_+, \theta'_+, u'_+) = \mathrm{sgn}(\tau_-\tau_+) \times Z_{\mathcal{C}}^{\mathrm{AdS}_3}(\tau_-, \theta_-, u_-; \tau_+, \theta_+, u_+), \tag{3.4}$$

where the primes denote the S modular transformed parameters. This suggests that the block $S_{d_i}{}^{d_j}$, d_i labeling discrete representations, is given by $\mathrm{sgn}(\tau)\mathcal{S}_{d_i}{}^{d_j}$ with $\mathcal{S}_{d_i}{}^{d_j}$ being unitary. Moreover, since the characters of the continuous representations contain purely contact terms, one expects that they close among themselves. This together with (3.4) suggest that the block $S_{c_i}{}^{c_j}$, c_i labeling continuous representations, is given by $\mathrm{sgn}(\tau)\mathcal{S}_{c_i}{}^{c_j}$ with $\mathcal{S}_{c_i}{}^{c_j}$ being unitary. We will explicitly show these features of the generalized modular transformations in the next section. In this sense, the characters of the AdS_3 model on the Lorentzian torus are pseudovectors with respect to the standard modular S transformations.

A naive treatment of the Lorentzian partition function as a Wick rotation of the Euclidean path integral, would suggest the appearance of this sign after an S transformation from the measure, when one takes into account the change in the metric (see Appendix A). However, it will be clear from the results of the next section, that the failure in the modular invariance of $Z_{\mathcal{D}}^{\mathrm{AdS}_3}$ is less subtle than just the sign appearing in (3.3).

1. Continuous representations

The S transformed characters of continuous representations can be written as

$$\chi_j^{\alpha,w}\left(\frac{\theta}{\tau}, -\frac{1}{\tau}, 0\right) = \frac{e^{-2\pi i((s^2/k-2)+(k/4)w^2)1/\tau}}{(-i\tau)^{3/2}\eta^3(\tau+i\epsilon_1)} \times \sum_{m=-\infty}^{\infty} e^{2\pi i m(\alpha+(k/2)w)} \delta\left(\frac{\theta}{\tau} + \frac{w}{\tau} - m\right), \tag{3.5}$$

where $\eta(-\frac{1}{\tau}+i\epsilon_1) \equiv \eta(-\frac{1}{\tau+i\epsilon_1}) = e^{\mp(i\pi/4)}\sqrt{|\tau|}\eta(\tau+i\epsilon_1)$, the upper (lower) sign holding for $\tau > 0$ ($\tau < 0$).

Using

$$e^{-2\pi i(s^2/k-2)(1/\tau)} = e^{\mp(i\pi/4)}\sqrt{\frac{2|\tau|}{k-2}}\int_{-\infty}^{+\infty} ds' e^{-4\pi i(ss'/k-2)} e^{2\pi i\tau(s'^2/k-2)}, \tag{3.6}$$

[6]Some authors use the $\tilde{S}$ matrix generating $\chi_\mu(-\frac{\theta}{\tau}, -\frac{1}{\tau}, u + \frac{\theta^2}{2\tau})$. This is given by $\tilde{S}_\mu{}^\nu = S_\mu{}^{\nu^+}$, where ν^+ labels the conjugate ν representation.

[7]$\tilde{Z}_{\mathcal{D}}^{\mathrm{AdS}_3}$ is the contribution to the partition function for θ and τ far from $\theta + n\tau \in \mathbb{Z}$, $\forall n \in \mathbb{Z}$.

we find

$$\chi_j^{\alpha,w}\left(\frac{\theta}{\tau},-\frac{1}{\tau},0\right)=\frac{e^{-2\pi i(k/4)(\theta^2/\tau)}}{\tau}\times\int_{-\infty}^{+\infty}ds'\tilde{\mathcal{S}}_s{}^{s'}\frac{e^{(2\pi i/k-2)\tau s'^2}}{\eta^3(\tau+i\epsilon_1)}\times\sum_{m=-\infty}^{\infty}e^{2\pi i(k/4)\tau m^2}e^{2\pi i m\alpha}\delta\left(\frac{\theta}{\tau}+\frac{w}{\tau}-m\right), \tag{3.7}$$

with $\tilde{\mathcal{S}}_s{}^{s'}=i\sqrt{\frac{2}{k-2}}e^{-4\pi i(ss'/k-2)}$.

From $\delta(\frac{\theta}{\tau}+\frac{w}{\tau}-m)=|\tau|\delta(\theta+w-m\tau)$ and renaming variables, one gets

$$\chi_j^{\alpha,w}\left(\frac{\theta}{\tau},-\frac{1}{\tau},0\right)=e^{-2\pi i(k/4)(\theta^2/\tau)}\mathrm{sgn}(\tau)\times\sum_{w'=-\infty}^{\infty}\int_{-\infty}^{+\infty}ds'\tilde{\mathcal{S}}_s{}^{s'}\frac{e^{2\pi i\tau((s'^2/k-2)+(k/4)w'^2)}}{\eta^3(\tau+i\epsilon_1)}\times e^{2\pi i w'\alpha}\delta(\theta-w'\tau+w). \tag{3.8}$$

In order to reconstruct the character $\chi_{j'}^{\alpha',w'}$ in the right-hand side (r.h.s.), we use the identity

$$\delta(\theta-w'\tau+w)=\sum_{m'=-\infty}^{\infty}\int_0^1 d\alpha' e^{2\pi i(w\alpha'+(k/2)ww')}\times e^{-2\pi i m'(\alpha'+(k/2)w')}\delta(\theta-w'\tau+m'), \tag{3.9}$$

and exchanging summation and integration,[8] (3.8) can be rewritten as

$$\chi_j^{\alpha,w}\left(\frac{\theta}{\tau},-\frac{1}{\tau},0\right)=e^{-2\pi i(k/4)(\theta^2/\tau)}\mathrm{sgn}(\tau)\times\sum_{w'=-\infty}^{\infty}\int_0^{+\infty}ds'\int_0^1 d\alpha'\mathcal{S}_{s,\alpha,w}{}^{s',\alpha',w'}\chi_{j'=-(1/2)+is'}^{\alpha',w'}(\theta,\tau,0),$$

with

$$\mathcal{S}_{s,\alpha,w}{}^{s',\alpha',w'}=2i\sqrt{\frac{2}{k-2}}\cos\left(4\pi\frac{ss'}{k-2}\right)e^{2\pi i(w\alpha'+w'\alpha+(k/2)ww')}, \tag{3.10}$$

which is symmetric and, as expected from (3.4), unitary, i.e.

$$\sum_{w'=-\infty}^{\infty}\int_0^{\infty}ds'\int_0^1 d\alpha'\mathcal{S}_{s_1,\alpha_1,w_1}{}^{s',\alpha',w'}\mathcal{S}^{\dagger}_{s',\alpha',w'}{}^{s_2,\alpha_2,w_2}=\delta(s_1-s_2)\delta(\alpha_1-\alpha_2)\delta_{w_1,w_2}. \tag{3.11}$$

2. Discrete representations

The structure of the characters of the discrete representations is more involved than that of the continuous ones. *A priori*, we expect that characters of both discrete and continuous representations appear in the generalized modular transformations. So, generically we can assume

$$\chi_j^{+,w}\left(\frac{\theta}{\tau},-\frac{1}{\tau},0\right)=e^{-2\pi i(k/4)(\theta^2/\tau)}\mathrm{sgn}(\tau)\times\sum_{w'=-\infty}^{\infty}\left\{\int_{-(k-1/2)}^{-(1/2)}\mathcal{S}_{j,w}{}^{j',w'}\chi_{j'}^{+,w'}(\theta,\tau,0)+\int_0^1 d\alpha'\int_0^{\infty}ds'\mathcal{S}_{j,w}{}^{s',\alpha',w'}\chi_{j'=-(1/2)+is'}^{\alpha',w'}(\theta,\tau,0)\right\}.$$

Fortunately, it is easy to separate the contributions from discrete and continuous representations. If one considers generic values of θ and τ far from $\theta+n\tau\in\mathbb{Z}$ for $n\in\mathbb{Z}$, the contributions of the continuous series in the r.h.s. can be neglected as well as all contact terms and ϵ's. On the other hand, if $\theta+n\tau\in\mathbb{Z}$, $\forall n\in\mathbb{Z}$ then $\frac{\theta}{\tau}-p\frac{1}{\tau}\in\mathbb{Z}$, $\forall p\in\mathbb{Z}$ and all contact terms and ϵ's of the left-hand side (l.h.s.) can be neglected too. Thus, we obtain

$$\chi_j^{+,w}\left(\frac{\theta}{\tau},-\frac{1}{\tau},0\right)=\frac{(-)^w e^{(2\pi i/k-2)1/\tau(j+(1/2)-w(k-2/2))^2}e^{-2\pi i(\theta/\tau)(j+(1/2)-w(k-2)/2)}}{i\vartheta_{11}(\frac{\theta}{\tau},-\frac{1}{\tau})}=(-)^{w+1}\frac{e^{(2\pi i/k-2)1/\tau(j+(1/2)-(w+\theta)k-2/2)^2}e^{-2\pi i(k/4)\theta^2/\tau}}{i\sqrt{i\tau}\vartheta_{11}(\theta,\tau)}, \tag{3.12}$$

where the following identity was used for $\tau\in\mathbb{R}$:

$$\vartheta_{11}\left(\frac{\theta}{\tau},-\frac{1}{\tau}\right)=\mp e^{\pi i(\theta^2/\tau)}e^{\pm i(\pi/4)}\sqrt{|\tau|}\vartheta_{11}(\theta,\tau), \tag{3.13}$$

[8]Here, summation and integration can be exchanged because, for a fixed w', the series always reduces to a finite sum when it is considered as a distribution acting on a test function.

the upper (lower) sign holding for $\tau > 0$ ($\tau < 0$). Inserting

$$e^{(2\pi i/k-2)1/\tau(j+(1/2)-(w+\theta)(k-2/2))^2} = e^{\pm i(\pi/4)}\sqrt{\frac{2|\tau|}{k-2}} \times \int_{-\infty}^{+\infty} d\lambda' e^{(4\pi i/k-2)\lambda'(j+(1/2)-(w+\theta)k-2/2)} e^{-(2\pi i/k-2)\tau\lambda'^2} \tag{3.14}$$

into (3.12), changing the integration variable to $j' + \frac{1}{2} - w'\frac{k-2}{2}$ and using (3.13), we get

$$\chi_j^{+,w}\left(\frac{\theta}{\tau}, -\frac{1}{\tau}, 0\right) = e^{-2\pi i(k/4)(\theta^2/\tau)}\mathrm{sgn}(\tau) \times \sum_{w'=-\infty}^{\infty} \int_{-(k-1)/2}^{-(1/2)} dj' \mathcal{S}_{j,w}{}^{j',w'} \chi_{j'}^{+,w'}(\theta, \tau, 0), \tag{3.15}$$

with

$$\mathcal{S}_{j,w}{}^{j',w'} = (-)^{w+w'+1}\sqrt{\frac{2}{k-2}} \times e^{4\pi i/k-2(j'+(1/2)-w'(k-2)/2)(j+(1/2)-w(k-2)/2)}. \tag{3.16}$$

Notice that this block of the $\mathcal{S}$ matrix is symmetric and, again as expected from (3.3), unitary.[9]

While the identity (3.14), which is essential to reconstruct the discrete characters in the r.h.s. of (3.15), only makes sense for Im $\tau \leq 0$, the characters are only well defined for Im $\tau \geq 0$. Therefore, to determine the generalized S transformation, it is crucial that $\tau \in \mathbb{R}$.

Finding the block $\mathcal{S}_{j,w}{}^{s',\alpha',w'}$ mixing discrete with continuous representations is a much more technical issue, which we discuss in Appendix B. Here we simply display the result, namely

$$\mathcal{S}_{j,w}{}^{s',\alpha',w'} = -i\sqrt{\frac{2}{k-2}}e^{-2\pi i(w'j-w\alpha'-ww'(k/2))} \times \left[\frac{e^{(4\pi/k-2)s'(j+(1/2))}}{1+e^{-2\pi i(\alpha'-is')}} + \frac{e^{-(4\pi/k-2)s'(j+(1/2))}}{1+e^{-2\pi i(\alpha'+is')}}\right]. \tag{3.18}$$

[9]Changing $e^{\pm i(\pi/4)}\sqrt{\tau}$ by $\sqrt{i\tau}$, the validity of (3.14) can be extended to the full lower half plane and that of (3.13) can be extended to the upper half plane, giving

$$\vartheta_{11}\left(\frac{\theta}{\tau}, -\frac{1}{\tau}\right) = -e^{\pi i(\theta^2/\tau)}\sqrt{i\tau}\vartheta_{11}(\theta, \tau). \tag{3.17}$$

If one naively cancels the $\sqrt{i\tau}$ terms and ignores the sign factor due to the different branches, a τ independent expression is obtained for the S matrix. However, such S matrix does not obey the properties $S^2 = (ST)^3 = C$, C being the charge conjugation matrix, but the opposite ones.

This block prevents the full $\mathcal{S}$ matrix from being unitary. Instead, we find $\mathcal{S}^*\mathcal{S} = id$. This implies that the full partition function defined from the product of characters is not modular invariant, not only due to the sign of the modular parameters. Actually, after a modular transformation, the mixing block introduces terms where the left modes are in discrete representations and the right ones in continuous series, and vice versa, as well as new terms containing left and right continuous representations.

