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Constructing local bulk observables in interacting AdS/CFT

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Local operators in the bulk of anti-de Sitter can be represented as smeared operators in the dual conformal field theory. We show how to construct these bulk observables by requiring that the bulk operators commute at spacelike separation. This extends our previous work by taking interactions into account. Large-*N* factorization plays a key role in the construction. We show diagrammatically how this procedure is related to bulk Feynman diagrams.

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I. INTRODUCTION

A fundamental question in quantum gravity is whether one can define local observables [1,2]. The development of AdS/CFT [3] places this question in a new context. AdS/CFT makes it clear that, with asymptotic AdS boundary conditions, the physical degrees of freedom of quantum gravity are completely encoded in the dual conformal field theory (CFT). In this setting a complete set of observables is provided by local operators in the CFT. But local operators in the CFT only directly describe excitations near the anti-de Sitter (AdS) boundary, so the fundamental question becomes: is there a way to represent local observables in the bulk using the CFT?

As a closely related question, one of the most puzzling aspects of the duality between conformal field theories and gravity is how bulk locality on distance scales shorter than the anti-de Sitter radius of curvature can be recovered.[1] At the level of two-point functions there is no obstacle to constructing local observables in an AdS background [4–8]. So to probe further one may consider backgrounds that break conformal symmetry [9], or consider interactions around backgrounds with exact conformal symmetry [10,11].

The original dictionary [12,13] is best thought of as a mapping from bulk AdS correlators to boundary CFT correlators, in a limit where the bulk operators approach the boundary.[2] In [5–8] we formulated Lorentzian signature versions of the inverse map, allowing CFT correlators to be mapped back to bulk correlators. We worked in the large N limit, which meant we mapped the CFT to a free theory in the bulk.[3] The large-N limit is rather simple since it sends the bulk Planck length to zero. Can one construct local observables beyond this limit? Certainly the usual lore is that holography forbids the existence of truly local bulk operators. However since at zero Planck length one can represent the creation and annihilation operators of the supergravity fields using CFT data [16,17], one may expect that at least in some perturbative scheme one can extend the construction to subleading orders in $1/N$.

In the present paper we address the issue of defining local bulk observables from CFT data, at subleading orders in $1/N$, by generalizing the map in [5–8]. First we show that simply applying the linear smearing transformation of [5–8] to a local operator in the CFT leads to correlators with unwanted singularities, beyond the expected bulk light-cone singularities. These unwanted singularities imply that the would-be bulk observables do not commute at spacelike separation, and are not local from the bulk point of view once interactions are included. However, as we will show, corrected bulk observables which do commute at spacelike separation may be constructed by mixing in multitrace CFT operators with higher conformal dimensions. The relevance of higher-dimension primary fields to bulk locality was discussed in [10,11,18], while the appearance of double-trace operators in internal lines of Witten diagrams was discussed in [19,20]. The condition that the unwanted singularities can be canceled yields constraints on the CFT, which appear to be satisfied order by order in a $1/N$ expansion. It is possible the cancellation works for a large class of large N conformal field theories, in line with the conjecture of [10].

In Sec. II we describe the problem of unwanted singularities, and in Sec. III we give a proposed solution. In Sec. IV we carry out the construction of local bulk observables in AdS_2 and in Sec. V we present a similar construction in AdS_3. In Sec. VI we show how these results are compatible with bulk perturbation theory (assuming that

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[1]By bulk locality we mean the existence of local observables which are causal, i.e. which commute at spacelike separation.

[2]The map is essentially a change of basis of group theory representations, with one set of representations realized as functions on the boundary, and the other realized as harmonic functions in the bulk. This is discussed in [14].

[3]The inverse map in the opposite limit, from a free CFT to higher-spin gravity in the bulk, has been constructed in [15].

such a perturbative description of the bulk is available). We present some extensions and generalizations of our results in Sec. VII and we conclude in Sec. VIII.

II. BREAKDOWN OF LOCALITY

In this section we show that the definition of a bulk observable given in [5–8] captures the correct bulk two-point function. However when interactions are taken into account it fails to give bulk observables which commute at spacelike separation.

For concreteness we consider primary operators $\mathcal{O}_i$ of dimension Δ_i in a two-dimensional CFT. The two- and three-point functions of these operators are fixed by conformal invariance.

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \frac{\delta_{ij}}{(X^2 - T^2)^{\Delta_i}}, \tag{1}$$

$$\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\mathcal{O}_k(x_k)\rangle = \frac{c_{ijk}}{|x_i - x_j|^{\Delta_i+\Delta_j-\Delta_k}|x_i - x_k|^{\Delta_i+\Delta_k-\Delta_j}|x_j - x_k|^{\Delta_j+\Delta_k-\Delta_i}}. \tag{2}$$

In previous work [5–8] we considered a free scalar field ϕ in the bulk, dual to an operator $\mathcal{O}$ of dimension Δ in the CFT, and showed that the bulk scalar could be reconstructed from the boundary operator via the linear smearing transformation

$$\begin{aligned}\phi(Z,X,T) &= \int dX'dT' K_\Delta(Z,X,T|X',T')\mathcal{O}(X',T') \\ &= \frac{\Delta-1}{\pi}\int_{Y'^2+T'^2<Z^2} dY'dT'\left(\frac{Z^2 - Y'^2 - T'^2}{Z}\right)^{\Delta-2} \\ &\quad \times \mathcal{O}(X+iY', T+T').\end{aligned} \tag{3}$$

Applying this transformation to the first operator on the left-hand side of (1) generates the expected correlator between one bulk and one boundary point. To see this we compute (setting $X=0$ and $T' = r\cos\theta$, $Y' = r\sin\theta$)

$$\begin{aligned}&\langle\phi(Z,0,T)\mathcal{O}(0,0)\rangle \\ &\quad = \frac{\Delta-1}{\pi}\int_{Y'^2+T'^2<Z^2} dY'dT'\left(\frac{Z^2 - Y'^2 - T'^2}{Z}\right)^{\Delta-2} \\ &\qquad \times \langle\mathcal{O}(iY', T+T')\mathcal{O}(0,0)\rangle\end{aligned} \tag{4}$$

$$\begin{aligned}&= \frac{\Delta-1}{\pi}(-1)^\Delta \int_0^Z dr \int_0^{2\pi} d\theta\left(\frac{Z^2 - r^2}{Z}\right)^{\Delta-2} \\ &\quad \times \frac{r}{(r^2+T^2+2rT\cos\theta)^\Delta}.\end{aligned} \tag{5}$$

For the moment we assume $T > Z$ and later use analytic continuation to generalize. We use the result

$$\int_0^{2\pi} d\theta \frac{1}{(r^2+T^2+2rT\cos\theta)^\Delta} = 2\pi T^{-2\Delta} {}_2F_1\left(\Delta,\Delta;1;\frac{r^2}{T^2}\right) \tag{6}$$

to perform the θ integral. The r integral then becomes, defining $q = r^2/Z^2$ and $y = Z^2/T^2$,

$$\begin{aligned}&\langle\phi(Z,0,T)\mathcal{O}(0,0)\rangle \\ &\quad = \frac{\Delta-1}{2\pi R}(-1)^\Delta T^{-2\Delta}Z^\Delta \int_0^1 dq(1-q)^{\Delta-2} {}_2F_1(\Delta,\Delta;1;qy) \\ &\quad = \frac{Z^\Delta}{2\pi R}\frac{1}{(Z^2-T^2)^\Delta}.\end{aligned}$$

The result for general X, T may be obtained using Lorentz invariance and analytic continuation. With a Wightman $i\epsilon$ prescription

$$\langle\phi(Z,X,T)\mathcal{O}(0,0)\rangle = \frac{Z^\Delta}{2\pi R}\frac{1}{(Z^2+X^2-(T-i\epsilon)^2)^\Delta}. \tag{7}$$

This is the expected bulk-boundary two-point function for AdS_3 in Poincaré coordinates. Note that inside the two-point function the operators will commute at bulk spacelike separation. So at this level bulk locality appears to be compatible with the definition (3) of a bulk observable. As we now show, interactions change this conclusion.

To take interactions into account we study a three-point function. For simplicity we consider three operators of dimension $\Delta = 2$. Up to an overall coefficient, their three-point function reads

$$\langle\mathcal{O}(x_0)\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{1}{|x_0-x_1|^2|x_0-x_2|^2|x_1-x_2|^2}. \tag{8}$$

We now smear the first operator using (3), to turn it into a bulk operator. We need to do the integral (note that the last term on the right-hand side just comes along for the ride, so will be dropped)

$$\langle\phi(Z,X_0,T_0)\mathcal{O}(X_1,T_1)(X_2,T_2)\rangle \sim \int_{Y'^2+T'^2<Z^2}\frac{dY'dT'}{[(T_1-T_0-T')^2-(X_1-X_0-iY')^2][(T_2-T_0-T')^2-(X_2-X_0-iY')^2]}. \tag{9}$$

Defining $T' = r\cos\theta$ and $Y' = r\sin\theta$ with $\alpha = e^{i\theta}$, and denoting $X^+_{kl} = (T+X)_k - (T+X)_l$ and $X^-_{kl} = (T-X)_k - (T-X)_l$, we get

$$\int_0^Z rdr\int \alpha d\alpha[(X^+_{10} - r\alpha)(\alpha X^-_{10} - r)(X^+_{20} - r\alpha)(\alpha X^-_{20} - r)]^{-1}, \tag{10}$$

where the integral over α is a contour integral around the poles at $\alpha = \frac{r}{X_{10}^-}$ and $\alpha = \frac{r}{X_{20}^-}$. Doing the integrals gives

$$\frac{1}{(X_1-X_2)^2-(T_1-T_2)^2}\left[\ln\frac{Z^2-X_{10}^+X_{10}^-}{Z^2-X_{20}^+X_{10}^-}+\ln\frac{Z^2-X_{20}^+X_{20}^-}{Z^2-X_{10}^+X_{20}^-}\right]. \tag{11}$$

This result is AdS covariant, as we show in Appendix C. However the terms $\ln(Z^2 - X_{20}^+X_{10}^-)$ and $\ln(Z^2 - X_{10}^+X_{20}^-)$ give rise to singularities (and hence nonzero commutators) even when all three operators are spacelike separated. This means the prescription (3) for defining bulk operators in the CFT cannot be used beyond the leading large-N limit (that is, when the bulk theory is not free).