In Sec. III C, we explicitly check that the blocks of the S matrix determined here have the correct properties.

3. Degenerate representations

The modular properties discussed above can be used to write the S transformation of the characters of the degenerate representations with $1 + 2J \in \mathbb{N}$ as

$$\chi_J\left(\frac{\theta}{\tau}, -\frac{1}{\tau}, 0\right) = e^{-2\pi i(k/4)(\theta^2/\tau)}\mathrm{sgn}(\tau) \times \sum_{w=-\infty}^{\infty}\left\{\int_{-(k-1)/2}^{-(1/2)} dj \mathcal{S}_J{}^{j,w}\chi_j^{+,w}(\theta,\tau,0) + \int_0^1 d\alpha \int_{-(k-1/2)}^{-(1/2)} ds \mathcal{S}_J{}^{s,\alpha,w}\chi_{j=-(1/2)+is}^{\alpha,w}(\theta,\tau,0)\right\},$$

where

$$\mathcal{S}_J{}^{j,w} = 2i\sqrt{\frac{2}{k-2}}(-)^{w+1}\sin\left[\frac{\pi}{k-2}(1+2j-w(k-2)) \times (2J+1)\right], \tag{3.19}$$

and

$$\mathcal{S}_J{}^{s,\alpha,w} = -i(-)^{2Jw}\sqrt{\frac{2}{k-2}}e^{(4\pi/k-2)s(J+(1/2))} \times \left(1 + \frac{1}{1+e^{-2\pi i(\alpha-is)}} + \frac{1}{1+e^{2\pi i(\alpha+is)}}\right) + (s \leftrightarrow -s). \tag{3.20}$$

B. The T matrix

Together with the S matrix, the T matrix defines a basis over the space of modular transformations. Using

$$\vartheta_{11}(\theta, \tau+1) = e^{(\pi i/4)}\vartheta_{11}(\theta,\tau), \quad \eta(\tau+1) = e^{(\pi i/12)}\eta(\tau), \tag{3.21}$$

the characters of the discrete and continuous representations transform, respectively, with

$$T_{j,w}{}^{j',w'} = \delta_{w,w'}\delta(j-j')e^{-(2\pi i/k-2)(j'+(1/2)-w'(k-2)/2)^2-(\pi i/4)} \tag{3.22}$$

and

$$T_{s,\alpha,w}{}^{s',\alpha',w'} = \delta_{w,w'}\delta(\alpha - \alpha')\delta(s - s') \times e^{2\pi i((s^2/k-2)-(k/4)w^2-w\alpha-(1/8))}, \quad (3.23)$$

while the T transformation of the characters of the degenerate representations is given by

$$\chi_J(\theta, \tau + 1, 0) = e^{-(2\pi i/k-2)(J+(1/2))^2} e^{-(\pi i/4)} \chi_J(\theta, \tau, 0). \quad (3.24)$$

C. Properties of the S and T matrices

The expressions $(ST)^3$ and S^2 must give the conjugation matrix, C. We have found above that the characters of the AdS_3 model do not expand a representation space for the modular group since the generators depend on the sign of τ. Nevertheless, in terms of the τ independent part of S, that we have denoted $\mathcal{S}$, these identities read $C = (ST)^3 = \mathrm{sgn}(\tau + 1)\mathrm{sgn}(\frac{\tau}{\tau+1})\mathrm{sgn}(-\frac{1}{\tau})(\mathcal{S}T)^3 = -(\mathcal{S}T)^3$ and $C = S^2 = \mathrm{sgn}(\tau)\mathrm{sgn}(-\frac{1}{\tau})\mathcal{S}^2 = -\mathcal{S}^2$.

As a consistency check on the expressions found above for S and T, an explicit computation gives

$$-(\mathcal{S}T)^3_{j_1,w_1}{}^{j_2,w_2} = -\mathcal{S}^2_{j_1,w_1}{}^{j_2,w_2} = \delta_{w_1+w_2+1,0}\,\delta\left(j_1 + j_2 + \frac{k}{2}\right), \quad (3.25)$$

which corresponds to the conjugation matrix restricted to the discrete sector, since $\hat{\mathcal{D}}_j^{+,w}$ is the conjugate representation of $\hat{\mathcal{D}}_j^{-,-w}$, which in turn can be identified with $\hat{\mathcal{D}}^{+,-w-1}_{-(k/2)-j}$ using the spectral flow symmetry. Similarly, for the block of continuous representations we get

$$-(\mathcal{S}T)^3_{s_1,\alpha_1,w_1}{}^{s_2,\alpha_2,w_2} = -\mathcal{S}^2_{s_1,\alpha_1,w_1}{}^{s_2,\alpha_2,w_2} = \delta_{w_1,-w_2}\delta(s_1 - s_2)\delta(\alpha_1 + \alpha_2 - 1), \quad (3.26)$$

which is again the charge conjugation matrix, since $\hat{\mathcal{C}}_j^{1-\alpha,-w}$ is the conjugate representation of $\hat{\mathcal{C}}_j^{\alpha,w}$.

Of course, one also needs to show that the non diagonal terms vanish. The equalities $(\mathcal{S}T)^3_{s_1,\alpha_1,w_1}{}^{j_2,w_2} = \mathcal{S}^2_{s_1,\alpha_1,w_1}{}^{j_2,w_2} = 0$ are trivially satisfied as a consequence of $\mathcal{S}_{s_1,\alpha_1,w_1}{}^{j_2,w_2} = 0$. One can also show that $(\mathcal{S}T)^3_{j_1,w_1}{}^{s_2,\alpha_2,w_2} = \mathcal{S}^2_{j_1,w_1}{}^{s_2,\alpha_2,w_2} = 0$, but this computation is more involved, so the details are left to Appendix B.

IV. REVISITING D-BRANES IN AdS_3

D-branes can be characterized by the one-point functions of the states in the bulk, living on the upper half plane. In RCFT, these one-point functions can be determined from the entries of the S matrix, a property that we will call a *Cardy structure*. This property is closely related to the Verlinde formula and, *a priori*, there is no reason for it to hold in non RCFT. In this section we explore this relation in the AdS_3 model.

D-branes in AdS_3 and related models have been studied in several works (see for instance [21,22,24–36] and references therein). Here, we shall restrict to the maximally symmetric D-branes discussed in [26]. Because the Lorentzian AdS_3 geometry is obtained by sewing an infinite number of $SL(2,\mathbb{R})$ group manifolds, these D-brane solutions can be trivially obtained from those of $SL(2,\mathbb{R})$. Their geometry was considered semiclassically in [26], where it was found that solutions of the Dirac-Born-Infeld action stand for regular and twined conjugacy classes of $SL(2,\mathbb{R})$. The model also has symmetry breaking D-brane solutions, but in this case, the open string spectrum is not a sum of $sl(2,\mathbb{R})$ representations and then we do not expect the one-point functions to be determined by the S matrix.

We begin this section with a short introduction to the geometry of D-branes in AdS_3. A very comprehensive study about the (twined) conjugacy classes of $SL(2,\mathbb{R})$ and a semiclassical analysis of branes can be found in [24,26]. Both can be easily extended to the universal covering. Here, we review the analysis of the conjugacy classes in order to make the discussion self contained and discuss the extension to the universal covering.

Then we turn to the explicit construction of the Ishibashi states for regular and twisted boundary gluing conditions which give rise to the maximally symmetric D-branes. These equations were solved in the past for the single cover of $SL(2,\mathbb{R})$ (see [31] for twisted gluing conditions) with different amounts of spectral flow in the left and right sectors, namely $w_L = -w_R$, and therefore, these solutions are not contained in the spectrum of the AdS_3 model (with the obvious exception of $w = 0$ discrete and $w = 0$, $\alpha = 0, \frac{1}{2}$ continuous representations).

We find that the one-point functions of states in discrete representations coupled to pointlike and H_2 branes exhibit a Cardy structure and we propose a generalized Verlinde formula giving the fusion rules of the degenerate representations with $1 + 2J \in \mathbb{N}$.

A. Conjugacy classes in AdS_3

Elements of $SL(2,\mathbb{R})$ can be parametrized by four real parameters $X_0, \ldots, X_3$ as

$$g = \frac{1}{\ell}\begin{pmatrix} X_0 + X_1 & X_2 + X_3 \\ X_2 - X_3 & X_0 - X_1 \end{pmatrix}, \quad (4.1)$$

with $X_0^2 - X_1^2 - X_2^2 + X_3^2 = \ell^2$. This gives a representation of the $SL(2,\mathbb{R})$ group manifold embedded in a 4 dimensional flat space. When the signature of this embedding space is $(-1, 1, 1, -1)$, it corresponds to a pseudosphere whose covering space is AdS_3.

A more convenient coordinate system is given by

$$X_0 + iX_3 = \ell e^{it}\cosh\rho, \quad X_1 + iX_2 = \ell e^{i\theta}\sinh\rho, \quad (4.2)$$

where AdS_3 is simply obtained by decompactifying the timelike direction t.

As is well known [36], the world volume of a symmetric D-brane on the $SL(2,\mathbb{R})$ group manifold is given by the (twined) conjugacy classes

$$\mathcal{W}_g^\omega = \{\omega(h)gh^{-1}, \forall h \in SL(2,\mathbb{R})\}, \tag{4.3}$$

where ω determines the gluing condition connecting left- and right-moving currents, $\omega(g) = \omega^{-1}g\omega$. When ω is an inner automorphism, $\mathcal{W}_g^\omega$ can be seen as left group translations of the regular conjugacy class (of the element ωg). So, one can restrict attention to the case $\omega = id.$, and the conjugacy classes are simply given by the solution to

$$trg = 2\frac{X_0}{\ell} = 2\tilde{C}. \tag{4.4}$$

The geometry of the world volume is then parametrized by the constant $\tilde{C}$ as

$$-X_1{}^2 - X_2{}^2 + X_3{}^2 = \ell^2(1 - \tilde{C}^2). \tag{4.5}$$

Different geometries can be distinguished for $\tilde{C}^2$ bigger, equal or smaller than 1. The former gives rise to a two dimensional de Sitter space, dS_2, the latter to a two dimensional hyperbolic space, H_2, and the case $|\tilde{C}| = 1$ splits into three different geometries: the apex, the future and the past of a light cone.

A more convenient way to parametrize these solutions is given by the redefinition

$$\tilde{C} = \cos\sigma. \tag{4.6}$$

For $|\tilde{C}| > 1$, $\sigma = ir + \pi v$, $r \in \mathbb{R}^+$, $v \in \mathbb{Z}_2$. The world volumes are given by

$$\cosh\rho\cos t = \pm\cosh r. \tag{4.7}$$

Each circular D-string is emitted and absorbed at the boundary in a time interval of width π but does not reach the origin unless $r = 0$. Their lifetime is determined by v.

For $|\tilde{C}| < 1$, σ is real and

$$\cosh\rho\cos t = \cos\sigma. \tag{4.8}$$

If one restricts $\sigma \in (0, \pi)$, there are two different solutions for each σ, for instance one with $t \in (-\frac{\pi}{2}, -\sigma]$ and another one with $t \in [\sigma, \frac{3\pi}{2})$. To distinguish between these two solutions we can take $\sigma = \lambda + \pi v$, $\lambda \in (-\pi, 0)$, $v \in \mathbb{Z}_2$, such that $t = \arccos(\cos\sigma/\cosh\rho)$, taking the branch where $t = \sigma$ when it crosses over the origin. Because these solutions have Euclidean signature, they are identified as instantons in AdS_3. In fact, they represent constant time slices in hyperbolic coordinates.

For $|\tilde{C}| = 1$, $\sigma = 0$ or π and

$$\cosh\rho\cos t = \pm 1. \tag{4.9}$$

For example, for $\tilde{C} = 1$, this corresponds to a circular D-string at the boundary at $t = -\pi/2$ collapsing to the instantonic solution in $\rho = 0$ at $t = 0$, and then expanding again to a D-string reaching the boundary at $t = \pi/2$.

All of these solutions are restricted to the single covering of $SL(2,\mathbb{R})$. In the universal covering, t is decompactified and the picture is periodically repeated. The general solutions can be parametrized by a pair (σ, q), $q \in \mathbb{Z}$, or equivalently, the range of σ can be extended to $\sigma = ir + q\pi$ for dS_2 branes, $\sigma = \lambda + q\pi$ for H_2 branes or $\sigma = q\pi$ for pointlike and light-cone branes.

Preparing for the discussions on one-point functions and Cardy structure, it is interesting to note that these parameters can be naturally identified with representations of the model. For instance, one can label the D-brane solutions as

$$\sigma = \frac{2\pi}{k-2}\left(j + \frac{1}{2} - w\frac{k-2}{2}\right), \tag{4.10}$$

with $j = -\frac{1}{2} + is$, $s \in \mathbb{R}^+$, $w \in \mathbb{Z}$ for dS_2 branes, $j \in (-\frac{k-1}{2}, -\frac{1}{2})$, $w \in \mathbb{Z}$ for H_2 branes and finally $\sigma = n\pi$, $n \in \mathbb{Z}$ for the pointlike and light cone D-brane solutions.

The appearance of the level k in a classical regime could seem awkward. However, it is useful to recall that σ is just a parameter labeling the conjugacy classes, and the factor $k - 2$ can be eliminated by simply redefining j through a change of variables. The important observation is that this suggests σ labels the exact solutions, e.g. the one-point functions at finite k will be found to be parametrized exactly by (4.10) and in fact, in the semiclassical regime $k \to \infty$, the domain of σ does not change at all.

When ω is an outer automorphism, one can take

$$\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

up to group translations. In this case, the twined conjugacy classes are given by

$$tr\omega g = 2\frac{X_2}{\ell} = 2C. \tag{4.11}$$

The world volume geometry now describes an AdS_2 space for all C since

$$X_0^2 - X_1^2 + X_3^2 = \ell^2(1 + C^2). \tag{4.12}$$

These are static open D-strings with endpoints fixed at the boundary. This is obvious in cylindrical coordinates, i.e.

$$\sinh\rho\sin\theta = \sinh r, \tag{4.13}$$

where we have renamed $C = \sinh r$. So, after decompactifying the timelike direction t, there is no need to extend the domain of r.

Let us end this brief review with a word of caution. In this section we have reviewed the twined conjugacy classes and, although branes wrap conjugacy classes, extra restrictions appear when studying the semiclassical or exact solutions. In particular, it was found in [26] that r becomes a positive quantized parameter at the semiclassical level.

B. Coherent states

Boundary states play a fundamental role in understanding boundary conformal field theories. They store all the information about possible D-brane solutions and their couplings to bulk states. Even though there is no systematic method to obtain all possible boundary states of an arbitrary model, if one works in the boundary theory of a given WZNW model and looks for special D-brane configurations with more symmetries than the conformal one, e.g. the symmetry generated by a given subalgebra of the original current algebra, then the procedure is more tractable because these symmetries impose extra restrictions, which together with certain *sewing constraints*, can be used to obtain exact solutions. Following these ideas, one can study different gluing conditions for the left and right current modes, consistent with the affine algebra [37] as well as with the conformal symmetry via the Sugawara construction [38].

In the case of AdS_3, much of the progress reached in this direction is based on the analytic continuation from H_3^+ [22]. Gluing conditions were imposed as differential equations applied directly to find, with the help of certain sewing constraints, the one-point functions of maximally symmetric D-branes. It would be interesting to get the one-point functions of the AdS_3 model without reference to other models, but the approach used so far cannot be easily extended. In the first place, it was developed in the x basis of the H_3^+ model, which is not a good basis for the representations of the universal covering of $SL(2,\mathbb{R})$. Suitable bases instead are the m- or t basis [1,7]. Moreover, there are still some open questions about the fusion rules of the AdS_3 model [4] which deserve further attention before analyzing the sewing constraints. Therefore, we will not compute the one-point functions in this way, but will give the first step in this direction by finding the explicit expressions for the Ishibashi states in the m basis for all the representations of the Hilbert space of the bulk theory.

1. Coherent states for regular gluing conditions

Boundary states associated to dS_2, H_2, light cone and pointlike D-branes in AdS_3 must satisfy the following regular gluing conditions [31]

$$(J_n^3 - \bar{J}_{-n}^3)|\mathbf{s}\rangle = 0, \qquad (J_n^\pm + \bar{J}_{-n}^\mp)|\mathbf{s}\rangle = 0, \qquad (4.14)$$

where $\mathbf{s}$ labels the members of the family of branes allowed by the gluing conditions.

These constraints are linear and leave each representation invariant, so that the boundary states must be expanded as a sum of solutions in each module. The solutions represent coherent states, usually called Ishibashi states [38].

Let us begin introducing the following notation which will be useful in the subsequent discussions. Let

$$\begin{aligned}|j,w,\alpha;\mathrm{n},\mathrm{m}\rangle &= |j,w,\alpha\rangle\{|\mathrm{n}\rangle\otimes\overline{|\mathrm{m}\rangle}\},\\ |j,w,+;\mathrm{n},\mathrm{m}\rangle &= |j,w,+\rangle\{|\mathrm{n}\rangle\otimes\overline{|\mathrm{m}\rangle}\},\end{aligned} \qquad (4.15)$$

denote orthonormal bases for $\hat{\mathcal{C}}_j^{\alpha,w}\otimes\hat{\mathcal{C}}_j^{\alpha,w}$ and $\hat{\mathcal{D}}_j^{+,w}\otimes\hat{\mathcal{D}}_j^{+,w}$, respectively. They satisfy[10]

$$\begin{aligned}&\langle j,w,\alpha;\mathrm{n},\mathrm{m}|j',w',\alpha';\mathrm{n}',\mathrm{m}'\rangle\\ &\quad=\langle j,w,\alpha|j,w,\alpha\rangle\times\langle\mathrm{n}|\mathrm{n}'\rangle\times\overline{\langle\mathrm{m}|\mathrm{m}'\rangle}\\ &\quad=\delta(s-s')\delta_{w,w'}\delta(\alpha-\alpha')\epsilon_\mathrm{n}\delta_{\mathrm{n},\mathrm{n}'}\epsilon_\mathrm{m}\delta_{\mathrm{m},\mathrm{m}'},\\ &\langle j,w,+;\mathrm{n},\mathrm{m}|j',w',+;\mathrm{n}',\mathrm{m}'\rangle\\ &\quad=\langle j,w,+|j,w,+\rangle\langle\mathrm{n}|\mathrm{n}'\rangle\overline{\langle\mathrm{m}|\mathrm{m}'\rangle}\\ &\quad=\delta(j-j')\delta_{w,w'}\epsilon_\mathrm{n}\delta_{\mathrm{n},\mathrm{n}'}\epsilon_\mathrm{m}\delta_{\mathrm{m},\mathrm{m}'},\end{aligned} \qquad (4.16)$$

$\{|\mathrm{n}>\}$ is an orthonormal basis in $\hat{\mathcal{C}}_j^{\alpha,w}$ (or $\hat{\mathcal{D}}_j^{+,w}$) for which the expectation values of J_n^3, $J_n^\pm$ are real numbers and $\epsilon_\mathrm{n}=\pm1$ is its norm squared. It is constructed by the action of the affine currents over the ket $|j, m=\alpha, w> = U_w|j,m=\alpha>$ ($|j,m=-j,w>$).

The Ishibashi states for continuous and discrete representations are found to be

$$\begin{aligned}|j,w,\alpha\gg &= \sum_\mathrm{n}\epsilon_\mathrm{n}\bar{V}|j,w,\alpha;\mathrm{n},\mathrm{n}\rangle \quad \text{and}\\ |j,w,+\gg &= \sum_\mathrm{n}\epsilon_\mathrm{n}\bar{V}|j,w,+;\mathrm{n},\mathrm{n}\rangle,\end{aligned} \qquad (4.17)$$

respectively, where V is defined as the linear operator satisfying

$$\begin{aligned}V\prod_I J_{n_I}^{a_I}|j,m=-j,w\rangle &= \prod_I\eta_{a_Ib_I}J_{n_I}^{b_I}|j,m=-j,w\rangle,\\ V\prod_I J_{n_I}^{a_I}|j,m=\alpha,w\rangle &= \prod_I\eta_{a_Ib_I}J_{n_I}^{b_I}|j,m=\alpha,w\rangle,\end{aligned} \qquad (4.18)$$

with $a=1,2,3$, $\eta_{ab}=\mathrm{diag}(-1,-1,1)$ and the bar denotes action restricted to the antiholomorphic sector. It is easy to see that this defines a unitary operator. The proof that they are solutions to (4.14) follows similar lines as those of [38]. As an example, let us consider an arbitrary base state $|j',w',\alpha';\mathrm{n}',\mathrm{m}'>$:

$$\begin{aligned}&<j',w',\alpha';\mathrm{n}',\mathrm{m}'|J_r^3-\bar{J}_{-r}^3|j,\alpha,w\gg\\ &\quad=\delta(s-s')\delta_{w,w'}\delta(\alpha-\alpha')\sum_\mathrm{n}\epsilon_\mathrm{n}\langle\mathrm{n}'|J_n^3|\mathrm{n}\rangle\overline{\langle\mathrm{m}'|}\,\bar{V}\,\overline{|\mathrm{n}\rangle}\\ &\qquad-\epsilon_\mathrm{n}\langle\mathrm{n}'|\mathrm{n}\rangle\overline{\langle\mathrm{m}'|}\bar{J}_{-n}^3\bar{V}\,\overline{|\mathrm{n}\rangle}\\ &\quad=\delta(s-s')\delta_{w,w'}\delta(\alpha-\alpha')\sum_\mathrm{n}\epsilon_\mathrm{n}\langle\mathrm{n}'|J_n^3|\mathrm{n}\rangle\langle\mathrm{n}|V|\mathrm{m}'\rangle\\ &\qquad-\epsilon_\mathrm{n}\langle\mathrm{n}'|\mathrm{n}\rangle\langle\mathrm{n}|VJ_n^3|\mathrm{m}'\rangle=0.\end{aligned}$$

[10]The separation between $|j,w,\alpha\rangle$ or $|j,w,+\rangle$ and $|\mathrm{n}\rangle$, $\overline{|\mathrm{m}\rangle}$ in different kets is simply a matter of useful notation for calculus and does not denote tensor product.

The normalization fixed above for the Ishibashi states implies

$$\begin{aligned} &\ll j, w, \alpha|e^{\pi i\tau(L_0+\bar{L}_0-(c/12))}e^{\pi i\theta(J_0^3+\bar{J}_0^3)}|j', w', \alpha' \gg \\ &\quad = \delta(s-s')\delta_{w,w'}\delta(\alpha-\alpha')\chi_j^{\alpha,w}(\tau,\theta), \\ &\ll j, w, +|e^{\pi i\tau(L_0+\bar{L}_0-(c/12))}e^{\pi i\theta(J_0^3+\bar{J}_0^3)}|j', w', + \gg \\ &\quad = \delta(j-j')\delta_{w,w'}\chi_j^{+,w}(\tau,\theta). \end{aligned} \tag{4.19}$$

2. Cardy structure and one-point functions for pointlike branes

Assuming that after Wick rotation the open string partition function in AdS_3 reproduces that of the H_3^+ model and a generalized Verlinde formula, we show in this section that the one-point functions on localized branes in AdS_3 previously found in [19] can be recovered. We also verify that the one-point functions on pointlike and H_2 D-branes exhibit a Cardy structure. Usually, this structure is accompanied by a Verlinde formula for the representations appearing in the boundary spectrum. In fact, the Cardy structure is a natural solution to the *Cardy condition* when the Verlinde theorem holds. However, as we shall discuss, the latter does not hold in the AdS_3 WZNW model. The generalized Verlinde formula proposed in Appendix C reproduces the fusion rules of the degenerate representations, but it gives contributions to the fusion rules of the discrete representations with an arbitrary amount of spectral flow, thus contradicting the selection rules determined in [3]. Nevertheless, we find a Cardy structure.

Boundary states: World-sheet duality allows to write the one-loop partition function for open strings ending on pointlike branes labeled by $\mathbf{s}_1$ and $\mathbf{s}_2$ as

$$\begin{aligned} &e^{-2\pi i(k/4)(\theta^2/\tau)}Z^{AdS_3}_{\mathbf{s}_1\mathbf{s}_2}(\theta,\tau,0) = \langle\Theta\mathbf{s}_1|\tilde{q}^{H^{(P)}}\tilde{z}^{J_0^3}|\mathbf{s}_2\rangle \\ &= \sum_{w=-\infty}^{\infty}\int_{-(k-1/2)}^{-(1/2)} dj\,\mathcal{A}^{\mathbf{s}_1}_{(j,w)}\mathcal{A}^{\mathbf{s}_2}_{(j^+,w^+)}\chi_j^{+,w}(\tilde{\theta},\tilde{\tau},0) + ccr, \end{aligned}$$

where Θ denotes the world-sheet *CPT* operator in the bulk theory, $\tilde{q} = e^{2\pi i\tilde{\tau}}$, $\tilde{z} = e^{2\pi i\tilde{\theta}}$, $\tilde{\tau} = -1/\tau$, $\tilde{\theta} = \theta/\tau$, (j^+, w^+) refer to the labels of the (j, w)-conjugate representations, ccr denotes the contributions of continuous representations and $\mathcal{A}^{\mathbf{s}}_{(j,w)}$ are the Ishibashi coefficients of the boundary states.

The open string partition function for the "spherical branes" of the H_3^+ model was found in [21] for $\theta = 0$ and extended to the case $\theta \neq 0$ in [34]. It reads

$$Z^{H_3^+}_{\mathbf{s}_1\mathbf{s}_2}(\theta,\tau,0) = \sum_{J_3=|J_1-J_2|}^{J_1+J_2}\chi_{J_3}(\theta,\tau,0), \tag{4.20}$$

where $\mathbf{s}_i = \frac{\pi}{k-2}(1+2J_i)$ and $1+2J_i \in \mathbb{N}$. This reveals an open string spectrum of discrete degenerate representations.

The Lorentzian partition function is expected to reproduce that of the H_3^+ model after analytic continuation in θ and τ. Then, if we concentrate on the one-point functions of fields in discrete representations, we only need to consider the case $\theta + n\tau \notin \mathbb{Z}$. Thus, using the generalized Verlinde formula (see Appendix C for details), namely

$$\begin{aligned} &\sum_{J_3=|J_1-J_2|}^{J_1+J_2}\chi_{J_3}(\theta,\tau,0) \\ &\quad = \sum_{w=-\infty}^{\infty}\int_{-(k-1)/2}^{-(1/2)} dj\frac{S_{J_1}{}^{j,w}S_{J_2}{}^{j,w}}{S_0^{j,w}} \\ &\qquad \times e^{2\pi i(k/4)(\theta^2/\tau)}\chi_j^{+,w}\left(\frac{\theta}{\tau}, -\frac{1}{\tau}, 0\right), \end{aligned} \tag{4.21}$$

we obtain the following expression for the coefficients of the boundary states:

$$\begin{aligned} \mathcal{A}^{\mathbf{s}}_{(j,w)} &= f(j,w)(-)^w\sqrt{\frac{2}{i}}\left(\frac{2}{k-2}\right)^{1/4} \\ &\quad \times \frac{\sin[\mathbf{s}(1+2j-w(k-2))]}{\sqrt{\sin[\frac{\pi}{k-2}(1+2j)]}}, \end{aligned} \tag{4.22}$$

defined up to a function $f(j,w)$ satisfying $f(j,w)\times f(-\frac{k}{2}-j, -w-1) = 1$.