Another way to reach the same conclusion is to study the operator product expansion (OPE) between quasiprimary operators.

$$\mathcal{O}_i(X,T)\mathcal{O}_j(0)=\frac{\delta_{ij}}{(X^2-T^2)^{\Delta_i}}+\sum_k\frac{c_{ijk}}{(X^2-T^2)^{(\Delta_i+\Delta_j-\Delta_k)/2}}\mathcal{O}_k(0)+\cdots \tag{12}$$

For simplicity we specialize to a dimension-two operator with

$$\mathcal{O}(X,T)\mathcal{O}(0)=\frac{1}{(X^2-T^2)^2}+\frac{1}{N}\frac{\mathcal{O}(0)}{(X^2-T^2)}+\cdots \tag{13}$$

(the $1/N$ coefficient reflects large-N counting). Let us try to use the smearing transformation (3) to turn this into a bulk-boundary OPE. The first term in (13) just gives the bulk-boundary two-point function, but the second term gives

$$\frac{\mathcal{O}(0)}{\pi N}\int_{Y'^2+T'^2<Z^2}\frac{1}{(X+iY')^2-(T+T')^2}. \tag{14}$$

Unlike the three-point correlator considered above this integral is not AdS covariant.[4] We can do the integral by going to r, α variables as before, and we get

$$\sim\frac{\mathcal{O}(0)}{N}\int_0^Z rdr\oint_{|\alpha|=1}d\alpha\frac{1}{(T+X+r\alpha)(\alpha(T-X)+r)}=\frac{\mathcal{O}(0)}{2N}\ln\frac{X^2-T^2}{X^2+Z^2-T^2}. \tag{15}$$

Besides the expected bulk light-cone singularity, at $X^2+Z^2=T^2$, there is a boundary light-cone singularity at $X^2=T^2$. Again these unwanted singularities (which are not even AdS covariant) mean that operators will not commute at bulk spacelike separation.

This means the boundary-to-bulk map constructed in [5–8], if applied to an interacting CFT, gives rise to a set of bulk observables which are nonlocal in the sense that they do not commute at spacelike separation. These nonlocal observables could still be used to study bulk physics. However it is natural to ask if there is a way of constructing bulk observables in an interacting CFT. This is the question we address in the remainder of the paper.

III. A POSSIBLE CURE

Since we still want a bulk scalar field, there are only a limited number of ways of changing the original construction (3). Given an operator of dimension Δ the smearing function we used is the unique way of mapping it to a bulk scalar field. So the only possible deformation of our construction is to add higher dimension, appropriately smeared primary operators (assuming such operators are available). With a sum over CFT primaries, the definition of a bulk operator becomes

$$\phi(Z,X,T)=\int K_\Delta(Z,X,T|X',T')\mathcal{O}(X',T')+\sum_k d_k\int K_{\Delta_k}(Z,X,T|X',T')\mathcal{O}_{\Delta_k}(X',T'). \tag{16}$$

Here K_{Δ_k} is the appropriate AdS covariant smearing function for an operator of dimension Δ_k. As we will see later the terms we have added produce log singularities of the type we found in the previous section, times a polynomial in $(X^2-T^2)/Z^2$. The coefficients d_k can be fixed (or at least constrained) by demanding that the unwanted log singularities appearing in (11) and (15) are canceled to some order in $(X^2-T^2)/Z^2$ (or perhaps to all orders). Of course this cancellation requires the existence of primary fields with increasing dimensions, with appropriate OPE's. If such operators are unavailable then bulk locality is destroyed on macroscopic scales.

The two-point function one recovers from this procedure is consistent with the general form of a two-point function one would expect based on a spectral decomposition

$$\langle\phi(Z,X,T)\phi(Z',X',T')\rangle_{\text{bulk}}=\int dm^2\rho(m^2)G_0(Z,X,T|Z',X',T';m^2), \tag{17}$$

where $G_0(.|.;m^2)$ is the free two-point function for a scalar field of mass m^2 and $\rho(m^2)$ is the positive semidefinite spectral density. It is worth making a few remarks on the formula (17) that are somewhat surprising from the viewpoint of flat space quantum field theory. In general, the bounds of the integral range over all possible values of the mass allowed by unitarity. However this is puzzling from the CFT viewpoint, since it appears to require a continuous spectrum of quasiprimary operators. Typically well-behaved conformal field theories have a discrete tower of primary operators. The puzzle is resolved once one realizes that at least in bulk perturbation theory, the density of states $\rho(m^2)$ is typically not a continuous function. Rather the AdS symmetries pick out discrete towers of masses that

[4]Since the OPE is a short-distance expansion in the CFT it does not lift to a covariant OPE in the bulk.

arise when, for example, an interaction polynomial in a scalar field is expanded in perturbation theory [21]. Thus, for example, a scalar field ϕ with mass m and interaction $\lambda\phi^3$ would give rise to terms in (17) dual to CFT operators of conformal weight $\Delta + n$, as well as weights $2\Delta + n, 3\Delta + n, \cdots$, with n a non-negative integer.

In the remainder of this paper we show in explicit examples that, at least in CFT's with a $1/N$ expansion, it seems possible to construct the higher-dimension operators which are necessary for bulk locality, as multitrace operators with derivatives.

IV. CFT CONSTRUCTION: AdS_2

In this section we show that one can correct the definition of a bulk observable in such a way as to restore bulk locality. For simplicity we begin with AdS_2; in the next section we treat AdS_3.

As a guide, in Sec. IV A we review correlators in AdS_2/CFT_1. In Sec. IV B we apply the linear smearing transformation to two- and three-point functions in the CFT and show that the resulting bulk operators fail to commute at spacelike separation. In Sec. IV C we argue that this can be cured by adding an infinite sequence of higher-dimension operators; we construct the necessary operators using a $1/N$ expansion. In Sec. IV D we show that another way to restore spacelike commutativity at $\mathcal{O}(1/N)$ is to add a bilocal correction term. This bilocal correction can be thought of as resumming the tower of higher-dimension operators.

A. AdS correlators

We work in the Poincaré patch of AdS_2 with metric

$$ds^2 = \frac{R^2}{Z^2}(-dT^2 + dZ^2).$$

We consider a massless scalar field ϕ, dual to a dimension-one operator $\mathcal{O}$ in the CFT. That is, as $Z \to 0$ we have

$$\phi(T, Z) \to Z\mathcal{O}(T).$$

The free bulk two-point function is [22]

$$\langle\phi(T, Z)\phi(T', Z')\rangle = \frac{1}{2\pi}\tanh^{-1}(1/\sigma), \tag{18}$$

where the invariant distance

$$\sigma = \frac{Z^2 + Z'^2 - (T - T')^2}{2ZZ'}. \tag{19}$$

Sending one point to the boundary gives the mixed bulk-boundary correlator

$$\langle\phi(T, Z)\mathcal{O}(T')\rangle = \frac{1}{\pi}\frac{Z}{Z^2 - (T - T')^2}, \tag{20}$$

while sending both points to the boundary gives the CFT correlator

$$\langle\mathcal{O}(T)\mathcal{O}(T')\rangle = -\frac{1}{\pi}\frac{1}{(T - T')^2}. \tag{21}$$

So far we have not given a prescription for handling light-cone singularities. The correct prescription depends on which Green's function you want. The Wightman function is defined by $T \to T - i\epsilon$, while the Feynman function is defined by $(T - T')^2 \to (T - T')^2 - i\epsilon$. So for instance

$$\langle 0|\mathcal{O}(T)\mathcal{O}(T')|0\rangle = -\frac{1}{\pi}\frac{1}{(T - T' - i\epsilon)^2}, \tag{22}$$

$$\langle 0|T\{\mathcal{O}(T)\mathcal{O}(T')\}|0\rangle = -\frac{1}{\pi}\frac{1}{(T - T')^2 - i\epsilon}. \tag{23}$$

We will also need the three-point correlator in the CFT. Provided that $T_1 > T_2 > T_3$ this is given by

$$\langle 0|\mathcal{O}(T_1)\mathcal{O}(T_2)\mathcal{O}(T_3)|0\rangle = -\frac{i\lambda R^2}{\pi}\frac{1}{(T_1 - T_2)(T_1 - T_3)(T_2 - T_3)}. \tag{24}$$

Here λR^2 is a dimensionless coefficient. As we will discuss in Appendix A this is induced at tree level by a bulk $\lambda\phi^3$ interaction. However aside from the coefficient the form of this result is fixed by conformal invariance. It can be continued outside the range $T_1 > T_2 > T_3$ with suitable $i\epsilon$ prescriptions. For instance suppose we wanted to extend (24) past the singularity at $T_1 = T_2$ without changing the operator ordering. This can be done with a $T_1 \to T_1 - i\epsilon$ prescription:

$$\langle 0|\mathcal{O}(T_1)\mathcal{O}(T_2)\mathcal{O}(T_3)|0\rangle = -\frac{i\lambda R^2}{\pi}\frac{1}{(T_1 - T_2 - i\epsilon)(T_1 - T_3)(T_2 - T_3)}. \tag{25}$$

This is the same prescription used to handle singularities in the Wightman function (22). It can be understood as a way to regulate the time evolution operator $e^{-iH(T_1 - T_2)}$. Other choices are possible, for instance the time-ordered three-point function is given in (A2).

B. Linear smearing

At lowest order we have the linear smearing relation [5]

$$\phi^{(0)}(T, Z) = \frac{1}{2}\int_{T-Z}^{T+Z} dT_1\mathcal{O}(T_1). \tag{26}$$

In this section we use this relation to generate candidate bulk observables. We will show that everything works fine at the level of two-point functions. But when we consider three-point functions we will see that the bulk operators we construct fail to commute at spacelike separation.

To illustrate the procedure, consider smearing one leg of the CFT three-point function (22). This should give a mixed bulk-boundary correlator. Using a Wightman $i\epsilon$ prescription we find

$$\langle 0|\phi^{(0)}(T,Z)\mathcal{O}(T')|0\rangle = \frac{1}{2}\int_{T-Z}^{T+Z} dT_1\left(-\frac{1}{\pi}\right)\frac{1}{(T_1 - T' - i\epsilon)^2} = \frac{1}{\pi}\frac{Z}{Z^2-(T-T'-i\epsilon)^2}, \tag{27}$$

which reproduces the exact result (20). Likewise smearing the second leg gives

$$\langle 0|\phi^{(0)}(T,Z)\phi^{(0)}(T',Z')|0\rangle = \frac{1}{2}\int_{T'-Z'}^{T'+Z'} dT_1'\frac{1}{\pi}\frac{Z}{Z^2-(T-T_1'-i\epsilon)^2} = \frac{1}{2\pi}\tanh^{-1}\left(\frac{2ZZ'}{Z^2+Z'^2-(T-T'-i\epsilon)^2}\right) \tag{28}$$

in agreement with the bulk Wightman function (18). The $i\epsilon$ prescriptions here cause no difficulty: smearing the Wightman function in the CFT gives the correct bulk Wightman function. As we will see in Sec. VI B, the story is more complicated for Feynman propagators.