One-point functions To find the one-point functions associated to these pointlike branes, let us make use of the following definition of boundary states (see for instance [39])[11]:

$$\begin{aligned} &\langle\Phi^{(H)}(|j,m,\bar{m},w\rangle; z,\bar{z})\rangle_{\mathbf{s}} \\ &\quad = \left(\frac{d\xi}{dz}\right)^{\Delta_j}\left(\frac{d\bar{\xi}}{d\bar{z}}\right)^{\bar{\Delta}_j}\langle 0|\Phi^{(P)}(|j,m,\bar{m},w\rangle;\xi,\bar{\xi})|\mathbf{s}\rangle, \end{aligned} \tag{4.23}$$

where $\Phi^{(H)}(|j,m,\bar{m},w\rangle; z,\bar{z})$ $(\Phi^{(P)}(|j,m,\bar{m},w\rangle;\xi,\bar{\xi}))$ is the bulk field of the boundary (bulk) CFT corresponding to the state inside the brackets,[12] $z, \bar{z}$ denote the coordinates of the upper half plane and $\xi, \bar{\xi}$ those of the exterior of the unit disc.

Conformal invariance forces the l.h.s. of (4.23) to be

$$\langle\Phi^{(H)}(|j,m,\bar{m},w\rangle; z,\bar{z})\rangle_{\mathbf{s}} = \frac{\mathcal{B}(\mathbf{s})^{j,w}_{m,\bar{m}}}{|z-\bar{z}|^{\Delta_j+\bar{\Delta}_j}}, \tag{4.24}$$

where the z-independent factor $\mathcal{B}(\mathbf{s})^{j,w}_{m,\bar{m}}$ is not fixed by the conformal symmetry. The solution (4.17) and (4.18) implies

[11]Strictly speaking, this identity is valid on a Euclidean world sheet. However, it is appropriate to use it here since we want to explore the relation of our results with those of the Euclidean model defined in [19] where the coefficients of the one-point functions are assumed to coincide with those of the Lorentzian AdS_3.

[12]Here $|j,m,\bar{m},w>$ is a shorthand notation for $|j,m,w> \otimes|j,\bar{m},w>$ and it must be distinguished from the orthonormal basis introduced in Sec. IV B 1.

$$\mathcal{B}(\mathbf{s})^{j,w}_{m,\bar{m}} = (-)^{j+m}\delta_{m,\bar{m}}\mathcal{A}^{\mathbf{s}}_{j,w}, \tag{4.25}$$

from which the spectral flow symmetry determines $f = 1$.

It is important to note that the normalization used here differs from the one usually considered in the literature. Our normalization is such that the spectral flow image of the primary operator corresponding to the state $|j, m, \bar{m}, w>$ is normalized to 1. In particular, it implies the following operator product expansions

$$J^3(\zeta)\Phi^{(P)}(|j,m,\bar{m},w\rangle;\xi,\bar{\xi}) = \frac{m+\frac{k}{2}w}{\zeta-\xi}\Phi^{(P)}(|j,m,\bar{m},w\rangle;\xi,\bar{\xi}) + \ldots$$
$$J^{\pm}(\zeta)\Phi^{(P)}(|j,m,\bar{m},w\rangle;\xi,\bar{\xi}) = \frac{\sqrt{-j(1+j)+m(m\pm1)}}{(\zeta-\xi)^{1\pm w}}\Phi^{(P)}(|j,m\pm1,\bar{m},w\rangle;\xi,\bar{\xi}) + \ldots \tag{4.26}$$

In Appendix D 1, we show that (4.25) agrees with the one-point function obtained in [19].

3. *Cardy structure in* H_2 *branes*

In Appendix D we review the results for the one-point functions in maximally symmetric *D*-branes obtained by applying the method of [19]. From the one-point functions of fields in discrete representations on H_2 branes we find the following Ishibashi coefficients (see (D10) and (D12))

$$\mathcal{A}^{\sigma'\equiv(j',w')}_{(j,w)} = \frac{\pi}{\sqrt{k}}\left(\frac{2}{k-2}\right)^{3/4} \times \frac{(-)^w e^{4\pi i/k-2(j'+(1/2)-w'(k-2)/2)(j+(1/2)-w(k-2)/2)}}{\sqrt{\sin[\frac{\pi}{k-2}(2j+1)]}}, \tag{4.27}$$

satisfying

$$\mathcal{A}^{j_1,w_1}_{(j,w)}\mathcal{A}^{j_2,w_2}_{(j^+,w^+)} \sim (-)^{w_1+w_2}\frac{\mathcal{S}_{j_1w_1}{}^{jw}\mathcal{S}_{j_2w_2}{}^{j^+w^+}}{\mathcal{S}_0^{jw}}, \tag{4.28}$$

where $\sim$ stands for equal up to the $k-$ dependent factor $\frac{-i4\pi^2}{k(k-2)}$. This expression leads to the following degeneracy for the open string spectrum of discrete representations

$$\mathcal{N}_{j_1,w_1 j_2,w_2}{}^{j_3,w_3} = \frac{-i2\pi^2(-)^{w_3}}{k(k-2)}\sum_{m=-\infty}^{\infty}\delta\left(j_2+j_3-j_1 - (w_2+w_3-w_1)\frac{k-2}{2}+m\right),$$

where the divergent integral $\int_0^1 d\lambda \frac{e^{-2\pi i(m+(1/2))\lambda}}{2i\sin(\pi\lambda)}$ has been replaced by its principal value, $\frac{1}{2}$.

Two comments are in order. First, a non negative integer times a Kronecker or Dirac delta function would be expected for the degeneracy. An integer can be obtained through a small modification by an overall k-dependent factor in the one-point functions, but the sign factor $(-)^{w_3}$ cannot be removed in this way, and it inevitably leads to negative degeneracies. The second comment is about the Verlinde theorem. Contrary to what happens in RCFT, here the Cardy structure is not accompanied by a Verlinde formula. Even, if we ignore the problems mentioned in the first comment, the naive application of this formula gives contributions to the fusion rules violating the spectral flow number conservation by an arbitrary amount, in contradiction with the selection rules determined in [3].

4. *Coherent states for twined gluing conditions*

The gluing conditions defining the coherent states $|j, w\gg$ for AdS_2 branes [31], frequently called twisted boundary conditions, are

$$(J^3_n + \bar{J}^3_{-n})|j,w\gg = 0, \quad (J^{\pm}_n + \bar{J}^{\pm}_{-n})|j,w\gg = 0. \tag{4.29}$$

These constraints are highly restrictive. As we show below, coherent states satisfying these conditions can only be found for representations where the holomorphic and antiholomorphic sectors are conjugate of each other, i.e. only for $w = 0$, $\alpha = 0, \frac{1}{2}$ continuous representations in the AdS_3 model.

Let us assume $|j, w\gg$ is an Ishibashi state associated to the spectral flow image of a discrete or continuous representation. The spectral flow transformation (2.3) allows to translate the problem of solving (4.29) to that of solving

$$(J^3_n + \bar{J}^3_{-n} + kw\delta_{n,0})|j\rangle^w = 0,$$
$$(J^{\pm}_n + \bar{J}^{\pm}_{-n\mp2w})|j\rangle^w = 0, \tag{4.30}$$

where $|j\rangle^w = U_{-w}\bar{U}_{-w}|j,w\gg$ is in an unflowed representation.[13]

The special case $n = 0$ in (4.30) implies $2\alpha + kw \in \mathbb{Z}$ and $-2j + kw \in \mathbb{Z}$ for continuous and discrete representations, respectively. In particular, for $w = 0$ continuous representations there are two solutions with $\alpha = 0, \frac{1}{2}$, given by

$$|j,0,\alpha\gg = \sum_{\mathrm{n}}\epsilon_{\mathrm{n}}\bar{U}|j,w,\alpha;\mathrm{n},\mathrm{n}\rangle, \tag{4.31}$$

[13]Notice that in the case $w = -\bar{w}$ discussed in [31] for the single covering of $SL(2,\mathbb{R})$, one gets (4.29) with the unflowed $|j>^{w,\bar{w}}$ state replacing $|j, w\gg$ instead of (4.30). Then, once an Ishibashi state is found for $w = -\bar{w} = 0$, the solutions for generic representations with $w = -\bar{w}$ are trivially obtained applying the spectral flow operation, and coherent states in arbitrary spectral flow sectors are found. This fails in AdS_3 and thus the discussion in *loc. cit.* does not apply here, except for $w = 0$ discrete or $w = 0$, $\alpha = 0$, $\frac{1}{2}$ continuous representations.

where the antilinear operator U is defined by

$$U\prod_I J^{a_I}_{n_I}|j,m=\alpha,w=0\rangle=\prod_I -J^{a_I}_{n_I}|j,m=-\alpha,w=0\rangle. \tag{4.32}$$

It can be easily verified that this defines an anti-unitary operator and it is exactly the same Ishibashi state found in SU(2) [38].

To understand why there are no solutions in other modules, let us expand the hypothetical Ishibashi state in the orthonormal base $|j,w,\zeta>\{|n>\otimes\overline{|m>}\}$, with $\zeta=\alpha$ or $+$ and $|n>$, $\overline{|m>}$ eigenvectors of J^3_0, L_0 and $\bar{J}^3_0$, $\bar{L}_0$ respectively. The constraint that Ishibashi states are annihilated by $L_0-\bar{L}_0$ forces $|n>$, $\overline{|m>}$ to be at the same level. But taking into account that all modules at a given level are highest or lowest-weight representations of the zero modes of the currents (with the only exception of $w=0$ continuous representations) and the fact that the eigenvalues of the highest (lowest) weight operators decrease (increase) after descending a finite number of levels, the first equation in (4.29) with $n=0$ has no solution below certain level. This implies that below that level there are no contributions to the Ishibashi states and so, using for instance the constraint $(J^a_1+\bar{J}^a_{-1})|j,w\gg=0$, it is easy to show by induction that no level contributes to the coherent states.

The coherent states defined above are normalized as

$$\begin{aligned}&\ll j,0,\alpha|e^{\pi i\tau(L_0+\bar{L}_0-(c/12))}e^{\pi i\theta(J^3_0-\bar{J}^3_0)}|j',0,\alpha'\gg\\&\quad=\delta(s-s')\delta(\alpha-\alpha')\chi^{\alpha,0}_j(\tau,\theta),\end{aligned} \tag{4.33}$$

for $\alpha=0,\frac{1}{2}$. The fact that it is only possible to construct Ishibashi states associated to $w=0$ continuous representations is again in agreement with the one-point functions found in [19] and the conjecture in [32] that only states in these representations couple to AdS_2 branes.

V. CONCLUSIONS

To conclude, let us summarize our results and contrast them with previous works in the literature.

We have computed the characters of the relevant representations of the AdS_3 model on the Lorentzian torus and studied their modular transformations. We fully determined the generalized S matrix, which depends on the sign of τ, and showed that real modular parameters are crucial to find the modular maps.

We have seen that the characters of continuous representations transform among themselves under S while both kinds of characters appear in the S transformation of the characters of discrete representations. An important consequence of this fact is that the Lorentzian partition function is not modular invariant [and the departure from modular invariance is not just the sign appearing in (3.3)]. The analytic continuation to obtain the Euclidean partition function (which must be invariant) is not fully satisfactory. Following the road of [1] and simply discarding the contact terms, one recovers the partition function of the H^+_3 model obtained in [18]. But even though modular invariant, this expression has poor information about the spectrum. Not only the characters of the continuous representations vanish in all spectral flow sectors but also those of the discrete representations are only well defined in different regions of the moduli space, depending on the spectral flow sector, so that it makes no mathematical sense to sum them in order to find the modular S transformation. An alternative approach was followed in [20], where an expression for the partition function was found starting from that of the $SL(2,\mathbb{R})/U(1)$ coset computed in [40] and using path integral techniques. Although formally divergent, it is modular invariant and allows to read the spectrum of the model.[14] It was shown that the partition function obtained in [1,18] is recovered after some formal manipulations. It would be interesting to better understand how the information is lost in the procedure implemented in [20] and to explore if it is possible to find an analytic continuation of the Lorentzian partition function leading to the integral expression obtained in *loc. it.* (or an equivalent one), in a controlled way in which the knowledge on the spectrum is not removed.

The treatment of the boundary states presented in Sec. IV differs from previous works. While we have expressed them as a sum over Ishibashi states, in other related models such as H^+_3 [22], Liouville [41] or the Euclidean black hole [34], the boundary states have been expanded, instead, in terms of primary states and their descendants. The coefficients in the latter expansions directly give the one-point functions of the primary fields. For instance, in the H^+_3 model, the gluing conditions were imposed in [22] not over the Ishibashi states but over the one-point functions. One of the reasons why this approach seems more suitable for H^+_3 is the observation that the expectation values used to fix the normalization of the Ishibashi states diverge in the hyperbolic model.[15] As we have seen, this is not the case in AdS_3.

The generalization of the Verlinde formula proposed in Sec. IV gives the fusion rules of the degenerate representations of $SL(2,\mathbb{R})$ appearing in the spectrum of open strings attached to the pointlike D-branes of the model and the coefficients of their boundary states. The formula holds for generic θ, τ far from $\theta+n\tau\in\mathbb{Z}$. It would be interesting to study the extension to generic θ, τ

[14]The spectrum was also obtained from a computation of the Free Energy in [2].

[15]Notice that, contrary to the AdS_3 or SU(2) models, the continuous representations appearing in the Hilbert space of the H^+_3 model do not factorize as tensor products of a holomorphic times an antiholomorphic representation. So, instead of the characters of the holomorphic sector appearing for instance in (4.19), the analog ones in the hyperbolic model have a trace over certain subspace of states satisfying $J^3_0=\pm\bar{J}^3_0$, depending on the gluing conditions considered. And this trace is divergent.

which requires to consider the S matrix block (3.20). Furthermore, one could also study the modular transformations of the characters of other degenerate representations and their spectral flow images and explore the validity of generalized Verlinde formulas in these cases.

We have shown that the one-point functions of fields in discrete representations coupled to H_2 branes are determined by one of the diagonal blocks of the generalized S matrix, as usual in RCFT. However, a puzzle arises when considering the open/closed duality which gives negative degeneracies in the open string spectrum of these branes. In contrast to general expectations, here the Cardy structure is not accompanied by a Verlinde theorem. Moreover, the Verlinde-like formula does not give the fusion rules of the bulk AdS_3 model. In particular, besides some undesirable negative signs, it gives contributions of arbitrary spectral flow numbers to the fusion of states in discrete representations, thus violating the selection rules established in [3].

Much remains to be understood on the modular properties and the role of the Verlinde theorem (or suitable generalizations) in this non RCFT. In particular, more work is necessary to understand what properties of the physical theory determine the relations that we have found between microscopic data and modular transformations. It would also be interesting to put the fusion rules of the AdS_3 WZNW model on a firmer ground, as puzzles such as the absence of the trivial representation [42] or the mechanism determining the truncation of states in the operator algebra [4] are far from elucidated.

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APPENDIX A: THE LORENTZIAN TORUS

In this appendix we present a description of the moduli space of the torus with Lorentzian metric.[16] Although it can be easily obtained from the Euclidean case, we include it here for completeness.

Consider the two dimensional torus with world-sheet coordinates σ^1, σ^2 obeying the identifications

$$(\sigma^1, \sigma^2) \cong (\sigma^1 + 2\pi n, \sigma^2 + 2\pi m), \qquad n, m \in \mathbb{Z}. \quad \text{(A1)}$$

By diffeomorphisms and Weyl transformations that leave invariant the periodicity, a general two dimensional Lorentzian metric can be taken to the form

$$ds^2 = (d\sigma^1 + \tau_+ d\sigma^2)(d\sigma^1 + \tau_- d\sigma^2), \qquad \text{(A2)}$$

where τ_+, τ_- are two real independent parameters. Recall that the metric of the Euclidean torus, namely $ds^2 = |d\sigma^1 + \tau d\sigma^2|^2$, is degenerate for $\tau \in \mathbb{R}$ since $detg = (\tau - \tau^*)^2$. In contrast, here it is degenerate for $\tau_- = \tau_+$.

The linear transformation

$$\tilde{\sigma}^1 = \sigma^1 + \tau^+ \sigma^2, \qquad \tilde{\sigma}^2 = \tau^- \sigma^2, \qquad \tau^{\pm} = \frac{\tau_- \pm \tau_+}{2}, \qquad \text{(A3)}$$

takes (A2) to the Minkowski metric. The new coordinates obey the periodicity conditions

$$(\tilde{\sigma}^1, \tilde{\sigma}^2) \cong (\tilde{\sigma}^1 + 2\pi n + 2\pi m\tau^+, \tilde{\sigma}^2 + 2\pi\tau^- m), \quad n, m \in \mathbb{Z}, \qquad \text{(A4)}$$

while the light-cone coordinates $\tilde{\sigma}_{\pm} = \tilde{\sigma}^1 \pm \tilde{\sigma}^2$, obey

$$\tilde{\sigma}_{\pm} \cong \tilde{\sigma}_{\pm} + 2\pi n + 2\pi m\tau_{\mp}. \qquad \text{(A5)}$$

In the Euclidean case, there are in addition global transformations that cannot be smoothly connected to the identity, generated by Dehn twists. A twist along the a cycle of a Lorentzian torus preserves the metric (A2) but changes the periodicity to

$$(\tilde{\sigma}^1, \tilde{\sigma}^2) \cong (\tilde{\sigma}^1 + 2\pi n + 2\pi m(1 + \tau^+), \tilde{\sigma}^2 + 2\pi m\tau^-), \quad n, m \in \mathbb{Z}, \qquad \text{(A6)}$$

or

$$\tilde{\sigma}_{\pm} \cong \tilde{\sigma}_{\pm} + 2\pi n + 2\pi m(\tau_{\mp} + 1). \qquad \text{(A7)}$$

Thus it gives a torus with modular parameters $(\tau'_+, \tau'_-) = (\tau_+ + 1, \tau_- + 1)$. A twist along the b cycle leads to the following periodicity conditions:

$$(\tilde{\sigma}^1, \tilde{\sigma}^2) \cong (\tilde{\sigma}^1 + 2\pi n(1 + \tau^+) + 2\pi m\tau^+, \tilde{\sigma}^2 + 2\pi n\tau^- + 2\pi m\tau^-), \qquad n, m \in \mathbb{Z}, \qquad \text{(A8)}$$

or

$$\tilde{\sigma}_{\pm} \cong \tilde{\sigma}_{\pm} + 2\pi n(1 + \tau_{\mp}) + 2\pi m\tau_{\mp}. \qquad \text{(A9)}$$

As in the Euclidean case, this is equivalent to a torus with $(\tau'_+, \tau'_-) = (\frac{\tau_+}{\tau_+ + 1}, \frac{\tau_-}{\tau_- + 1})$ and conformally flat metric. But there is a crucial difference. In the Euclidean case, the overall conformal factor multiplying the flat metric is positive definite, namely $\frac{1}{(1+\tau)(1+\tau^*)}$. On the contrary, in the Lorentzian torus, the conformal factor $\frac{1}{(1+\tau_-)(1+\tau_+)}$ is not positive definite and so, it can not be generically eliminated through a Weyl transformation.

Defining the modular S transformation as $S\tau_{\pm} = -\frac{1}{\tau_{\pm}}$, we can write $\tau'_{\pm} = \frac{\tau_{\pm}}{1+\tau_{\pm}} = TST\tau_{\pm}$, and then the problem

[16] Tori in 1 + 1 dimensions have been considered previously in [43–46] in the context of string propagation in time dependent backgrounds.

can be reformulated in the following way. The T transformation works as in the Euclidean case. Instead, under a modular S transformation, the torus defined by (A1) and (A2) is equivalent to a torus with the same periodicities but with the following metric (after diffeomorphisms and Weyl rescaling)

$$ds^2 = \mathrm{sgn}(\tau_-\tau_+)(d\sigma'^1 + \tau_+ d\sigma'^2)(d\sigma'^1 + \tau_- d\sigma'^2). \quad \text{(A10)}$$

1. The fundamental region

In the Euclidean torus, one can find a coordinate system preserving the periodicity conditions (A1), where the metric takes the form $ds^2 = |d\sigma^1 + \tau d\sigma^2|^2$, with $\tau \in \mathbb{C}$. Since it is invariant under complex conjugation, the complex τ plane can be restricted to Im $\tau > 0$ (discarding Im $\tau = 0$ because it gives a degenerate metric). Similarly, in the Lorentzian case, the metric (A2) is invariant under $\tau_+ \leftrightarrow \tau_-$ and one can take $\tau_+ > \tau_-$ (discarding $\tau_+ = \tau_-$).

Unlike the Euclidean case, where the S transformation maps the interior to the exterior of the unit circle, in the Lorentzian case it maps the interior of the hyperbola $\tau_+ = -\tau_-^{-1}$ in the second quadrant to the exterior of the hyperbola in the fourth quadrant. But the symmetry $\tau_+ \leftrightarrow \tau_-$, allows to identify this region of the fourth quadrant with the exterior of the hyperbola in the second quadrant. Similarly, using this symmetry, the S transformation maps the exterior to the interior of the hyperbola in the second quadrant (see Fig. 2) and leaves the points on the hyperbola fixed. One of these points is $(\tau_-, \tau_+) = (-1, 1)$ which corresponds to the Minkowski metric. (Recall that in the Euclidean case there is a single fixed point, $\tau = i$, giving a flat Euclidean metric).

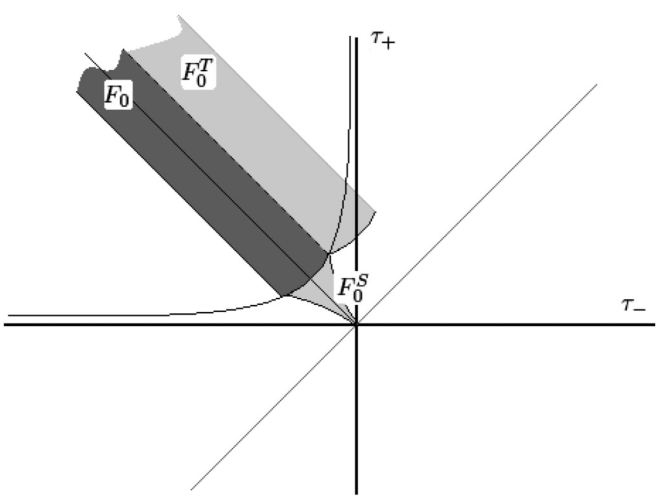

FIG. 2. A fundamental region F_0 can be defined as $-\tau_- - 1 \leq \tau_+ < -\tau_- + 1$, $\tau_- < 0$, $\tau_+ \geq -\tau_-^{-1}$ ($\tau_+ > -\tau_-^{-1}$) for $\tau_+ > -\tau_-$ ($\tau_+ < -\tau_-$). Other possible fundamental regions are the images of F_0 by S or T, denoted F_0^S, F_0^T respectively.

APPENDIX B: THE MIXING BLOCK OF THE S MATRIX

In this appendix we sketch the computation of the off-diagonal block of the S matrix mixing the characters of continuous and discrete representations.

1. A useful identity

It is convenient to begin displaying a useful identity.

Let $h(x; \epsilon_0) = \frac{1}{1-e^{2\pi i(x+i\epsilon_0)}}$, with $x \in \mathbb{R}$, be the distribution defined as the weak limit $\epsilon_0 \to 0$ and $G(x; \epsilon_1, \epsilon_2, \epsilon_3, \ldots)$ a generalized function having simple poles outside of the real line,[17] defined as the weak limit $\epsilon_i \to 0$, $i = 1, 2, 3, \ldots$. The non vanishing infinitesimals ϵ_i are allowed to depend on the x coordinate and they all differ from each other in an open set around each simple pole. Then, the following identity holds (in a distributional sense):

$$\begin{aligned}
&\frac{1}{1-e^{2\pi i(x+i\epsilon_0)}} G(x; \epsilon_1, \epsilon_2, \epsilon_3, \ldots) \\
&\quad = \frac{1}{1-e^{2\pi i(x+i\tilde{\epsilon}_0)}} G(x; \epsilon_1, \epsilon_2, \epsilon_3, \ldots) \\
&\qquad + \sum_{x_i^\downarrow} \delta(x - x_i^\downarrow) G(x; \epsilon_1 - \epsilon_0, \epsilon_2 - \epsilon_0, \epsilon_3 - \epsilon_0, \ldots) \\
&\qquad - \sum_{x_i^\uparrow} \delta(x - x_i^\uparrow) G(x; \epsilon_1 - \epsilon_0, \epsilon_2 - \epsilon_0, \epsilon_3 - \epsilon_0, \ldots), \quad \text{(B1)}
\end{aligned}$$

where $\tilde{\epsilon}_0$ is a new infinitesimal parameter, $x_i^\downarrow$ ($x_i^\uparrow$) is the real part of the pulled down (up) poles, i.e. those poles where $\epsilon_0(x_i^\downarrow) < 0 < \tilde{\epsilon}_0(x_i^\downarrow)$ ($\tilde{\epsilon}_0(x_i^\uparrow) < 0 < \epsilon_0(x_i^\uparrow)$). Of course, here $x_i^\downarrow, x_i^\uparrow \in \mathbb{Z}$, but (B1) can be trivially generalized to other functionals having simple poles, the only change being that the residue has to multiply each delta function.

The proof of this identity follows from multiplying (B1) by an arbitrary test function ($f(x) \in C_0^\infty$) and integrating over the real line.