So far, so good. But now let us see what happens when we apply the linear smearing relation (26) to the first operator in the CFT three-point function (24). Taking $T-Z>T_2>T_3$ so that we do not need to worry about $i\epsilon$ prescriptions, the integral gives

$$\langle 0|\phi^{(0)}(T,Z)\mathcal{O}(T_2)\mathcal{O}(T_3)|0\rangle = \frac{i\lambda R^2}{2\pi}\frac{1}{(T_2-T_3)^2}\log\frac{(T+Z-T_3)(T-Z-T_2)}{(T+Z-T_2)(T-Z-T_3)}. \tag{29}$$

This result has some nice properties. It only has singularities when the bulk point is lightlike-separated from one of the boundary points. Also it is covariant under $SO(1,2)$.[5]

Despite these nice properties, the bulk operators we have constructed do not commute at spacelike separation. To see this we first continue (29) into the regime $T+Z>T_2>T-Z>T_3$, using a $T_2\to T_2+i\epsilon$ prescription to avoid the singularity at $T_2=T-Z$. This gives

$$\langle 0|\phi^{(0)}(T,Z)\mathcal{O}(T_2)\mathcal{O}(T_3)|0\rangle = \frac{i\lambda R^2}{2\pi}\frac{1}{(T_2-T_3)^{|2}}\log\frac{(T+Z-T_3)(T-Z-T_2-i\epsilon)}{(T+Z-T_2)(T-Z-T_3)}. \tag{30}$$

Then we repeat the calculation, starting from

$$\langle 0|\mathcal{O}(T_2)\mathcal{O}(T_1)\mathcal{O}(T_3)|0\rangle = +\frac{i\lambda R^2}{\pi}\frac{1}{(T_1-T_2)(T_1-T_3)(T_2-T_3)}, \tag{31}$$

which is valid for $T_2>T_1>T_3$. Note the change of sign! Smearing the middle operator and continuing to $T+Z>T_2>T-Z>T_3$ with a $T_2\to T_2-i\epsilon$ prescription gives

$$\langle 0|\mathcal{O}(T_2)\phi^{(0)}(T,Z)\mathcal{O}(T_3)|0\rangle = -\frac{i\lambda R^2}{2\pi}\frac{1}{(T_2-T_3)^2}\log\frac{(T+Z-T_3)(T_2-T+Z)}{(T_2-T-Z-i\epsilon)(T-Z-T_3)}. \tag{32}$$

Taking the difference of (30) and (32) gives the commutator

$$\langle 0|i[\phi^{(0)}(T,Z),\mathcal{O}(T_2)]\mathcal{O}(T_3)|0\rangle = -\frac{\lambda R^2}{\pi}\frac{1}{(T_2-T_3)^2}\log\frac{(T+Z-T_3)(T_2-T+Z)}{(T+Z-T_2)(T-Z-T_3)}. \tag{33}$$

This is nonvanishing at spacelike separation.

C. Higher-dimension operators

Let us see if we can add something to the lowest-order bulk operator (26) that will restore spacelike commutativity. The only objects at our disposal would seem to be higher-dimension operators. For instance at large N we can build a dimension-two primary field[6]

$$\mathcal{O}_2(T) = :\mathcal{O}(T)\mathcal{O}(T):$$

and we could imagine adding a correction term

$$\phi^{(1)}_{\Delta=2}(T,Z) = A\int_{T-Z}^{T+Z} dT'\frac{Z^2-(T-T')^2}{Z}\mathcal{O}_2(T'). \tag{34}$$

Here A is a coefficient we need to determine, and we have used the smearing function $\sim(\sigma Z')^{\Delta-1}$ appropriate to a dimension-two operator. Likewise at dimension four we have a primary field

$$\mathcal{O}_4(T) = :\partial_T\mathcal{O}\partial_T\mathcal{O} - \frac{2}{3}O\partial_T^2 O:$$

and we could imagine adding a correction

$$\phi^{(1)}_{\Delta=4}(T,Z) = B\int_{T-Z}^{T+Z} dT'\left(\frac{Z^2-(T-T')^2}{Z}\right)^3\mathcal{O}_4(T').$$

In this way we have an infinite number of parameters $A, B, \ldots$ at our disposal. The idea is to fix these coefficients

[5]The prefactor $1/(T_2-T_3)^2$ has the right conformal weight, and you can check that the argument of the logarithm is invariant under the special conformal transformation

$$T\to\frac{T+b(T^2-Z^2)}{1+2bT+b^2(T^2-Z^2)},\quad Z\to\frac{Z}{1+2bT+b^2(T^2-Z^2)}.$$

The boundary points transform as $T_2\to T_2/(1+bT_2)$, $T_3\to T_3/(1+bT_3)$.

[6]The colons denote normal ordering, i.e. no self-contractions. The statement that $\mathcal{O}_2$ has dimension two is true at large N, where we can ignore anomalous dimensions and operator mixing.

so as to cancel off the commutator (33). It is useful to work in terms of

$$\psi = \frac{Z^2 - T^2 + TT_2 + TT_3 - T_2T_3}{Z(T_2 - T_3)}. \tag{35}$$

This is the unique $SO(2,1)$-invariant quantity associated with one bulk point (T, Z) and two boundary points T_2, T_3. The regime of interest, where the bulk point is spacelike separated from the first boundary point, corresponds to $T - Z < T_2 < T + Z$ or equivalently $-1 < \psi < 1$.

The lowest-order commutator calculated in (33) can be expressed in terms of ψ.

$$\begin{aligned}\langle 0|i[\phi^{(0)}(T,Z), \mathcal{O}(T_2)]\mathcal{O}(T_3)|0\rangle &= -\frac{\lambda R^2}{\pi}\frac{1}{(T_2-T_3)^2}\log\frac{1+\psi}{1-\psi}\\ &= -\frac{2\lambda R^2}{\pi}\frac{1}{(T_2-T_3)^2}\left(\psi + \frac{1}{3}\psi^3 + \frac{1}{5}\psi^5 + \cdots\right).\end{aligned}$$

In Appendix B we show that at leading order for large N

$$\begin{aligned}&\langle 0|i[\phi^{(1)}_{\Delta=2}(T,Z), \mathcal{O}(T_2)]\mathcal{O}(T_3)|0\rangle = \frac{8A}{\pi}\frac{1}{(T_2-T_3)^2}\psi\\ &\langle 0|i[\phi^{(1)}_{\Delta=4}(T,Z), \mathcal{O}(T_2)]\mathcal{O}(T_3)|0\rangle = \frac{96B}{\pi}\frac{1}{(T_2-T_3)^2}\left(\psi - \frac{5}{3}\psi^3\right)\end{aligned}$$

These results rely on large-N factorization: they were obtained from a disconnected product of two-point functions in the CFT, which makes the leading contribution at large N. The connected four-point correlator of the CFT, which is a subleading correction in the $1/N$ expansion, would modify these results.

Using just the dimension-two primary we could cancel the term linear in ψ by setting $A = \frac{1}{4}\lambda R^2$. Using both dimension-two and dimension-four primaries we could cancel the ψ and ψ^3 terms by setting $A = \frac{3}{10}\lambda R^2$, $B = -\frac{1}{240}\lambda R^2$. Assuming this pattern holds in general, by including operators up to dimension Δ we could cancel the first $\Delta/2$ terms in the Taylor series expansion of the commutator.

As we will see in the next section, it is possible to resum this infinite series to obtain a correction term which is bilocal in $\mathcal{O}(T)$. These results will show that the series converges, with $A \to \frac{3}{8}\lambda R^2$ as more and more operators are taken into account.

D. Bilinear smearing

It is not hard to write down a correction to the lowest-order smearing function (26) which fully restores spacelike commutativity at $\mathcal{O}(1/N)$. Consider the bilocal operator

$$\phi^{(1)}(T,Z) = \frac{\lambda R^2}{8}\int_0^Z \frac{dZ'}{Z'^2}\int_{T-Z+Z'}^{T+Z-Z'} dT' \times \int_{T'-Z'}^{T'+Z'} dT_1 dT_2 {:}\mathcal{O}(T_1)\mathcal{O}(T_2){:} \tag{36}$$

Here $:\cdots:$ denotes normal ordering (meaning no self-contractions). The (T', Z') integrals run over the right light cone of the bulk point. The claim is that, if one ignores four- and higher-point functions in the CFT, the operator $\phi^{(0)} + \phi^{(1)}$ commutes at spacelike separation. In the $1/N$ expansion, this corresponds to ignoring $\mathcal{O}(1/N^2)$ effects.[7]

To show that adding $\phi^{(1)}$ makes the commutator vanish we first take $T - Z > T_2 > T_3$, where

$$\begin{aligned}\langle 0|\phi^{(1)}(T,Z)\mathcal{O}(T_2)\mathcal{O}(T_3)|0\rangle &= \frac{\lambda R^2}{8}\int_0^Z \frac{dZ'}{Z'^2}\int_{T-Z+Z'}^{T+Z-Z'} dT' \int_{T'-Z'}^{T'+Z'} d\tilde{T}_1 d\tilde{T}_2\\ &\langle 0|{:}\mathcal{O}(\tilde{T}_1)\mathcal{O}(\tilde{T}_2){:}\mathcal{O}(T_2|)\mathcal{O}(T_3)|0\rangle\end{aligned}$$

At this stage we have to evaluate a four-point correlator in the CFT. Again we use large-N factorization, which tells us that at leading order for large N the correlator is given by a disconnected product of CFT two-point functions. This approximation gives

$$\begin{aligned}\langle 0|\phi^{(1)}(T,Z)\mathcal{O}(T_2)\mathcal{O}(T_3)|0\rangle &= \frac{\lambda R^2}{\pi^2}\int_0^Z dZ'\int_{T-Z+Z'}^{T+Z-Z'} dT',\\ &\frac{1}{(T'+Z'-T_2)(T'+Z'-T_3)(T'-Z'-T_2)(T'-Z'-T_3)}.\end{aligned} \tag{37}$$

Of course taking the connected four-point correlator of the CFT into account, which is a subleading effect in the $1/N$ expansion, would change this result.