As an example, let us consider the simplest case $G = 1$, $\epsilon_0 = 0^+$, $\tilde{\epsilon}_0 = 0^-$, where one recovers the well known formula

$$\frac{1}{1 - e^{2\pi i(x+i0^+)}} = \frac{1}{1 - e^{2\pi i(x+i0^-)}} - \sum_{m=-\infty}^{\infty} \delta(x + m). \quad \text{(B2)}$$

2. The mixing block

Let us first consider the modular transformation of the elliptic theta function

[17] $G(x; 0, 0, 0, \ldots)$ not necessarily has only simple poles. In the most general case, it will have poles of arbitrary order.

$$\frac{1}{i\vartheta_{11}(\theta+i\epsilon_2^w,\tau+i\epsilon_1)}\to\frac{1}{i\vartheta_{11}(\frac{\theta}{\tau}+i\epsilon_2^w,-\frac{1}{\tau}+i\epsilon_1)}\equiv\frac{1}{i\vartheta_{11}\Big(\frac{\theta+i\epsilon_2^{\prime w}}{\tau+i\epsilon_1'},-\frac{1}{\tau+i\epsilon_1'}\Big)}$$
$$=\frac{-\mathrm{sgn}(\tau)e^{-\pi i(\theta^2/\tau)}e^{-\mathrm{sgn}(\tau)i(\pi/4)}}{\sqrt{|\tau|}}\frac{1}{\vartheta_{11}(\theta+i\epsilon_2^{\prime w},\tau+i\epsilon_1')},\tag{B3}$$

$$\epsilon_1'=\tau^2\epsilon_1,\qquad \epsilon_2^{\prime w}=\tau(\epsilon_2^w+\theta\epsilon_1),\tag{B4}$$

and ϵ_1, ϵ_2^w satisfy (2.7). The identity (3.17) was used in the last line of (B3) and the limits ϵ_1', $\epsilon_2^{\prime w}\to 0$ were taken where it is allowed.

Let us now concentrate on the last term in (B3). It is explicitly given by (2.11), where now the ϵ's are replaced by ϵ_1', $\epsilon_3^{\prime n,w}$, $\epsilon_4^{\prime n,w}$ satisfying $\epsilon_1'>0$,

$$\epsilon_3^{\prime n,w}\begin{cases}<0, & \theta-n\tau\le -1-w\\ >0, & \theta-n\tau\ge -w\end{cases},\qquad \epsilon_4^{\prime n,w}\begin{cases}<0, & \theta+n\tau\ge -w\\ >0, & \theta+n\tau\le -1-w\end{cases},\qquad \tau<0,\tag{B5}$$

$$\epsilon_3^{\prime n,w}\begin{cases}<0, & \theta-n\tau\ge -w\\ >0, & \theta-n\tau\le -1-w\end{cases},\qquad \epsilon_4^{\prime n,w}\begin{cases}<0, & \theta+n\tau\ge -w\\ >0, & \theta+n\tau\le -1-w\end{cases},\qquad \tau>0.\tag{B6}$$

By comparing with (2.12) and using (B1), one finds, for instance in the case $w<0$, $\tau<0$, after a straightforward but tedious computation, the following identity:

$$\frac{1}{i\vartheta_{11}(\theta+i\epsilon_2^{\prime w},\tau+i\epsilon_1')}=\frac{1}{i\vartheta_{11}(\theta+i\epsilon_2^w,\tau+i\epsilon_1)}-\frac{1}{\eta^3(\tau+i\epsilon_1)}\Big[e^{-i\pi\theta}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{-w-1}(-)^n e^{\pi i\tau n(1+n)}\delta(-\theta+n\tau+m)$$
$$+e^{i\pi\theta}\Big(\sum_{n=1}^{-w-1}\sum_{m=w+1}^{\infty}-\sum_{n=-w}^{\infty}\sum_{m=-\infty}^{w}\Big)(-)^n e^{\pi i\tau n(1+n)}\delta(\theta+n\tau+m)\Big].$$

Repeating the same analysis for the other cases one finds, for arbitrary w,

$$\frac{1}{i\vartheta_{11}(\theta+i\epsilon_2^{\prime w},\tau+i\epsilon_1')}=\frac{1}{i\vartheta_{11}(\theta+i\epsilon_2^w,\tau+i\epsilon_1)}+\Bigg[\sum_{n=-\infty}^{w}\begin{cases}\sum_{m=-\infty}^{w}\delta(\theta-n\tau+m), & \tau<0\\ \sum_{m=1+w}^{\infty}\delta(\theta-n\tau+m), & \tau>0\end{cases}$$
$$-\sum_{n=1+w}^{\infty}\begin{cases}\sum_{m=1+w}^{\infty}\delta(\theta-n\tau+m), & \tau<0\\ \sum_{m=-\infty}^{w}\delta(\theta-n\tau+m), & \tau>0\end{cases}\Bigg]\frac{(-)^{n+m}e^{2i\pi\tau(n^2/2)}}{\eta^3(\tau+i\epsilon_1)}$$

Using (3.14) and summing or subtracting delta function terms like in (2.13) and (2.14), in order to construct the characters of discrete representations, one finds

$$\chi_j^{+,w}\Big(\frac{\theta}{\tau},-\frac{1}{\tau},0\Big)=e^{-2\pi i(k/4)(\theta^2/\tau)}\mathrm{sgn}(\tau)\Bigg\{\sum_{w'=-\infty}^{\infty}\int_{-(k-1)/2}^{-(1/2)}dj'\sqrt{\frac{2}{k-2}}(-)^{w+w'+1}e^{4\pi i/k-2(j'+(1/2)-w'(k-2)/2)(j+(1/2)-w(k-2)/2)}$$
$$\times\chi_{j'}^{+,w'}(\theta,\tau,0)+\sum_{w',n,m\in I(\tau)}\int_{-(k-1/2)}^{-(1/2)}dj'\sqrt{\frac{2}{k-2}}(-)^{w+1}e^{4\pi i/k-2(j'+(1/2)-w'(k-2)/2)(j+(1/2)-w(k-2)/2)}$$
$$\times\frac{e^{-(2\pi i/k-2)\tau(j'+(1/2)-w'(k-2)/2)^2}e^{-2\pi i\theta(j'+(1/2)-w'(k-2)/2)}}{\eta^3(\tau+i\epsilon_1)}(-)^{n+m}e^{2\pi i\tau(n^2/2)}\delta(\theta-n\tau+m)\Bigg\},$$

where $\sum_{w',n,m\in I(\tau)}$ is expected to reproduce the contribution from the continuous representations and is explicitly given by

$$\sum_{w',n,m\in I(\tau)}\equiv-\sum_{w'=-\infty}^{w-1}\sum_{n=1+w'}^{w}\sum_{m=-\infty}^{\infty}+\sum_{w'=1+w}^{\infty}\sum_{n=1+w}^{w'}\sum_{m=-\infty}^{\infty}+\sum_{w'=-\infty}^{\infty}\Bigg(\sum_{n=-\infty}^{w}\begin{cases}\sum_{m=-\infty}^{w}\\ \sum_{m=1+w}^{\infty}\end{cases}-\sum_{n=1+w}^{\infty}\begin{cases}\sum_{m=1+w}^{\infty}\\ \sum_{m=-\infty}^{w}\end{cases}\Bigg)$$
$$=\sum_{w'=-\infty}^{\infty}\Bigg(\sum_{n=-\infty}^{w'}\begin{cases}\sum_{m=-\infty}^{w}\\ \sum_{m=1+w}^{\infty}\end{cases}-\sum_{n=1+w'}^{\infty}\begin{cases}\sum_{m=1+w}^{\infty}\\ \sum_{m=-\infty}^{w}\end{cases}\Bigg)=\sum_{n=-\infty}^{\infty}\Bigg(\sum_{w'=n}^{\infty}\begin{cases}\sum_{m=-\infty}^{w}\\ \sum_{m=1+w}^{\infty}\end{cases}-\sum_{w'=-\infty}^{n-1}\begin{cases}\sum_{m=1+w}^{\infty}\\ \sum_{m=-\infty}^{w}\end{cases}\Bigg),$$

where the upper lines inside the brackets hold for $\tau < 0$ and the lower ones for $\tau > 0$. In the last line we have exchanged the order of summations. The sum over w' together with the integral over j', the spin of the states in discrete representations, match together to give, after analytic continuation, the integral over s', the imaginary part of the spin of the states in the principal continuous representations:

$$\sum_{w'=n}^{\infty}\int_{-(k-1)/2}^{-(1/2)} dj' e^{(4\pi i/k-2)(j'+(1/2)-w'(k-2)/2)(j+(1/2)-w(k-2)/2)} e^{-(2\pi i/k-2)\tau(j'+(1/2)-w'(k-2)/2)^2} e^{-2\pi i\theta(j'+(1/2)-w'(k-2)/2)}$$
$$= \int_{-\infty}^{0} d\lambda e^{4\pi i/k-2(\lambda-n(k-2)/2)(j+(1/2)-w(k-2)/2)} e^{-(2\pi i/k-2)\tau(\lambda-n(k-2)/2)^2} e^{-2\pi i\theta(\lambda-n(k-2)/2)}$$
$$= \begin{cases} i\int_0^\infty ds' e^{4\pi i/k-2(-is'-n(k-2)/2)(j+(1/2)-w(k-2)/2)} e^{-(2\pi i/k-2)\tau(-is'-n(k-2)/2)^2} e^{-2\pi i\theta(-is'-n(k-2)/2)}, & \tau<0, \\ -i\int_0^\infty ds' e^{4\pi i/k-2(is'-n(k-2)/2)(j+(1/2)-w(k-2)/2)} e^{-(2\pi i/k-2)\tau(is'-n(k-2)/2)^2} e^{-2\pi i\theta(is'-n(k-2)/2)}, & \tau>0. \end{cases}$$

After a similar analysis for the terms in the sum $\sum_{w'=-\infty}^{n-1}$ and relabeling the dummy index $n \to w'$, one finds the following contribution from the continuous series

$$\sum_{w'=-\infty}^{\infty} i\sqrt{\frac{2}{k-2}}\int_0^\infty ds'(-)^{w+w'+1}\Bigg[\sum_{m=-\infty}^{w} e^{(4\pi i/k-2)(-is'-w'(k-2)/2)(j+(1/2)-w(k-2)/2)} e^{-2\pi i m((1/2)+is'+w'(k-2)/2)}$$
$$-\sum_{m=1+w}^{\infty} e^{4\pi i/k-2(is'-w'(k-2)/2)(j+(1/2)-w(k-2)/2)} e^{-2\pi i m((1/2)-is'+w'(k-2)/2)}\Bigg]\frac{e^{2\pi i\tau((s'^2/k-2)+(k/4)w'^2)}}{\eta^3(\tau+i\epsilon_1)}\delta(\theta-w'\tau+m)$$

Finally, using (3.9), with the appropriate relabeling and performing the sum over m (which then simply reduces to a geometric series) one gets

$$\sum_{w'=-\infty}^{\infty}\int_0^\infty ds'\int_0^1 d\alpha' \mathcal{S}_{j,w}{}^{s',\alpha',w'}\chi_{s'}^{\alpha',w'}(\theta,\tau,0), \qquad \text{(B7)}$$

with

$$\mathcal{S}_{j,w}{}^{s',\alpha',w'} = -i\sqrt{\frac{2}{k-2}}e^{-2\pi i(w'j-w\alpha'-ww'(k/2))} \times\Bigg[\frac{e^{(4\pi/k-2)s'(j+(1/2))}}{1+e^{-2\pi i(\alpha'-is')}}+\frac{e^{-(4\pi/k-2)s'(j+(1/2))}}{1+e^{-2\pi i(\alpha'+is')}}\Bigg]. \qquad \text{(B8)}$$

It is interesting to note that (repeated indices denote implicit sum)

$$\mathcal{S}_{j,w}{}^{s_1,\alpha_1,w_1}\mathcal{S}_{s_1,\alpha_1,w_1}{}^{s',\alpha',w'} = -\mathcal{S}_{j,w}{}^{j_1,w_1}\mathcal{S}_{j_1,w_1}{}^{s',\alpha',w'}$$
$$=\frac{(-)^{w+w'+1}}{2\pi}\sum_{m=-\infty}^{\infty}\Bigg[\frac{1}{\frac{1}{2}+\alpha'-is'-m} + \frac{1}{\frac{1}{2}+\alpha'+is'-m}\Bigg]\delta\Big(j-\alpha'-(w+w')\frac{k-2}{2}+m\Big). \qquad \text{(B9)}$$

The first line implies $\mathcal{S}^2_{j,w}{}^{s',\alpha',w'} = 0$.

To show that $(\mathcal{S}T)^3{}_{j,w}{}^{s',\alpha',w'} = 0$ is a bit more involved. This block is explicitly given by

$$\mathcal{S}_{j,w}{}^{s_1,\alpha_1,w_1}[(T\mathcal{S}T\mathcal{S}T)_{s_1,\alpha_1,w_1}{}^{s',\alpha',w'}] + \mathcal{S}_{j,w}{}^{j_1,w_1}[(T\mathcal{S}T\mathcal{S}T)_{j_1,w_1}{}^{s',\alpha',w'}]. \qquad \text{(B10)}$$

The first term above coincides with the first one in (B9). This is a consequence of (3.26), which implies $(T\mathcal{S}T\mathcal{S}T)_{s_1,\alpha_1,w_1}{}^{s',\alpha',w'} = \mathcal{S}_{s_1,\alpha_1,w_1}{}^{s',\alpha',w'}$. So, in order for this block to vanish it is sufficient to show that the term inside the second bracket is exactly the $\mathcal{S}$ matrix mixing block.

The factor inside the last bracket splits into the sum

$$T_{j_1,w_1}{}^{j_2,w_2}\mathcal{S}_{j_2,w_2}{}^{s_3,\alpha_3,w_3}T_{s_3,\alpha_3,w_3}{}^{s_4,\alpha_4,w_4}\mathcal{S}_{s_4,\alpha_4,w_4}{}^{s_5,\alpha_5,w_5}T_{s_5,\alpha_5,w_5}{}^{s',\alpha',w'} + T_{j_1,w_1}{}^{j_2,w_2}\mathcal{S}_{j_2,w_2}{}^{j_3,w_3}T_{j_3,w_3}{}^{j_4,w_4}\mathcal{S}_{j_4,w_4}{}^{s_5,\alpha_5,w_5}T_{s_5,\alpha_5,w_5}{}^{s',\alpha',w'}. \qquad \text{(B11)}$$

These terms are very difficult to compute separately because each one gives the integral of a Gauss error function. So, we show here how the sums can be reorganized in order to cancel all the intricate integrals when summing both terms and one ends with the mixing block $\mathcal{S}_{j_1,w_1}{}^{s',\alpha',w'}$. In fact, after some few steps, the first line can be expressed as

$$\sqrt{\frac{2}{k-2}}\int_0^\infty ds\Big\{\tilde{S}^{s_2,\alpha_2,w_2}_{j_1,w_1}\Big[\sum_{w=-\infty}^{0} e^{-i(\pi/4)}e^{-(2\pi i/k-2)[-is-w(k-2)/2-(j_1+(1/2))+is_2]^2}e^{2\pi i w(\alpha_2+(1/2)-is_2)} - \sum_{w=1}^{\infty} e^{-i(\pi/4)}e^{2\pi i/k-2[-is-w(k-2)/2-(j_1+(1/2))+is_2]^2}e^{2\pi i w(\alpha_2+(1/2)-is_2)}\Big] + (s_2\to -s_2)\Big\}, \tag{B12}$$

where we have introduced $\tilde{S}_{j_1,w_1}{}^{s_2,\alpha_2,w_2} = -i\sqrt{\frac{2}{k-2}}e^{-2\pi i(w_2 j_1 - w_1\alpha_2 - w_1 w_2(k/2))}e^{4\pi s_2/k-2(j_1+(1/2))}$.

On the other hand, the second line in (B11) takes the form

$$\sum_{w=-\infty}^{\infty}\sqrt{\frac{2}{k-2}}\int_{-(k-/2)}^{-(1/2)} dj e^{i(\pi/2)}e^{-(2\pi i/k-2)[j+(1/2)-w(k-2)/2-(j_1+(1/2))+is_2]^2}e^{2\pi i w(\alpha_2+(1/2)-is_2)}\frac{\tilde{S}_{j_1,w_1}{}^{s_2,\alpha_2,w_2}}{1+e^{-2\pi i(\alpha_2-is_2)}} + (s_2\to -s_2). \tag{B13}$$

Now notice that, for $w\le -1$, the integral over j can be replaced by an integral over $-\frac{k-1}{2}+is$ minus an integral over $-\frac{1}{2}+is$ with $s\in(-\infty,0]$. For $w\ge 1$, the original integral splits into the same two integrals, but now with $s\in[0,\infty)$. Adding these terms to (B12) one ends, after some extra contour deformations in the remaining integrals, with $S_{j_1,w_1}{}^{s',\alpha',w'}$ and we can conclude that $(\mathcal{S}T)^3_{j,w}{}^{s',\alpha',w'} = 0$.

APPENDIX C: A GENERALIZED VERLINDE FORMULA

As is well known, the Verlinde theorem allows to compute the fusion coefficients in RCFT as

$$\mathcal{N}_{\mu\nu}{}^{\rho} = \sum_{\kappa}\frac{S_\mu{}^\kappa S_\nu{}^\kappa (S_\rho{}^\kappa)^{-1}}{S_0{}^\kappa}, \tag{C1}$$

where the index "0" refers to the representation containing the identity field. In the case of the fractional level admissible representations of the $\widehat{sl(2)}$ affine Lie algebra, the negative integer fusion coefficients obtained from (8) in [47] were interpreted as a consequence of the identification $j\to -1-j$ in [48],[18] where it was also shown that fusions are not allowed by the Verlinde formula if the fields involved are not highest- or lowest-weight. Applications to other non RCFT were discussed in [9], where generalizations of the theorem were proposed for certain representations in the Liouville theory, the H_3^+ model and the $SL(2,\mathbb{R})/U(1)$ coset.

In order to explore alternative expressions in the AdS$_3$ model, let us consider the more tractable finite dimensional degenerate representations. From the results for the characters obtained in Sec. II, it is natural to propose the following generalization of the Verlinde formula[19]

$$\sum_{J_3}\mathcal{N}_{J_1J_2}{}^{J_3}\chi_{J_3}(\theta,\tau,0) = \sum_{w=-\infty}^{\infty}\int_{-(k-1/2)}^{-(1/2)} dj\frac{S_{J_1}{}^{j,w}S_{J_2}{}^{j,w}}{S_0{}^{j,w}} \times e^{2\pi i(k/4)(\theta^2/\tau)}\chi_j^{+,w}\Big(\frac{\theta}{\tau},-\frac{1}{\tau},0\Big), \tag{C2}$$

which holds for generic (θ,τ) far from the points $\theta+n\tau\in\mathbb{Z}$, $\forall n\in\mathbb{Z}$. In order to prove it, notice that, in the region of the parameters where we claim it holds, one can neglect the $\epsilon' s$ and contact terms on both sides of the equation and show that the fusion coefficients $\mathcal{N}_{J_1J_2}{}^{J_3}$ coincide with those obtained in the H_3^+ model, namely

$$\mathcal{N}_{J_1J_2}{}^{J_3} = \begin{cases} 1 & |J_1-J_2|\le J_3\le J_1+J_2,\\ 0 & \text{otherwise.}\end{cases} \tag{C3}$$

Let us denote the r.h.s. of (C2) as $I(J_1,J_2)$ and rewrite it as (see (3.12))

$$I(J_1,J_2) = \sqrt{\frac{2}{k-2}}\frac{e^{(2\pi i/k-2)(k-2/2)^2(\theta^2/\tau)}}{\sqrt{i\tau}\,i\vartheta_{11}(\theta,\tau)} \times\int_{-\infty}^{\infty} d\lambda\frac{e^{(2\pi i/k-2)(\lambda^2/\tau)}e^{2\pi i(\theta/\tau)\lambda}}{e^{\pi i\sqrt{2/k-2}\lambda}-e^{-\pi i\sqrt{2/k-2}\lambda}} \times\big[e^{(2\pi i/k-2)N_1\lambda}+e^{-(2\pi i/k-2)N_1\lambda} - e^{(2\pi i/k-2)N_2\lambda}-e^{-(2\pi i/k-2)N_2\lambda}\big], \tag{C4}$$

where $N_1 = 2(J_1+J_2+1)$ and $N_2 = 2(J_1-J_2)$. Changing $\lambda\to-\lambda$ in the second and fourth terms, we get

$$I(J_1,J_2) = I(N_1) - I(N_2),\qquad I(N_i) = \tilde{I}(N_i,\theta,\tau)+\tilde{I}(N_i,-\theta,\tau), \tag{C5}$$

with

[18] Interestingly, it was shown in a recent detailed study of the $\hat{sl}(2)_{k=1/2}$ model [49], that the origin of the negative signs is the absence of spectral flow images of the admissible representations in the analysis of [48].

[19] A similar expression was obtained in [9] for the H_3^+ model applying the Cardy ansatz.

$$\tilde{I}(N_i, \theta, \tau) = \frac{\sqrt{\frac{2}{k-2}}}{\sqrt{i\tau} i\vartheta_{11}(\theta, \tau)} \int_{-\infty}^{\infty} d\lambda \frac{e^{(2\pi i/k-2)(1/\tau)(\lambda + \theta(k-2)/2)^2} e^{\pi i\sqrt{2/k-2}N_i\lambda}}{e^{\pi i\sqrt{2/k-2}\lambda} - e^{-\pi i\sqrt{2/k-2}\lambda}}. \tag{C6}$$

The divergent terms in this expression cancel in the sum (C5).

Without loss of generality, let us assume $J_1 \geq J_2$. To perform the λ integral in (C6), it is convenient to split the cases with odd and even N_i. Writing $N_i + 1 = 2m_i$, $m_i \in \mathbb{N}$, in the first case we get

$$\tilde{I}(N_i, \theta, \tau) = \sum_{L=0}^{m_i-1} \frac{e^{(-2\pi i/k-2)\tau L^2} e^{-2\pi i\theta L}}{i\vartheta_{11}(\theta, \tau)} - \frac{e^{\pi i(k-2)/2(\theta^2/\tau)}}{\sqrt{i\tau} i\vartheta_{11}(\theta, \tau)} \times \int_{-\infty}^{\infty} d\lambda \frac{e^{\pi i(\lambda^2/\tau)} e^{2\pi i\sqrt{k-2/2}(\theta\lambda/\tau)}}{1 - e^{2\pi i\sqrt{2/k-2}\lambda}}, \tag{C7}$$

where the second term diverges. For even N_i, take $N_i + 2 = 2n_i$ with $n_i \in \mathbb{N}$, and then

$$\tilde{I}(N_i, \theta, \tau) = \sum_{L=0}^{n_i-1} \frac{e^{(-2\pi i/k-2)\tau(L-(1/2))^2} e^{-2\pi i\theta(L-(1/2))}}{i\vartheta_{11}(\theta, \tau)} - \frac{e^{\pi i(k-2)/2(\theta^2/\tau)}}{\sqrt{i\tau} i\vartheta_{11}(\theta, \tau)} \times \int_{-\infty}^{\infty} d\lambda \frac{e^{\pi i(\lambda^2/\tau)} e^{2\pi i\sqrt{(k-2)/2}(\theta\lambda/\tau)} e^{-\pi i\sqrt{2/k-2}\lambda}}{1 - e^{2\pi i\sqrt{2/k-2}\lambda}}, \tag{C8}$$

where again the second term diverges.

Notice that N_1 and N_2 are either both even or odd, and since the divergent term is the same in $I(N_1)$ and $I(N_2)$, it cancels in the sum $I(J_1, J_2)$. Thus, putting all together we get

$$I(J_1, J_2) = \sum_{J_3=J_1-J_2}^{J_1+J_2} \frac{-e^{(-2\pi i/4(k-2))\tau(2J_3+1)^2} 2\sin(\pi i\theta(2J_3+1))}{\vartheta_{11}(\theta, \tau)} = \sum_{J_3=J_1-J_2}^{J_1+J_2} \chi_{J_3}(\theta, \tau, 0). \tag{C9}$$

where we have defined $J_3 = L - \frac{1}{2}$ for odd N_1 and N_2 and $J_3 = L - 1$ for even N_1 and N_2.

From a similar analysis of the case $J_2 > J_1$, we obtain (C2) and (C3).

In conclusion, consistently with the assumption that correlation functions of fields in degenerate representations in the H_3^+ and AdS_3 models are related by analytic continuation, the generalized Verlinde formula (C2) reproduces the fusion rules of degenerate representations previously obtained in the Euclidean model. However, even if it is not expected to reproduce the fusion rules of continuous representations [48], applying it for discrete representations also fails.

APPENDIX D: ONE-POINT FUNCTIONS

In this appendix we summarize the results for one-point functions in maximally symmetric D-branes, obtained by applying the method of [19]. The solution for one-point functions in H_2 D-branes found in *loc. cit.* holds for integer level k. Here we work with an alternative expression, equivalent to the one obtained in [19], but with a different extension for generic $k \in \mathbb{R}$.

The method rests on the observation that, after doing a T duality in the timelike direction, the N-th cover of $SL(2,\mathbb{R})$, i.e. $SL(2,\mathbb{R})_k^N$, is given by the orbifold

$$\frac{SL(2,\mathbb{R})_k/U(1) \times U(1)_{-k}}{\mathbb{Z}_{Nk}}. \tag{D1}$$

Because now the timelike direction is a free compact boson, the analytic continuation to Euclidean space is simply obtained by replacing $U(1)_{-k} \to U(1)_{R^2k}$. Thus, one can construct arbitrary correlation functions in AdS_3 from those in the cigar and the free compact boson theories, after taking the limits $N \to \infty$, $R^2 \to -1$. The effect of the orbifold is to produce new (twisted) sectors. These can be read in the following modification of the left and right momentum modes in the coset and the free boson models, respectively,

$$\frac{(n + k\omega, n - k\omega)}{\sqrt{2k}} \to \frac{(n + k\omega - \frac{\gamma}{N}, n - k\omega + \frac{\gamma}{N})}{\sqrt{2k}}, \qquad \gamma \in \mathbb{Z}_{kN},$$
$$\frac{(\tilde{n} + R^2k\tilde{\omega}, \tilde{n} - R^2k\tilde{\omega})}{R\sqrt{2k}} \to \frac{(n + kNp + R^2k\tilde{\omega} + \frac{R^2\gamma}{N}, n + kNp - R^2k\tilde{\omega} - \frac{R^2\gamma}{N})}{R\sqrt{2k}}, \tag{D2}$$

with $p \in \mathbb{Z}$ and $\omega, \tilde{\omega}$ being the winding numbers in the cigar and $U(1)$ respectively. In the N-th cover, k has to be an integer, but in the universal covering, the theory can be defined for arbitrary real level $k > 2$ [19].