The next step is to continue (37) into the regime $T + Z > T_2 > T - Z > T_3$ using a $T_2 \to T_2 + i\epsilon$ prescription. A similar calculation of $\langle 0|\mathcal{O}(T_2)\phi^{(1)}(T,Z)\mathcal{O}(T_3)|0\rangle$ leads to exactly the same expression, but with a $T_2 \to T_2 - i\epsilon$ prescription. Taking the difference, the commutator is given by integrating T' over a closed contour. The contour encircles the pole at $T' = T_2 + Z'$ provided $0 < Z' < (T + Z - T_2)/2$, and it encircles the pole at $T' = T_2 - Z'$ provided $0 < Z' < (T_2 - T + Z)/2$. So

[7] At this order in $1/N$ the operator ordering does not matter. But the results of Sec. VI B suggest that it is natural to time order the operators appearing on the right-hand side of (36).

$$\langle 0|i[\phi^{(1)}(T,Z),\mathcal{O}(T_2)]\mathcal{O}(T_3)|0\rangle$$
$$= -\frac{2\lambda R^2}{\pi}\frac{1}{(T_2-T_3)}\left[\int_0^{(T+Z-T_2)/2}\frac{dZ'}{2Z'(2Z'+T_2-T_3)} + \int_0^{(T_2-T+Z)/2}\frac{dZ'}{2Z'(2Z'+T_3-T_2)}\right]$$
$$= \frac{\lambda R^2}{\pi}\frac{1}{(T_2-T_3)^2}\log\frac{(T+Z-T_3)(T_2-T+Z)}{(T+Z-T_2)(T-Z-T_3)}. \qquad (38)$$

This exactly cancels (33).

To make contact with the results of the previous section, consider expanding (36) in powers of Z. Near the boundary the leading behavior is

$$\phi^{(1)}(T,Z) \sim \tfrac{1}{2}\lambda R^2 Z^2 {:}(\mathcal{O}(T))^2{:} \quad \text{as} \quad Z\to 0.$$

The interpretation is that we have corrected the lowest-order smearing function (26) by mixing in a dimension-two operator. Matching to the behavior of (34) near the boundary, namely

$$\phi^{(1)}_{\Delta=2}(T,Z) \sim \tfrac{4}{3}AZ^2{:}(\mathcal{O}(T))^2{:}$$

fixes $A = \frac{3}{8}\lambda R^2$. Subleading terms in the expansion of $\phi^{(1)}$ correspond to the infinite sequence of higher-dimension operators considered in the previous section.

V. CFT CONSTRUCTION: AdS_3

We now consider the construction of bulk observables in AdS_3. As we showed in Sec. II, once interactions are taken into account the bulk observables defined in [5–8] do not commute with boundary operators, even when the bulk and boundary points are at spacelike separation. As in Sec. IV C we will cure this problem by adding higher-dimension operators to our definition of a bulk observable. Our conclusions in this section are based on smearing the OPE in the CFT. Analogous results, based on smearing CFT correlators, are obtained in Appendix D.

Imagine we have an infinite set of primary operators $\mathcal{O}_i$ with dimension Δ_i, with OPE

$$\mathcal{O}_i(X,T)\mathcal{O}_j(0,0) = \frac{\delta_{ij}}{(X^2-T^2)^{\Delta_i}} + c_{ijk}\frac{\mathcal{O}_k(0,0)}{(X^2-T^2)^{\tilde\Delta}} + \cdots \qquad (39)$$

Here $\tilde\Delta = (\Delta_i+\Delta_j-\Delta_k)/2$. Using (3) we smear $\mathcal{O}_i$ to turn it into a bulk operator $\phi_i(Z,X,T)$. The first term in the OPE gives the free bulk-boundary two-point function, while the second gives

$$\phi_i(Z,X,T)\mathcal{O}_j(0,0) = c_{ijk}f(Z,X,T;0,0)\mathcal{O}_k(0,0) + \cdots \psi, \qquad (40)$$

where

$$f(Z,X,T;0,0) = \frac{\Delta_i-1}{\pi}(-1)^{\tilde\Delta}\int_{Y'^2+T'^2<Z^2} dY'dT'\left(\frac{Z^2-Y'^2-T'^2}{Z}\right)^{\Delta_i-2} \times\frac{1}{((T+T')^2-(X+iY')^2)^{\tilde\Delta}}.$$

As before we begin by working in the regime $T>Z$ with $X=0$. Switching to polar coordinates

$$f(Z,0,T;0,0) = \frac{\Delta_i-1}{\pi}(-1)^{\tilde\Delta}\int_0^Z dr\int_0^{2\pi}d\theta\left(\frac{Z^2-r^2}{Z}\right)^{\Delta_i-2}\times\frac{r}{(r^2+T^2+2rT\cos\theta)^{\tilde\Delta}}.$$

Compared to the two-point function (5), the only difference is the relative power of the two factors in the integrand. The integral in (5) reflected the casual structure of AdS and only had singularities on the bulk light cones. Here things will be different.

The integral over θ is performed as before using (6). Again defining $q=r^2/Z^2$ and $y=Z^2/T^2$ we obtain

$$f(Z,0,T;0,0) = (\Delta_i-1)(-1)^{\tilde\Delta}T^{-2\tilde\Delta}Z^{\Delta_i}\times\int_0^1 dq(1-q)^{\Delta_i-2}{}_2F_1(\tilde\Delta,\tilde\Delta;1;qy) = (-1)^{\tilde\Delta}T^{-2\tilde\Delta}Z^{\Delta_i}{}_2F_1\left(\tilde\Delta,\tilde\Delta;\Delta_i;\frac{Z^2}{T^2}\right).$$

We can extend this to general X, T using analytic continuation and Lorentz invariance.

$$f(Z,X,T;0,0) = (X^2-(T-i\epsilon)^2)^{-\tilde\Delta}Z^{\Delta_i}{}_2F_1\left(\tilde\Delta,\tilde\Delta;\Delta_i;\frac{Z^2}{(T-i\epsilon)^2-X^2}\right). \qquad (41)$$

Let us look at a few relevant limits of this expression. First, in the limit $Z\to 0$ with X, T fixed, we have $\phi(Z,X,T)\to Z^{\Delta_i}\mathcal{O}(X,T)$ by construction [6]. In this limit

$$f(Z,X,T;0,0) = (X^2-(T-i\epsilon)^2)^{-\tilde\Delta}Z^{\Delta_i}. \qquad (42)$$

So indeed in this limit the mixed bulk-boundary OPE (40) goes over to the CFT OPE (39).

To see the failure of bulk locality we need to look at a different limit where we approach a boundary light cone. Let us first look at the case where all operators have even dimensions. Then the hypergeometric function has a simple form in terms of elementary functions,

$${}_2F_1(\tilde\Delta,\tilde\Delta;\Delta_i;z) = R_1(z) + R_2\ln(1-z), \qquad (43)$$

where $R_i(z)$ are rational functions of $z=\frac{Z^2}{(T-i\epsilon)^2-X^2}$. In fact one can show that

$$f(Z,X,T;0,0) = g_1(z) + Z^{\Delta_k-\Delta_j} g_2(z)\ln(1-z), \quad (44)$$

where $g_1(z)$ is a rational function which has no singularities as $z \to \infty$, but which may have singularities as $z \to 1$, while $g_2(z)$ is a polynomial in $1/z$ of rank $\Delta_i - \tilde{\Delta} - 1$. From this we see that the two operators $\phi_i(Z,X,T)$ and $\mathcal{O}_j(0,0)$ will not commute once $X^2 - T^2 < 0$. That is, they will not commute when they are timelike separated on the boundary, even though they are spacelike separated in the bulk. The nonvanishing commutator comes only from the $\ln(1-z)$ term and is thus proportional to $Z^{\Delta_k-\Delta_j} g_2(z)$.

One can define a new bulk operator

$$\phi_i(Z,X,T) = \int K_{\Delta_i}(Z,X,T|X',T')\mathcal{O}_i(X',T') + \sum_n d_n \int K_{\Delta_n}(Z,X,T|X',T')\mathcal{O}_n(X',T'), \quad (45)$$

where Δ_n is an even number. Given the structure of the commutator we found above, each term in the sum contributes a polynomial in $\frac{1}{z} = \frac{T^2-X^2}{Z^2}$ of some rank. One can adjust the coefficients d_n in such a way as to cancel the commutator up to any desired power of $1/z$. The problem with bulk locality arises when the points are timelike separated on the boundary but spacelike separated in the bulk. This corresponds to $|1/z| < 1$. So canceling the commutator to a high power in $1/z$ means the commutator can be made very small, except near the bulk light cone. Depending on the operator content, it may even be possible to cancel the commutator to all orders in $1/z$.

One might worry that this is all special to operators of even conformal dimension, but this is not the case. For noninteger conformal dimensions (as arises for nonprotected operators) the appropriate analytic continuation (that is, analytic continuation of ${}_2F_1(\alpha,\alpha,\gamma,z)$ to $|z| > 1$) gives

$$\begin{aligned} f(Z,X,T,0,0) &= \frac{\Gamma(\Delta_i)Z^{\Delta_k-\Delta_j}}{\Gamma(\tilde{\Delta})\Gamma(\Delta_i-\tilde{\Delta})} \times \sum_{l=0}^{\infty} \frac{(\tilde{\Delta})_l(1+(\Delta_j-\Delta_k-\Delta_i)/2)_l}{(l!)^2} \\ &\quad \times \left(\frac{(T-i\epsilon)^2-X^2}{Z^2}\right)^l \left(2\psi(l+1) + \log\left(\frac{Z^2}{X^2-(T-i\epsilon)^2}\right) \right. \\ &\quad \left. - \psi(\Delta_i-\tilde{\Delta}-l) - \psi(\tilde{\Delta}+l)\right). \quad (46) \end{aligned}$$

Here $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ and $(n)_l = \frac{\Gamma(n+l)}{\Gamma(n)}$. Again the log term gives rise to a nonzero commutator when $X^2 - T^2 < 0$, i.e. timelike separation on the boundary, even if the points are spacelike separated in the bulk. The commutator has an expansion in $\frac{(T-i\epsilon)^2-X^2}{Z^2}$ which for bulk spacelike separation is less that 1. Thus the structure is such that by using (45) with appropriate d_n's one can make the commutator arbitrarily small, provided appropriate higher-dimension operators exist. Of course being able to carry out this procedure simultaneously for different pairs of operators ϕ_i, $\mathcal{O}_j$ will place stringent constraints on the operator content and interactions of the CFT.

We have thus found that by adding higher-dimension operators we can define local observables in the bulk. In Appendix D we reach the same conclusion by smearing three-point correlators in the CFT.

VI. BULK CONSTRUCTION

So far our approach has been to work purely within the CFT, seeking to define bulk observables which commute at spacelike separation. But let us imagine that, at least in some approximation, we have access to a local description of bulk physics. Then we should be able to rederive our results from the bulk point of view. Here we show how this works, using AdS_2 as our main example.

A. Bulk equations of motion

To illustrate how this works, take a massless ϕ^3 theory in the bulk.

$$S = \int d^2x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{3}\lambda\phi^3\right). \quad (47)$$

The bulk field is dual to an operator $\mathcal{O}$ with dimension one on the boundary. The bulk equation of motion $\nabla\phi = \lambda\phi^2$ can be solved perturbatively in λ.

$$\phi = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \cdots$$

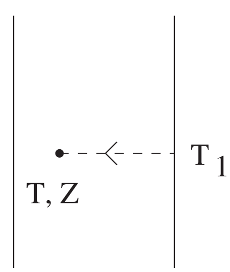
where

$$\nabla\phi^{(0)} = 0 \qquad \nabla\phi^{(1)} = \lambda(\phi^{(0)})^2$$
$$\nabla\phi^{(2)} = 2\lambda\phi^{(0)}\phi^{(1)} \qquad \ldots$$

We already know how to solve the zeroth-order equation.

$$\phi^{(0)}(T,Z) = \frac{1}{2}\int_{T-Z}^{T+Z} dT_1\mathcal{O}(T_1).$$

This can be represented diagrammatically as



The dashed propagator is nonzero and equal to $1/2$ only in the right light cone of the bulk point (T,Z). The arrow on the dashed propagator points towards the vertex of the light cone.

The first-order equation is solved by

$$\phi^{(1)}(x) = \int d^2x'\sqrt{-g}G(x|x')\lambda(\phi^{(0)}(x'))^2,$$

where a suitable Green's function is

$$G(T,Z|T',Z') = \tfrac{1}{2}\theta(Z-Z')\theta(Z-Z'-|T-T'|)$$

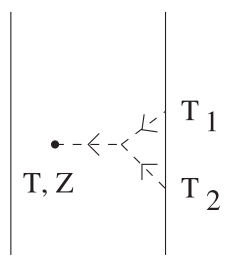
(nonzero and equal to $1/2$ only in the right light cone of (T,Z)). In defining the composite operator $(\phi^{(0)})^2$ there is a self-contraction one can make. This generates a tadpole diagram that we will ignore. More precisely, we have in mind canceling the tadpole against a linear term in the action. Dropping the tadpole amounts to normal ordering $(\phi^{(0)})^2$, so

$$\phi^{(1)}(x) = \frac{\lambda}{2}\int_{\text{right l.c. of } x} d^2x'\sqrt{-g}{:}(\phi^{(0)}(x'))^2{:}$$

Writing this out explicitly

$$\phi^{(1)}(T,Z) = \frac{\lambda R^2}{8}\int_0^Z \frac{dZ'}{(Z')^2}\int_{T-(Z-Z')}^{T+(Z-Z')} dT' \times \int_{T'-Z'}^{T'+Z'} dT_1 dT_2 {:}\mathcal{O}(T_1)\mathcal{O}(T_2){:}$$

This is the first-order correction introduced in (36). By construction it is AdS covariant and satisfies the bulk equation of motion to first order in λ. It can be represented diagrammatically as

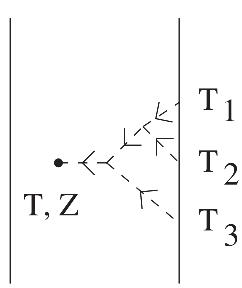


In this diagram we are using the dashed propagator, and the vertex factor for three dashed lines is $\lambda R^2/(Z')^2$.

Likewise the second-order equation is solved by

$$\phi^{(2)}(x) = 2\lambda\int d^2x'\sqrt{-g}G(x|x')\phi^{(0)}(x')\phi^{(1)}(x'), \quad (48)$$

which can be represented diagrammatically as

2 

There is an important difference between the procedure we have outlined here and conventional perturbation theory. In conventional perturbation theory one begins with a free field that is local and causal and uses it as a basis for building up an interacting field. Superficially our construction is similar: we use $\phi^{(0)}$ as a basis for constructing an interacting local bulk operator. But note that, although $\phi^{(0)}$ obeys a free wave equation, it is not a local field when interactions are taken into account in the CFT: as shown in Sec. II $\phi^{(0)}$ fails to commute with itself at spacelike separation.

B. Bulk Feynman diagrams

In this section we show how the Feynman diagrams associated with a local theory in the bulk can be mapped over to CFT calculations. This will provide yet another way of deriving the CFT operators which are dual to local bulk observables. It will also show that, in a $1/N$ expansion of the CFT, these operators have correlation functions which reproduce bulk perturbation theory. As in the previous section, we work with massless ϕ^3 theory in the bulk as described by (47).

We begin with a lemma. From (18) the bulk Feynman propagator is

$$\begin{aligned} iG_F(x|x') &= \langle 0|T\{\phi(x)\phi(x')\}|0\rangle \\ &= \frac{1}{2\pi}\tanh^{-1}\left(\frac{2ZZ'}{Z^2+Z'^2-(T-T')^2+i\epsilon}\right) \\ &= \frac{1}{4\pi}\log\frac{(Z+Z')^2-(T-T')^2+i\epsilon}{(Z-Z')^2-(T-T')^2+i\epsilon}. \end{aligned} \quad (49)$$

Sending $Z \to 0$ gives the bulk-boundary Feynman propagator

$$\begin{aligned} iG_F(T|x') &= \langle 0|T\{\mathcal{O}(T)\phi(x')\}|0\rangle \\ &= \frac{Z'}{\pi}\frac{1}{Z'^2-(T-T')^2+i\epsilon}. \end{aligned} \quad (50)$$

Consider applying the linear smearing relation (26) to the boundary operator $\mathcal{O}(T)$ which appears here, in an attempt to recover the bulk Feynman propagator. This gives

$$\begin{aligned} &\frac{1}{2}\int_{T-Z}^{T+Z} dT_1 iG_F(T_1|x') \\ &\quad = \frac{1}{4\pi}\log\frac{(Z+Z'+i\epsilon)^2-(T-T')^2}{(Z-Z'-i\epsilon)^2-(T-T')^2}. \end{aligned} \quad (51)$$

Compared to the bulk Feynman propagator (49), this has a different $i\epsilon$ prescription. So—unlike the Wightman functions considered in Sec. IV B—smearing the bulk-boundary

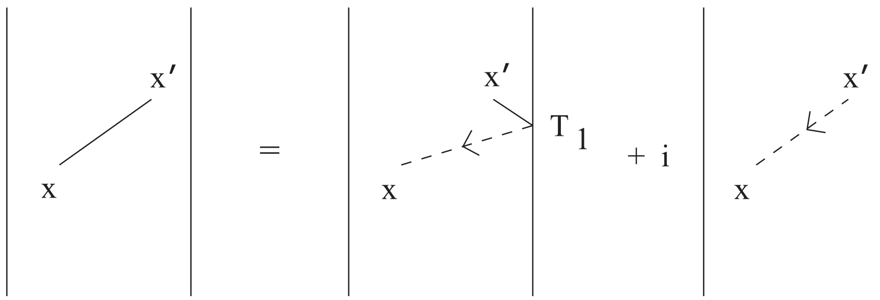
Feynman propagator does not give the bulk-bulk Feynman propagator. Instead we find that the two expressions differ in the right light cone of the bulk point (T, Z):

$$iG_F(x|x') = \int dT_1 K(x|T_1, 0) iG_F(T_1|x') + iK(x|x'). \quad (52)$$

Here

$$K(T, Z|T', Z') = \tfrac{1}{2}\theta(Z - Z')\theta(Z - Z' - |T - T'|) \quad (53)$$

is nonzero and equal to 1/2 only when (T', Z') lies in the right light cone of the point (T, Z). Note that K is exactly the Green's function we introduced in Sec. VI! So (52) can be represented diagrammatically as

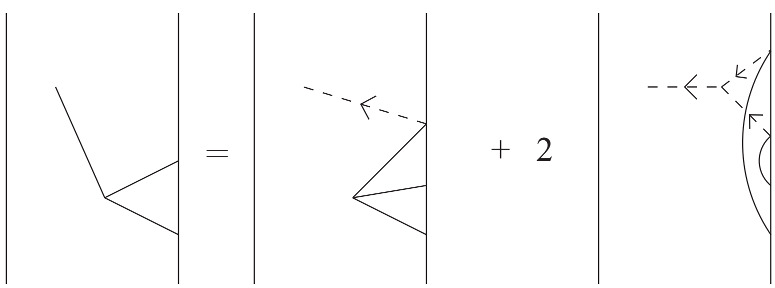


Here solid lines represent Feynman propagators iG_F and dashed lines represent K. This is the lemma we wished to prove.

With this lemma it is straightforward to map bulk Feynman diagrams to CFT calculations. For instance consider the lowest-order Feynman diagram which contributes to $\langle\phi(T_1, Z_1)\mathcal{O}(T_2)\mathcal{O}(T_3)\rangle$. Assuming

$$T_1 - Z_1 > T_2 > T_3 \quad (54)$$

so that the operators are time ordered and their right light cones do not overlap on the boundary, we have[8]



(55)

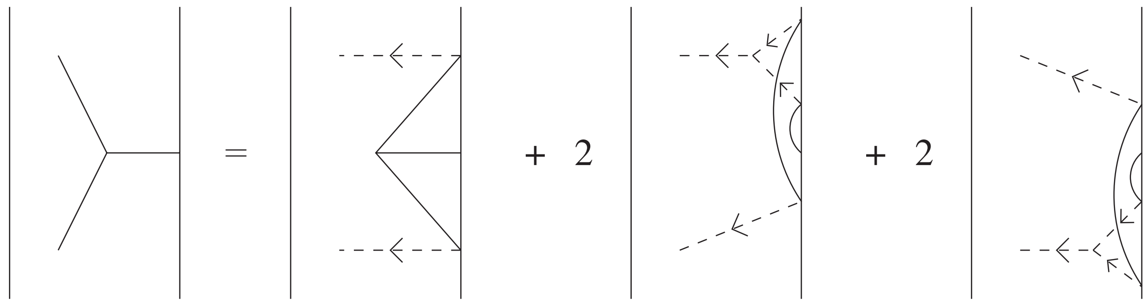
[We have dropped some diagrams involving dashed propagators from the bulk point to the boundary points. They do not contribute since, given (54), the boundary points are not in the right light cone of the bulk point.] The first diagram on the right-hand side of (55) involves the CFT three-point function, as induced by a bulk Feynman diagram (see Appendix A). The second diagram on the right involves a disconnected product of CFT two-point functions. In terms of correlators (55) means

$$\langle\phi(T_1, Z_1)\mathcal{O}(T_2)\mathcal{O}(T_3)\rangle = \langle\phi^{(0)}(T_1, Z_1)\mathcal{O}(T_2)\mathcal{O}(T_3)\rangle + \langle\phi^{(1)}(T_1, Z_1)\mathcal{O}(T_2)\mathcal{O}(T_3)\rangle. \quad (56)$$

In other words, at this order computing $\langle(\phi^{(0)} + \phi^{(1)})\mathcal{O}\mathcal{O}\rangle$ in the CFT exactly reproduces the tree-level correlator between one bulk point and two boundary points. Moreover, from the last diagram in (55) you can read off the need to include $\phi^{(1)}$ in the definition of a bulk observable.