The vertex operators for the orbifold theory are the product of the vertices in each space, namely

$$V^j_{n\omega\gamma p\tilde{\omega}}(z,\bar{z}) = \Phi^{sl(2)/u(1)}_{j,n,\omega-\frac{\gamma}{kN}}(z,\bar{z})\Phi^{u(1)}_{n+kNp,\tilde{\omega}+(\gamma/kN)}(z,\bar{z}). \quad \text{(D3)}$$

In the universal covering, the discrete momentum $\frac{\gamma}{kN}$ becomes a continuous parameter $\lambda \in [0,1)$, the J_0^3, $\bar{J}_0^3$ quantum numbers read

$$M = -\frac{n}{2} + \frac{k}{2}(\tilde{\omega}+\lambda), \qquad \bar{M} = \frac{n}{2} + \frac{k}{2}(\tilde{\omega}+\lambda), \quad \text{(D4)}$$

and the winding number is given by

$$w = \omega + \tilde{\omega}. \quad \text{(D5)}$$

1. One-point functions for pointlike instanton branes

To obtain the one-point functions for the pointlike branes, we simply take the $\mathbb{Z}_{kN}$ orbifold action on the product of the one-point functions associated to D branes in the cigar [34] and to Neumann boundary conditions in the $U(1)$ theories, respectively

$$\langle\Phi^{sl(2)/u(1)}_{j,n,\omega}(z,\bar{z})\rangle^{D0}_{\mathbf{s}} = \frac{\delta_{n,0}(-)^{r\omega}}{|z-\bar{z}|^{h^j_{nr}+\bar{h}^j_{nr}}}\frac{\Gamma(-j+\frac{k}{2}\omega)\Gamma(-j-\frac{k}{2}\omega)}{\Gamma(-2j-1)} \times\left(\frac{k}{k-2}\right)^{1/4}\left(\frac{\sin[\pi b^2]}{4\pi}\right)^{1/2} \times\frac{\sin[\mathbf{s}(2j+1)]}{\sin[\pi b^2(2j+1)]}\frac{\Gamma(1+b^2)\nu^{1+j}}{\Gamma(1-b^2(2j+1))}, \quad \text{(D6)}$$

and

$$\langle\Phi^{u(1)}_{\tilde{n},\tilde{\omega}}(z,\bar{z})\rangle^{\mathcal{N}}_{x_0} = \frac{\delta_{\tilde{n},0}e^{i\tilde{\omega}x_0}(\sqrt{k/2}R)^{1/2}}{|z-\bar{z}|^{(k/2)\tilde{\omega}^2}}.$$

Here $\mathbf{s} = \pi r b^2$, $r\in\mathbb{N}$, $b^2 = \frac{1}{k-2}$, $\nu = \pi\frac{\Gamma(1-\frac{1}{k-2})}{\Gamma(1+\frac{1}{k-2})}$, $\mathcal{N}$ refers to Neumann boundary conditions[20] and x_0 is the position of the $D0$ brane in the timelike direction. In the single covering of $SL(2,\mathbb{R})$, the only possibilities are $x_0 = 0$ and π, which represent the center of the group $\mathbb{Z}_2$ (see [24]). But in the universal covering, one can take $x_0 = q\pi$ with $q\in\mathbb{Z}$ (see Sec. IV A).

To compare these one-point functions with those obtained in Sec. IV, it is convenient to consider the conventions used in [4].[21] There, the fields $\Phi^{j,w}_{m,\bar{m}}$ represent the spectral flow images of the primary fields $\Phi^{j,0}_{m,\bar{m}}$, i.e. they are in correspondence with highest or lowest-weight states depending if $w<0$ or $w>0$, and have J_0^3, $\bar{J}_0^3$ eigenvalues $M = m+\frac{k}{2}w$, $\bar{M} = \bar{m}+\frac{k}{2}w$. They are related to the vertex operators (D3) as

$$\Phi^{j,w}_{m,\bar{m}}(z,\bar{z}) = (-)^w\sqrt{B(j)}V^{-1-j}_{n\omega\gamma p\tilde{\omega}}(z,\bar{z}),$$
$$B(j) = \frac{k-2}{\pi}\frac{\Gamma(1+\frac{1+2j}{k-2})}{\Gamma(-\frac{1+2j}{k-2})}\nu^{(1/2)+j}, \quad \text{(D7)}$$

When looking for $w=0$ solutions, i.e. $\omega = -\tilde{\omega}$, one expects to reproduce the one-point functions of pointlike D-branes in the H_3^+ model, which forces $x_0 = r\pi$. So,

$$\langle\Phi^{j,w}_{m,\bar{m}}(z,\bar{z})\rangle_{\mathbf{s}} = \frac{\delta_{m,\bar{m}}}{|z-\bar{z}|^{\Delta_j+\bar{\Delta}_j}}\frac{\Gamma(1+j-m)\Gamma(1+j+m)}{\Gamma(2j+1)} \times\frac{i\sqrt{k}(-)^{w+1}}{2^{\frac{5}{4}}}\frac{\sin[\mathbf{s}((2j+1)-w(k-2))]}{\sqrt{\sin[\frac{\pi}{k-2}(2j+1)]}}, \quad \text{(D8)}$$

with the parameter $\mathbf{s}$ labeling the positions of the instanton solutions.

Comparing the operator product expansions

$$J^3(\zeta)\Phi^{j,w}_{m,\bar{m}}(\xi,\bar{\xi}) = \frac{m+\frac{k}{2}w}{\zeta-\xi}\Phi^{j,w}_{m,\bar{m}}(\xi,\bar{\xi}) + \ldots$$
$$J^{\pm}(\zeta)\Phi^{j,w}_{m,\bar{m}}(\xi,\bar{\xi}) = \frac{m\mp j}{(\zeta-\xi)^{1\pm w}}\Phi^{j,w}_{m,\bar{m}}(\xi,\bar{\xi}) + \ldots \quad \text{(D9)}$$

and the antiholomorphic ones with those of the fields $\Phi^{(P)}$ of Sec. IV, namely (4.26), we obtain the following relation, valid for $m=\bar{m}\in -j+\mathbb{Z}_{\geq 0}$,

$$\Phi^{j,w}_{m,\bar{m}}(\xi,\bar{\xi}) = \Omega(-)^{j+m}\frac{\Gamma(1+j-m)\Gamma(1+j+m)}{\Gamma(1+2j)} \times\Phi^{(P)}(|j,m,\bar{m},w\rangle;\xi,\bar{\xi}), \quad \text{(D10)}$$

where Ω is the normalization of $\Phi^{j,w}_{-j,-j}$. We find perfect agreement between the expressions (4.25) and (D8) for one-point functions, as long as $\Omega = -\sqrt{\frac{-ik(k-2)}{16}}$.

2. One-point functions for H_2, dS_2 and light-cone branes

All of the H_2, dS_2 and light-cone branes can be constructed from a $D2$-brane in the cigar and taking Neumann boundary conditions in the $U(1)$. They are simply related to each other by analytic continuation of a parameter labeling the scale of the branes. Here, we discuss in detail the case of the one-point functions of fields in discrete representations on H_2 branes and show that the Cardy structure is realized in this case. These one-point functions correspond to H_2 branes at $X^3 = $ cons rather than $X^0 = $ cons, so we

[20]Recall that we considered Dirichlet gluing conditions when constructing the coherent states. Here we take Neumann boundary conditions because this is the T dual version in the time direction.

[21]Notice that here we take a different normalization in order to explicitly realize the relation between the spectral flow image of highest and lowest-weight representations.

have to translate these solutions before comparing with the results of Sec. IV.

The one-point functions for the $D2$-branes in the cigar are given by [34]

$$\langle\Phi^{sl(2)/u(1)}_{j,n,\omega}(z,\bar z)\rangle^{D2}_{\tilde\sigma}=\frac{\frac{1}{2}\delta_{n,0}(-)^{\omega}e^{-i\tilde\sigma\omega(k-2)}(\frac{k-2}{k})^{1/4}}{|z-\bar z|^{h^j_{nr}+\bar h^j_{nr}}}\times\Gamma(1+2j)\Gamma\left(1+\frac{1+2j}{k-2}\right)\nu^{(1/2)+j}\times\left(\frac{\Gamma(-j+\frac{k}{2}\omega)}{\Gamma(1+j+\frac{k}{2}\omega)}e^{i\tilde\sigma(1+2j)}+\frac{\Gamma(-j-\frac{k}{2}\omega)}{\Gamma(1+j-\frac{k}{2}\omega)}e^{-i\tilde\sigma(1+2j)}\right).\tag{D19}$$

Notice that this differs from the result in [34] by the ω dependent phase $(-)^{\omega}e^{-i\tilde\sigma\omega(k-2)}$.[22] The position of the D-brane over the $U(1)$ is again fixed by the one-point function of the H_3^+ model. We find

$$\langle\Phi^{j,w}_{m,\bar m}(z,\bar z)\rangle^{H_2,X^3}_{\tilde\sigma}=\frac{\delta_{m,\bar m}}{|z-\bar z|^{\Delta_j+\bar\Delta_j}}\frac{-1}{2^{5/4}\sqrt{i}(k-2)^{1/4}}\times\frac{\pi e^{-i\tilde\sigma w(k-2)}}{\sqrt{\sin[\frac{\pi}{k-2}(2j+1)]}}\times\left(\frac{\Gamma(1+j-m)}{\Gamma(-j-m)}e^{-i\tilde\sigma(1+2j)}+\frac{\Gamma(1+j+m)}{\Gamma(-j+m)}e^{i\tilde\sigma(1+2j)}\right).\tag{D11}$$

For fields in discrete representations with $m=-j+\mathbb{Z}_{\geq 0}$ and $j\notin\mathbb{Z}$, only one factor survives in the last line. Here $\tilde\sigma$ is a real parameter, determining the embedding of the brane in AdS_3 as $X^3=\cosh\rho\sin\tau=\sin\tilde\sigma$. So, in order to compare with the solutions discussed in Sec. IV, the identification $\tilde\sigma=\sigma+\frac{\pi}{2}$ and the global shift in the timelike coordinate on the cylinder, namely $t\to t+\frac{\pi}{2}$, must be performed. The latter simply adds a phase $e^{i\frac{\pi}{2}(M+\bar M)}$ (in fact, $J^3_0+\bar J^3_0$ gives the energy in AdS_3 and so this combination is the generator of t translations).

[22]This phase that we added by hand is required by the spectral flow symmetry, when used to construct the one-point functions for H_2 branes, which demands $\langle\Phi^{j,w}_{j,j}\rangle^{H_2}=\langle\Phi^{-(k/2)-j,w-1}_{(k/2)+j,(k/2)+j}\rangle^{H_2}$, in our conventions. The one-point function for $D2$ branes was constructed in [34] beginning from the parent H_3^+ model and was found to have some sign problems. We claim this phase cannot be deduced from the H_3^+ model because of the absence of spectral flowed states. It would be interesting to investigate the implications of this modification in the sign. Unfortunately, this information cannot be obtained from the w independent semiclassical limit of the one-point functions.

From the analysis of conjugacy classes, it is natural to relabel $\sigma=\frac{\pi}{k-2}(2j'+1)-w'\pi$, with $j'\in(-\frac{k-1}{2},-\frac{1}{2})$, $w'\in\mathbb{Z}$,[23] and

$$\langle\Phi^{j,w}_{m,\bar m}(z,\bar z)\rangle^{H_2,X^0}_{\sigma(j',w')}=\frac{\delta_{m,\bar m}}{|z-\bar z|^{\Delta_j+\bar\Delta_j}}\frac{\Gamma(1+j+m)\Gamma(1+j-m)}{\Gamma(1+2j)}\times\frac{-\pi\sqrt{-i}}{2^{5/4}(k-2)^{1/4}}\times\frac{(-)^w e^{(4\pi i/k-2)(j'+(1/2)-w'(k-2)/2)(j+(1/2)-w(k-2)/2)}}{\sqrt{\sin[\frac{\pi}{k-2}(2j+1)]}}.\tag{D12}$$

3. One-point functions for AdS_2 branes

For completeness, we display here the one-point functions for AdS_2 branes obtained in [19], in our conventions. These are constructed by gluing two one-point functions: one for a $D1$-brane in the coset model and another one with Dirichlet boundary conditions in the $U(1)$ model. The result is

$$\langle\Phi^{j,w}_{m,\bar m}(z,\bar z)\rangle^{AdS_2}_r=\frac{\delta_{w,0}\delta_{m,-\bar m}e^{-i(\pi/4)}e^{in(\theta_0+x_0)}(\frac{k-2}{2})^{1/4}}{|z-\bar z|^{\Delta_j+\bar\Delta_j}}\times\frac{\Gamma(-1-2j)}{\Gamma(-j-m)\Gamma(-j+m)}\times\cos\left(ir\left(j+\frac{1}{2}\right)+m\pi\right)\times\Gamma\left(1-\frac{1+2j}{k-2}\right)\nu^{-(1/2)-j},\tag{D13}$$

where θ_0 is related to the angles (in cylindrical coordinates) to which the branes asymptote when they get close to the boundary of AdS_3, x_0 is the location of the brane and r determines their scale. From the geometrical point of view, r seems to be an arbitrary real number, but as shown in [26], it becomes quantized at the semiclassical level.

Let us end this appendix by noticing the perfect agreement with the analysis of the coherent states presented in Sec. IV. Because of the Gamma functions in the denominator of (D13), only states in the continuous representations couple to the AdS_2 branes and, due to the delta functions, only those with $w=0$ and $m=-\bar m$ have non vanishing expectation values.

[23]The one-point functions for dS_2 branes are given by (D11) with $j'\in\{-\frac{1}{2}+i\mathbb{R}^+\}$ and for light-cone branes, they are given by $\sigma=n\pi$, $n\in\mathbb{Z}$.

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