As a more involved example, consider the correlator between two bulk points and one boundary point, $\langle\phi(T_1, Z_1)\mathcal{O}(T_2)\phi(T_3, Z_3)\rangle$. Taking $T_1 - Z_1 > T_2 > T_3 + Z_3$, so that again the points are time ordered and their right light cones do not overlap, we have

[8]Note that the vertex factor for three solid lines is $-i2\lambda R^2/Z^2$ while the vertex factor for three dashed lines is $\lambda R^2/Z^2$.



$$= \langle \phi^{(0)}(x_1)\mathcal{O}(T_2)\phi^{(0)}(x_3)\rangle + \langle \phi^{(1)}(x_1)\mathcal{O}(T_2)\phi^{(0)}(x_3)\rangle + \langle \phi^{(0)}(x_1)\mathcal{O}(T_2)\phi^{(1)}(x_3)\rangle. \tag{57}$$

In the first diagram on the right the lowest-order smearing functions are tied together with a three-point correlator in the CFT. In the second and third diagrams the first-order correction to the smearing function is combined with a disconnected product of CFT two-point correlators. So again we see that to this order in λ we can identify the combination $\phi^{(0)} + \phi^{(1)}$ with a local operator in the bulk.

VII. GENERALIZATIONS AND EXTENSIONS

In this section we discuss the extension of our results, first to general CFT's, then to higher orders in $1/N$, using AdS_2 to illustrate the ideas.

Consider a general one-dimensional CFT, with primary fields $\mathcal{O}_i$ of dimension Δ_i. The three-point correlator is a generalization of (24).

$$\langle 0|\mathcal{O}_i(T_1)\mathcal{O}_j(T_2)\mathcal{O}_k(T_3)|0\rangle = c_{ijk}\frac{1}{(T_1-T_2)^{\Delta_i+\Delta_j-\Delta_k}(T_1-T_3)^{\Delta_i+\Delta_k-\Delta_j}(T_2-T_3)^{\Delta_j+\Delta_k-\Delta_i}}. \tag{58}$$

The simplest way to construct CFT operators dual to bulk observables is to generalize the construction of Sec. VI and note that this correlator is induced at tree level by a cubic coupling between bulk scalar fields. The bulk action is a generalization of (47),

$$S = \int d^2x\sqrt{-g}\Big(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i - \frac{1}{2}m_i^2\phi_i^2 - \frac{1}{3}\lambda_{ijk}\phi_i\phi_j\phi_k\Big), \tag{59}$$

where $\Delta_i = \frac{1}{2}(1+\sqrt{1+4m_i^2R^2})$, and the coefficient of proportionality relating c_{ijk} and λ_{ijk} could be worked out as in Appendix A. At lowest order we have the expression for bulk observables worked out in Sec. 3.1 of [5],

$$\phi_i^{(0)}(x) = \int dT' K_{\Delta_i}(x|T')\mathcal{O}_i(T'), \tag{60}$$

where the smearing function for an operator of dimension Δ is

$$K_\Delta(T,Z|T') = \frac{\Gamma(\Delta+1/2)}{\sqrt{\pi}\Gamma(\Delta)}\Big(\frac{Z^2-(T-T')^2}{Z}\Big)^{\Delta-1} \times \theta(Z-|T-T'|). \tag{61}$$

These lowest-order operators satisfy a free equation of motion, $(\nabla - m_i^2)\phi_i^{(0)} = 0$. The first-order correction, satisfying $(\nabla - m_i^2)\phi_i^{(1)} = \lambda_{ijk}\phi_j^{(0)}\phi_k^{(0)}$, is given by

$$\phi_i^{(1)}(x) = \lambda_{ijk}\int d^2x'\sqrt{-g}G_{\Delta_i}(x|x')\phi_j^{(0)}(x')\phi_k^{(0)}(x'), \tag{62}$$

where an appropriate Green's function, satisfying $(\nabla - m^2)G_\Delta(x|x') = \frac{1}{\sqrt{-g}}\delta^2(x-x')$, is

$$G_\Delta(x|x') = \tfrac{1}{2}P_{\Delta-1}(\sigma)\theta(Z-Z')\theta(Z-Z'-|T-T'|). \tag{63}$$

This Green's function was worked out in Sec. 2.2 of [5]. It is nonzero only in the right light cone of the point x. σ is the invariant distance (19) between x and x', and $P_{\Delta-1}$ is a Legendre function.

By construction the operators $\phi_i^{(0)} + \phi_i^{(1)}$ satisfy the bulk equations of motion to first order in λ. They will commute at spacelike separation, along the lines of Sec. IV D, provided that four-point and higher-point correlators are ignored in the CFT. Thus (60) and (62) define a local bulk observable in any one-dimensional CFT, to the extent that four- and higher-point correlators can be ignored.

A natural conjecture is that this pattern continues order by order when higher-point correlators are taken into account. For instance, to build commuting bulk observables when four-point correlators are taken into account, we should add a correction $\phi_i^{(2)}$ which is trilocal in the CFT primaries. For an explicit example of a second-order correction, in the CFT dual to ϕ^3 theory in the bulk, see (48).

This conjecture is consistent with leading large-N counting. Recall that in the 't Hooft large-N limit the connected correlation function of k single-trace operators scales as

$$\langle \mathcal{O}_1\cdots\mathcal{O}_k\rangle_C \sim 1/N^{k-2}.$$

In the ϕ^3 theory of Sec. VI, tree diagrams with k external legs scale as λ^{k-2}, so we can identify the bulk coupling

$\lambda \sim 1/N$. The idea is that a bulk observable has an expansion

$$\phi = \sum_{n=0}^{\infty} \phi^{(n)}, \tag{64}$$

where $\phi^{(n)}$ carries an explicit factor of λ^n and is a multilocal expression involving $n+1$ single-trace operators, defined so that there are no self-contractions. To fix $\phi^{(n+1)}$ the recipe is as follows. Suppose we have already constructed $\phi^{(0)}, \cdots, \phi^{(n)}$ so that bulk operators commute at the level of $(n+2)$-point functions. Taking the connected $(n+3)$-point correlator into account will lead to a nonzero commutator in

$$\langle [\phi^{(n)}, \mathcal{O}]\mathcal{O}\rangle \sim \lambda^n \frac{1}{N^{n+1}} \sim \frac{1}{N^{2n+1}}. \tag{65}$$

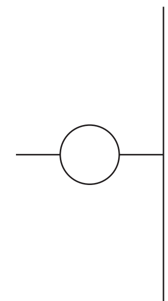
There is no reason to expect this to vanish, so we need to further correct our definition of a bulk observable. We conjecture that $\phi^{(n+1)}$ can be chosen to cancel (65), at least at spacelike separation. As a consistency check, at least the powers of N come out the same:

$$\langle [\phi^{n+1}, \mathcal{O}]\mathcal{O}\rangle \sim \sum_k \lambda^{n+1} \frac{1}{N^{k-2}} \frac{1}{N^{n+2-k}} \sim \frac{1}{N^{2n+1}} \tag{66}$$

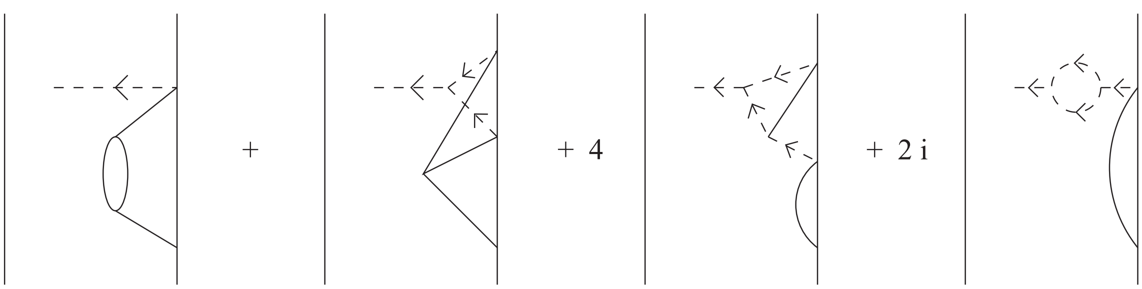
(the CFT correlator is a sum of disconnected products of k-point and $(n+4-k)$-point correlators).

This argument shows that the conjecture is consistent with planar large-N counting. Of course there are subleading nonplanar corrections to CFT correlators: recall that a CFT diagram with k punctures and L handles, dual to a bulk diagram with k legs and L loops, scales as $1/N^{2L+k-2}$. It should be possible to take these nonplanar corrections into account by making subleading corrections to the bulk operators.

To get a feel for the sort of corrections which arise from nonplanar diagrams, consider the following diagram in the ϕ^3 theory of Sec. VI.



From the bulk point of view this diagram is $\mathcal{O}(\lambda^2)$; it is a one-loop correction to the bulk-boundary propagator. The form of the bulk-boundary propagator is fixed by AdS invariance, so this diagram can be absorbed into a mass and wave function renormalization of the bulk field. From the CFT point of view this diagram is an $\mathcal{O}(1/N^2)$ effect. Mapping it to the CFT as in Sec. VI B gives[9]



A few comments are in order.

(i) The first diagram is a contribution to $\langle \phi^{(0)}\mathcal{O}\rangle$. It makes an $\mathcal{O}(1/N^2)$ correction to the CFT two-point function, correcting the conformal dimension of the boundary operator.[10]

(ii) The second diagram is a contribution to $\langle \phi^{(1)}\mathcal{O}\rangle$, involving the CFT three-point function evaluated at $\mathcal{O}(1/N)$. Note that the CFT correlator is induced by a conventional bulk Feynman diagram, which means it is a time-ordered product. This is the first place where operator ordering is important, and it suggests that the CFT operators appearing in (36) should be time ordered.

(iii) The final two diagrams involve a two-point function in the CFT. They can be thought of as making an $\mathcal{O}(1/N^2)$ correction to the lowest-order smearing function (26), appropriate for the corrected conformal dimension coming from the first diagram.

[9]The bulk Feynman diagram has a symmetry factor of 1/2, which we write out explicitly in the coefficients of the diagrams with a dashed propagator.

[10]Bulk perturbation theory corresponds to a large-N expansion of CFT correlators. This expansion in powers of λ should not be confused with (24), where λR^2 was defined as the coefficient of the exact CFT three-point function.

VIII. CONCLUSIONS

In this paper we described the construction of local bulk operators from the CFT beyond leading order in $1/N$. This provides a working definition of Heisenberg bulk operators, which may be used to construct new off-shell bulk quantum gravity amplitudes directly from conformal field theory correlators. We showed that using the naive smeared operator beyond the leading large-N limit results in bulk operators which do not commute at spacelike separation. We then showed that this problem can be cured in perturbation theory, by changing the definition of bulk operators. We presented several derivations of the corrected operators. The most interesting constructions—adding higher-dimension operators, and adding multilocal corrections—could be carried out completely within the CFT. In these constructions one seems to need a large number of primary operators with prescribed properties to make the bulk theory local. The requisite properties seem to be satisfied in a large N CFT, to all orders in $1/N$ perturbation theory, through the presence of multitrace operators with appropriate insertions of derivatives. An alternate construction uses the radial Hamiltonian from a local bulk theory in AdS.[11] The different constructions agree in perturbation theory, but the CFT construction may make it possible to understand how bulk locality breaks down.

One might be surprised that such a construction could be carried out at all, since the diffeomorphism constraints of quantum gravity would seem to rule out the existence of local observables [1,2]. Of course we constructed our observables purely within the CFT, where we never really had to face up to this issue: normalizable diffeomorphisms in the bulk act trivially on the CFT. This means our bulk observables are by construction diffeomorphism invariant. This suggests that, from the bulk point of view, we have managed to construct local observables using a particular choice of gauge (corresponding to our use of Poincaré coordinates to label points in the bulk).

To better understand these results let us look at a local quantum field theory on AdS (without gravity). Then the limit of bulk correlation functions as you approach the boundary still look like those of a CFT. The bulk operators (which are all independent at some fixed time) can be written as integrals over the boundary operators (at different times) using the radial Hamiltonian approach. This gives local bulk operators, but this is not a surprise since there really is a local bulk theory and we have just exchanged the initial data surface (at fixed time) with an initial data surface on the timelike boundary. In this case one can either use the radial Hamiltonian approach or regular perturbation theory around local free fields, both should give the same answer. The key difference between the perturbative expansion using the radial Hamiltonian which we used in the previous sections and the usual perturbative expansion of Green functions in quantum field theory, is that beyond leading order in the perturbation expansion the operator $\phi^{(0)}$ which we used does not commute with itself at bulk spacelike separation. However $\phi^{(0)}$ does satisfy the free wave equation, so for a given bulk Lagrangian, it will produce correlators that agree order by order in the coupling with correlators constructed in the usual interaction picture approach. This expansion may be viewed as a choice of nonlocal interpolating field being used to set up the perturbative expansion. Since the operators we construct satisfy the correct operator equation of motion, they provide the same approximation to the full Heisenberg operator as standard perturbation theory. It is important to note that in this setup the boundary theory is not unitary. Things can come in from the bulk, since there really are extra degrees of freedom in the bulk not accounted for on the boundary at some fixed time. Nevertheless, boundary correlation functions do look like those of a CFT.

Now add gravity. If we just do perturbation theory to some order in $1/N$ around an AdS background things work as above (as long as the theory is renormalizable). The bulk theory is local, the boundary operators look like a sector in a CFT, and writing bulk operators using the radial Hamiltonian will of course give local bulk operators.

However in the full quantum gravity theory (meaning finite N, and not perturbation around a fixed background to some order) things are different. The only local operators are at the boundary, which means there are fewer degrees of freedom. This is manifested by the fact that the boundary theory is now unitary. A unitary theory on the boundary cannot describe a local QFT in the bulk.

From the CFT point of view the most plausible way for bulk locality to fail is if the constraints on the CFT primaries, that we needed to construct local bulk observables in Sec. V, cannot be satisfied beyond some conformal dimension $\Delta_{\max}$. Let us say $\Delta_{\max}$ is of order N. What are the consequences of this? The infinite sum over primaries (45) that is necessary for locality is truncated, so bulk operators will not commute at spacelike separation. Take a bulk operator at a point $(Z, X = 0, T = 0)$ and a boundary operator at $(X = 0, T)$. The commutator of the two operators inside a correlation function, as long as all other operators are far away, is $[\phi(Z, 0, 0), \mathcal{O}(0, T)] \sim \frac{1}{N}(\frac{T^2}{Z^2})^N$. This means the causal structure of the bulk space-time has been destroyed. However away from the bulk light cone the commutator is of order e^{-N}, which is invisible in perturbation theory. Very near the light cone, say $Z^2 - T^2 \sim (1 - \frac{a}{N})$ with a independent of N, the commutator will be nonzero even in perturbation theory. But the interpretation, in perturbation theory, is just that one has a slightly nonlocal bulk theory, with nonlocality on the scale of $1/N$.

[11] A method for defining bulk operators, based on identifying the radial Hamiltonian with the Fokker-Planck Hamiltonian of the boundary theory, was developed in [23].

These represent the expected light-cone fluctuations, and not complete destruction of bulk space-time locality, even though nonperturbatively the whole bulk causal structure is destroyed. When other operators are nearby the condition for noncommutativity changes somewhat. For a three-point function the condition is $\chi < 1$ where χ is given in (D1).

How far one can venture from the light cone and still see a large commutator? The answer depends on Z. It is of order $\delta T \sim aZ/N$, or $\delta X \sim Z\sqrt{a/N}$. So for very large Z (i.e. near the Poincaré horizon) there is a large region on the boundary, which is spacelike to a bulk point, and in which operators will have a large commutator with the bulk point. This is due to the large redshift from the boundary to the Poincaré horizon. This is just the old argument about small nonlocality near the horizon getting transmitted to large scales on the boundary and giving rise to a stretched horizon.

Finally, it is worth trying to draw conclusions from these results regarding generic predictions which might be used to motivate future experimental tests of theories of quantum gravity. An important observation is that to all orders in the $1/N$ expansion we have constructed local bulk observables whose n-point correlators respect both causality and AdS covariance. At finite N presumably causality is violated, along the lines discussed above, but exact AdS covariance is maintained. Other models of quantum gravity predict modified dispersion relations arising from violation of Lorentz invariance on short distance scales [24]. Using AdS covariance as a proxy for Lorentz invariance, the present work predicts there should be no sign of such modified dispersion relations. This is compatible with recent experimental results [25] which bound the scale of such corrections at well above the Planck scale. Instead the results of the present work indicate that new quantum gravity effects are only to be expected once one looks for signs of causality violation—operators that fail to commute at spacelike separation—in three- and higher-point functions.

ACKNOWLEDGMENTS

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APPENDIX A: CUBIC COUPLINGS IN AdS_2

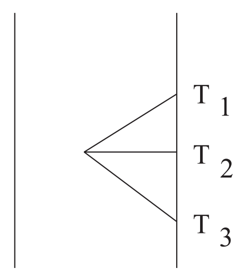
The ϕ^3 interaction of Sec. VI induces a tree-level three-point coupling

$$\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\}|0\rangle = -\frac{i\lambda R^2}{4\pi^3}\int_{-\infty}^{\infty} dT_4 \int_0^{\infty}\frac{dZ_4}{Z_4^2}\tanh^{-1}\left(\frac{1}{\sigma_{14}+i\epsilon}\right) \times \tanh^{-1}\left(\frac{1}{\sigma_{24}+i\epsilon}\right)\tanh^{-1}\left(\frac{1}{\sigma_{34}+i\epsilon}\right). \tag{A1}$$

Sending the bulk points to the boundary gives

$$\langle 0|T\{\mathcal{O}(T_1)\mathcal{O}(T_2)\mathcal{O}(T_3)\}|0\rangle = -\frac{i\lambda R^2}{\pi}\frac{1}{|T_1-T_2|\cdot|T_1-T_3|\cdot|T_2-T_3|}. \tag{A2}$$

This agrees with (24) for $T_1 > T_2 > T_3$. So we can think of our three-point coupling in the CFT as coming from a bulk Feynman diagram



Since the form of the three-point function is fixed by conformal invariance, the only real lesson here is that our bulk and boundary conventions for normalizing λ are the same.

APPENDIX B: COMMUTATORS AT LARGE N

We wish to calculate

$$\langle 0|i[\phi^{(1)}_{\Delta=2}(T,Z),\mathcal{O}(T_2)]\mathcal{O}(T_3)|0\rangle,$$

where

$$\phi^{(1)}_{\Delta=2}(T,Z) = A\int_{T-Z}^{T+Z} dT'\frac{Z^2-(T-T')^2}{Z}:\mathcal{O}(T')\mathcal{O}(T'):$$

We begin by studying the correlator

$$\langle 0|\phi^{(1)}_{\Delta=2}(T,Z)\mathcal{O}(T_2)\mathcal{O}(T_3)|0\rangle$$

with $T - Z > T_2 > T_3$ so the operators do not overlap. At leading order for large N the correlator is given by a disconnected product of two CFT two-point functions (22). This gives

$$\begin{aligned}
&\langle 0|\phi^{(1)}_{\Delta=2}(T,Z)\mathcal{O}(T_2)\mathcal{O}(T_3)|0\rangle \\
&\quad = \frac{2A}{\pi^2}\int_{T-Z}^{T+Z} dT' \frac{Z^2-(T-T')^2}{Z}\frac{1}{(T'-T_2)^2}\frac{1}{(T'-T_3)^2} \\
&\quad = -\frac{8A}{\pi^2(T_2-T_3)^2} + \frac{4A(Z^2-T^2+TT_2+TT_3-T_2T_3}{\pi^2 Z(T_2-T_3)^3} \\
&\qquad \times \log\frac{(T+Z-T_3)(T-Z-T_2)}{(T-Z-T3)(T+Z-T_2)},
\end{aligned}$$

This can be continued into the regime $T+Z > T_2 > T - Z > T_3$ with a $T_2 \to T_2 + i\epsilon$ prescription. We then repeat the calculation, starting from

$$\langle 0|\mathcal{O}(T_2)\phi^{(1)}_{\Delta=2}(T,Z)\mathcal{O}(T_3)|0\rangle,$$

with $T_2 > T + Z$. Continuing to the same regime as before gives exactly the same expression, but with a $T_2 \to T_2 - i\epsilon$ prescription. Taking the difference gives the commutator

$$\langle 0|i[\phi^{(1)}_{\Delta=2}(T,Z),\mathcal{O}(T_2)]\mathcal{O}(T_3)|0\rangle = \frac{8A}{\pi}\frac{1}{(T_2-T_3)^2}\psi,$$

where the AdS invariant cross-ratio ψ is defined in (35). The calculation of

$$\langle 0|i[\phi^{(1)}_{\Delta=4}(T,Z),\mathcal{O}(T_2)]\mathcal{O}(T_3)|0\rangle$$

proceeds along the same lines.

APPENDIX C: MIXED BULK-BOUNDARY CORRELATORS AND CONFORMAL INVARIANCE

We work in Poincaré coordinates where the three-dimensional AdS metric takes the form

$$ds^2 = (dZ^2 + dX^2 - dT^2)/Z^2$$

The isometries of this metric form the group $SO(2,2)$ which is generated by the following symmetry transformations: $SO(1,1)$ Lorentz transformations on $x^\mu = (T,X)$; dilatations, acting as

$$Z \to \lambda Z, \qquad x^\mu \to \lambda x^\mu$$

and special conformal transformations, parametrized by b^μ, acting as

$$\begin{aligned}
x^\mu &\to \frac{x^\mu - b^\mu(x^2+Z^2)}{1-2b\cdot x + b^2(x^2+Z^2)} \\
Z &\to \frac{Z}{1-2b\cdot x + b^2(x^2+Z^2)}.
\end{aligned} \tag{C1}$$

The "bulk" distance function transforms as

$$|x_1-x_2|^2 + Z_1^2 + Z_2^2 \to \frac{|x_1-x_2|^2+Z_1^2+Z_2^2}{(1-2b\cdot x_1 + b^2(x_1^2+Z_1^2))(1-2b\cdot x_2+b^2(x_2^2+Z_2^2))}.$$

In the limit that $Z \to 0$ these expressions reduce to the familiar global conformal transformations of two-dimensional conformal field theory. In the following, it will be helpful to define $\gamma_{x,z} = 1 - 2b\cdot x + b^2(x^2+Z^2)$.

Let $\mathcal{O}(X,T)$ be a CFT primary operator with conformal dimension Δ. The dual bulk scalar operator according to the prescription of [6] is

$$\begin{aligned}
\phi(Z,X,T) &= \int dx'dt' K_\Delta(Z,X,T|X',T')\mathcal{O}(X+ix',T+t') \\
&= \frac{\Delta-1}{\pi}\int_{x'^2+t'^2<Z^2} dx'dt'\left(\frac{Z^2-x'^2-t'^2}{Z}\right)^{\Delta-2} \\
&\quad \times \mathcal{O}(X+ix',T+t').
\end{aligned} \tag{C2}$$

Correlators of this operator with other CFT primary operators transform covariantly under the group $SO(2,2)$. To see this consider acting with such a transformation on the mixed bulk-boundary correlator

$$\begin{aligned}
&\left\langle \phi(Z,X,T)\prod_k \mathcal{O}_k(x_k^\mu)\right\rangle \\
&\quad = \frac{\Delta-1}{\pi}\int_{x'^2+t'^2<Z^2} dx'dt'\left(\frac{Z^2-x'^2-t'^2}{Z}\right)^{\Delta-2} \\
&\qquad \times \left\langle \mathcal{O}(X+ix',T+t')\prod_k \mathcal{O}_k(x_k^\mu)\right\rangle.
\end{aligned} \tag{C3}$$

The expression is manifestly dilatation covariant, and Lorentz invariant, so it remains to check special conformal transformations (C1). The CFT correlator transforms covariantly under such a transformation. We wish to check whether

$$\left\langle \phi(\tilde Z,\tilde X,\tilde T)\prod_k \mathcal{O}_k(\tilde x_k^\mu)\right\rangle = \left\langle \phi(Z,X,T)\prod_k \mathcal{O}_k(x_k^\mu)\right\rangle \prod_j \gamma_{x_j,0}^{\Delta_j}, \tag{C4}$$

where $\tilde Z$, etc. are related to Z, etc. via the transformation (C1). Using (C3) the left-hand side of (C4) is

$$\left\langle \phi(\tilde{Z}, \tilde{X}, \tilde{T}) \prod_k \mathcal{O}_k(\tilde{x}_k^\mu) \right\rangle = \prod_k \gamma_{x_k,0}^{\Delta_k} \int_{a^2+b^2<\tilde{Z}^2} dadb \left(\frac{\tilde{Z}^2 - a^2 - b^2}{\tilde{Z}} \right)^{\Delta-2} \times \gamma_{Y,0}^{\Delta} \left\langle \mathcal{O}(Y^\mu) \prod_k \mathcal{O}_k(x_k^\mu) \right\rangle \tag{C5}$$

using covariance of the CFT correlator, and defining $A = (ia, b)$ which are related to new dummy variables x'', y'' and $Y^\mu = (X + ix'', T + t'')$ by a special conformal transformation

$$(\tilde{x} + A)^\mu = \frac{Y^\mu - b^\mu Y^2}{1 - 2b \cdot Y + b^2 Y^2}. \tag{C6}$$

Now

$$\frac{\tilde{Z}^2 - a^2 - b^2}{\tilde{Z}} = \frac{1}{\gamma_{Y,0}} \frac{Z^2 - x''^2 - t''^2}{Z}$$

and

$$dadb = \frac{1}{\gamma_{Y,0}^2} dx'' dt''.$$

We therefore find that the $\gamma_{Y,0}$ factors in the integrand of (C5) cancel. However one must bear in mind that the change of variables (C6) makes x'' and t'' complex, though the surface of integration is still bounded by the locus $Z^2 - x''^2 - t''^2 = 0$. For infinitesimal transformations, it is clear the integral can be viewed as a double contour integration, and each contour can be deformed back to the disc, where x'' and t'' are real. Therefore we can finally switch dummy variables and recover (C4). Thus we conclude the mixed bulk-boundary correlator (C3) transforms covariantly under $SO(2, 2)$.

APPENDIX D: GENERAL BULK / BOUNDARY THREE-POINT FUNCTION

Let us consider the mixed bulk/boundary three-point function with a bulk operator (dual to operator of conformal weight Δ) and two boundary operators with conformal weights Δ_1 and Δ_2. We use the results of Appendix C to first express the general functional form of the correlator. A key point to note is that a cross-ratio, invariant under dilatations and special conformal transformations, can be constructed using two boundary points and one bulk point

$$\chi = \frac{((\vec{x} - \vec{x}_1)^2 + Z^2)((\vec{x} - \vec{x}_2)^2 + Z^2)}{Z^2(\vec{x}_2 - \vec{x}_1)^2}. \tag{D1}$$

Let us define

$$\langle \phi_\Delta(Z, \vec{x}) \mathcal{O}_{\Delta_1}(\vec{x}_1) \mathcal{O}_{\Delta_2}(\vec{x}_2) \rangle = c(Z, \vec{x}; \vec{x}_1; \vec{x}_2). \tag{D2}$$

Using dilatations, rotations, translations and special conformal transformations, the general three-point function may be fixed to be of the form

$$c(Z, \vec{x}; \vec{x}_1; \vec{x}_2) = |\vec{x}_1 - \vec{x}_2|^{-(\Delta_1+\Delta_2-\Delta)} \left[\frac{Z^2 + (\vec{x} - \vec{x}_1)^2}{Z} \right]^{-(\Delta+\Delta_1-\Delta_2)/2} \times \left[\frac{Z^2 + (\vec{x} - \vec{x}_2)^2}{Z} \right]^{-(\Delta+\Delta_2-\Delta_1)/2} f(\chi). \tag{D3}$$

The form of the function $f(\chi)$ may be fixed by performing a conformal transformation to send point $\vec{x}_1 \to 0$ and $\vec{x}_2 \to \infty$ and comparing to the results of Sec. V. In this limit

$$\chi \to \frac{|\vec{x}|^2 + Z^2}{Z^2}$$

so

$$c(Z, \vec{x}; 0; \infty) = Z^\Delta (|\vec{x}|^2 + Z^2)^{-(\Delta+\Delta_1-\Delta_2)/2} f\left(\frac{|\vec{x}|^2 + Z^2}{Z^2} \right), \tag{D4}$$

which should be matched with (41). This fixes

$$f(\chi) = \frac{1}{2\pi R} \left(\frac{\chi}{\chi - 1} \right)^{(\Delta+\Delta_1-\Delta_2)/2} {}_2F_1\left((\Delta + \Delta_{1|} - \Delta_2)/2, (\Delta + \Delta_1 - \Delta_2)/2; \Delta; \frac{1}{1-\chi} \right).$$

It is a nontrivial fact that this expression is symmetric under switching $1 \leftrightarrow 2$. This completes the derivation of the general three-point function.

The singularities of the three-point function occur at $\chi = 0, 1$. The locus $\chi = 0$ coincides with the bulk lightlike separations between the bulk point and one of the boundary points. However the locus $\chi = 1$ yields singularities at bulk spacelike separations in general. In the special limit when one of the boundary points moves off to infinity, this simply becomes the boundary light cone. More generally the position of the singularity is sensitive to the position of both boundary operators.

There are three interesting limits that (D4) may be expanded around. The basic CFT limit is extracted from the Z^Δ coefficient in the limit that $Z \to 0$ with $\vec{x}$ fixed. In this limit $f \to 1/2\pi R$ so the expected CFT behavior of $|\vec{x}|^{-(\Delta+\Delta_1-\Delta_2)}$ is recovered.

The OPE of the gravitational theory is recovered by expanding around the bulk light cone $\chi = 0$. This yields an expression of the form

$$c(Z, \vec{x}; 0; \infty) \sim c_1 Z^{\Delta_2-\Delta_1}(1 + \mathcal{O}(\chi)) + c_2 Z^{\Delta_2-\Delta_1} \chi^{\Delta_2-\Delta_1}(1 + \mathcal{O}(\chi)), \tag{D5}$$

where c_i are constants. The first term is analytic in χ, and hence respects bulk causality. The second term can lead to noncommutativity, but only at timelike bulk separations. Overall, the bulk OPE is of the form expected from Wilson's original paper [26], namely, an expansion in a function of the bulk geodesic distance.

The problem we need to attend to comes from examining the correlator (D4) around the locus $\chi = 1$. When point two is at infinity, this corresponds to boundary lightlike separations of points zero and one. More generally, this locus simply corresponds to bulk spacelike or timelike separations, depending on the position of operator two. Expanding around this locus we find

$$c(Z, \vec{x}; 0; \infty) \sim Z^{\Delta_2 - \Delta_1} c_4 \log(\chi - 1)(1 + \mathcal{O}(\chi - 1)) + c_3(1 + \mathcal{O}(\chi - 1)).$$

The presence of the log term leads to noncommutativity at bulk spacelike separations. But as in Sec. V it can be canceled by adding higher-dimension operators.

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